ON THE BAD REDUCTION OF CERTAIN $U(2,1)$ SHIMURA VARIETIES

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Contents

Introduction

Let E be a quadratic imaginary field and let p be a prime which is inert in E. This paper is concerned with the detailed study of three types of Picard modular surfaces in positive characteristic p and the morphisms between them. Deferring precise definitions to the body of the paper, the first Picard surface, denoted S,

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parametrizes triples (A, ϕ, ι) comprised of a certain abelian threefold A with an action ι of the ring of integers $\mathcal{O}_E$, and a principal polarization ϕ. Unlike the other two, S is smooth. The second surface, $S_0(p)$, parametrizes, in addition, a suitably restricted choice of a subgroup $H \subset A[p]$ of rank p^2. The third Picard surface, $\widetilde{S}$, parametrizes triples (A, ψ, ι) similar to those parametrized by S, but where ψ is a polarization of degree p^2. There are natural morphisms providing us with a diagram

$$\tilde{S} \xleftarrow{\ \tilde{\pi}\ } S_0(p) \xrightarrow{\ \pi\ } S.$$

From another perspective, there are three Shimura varieties associated with the unitary group of E of signature (2,1), having parahoric level structure at p. The above mentioned moduli spaces are the special fibers at p of the integral models of these Shimura varieties, studied by Rapoport and Zink in [Ra-Zi].

Before describing the main results of this article, we provide some background, context and motivation. Picard modular surfaces appear in many places in the literature; the book by Langlands and Ramakrishnan [La-Ra] provides a strong motivation for their study as a test case for the Langlands conjectures on modularity of L-functions, as well as a guide to the literature at the time. The local structure at p of $S_0(p)$ and related moduli spaces was studied in Bellaïche's thesis [Bel], and later in the work of Bültel-Wedhorn [Bu-We] and Koskivirta [Kos], where the authors applied it to lifting problems of Picard modular forms, Galois representations, and congruence relations for Hecke operators. However, the global structure of $S_0(p)$ and of the map $S_0(p) \to S$ remained opaque. Thus, one of our original motivations was to make this global structure precise.

Unlike $S_0(p)$, there is little information in the literature on $\widetilde{S}$, or in general on moduli spaces of abelian varieties with non-separable polarizations. The main examples we are aware of are [Cri, dJ1, Nor, N-O], and they tend to exhibit rather pathological phenomena. It is desirable to have additional examples available, and indeed $\widetilde{S}$, in contrast to loc. cit., has proven to be extremely well-behaved.

Our main reason for studying the three Picard modular surfaces, was however different. Motivated by questions on the canonical subgroup, or by the search for a geometric proof of the congruence relation (as in [Bu-We, Kos]), it is desirable to have a surface parametrizing tuples (A, ϕ, ι, H), where H is a finite flat subgroup scheme which may reduce mod p to the kernel of Frobenius. As this kernel has rank p^3 and in characteristic 0 the rank of a p-primary $\mathcal{O}_E$-subgroup must be an even power of p, such a surface does not exist. To remedy the situation, one is forced to consider a moduli space as above, but where H is now of rank p^6. In the context of modular curves this is akin to passing from $X_0(p)$ to $X_0(p^2)$; a process which is, of course, unnecessary for modular curves, but would be required for many Shimura varieties.

It turns out that it is beneficial to modify the moduli problem somewhat and following [dJ2] to consider a filtration of H as part of the datum. That is, (roughly) the following data: $(A, \phi, \iota, H_0 \subseteq H)$, where (A, ϕ, ι, H_0) is an object parametrized by $S_0(p)$ and H is a suitable rank p^6 finite flat subgroup scheme. We call this moduli problem $\mathcal{T}$, and one of our initial observations is that

$$\mathcal{T} \cong S_0(p) \times_{\widetilde{S}} S_0(p).$$

In characteristic 0, this surface is finite flat of degree $(p+1)(p^3+1)$ over S, and represents the Hecke operator $T(p)$. This, therefore, motivated both the introduction of $\widetilde{S}$ and the study of the morphism $\widetilde{\pi}$. The study of the moduli space $\mathcal{T}$ will be carried out in a subsequent paper. Nonetheless, the foundations are laid down here.

While studying the three moduli spaces $S, S_0(p)$ and $\widetilde{S}$, we discovered a new interesting phenomenon. The generic stratum of S in characteristic p parametrizes μ-ordinary abelian threefolds. Although their p-divisible groups are all isomorphic, studying their cotangent spaces we were able to distinguish in the tangent space of S a certain "foliation", amounting in this very simple example to a line sub-bundle closed under the operation of raising to power p (see §2.2). The link between the cotangent space of the universal abelian variety and that of S is supplied by the Kodaira-Spencer map. This foliation extends to the general supersingular locus of S, but fails to extend, in a way made precise, to the superspecial points there. Moreover, we found two other ways to characterize it: the first, as the foliation of "unramified directions" (in the sense of [Ru-Sh]) for a map $\overline{\pi} : S_0(p)^{(p)} \to S$ derived from the map π (Theorem 4.4). The second, in terms of Moonen's generalized Serre-Tate coordinates [Mo] (Proposition 2.4). Shimura curves embedded in S, as well as the supersingular curves in S, are integral curves of this foliation (Theorem 2.3). Does it have any other global integral curves? We expect this new phenomenon to generalize to other Shimura varieties of PEL type whose generic stratum is μ-ordinary but not ordinary.

A summary of the results. We now describe briefly the content of this paper. Chapter 1 reviews the three Shimura varieties and their integral models. We explain the precise relation between the moduli problem with parahoric level structure as in [Ra-Zi] and the Raynaud condition appearing in [Bel]. The last section reviews the embeddings of modular curves and Shimura curves in the Picard modular surface.

Chapter 2 deals with the Picard modular surface S, where the level at p is a hyperspecial maximal compact. The mod p fiber is smooth, and its stratification was studied by Vollaard in [Vo]. It consists of three strata. The dense open stratum S_μ parametrizes μ-ordinary abelian threefolds. Its complement S_{ss} parametrizes supersingular ones, and consists (at least when the tame level N is large, depending on p) of Fermat curves of degree $p+1$, intersecting transversally at their $\mathbb{F}_{p^2}$-rational points. These intersection points support superspecial abelian threefolds (isomorphic, not only isogenous, to a product of supersingular elliptic curves), and constitute the third stratum S_{ssp}. The non-singular locus of the curve S_{ss} supports supersingular, but not superspecial, abelian threefolds, and is denoted S_{gss}. This is the intermediate stratum. The number of its irreducible components was determined in [dS-G1] using intersection theory on S and a secondary Hasse invariant constructed there. It turns out to be related to the second Chern number of S, and via a result of Holzapfel, expressible as an L-value. Our contribution to the study of S in the present paper is: (a) We introduce the foliation TS^+ in the tangent bundle of S, outside S_{ssp}, and prove the results to which we alluded above; (b) We introduce the blow-up $S^\#$ of S at S_{ssp} and give it a modular interpretation. It has the advantage that the irreducible components of S_{ss} become, after blowing up, disjoint non-singular Fermat curves (even when N is small), i.e. all their intersections, including self-intersections, are resolved. The exceptional divisor at every blown-up point x is a projective line E_x defined over $\mathbb{F}_{p^2}$. The components of S_{ss} intersect E_x at points ζ satisfying $\zeta^{p+1} = -1$. Embedded Shimura curves, on the other hand, intersect E_x at $\mathbb{F}_{p^2}$-rational points satisfying $\zeta^{p+1} \neq -1$. The proofs of these results will have to wait until Theorem 4.11 and §4.3.3.

Chapter 3 is based on chapter III of Bellaïche's thesis [Bel] and describes the local models for the completed local rings of the three Shimura varieties, at any point of the special fiber. We are nevertheless interested not only in the completed local rings *per se,* but in the maps between them. The theory of local models yields these maps only modulo pth powers of the maximal ideal. This is evident already in the case of the germ of the map $X_0(p) \to X$ between two modular curves, with and without $\Gamma_0(p)$-level structure, at a supersingular point. In this "baby case" the map between the local models is

$$k[[x]] \hookrightarrow k[[x,y]]/(xy),$$

which is not even flat. The correct map, however, is known ever since Kronecker to be

$$k[[x]] \hookrightarrow k[[x,y]]/((x^p - y)(x - y^p)),$$

which is finite flat of degree $p+1$. Similar but more serious problems arise when we study the maps between the completed local rings of our three Picard surfaces. Luckily, a general theorem of Rudakov and Shafarevich [Ru-Sh] on the local structure of inseparable maps between smooth surfaces, allows us to give a partial answer to our question. In essence, it allows us to determine the maps between the completed local rings of the *analytic branches* through any given point. Once again, results of this type have to await the study of $S_0(p)$ and $\widetilde{S}$ in subsequent chapters, where we relate them also to the foliation TS^+ mentioned above.

Chapter 4 is the longest, and deals with the Picard surface $S_0(p)$ of Iwahori level structure, and the map π from $S_0(p)$ to S. We caution that π is neither finite nor flat. The special fiber of $S_0(p)$ consists of vertical and horizontal components intersecting transversally. There are two horizontal components, multiplicative and étale. The multiplicative component maps under π isomorphically onto $S^{\#}$. The map from the étale component is purely inseparable of degree p^3 and factors through Frobenius. The factored map $\overline{\pi}_{et}$ is inseparable of degree p, and we show that its "field of unramified directions" is just the foliation TS^+, which was defined before intrinsically on S. The vertical components of π are $\mathbb{P}^1$-bundles over Fermat curves, which we call the "supersingular screens". Above each superspecial point $x \in S_{ssp}$ lies in $S_0(p)$ a "comb", whose base F_x is a $\mathbb{P}^1$ along which the two horizontal sheets of $S_0(p)$ meet, and whose "teeth" $G_x[\zeta]$ belong to the supersingular screens. For a more precise description we refer to Theorems 4.1, 4.5 and 4.11 and their corollaries.

Chapter 5 deals with $\widetilde{S}$ and the map $\widetilde{\pi}$. Unlike π, this map is finite flat of degree $p+1$. Here again there are horizontal and vertical components. This time $\widetilde{\pi}$ is an isomorphism on the étale component of $S_0(p)$ and purely inseparable of degree p on the multiplicative component. The maps π and $\widetilde{\pi}$ allow us to go back and forth between S and $\widetilde{S}$ and produce maps that we are able to analyze easily in light of the modular interpretation. On the vertical components of $S_0(p)$ (the supersingular screens) the map $\widetilde{\pi}$ is pretty intricate. We collect some results on it in the last section of Chapter 5, but leave some other questions unanswered.

The appendix contains some ugly but unavoidable computations with Dieudonné modules, that would have interrupted the presentation, had they been left where needed.

Deformation theory of p-divisible groups clearly is a central tool in this work. Unfortunately, there are at least three traditional approaches to it: Grothendieck's theory of crystals, contravariant Dieudonné theory, and covariant Dieudonné-Cartier theory (not counting displays, p-typical curves etc.). We made every effort to remain faithful to the language and notation used by the various references cited by us. This resulted, however, in a mixture of the three approaches. A very useful guide, and a dictionary between the various languages, can be found in the appendix to [C-C-O].

Notation

- If A is an abelian scheme over a base S, A^t denotes its dual abelian scheme.
- If H is a finite flat group scheme over a base S, H^D denotes its Cartier dual.
- If S is a scheme over $\mathbb{F}_p$ we denote by $\Phi_S : S \to S$ the absolute Frobenius morphism of S. If $X \to S$ is any scheme, we denote by $X^{(p)/S}$, or simply by $X^{(p)}$, if no confusion may arise, the fiber product

$$X^{(p)} = S \times_{\Phi_S, S} X$$

  and by $Fr_{X/S} : X \to X^{(p)}$ the unique morphism over S such that

$$(\Phi_S \times 1) \circ Fr_{X/S} = \Phi_X.$$

- If A is an abelian scheme over S then $\mathrm{Fr} = Fr_{A/S} : A \to A^{(p)}$ is an isogeny (the Frobenius of A). The Verschiebung $\mathrm{Ver} : A^{(p)} \to A$ is the isogeny dual to the Frobenius of A^t.
- If $\lambda : A \to A^t$ is a polarization of an abelian scheme A and $K = \ker \lambda$, we denote by $e_\lambda : K \times K \to \mathbb{G}_m$ the Mumford pairing on K. If $\lambda = n\phi$ where ϕ is a principal polarization, then e_λ is Weil's e_n-pairing associated with ϕ.
- E is a quadratic imaginary field, $\mathcal{O}_E$ its ring of integers, p a prime that remains inert in E, $\kappa = \mathcal{O}_E/p\mathcal{O}_E$ and $\mathcal{O}_p$ is the completion of $\mathcal{O}_E$ at p. We write σ for the non-trivial automorphism of E, extended to $\mathcal{O}_p$.
- If R is an $\mathcal{O}_p$-algebra we denote by Σ the given homomorphism $\mathcal{O}_p \to R$ and $\overline{\Sigma} = \Sigma \circ \sigma$.
- If G is a commutative group scheme over a base S we denote by $\mathcal{O}_E \otimes G$ the S-group scheme representing the functor $S' \mapsto \mathcal{O}_E \otimes_{\mathbb{Z}} G(S')$. It has an obvious $\mathcal{O}_E$ action.
- If X is a non-singular algebraic variety over a field k we denote its tangent bundle by TX. The fiber of TX at $x \in X(k)$ (the tangent space at x) will be denoted by $T_xX = TX|_x$.
- If X is any scheme we denote by X^{red} the same underlying space, equipped with the reduced induced subscheme structure.

1. Three integral models with Parahoric level structure

1.1. **Shimura varieties.** Let E be a quadratic imaginary field. Let $\Lambda = \mathcal{O}_E^3$, equipped with the hermitian form

$$(u,v) = {}^t\bar{u} \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} v,$$

which is of signature $(2,1)$ over $\mathbb{R}$. We denote by e_0, e_1, e_2 the three standard basis vectors. Let $\boldsymbol{G}$ be the group of unitary similtudes $GU(\Lambda, (,))$, regarded as a linear algebraic group over $\mathbb{Z}$. The Shimura varieties in the title will be associated with $\boldsymbol{G}$. More precisely, $G_\infty = \boldsymbol{G}(\mathbb{R})$ acts by projective linear transformations on $\mathbb{P}^2(\mathbb{C})$. The bounded symmetric domain

$$\mathscr{D} = \{(z_0 : z_1 : z_2) |\ \overline{z}_0 z_2 + \overline{z}_1 z_1 + \overline{z}_2 z_0 < 0\},$$

biholomorphic to the unit ball in $\mathbb{C}^2$, is preserved by G_∞, which acts on it transitively. Denote by K_∞ the stabilizer of the "center" $(-1:0:1)$. For any compact open subgroup $K_f \subset \boldsymbol{G}(\mathbb{A}_f)$ we put $K = K_\infty K_f \subset \boldsymbol{G}(\mathbb{A})$.

The Shimura variety S_K is a quasi-projective variety over E whose complex points are identified, as a complex manifold, with

$$S_K(\mathbb{C}) \simeq \boldsymbol{G}(\mathbb{Q}) \setminus \boldsymbol{G}(\mathbb{A})/K \simeq \boldsymbol{G}(\mathbb{Q}) \setminus [\mathscr{D} \times \boldsymbol{G}(\mathbb{A}_f)/K_f].$$

Fix an odd prime p which is inert in E, and let $N \geq 3$ be an integer such that $p \nmid N$. Let $\kappa = \mathcal{O}_E/p\mathcal{O}_E$ and denote by $\mathcal{O}_p$ the ring of integers in the completion E_p. Assume that $K_f = K_p K^p$ where $K^p \subset \boldsymbol{G}(\mathbb{A}_f^p)$ is the principal level subgroup of level N, and $K_p \subset G_p = \boldsymbol{G}(\mathbb{Q}_p)$.

In this paper we are interested in three choices of K_p. As p is inert in E, G_p is non-split, and its semi-simple rank is 1. Its Bruhat-Tits building is a biregular tree of bi-degree $(p^3+1, p+1)$. The vertices of degree p^3+1 are stabilized by hyperspecial maximal compact subgroups of G_p, which are all conjugate to $K_p^0 = \boldsymbol{G}(\mathbb{Z}_p)$. This subgroup is the stabilizer of the standard self-dual lattice

$$\Lambda_0 = \Lambda \otimes \mathbb{Z}_p = \langle e_0, e_1, e_2 \rangle_{\mathcal{O}_p}. \tag{1.1}$$

The vertices of degree $p+1$ are stabilized by special, but not hyperspecial, maximal compact subgroups, which are all conjugate to the stabilizer $\widetilde{K}_p^0$ of the lattice

$$\Lambda_1 = \langle pe_0, e_1, e_2 \rangle_{\mathcal{O}_p}.$$

Note that this is also the stabilizer of $p^{-1}\Lambda_2$, the *dual* lattice with respect to the hermitian pairing, where

$$\Lambda_2 = \langle pe_0, pe_1, e_2 \rangle_{\mathcal{O}_p}.$$

We call the vertices of degree p^3+1 vertices of type (hs) and the ones of degree $p+1$ of type (s). The vertices v_0 and $\widetilde{v}_0$ corresponding to K_p^0 and $\widetilde{K}_p^0$ are called the *standard* vertices of the respective types. The oriented edge $(v_0, \widetilde{v}_0)$ is then stabilized by the standard Iwahori subgroup

$$K_p^1 = K_p^0 \cap \widetilde{K}_p^0.$$

We denote by S (resp. $\widetilde{S}$, resp. $S_0(p)$) the Shimura variety over E of level $K_f = K_p K^p$, where K^p is as above (of full tame level N) and $K_p = K_p^0$ (resp. $\widetilde{K}_p^0$, resp. K_p^1). The following result is well-known.

Proposition 1.1. *The Shimura varieties S, $\widetilde{S}$ and $S_0(p)$ are non-singular quasi-projective surfaces over E and the natural maps*

$$\pi : S_0(p) \to S,\ \ \widetilde{\pi} : S_0(p) \to \widetilde{S}$$

are finite étale of degrees p^3+1 and $p+1$ respectively.

We denote by $\mathscr{S}$ (resp. $\widetilde{\mathscr{S}}$, resp. $\mathscr{S}_0(p)$) the integral models of these varieties over $\mathcal{O}_p$ constructed in chapter 6 of [Ra-Zi]. They are of relative dimension 2, $\mathscr{S}$ is smooth over $\mathcal{O}_p$, but the other two are not. The relative surface $\mathscr{S}$ is the integral model of the Picard modular surface which has been studied in detail by Vollaard [Vo] §§4-6. See [dS-G1] for related results. The surface $\mathscr{S}_0(p)$ has been studied to some extent in Bellaïche's thesis [Bel]. Previous to this paper, little was known about $\widetilde{\mathscr{S}}$, apart from the general facts that follow from [Ra-Zi]. We review these three integral models in the next section.

From a general theorem of Görtz [Gö], or from the computations of the local models cited in §3.2, it follows that all three integral models are *flat* over $\mathcal{O}_p$, and their *special fibers are reduced*. As we shall later show, they are also *regular*.

1.2. **The moduli problems.**

1.2.1. *The Raynaud condition.* Let R be a commutative $\mathcal{O}_p$-algebra and H a finite flat group scheme over R of rank p^2. Assume that we are given a ring homomorphism $\iota : \mathcal{O}_E \to \mathrm{End}_R(H)$, and that H is killed by p, or, equivalently, ι factors through the field $\kappa = \mathcal{O}_E/p\mathcal{O}_E$. Locally on $Spec(R)$, $\mathcal{O}(H) = A$ is free of rank p^2 over R, and the zero section of H is given by an R-homomorphism $\epsilon : A \to R$ whose kernel I, the augmentation ideal, is free of rank p^2-1. Letting $a \in \kappa^\times$ act on A via $\iota(a)^*$, this becomes a group action, which preserves I. Let $\omega : \kappa^\times \to \mathcal{O}_p^\times \to R^\times$ be the Teichmüller character, and for $1 \le i \le p^2-1$ let

$$I^{(i)} = \{f \in I |\, \forall a \in \kappa^\times,\ \iota(a)^*(f) = \omega^i(a) f\}.$$

Thanks to the fact that p^2-1 is invertible in R, these are distinct R-submodules, and I is their direct sum. Following [Bel, Ray], we call H *Raynaud* if each $I^{(i)}$ is free of rank 1 over R. The following facts are easily checked.

- Let $R \to R'$ be any base change. Then if H is Raynaud, so is $H \times_{Spec(R)} Spec(R')$.
- The converse holds if $Spec(R)$ is connected. In particular, it is enough to check then the Raynaud condition at one geometric point.
- The constant group scheme $\mathcal{O}_E \otimes \mathbb{Z}/p\mathbb{Z}$ and its dual $\mathcal{O}_E \otimes \mu_p$ are Raynaud.

It follows from the three properties that étale and multiplicative (dual to étale) group schemes are automatically Raynaud.

Assume now that $R = k$ is a perfect field containing κ. Let $M = M(H)$ be the covariant Dieudonné module[1] of H. Since H is killed by p, M is a 2-dimensional vector space over k, equipped with linear maps

$$F\colon M^{(p)} \to M,\ V\colon M \to M^{(p)},$$

where $M^{(p)} = k \otimes_{\sigma,k} M$ and $\sigma(x) = x^p$ is the Frobenius on k. The action of κ on H induces an action of κ on M; we let $M(\Sigma)$ be the subspace on which κ acts through

[1]We adhere to the conventions of [C-C-O], Appendix B.3. Our $M(H)$ is denoted there $M_*(H)$. F and V can be regarded also as σ or σ^{-1}-linear maps on M. Recall that V is induced by $\mathrm{Fr} : H \to H^{(p)}$ and F is induced by $\mathrm{Ver} : H^{(p)} \to H$.

the natural embedding $\Sigma : \kappa \hookrightarrow k$, and $M(\overline{\Sigma})$ the subspace on which it acts via $\overline{\Sigma} = \sigma \circ \Sigma$. Then $M = M(\Sigma) \oplus M(\overline{\Sigma})$. Note that $(M^{(p)})(\Sigma) = (M(\overline{\Sigma}))^{(p)}$ and vice versa. We call M *balanced* if both $M(\Sigma)$ and $M(\overline{\Sigma})$ are 1-dimensional.

Lemma 1.2. *H is Raynaud if and only if $M(H)$ is balanced.*

Proof. We may assume that k is algebraically closed, as both conditions are invariant under passage to an algebraic closure. If H is étale, it is constant, and must then be isomorphic, with the $\mathcal{O}_E$ action, to $\mathcal{O}_E \otimes \mathbb{Z}/p\mathbb{Z}$, whose Dieudonné module is evidently balanced. Similarly, if H is multiplicative.

There remains the local–local case. As a *scheme,* stripped of the group structure, H is then either (i) $Spec(k[X]/(X^{p^2}))$ or (ii) $Spec(k[X,Y]/(X^p, Y^p))$, where the second case occurs if and only if H is killed by the Frobenius morphism $\mathrm{Fr}\colon H \to H^{(p)}$. Since I is of codimension 1 and, in the local case, also nilpotent, it coincides with the maximal ideal of $A = \mathcal{O}(H)$. The cotangent space at the origin, I/I^2, is then $k\overline{X}$ in case (i) and $k\overline{X} \oplus k\overline{Y}$ in case (ii).

In case (i) κ may act on the one-dimesional I/I^2 by Σ or $\overline{\Sigma}$, and so does the group $\kappa^\times$ act. Either way, $\kappa^\times$ acts on I^i/I^{i+1} $(1 \le i \le p^2 - 1)$ via Σ^i (or $\overline{\Sigma}^i$), so every character $\omega^i : \kappa^\times \to k^\times$ must occur in I with multiplicity 1, and H is automatically Raynaud. But in case (i) we also have an exact sequence of finite flat $\mathcal{O}_E$-group schemes

$$0 \to H_1 \to H \overset{\mathrm{Fr}}{\to} H_1^{(p)} \to 0.$$

Here $H_1 = \ker(\mathrm{Fr}\colon H \to H^{(p)})$ is a subgroup scheme of rank p, and $H_1^{(p)}$ is its image. It follows that in case (i) $M(H)$ is an extension of $M(H_1)^{(p)}$ by $M(H_1)$, so is automatically balanced.

Case (ii) is the only case where the "balanced" condition may fail. In this case Fr annihilates H so $V = 0$ on $M = M(H)$ and

$$\mathrm{Lie}(H) = M[V] = M$$

(see [C-C-O], B.3.5.6-3.5.7). We find that M is balanced if and only if $\mathrm{Lie}(H)$, equivalently its dual I/I^2, is balanced. If this is the case, i.e. both Σ and $\overline{\Sigma}$ occur in I/I^2, we may choose the variables X and Y so that $\kappa^\times$ acts on X via ω and on Y via ω^p, so on X^iY^j $(i, j < p$, not both 0) it acts via ω^{i+jp} and every character occurs with multiplicity 1 in I. Thus H is Raynaud in this case. If, on the contrary, I/I^2 is $\kappa^\times$-isotypical, we can not have $\dim I^{(i)} = 1$ for every i, and H is not Raynaud. □

Let H^D denote the Cartier dual of H, which is also finite flat of rank p^2, and endow it by an $\mathcal{O}_E$-action $\iota^D : \mathcal{O}_E \to \mathrm{End}_R(H^D)$ via the formula

$$\iota^D(a) = \iota(a)^t,$$

i.e. for any R-algebra R' and any $x \in H(R')$, $y \in H^D(R')$,

$$\langle x, \iota^D(a)y \rangle = \langle \iota(a)x, y \rangle \in (R')^\times.$$

Corollary 1.3. *H is Raynaud if and only if H^D is Raynaud.*

Proof. $M(H^D)$ (identified with the *contravariant* Dieudonné module of H) is the k-linear dual of $M(H)$, so one is balanced if and only if the other is. □

1.2.2. *The moduli problem* (S). We now define the three integral models for the Shimura varieties with parahoric level structure at p as moduli schemes for moduli problems of PEL type. It is well known and easy to check that in the generic fiber these moduli problems yield the given Shimura varieties. For the relation with the models defined by Rapoport and Zink, and the representability of the moduli problems, see 1.3 below.

The Picard modular surface S has a smooth integral model $\mathscr{S}$ over $\mathcal{O}_p$. It is a fine moduli scheme for the moduli problem which assigns to each $\mathcal{O}_p$-algebra R isomorphism classes of tuples $\underline{A} = (A, \phi, \iota, \eta)$, where

- A is an abelian 3-fold over R.
- $\phi\colon A \xrightarrow{\sim} A^t$ is a principal polarization.
- $\iota\colon \mathcal{O}_E \to \mathrm{End}_R(A)$ is a ring homomorphism, such that the Rosati involution induced by ϕ on $\mathrm{End}_R(A)$ preserves its image, and is given on it by $\iota(a) \mapsto \iota(\overline{a})$. We furthermore require that $\mathrm{Lie}(A)$ becomes an $\mathcal{O}_E$-module of type $(2,1)$ in the sense that it is the direct sum of a locally free R-module of rank 2 on which $\iota(a)_*$ acts like the image of a in R, and a locally free rank 1 module on which it acts like $\overline{a}$.
- $\eta{:}\Lambda/N\Lambda \simeq A[N]$ is a full level-N $\mathcal{O}_E$-structure (recall $p \nmid N \geq 3$).

Our reference to moduli problems and representability is the comprehensive volume by Lan. In particular, we refer the reader to the precise definition of level structure given there ([Lan] 1.3.6.2), and to the condition of *étale liftability*. In addition to being compatible with the $\mathcal{O}_E$-action, η should carry the polarization pairing

$$\langle , \rangle : \Lambda/N\Lambda \times \Lambda/N\Lambda \to \mathbb{Z}/N\mathbb{Z}$$

derived from $(,)$ to the Weil e_N-pairing induced by ϕ on $A[N] \times A[N]$. Part of the data involved in η is an isomorphism between the (étale) target groups of the two pairings: $\nu_N : \mathbb{Z}/N\mathbb{Z} \simeq \mu_N$, making the last condition meaningful. These isomorphisms form a torsor $\underline{\mathrm{Isom}}(\mathbb{Z}/N\mathbb{Z}, \mu_N)$ under $(\mathbb{Z}/N\mathbb{Z})^\times$, and in this way ν_N becomes a morphism form $\mathscr{S}$ to $\underline{\mathrm{Isom}}(\mathbb{Z}/N\mathbb{Z}, \mu_N)$, regarded as a scheme over $\mathcal{O}_p$ of relative dimension 0. We call ν_N the *multiplier morphism*.

1.2.3. *The moduli problem* $(\widetilde{S})$. The Shimura variety $\widetilde{S}$ has an integral model $\widetilde{\mathscr{S}}$ over $\mathcal{O}_p$. It is a fine moduli scheme for the moduli problem which assigns to each $\mathcal{O}_p$-algebra R isomorphism classes of tuples $\underline{A}' = (A', \psi, \iota', \eta')$, where

- A' is an abelian 3-fold over R.
- $\psi\colon A' \to A'^t$ is a polarization of degree p^2.
- $\iota'\colon \mathcal{O}_E \to \mathrm{End}_R(A')$ is a ring homomorphism, satisfying the same requirements as for (S). In addition, we require that $\ker(\psi)$ is preserved by $\iota'(\mathcal{O}_E)$ and is Raynaud.
- η' is a full level-N $\mathcal{O}_E$-structure.

1.2.4. *The moduli problem* $(S_0(p))$. The Shimura variety $S_0(p)$ has an integral model $\mathscr{S}_0(p)$ over $\mathcal{O}_p$. It is a fine moduli scheme for the moduli problem which assigns to each $\mathcal{O}_p$-algebra R isomorphism classes of tuples $(\underline{A}, H) = (A, \phi, \iota, \eta, H)$, where

- $\underline{A}$ is as in (S)
- $H \subset A[p]$ is a Raynaud $\mathcal{O}_E$-subgroup scheme of rank p^2, which is isotropic for the Weil pairing e_p (the Mumford pairing $e_{p\phi}$ attached to the polarization $p\phi$).

1.2.5. *The maps between the integral models.* There are projection maps

$$\pi : \mathscr{S}_0(p) \to \mathscr{S}, \quad \widetilde{\pi} : \mathscr{S}_0(p) \to \widetilde{\mathscr{S}}$$

extending the maps of Proposition 1.1. The map π is neither finite, nor flat anymore. On the moduli problem, it is simply "forget H".

The second map $\widetilde{\pi}$ is defined as follows. Pick $(\underline{A}, H) \in \mathscr{S}_0(p)(R)$. Let $A' = A/H$. Since H is isotropic for $e_{p\phi}$, its annihilator in this pairing is a finite flat subgroup scheme $H \subset H^\perp \subset A[p]$ and $A[p]/H^\perp \simeq H^D$. We claim that $H^\perp/H$ is Raynaud. We may assume that $R = k$ is an algebraically closed field of characteristic p. As both H and H^D are Raynaud, $M(H)$ and $M(A[p]/H^\perp)$ are balanced. It follows that $M(H^\perp/H)$ is also balanced, so $H^\perp/H$ is Raynaud. The polarization $p\phi$ descends canonically to a polarization $\psi : A' \to (A')^t$ whose kernel is $\ker(\psi) = H^\perp/H$. Its degree is p^2. Finally ι' and η' are defined naturally from ι and η. To check that we obtained a point of $\widetilde{\mathscr{S}}$, we need only check one non-trivial[2] point, that $\mathrm{Lie}(A')$ is indeed of type $(2,1)$. This can be seen, using the Raynaud condition, as follows. We may assume again that $R = k$ is an algebraically closed field containing κ. The exact sequence

$$0 \to H \to A \to A' \to 0$$

yields, in covariant Dieudonné theory, exact sequences[3] and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & M(A) & \to & M(A') & \to & M(H) & \to & 0 \\ & & \downarrow V & & \downarrow V & & \downarrow V & & \\ 0 & \to & M(A)^{(p)} & \to & M(A')^{(p)} & \to & M(H)^{(p)} & \to & 0 \end{array}$$

where we have abbreviated $M(A) = M(A[p^\infty])$ etc. The snake lemma yields

$$\begin{array}{ccccccc} 0 & \to & M(H)[V] & \to & M(A)^{(p)}/VM(A) & \to \\ & & \| & & \| & \\ 0 & \to & \mathrm{Lie}(H) & \to & \mathrm{Lie}(A) & \to \end{array}$$

$$\begin{array}{cccccc} \to & M(A')^{(p)}/VM(A') & \to & M(H)^{(p)}/VM(H) & \to & 0 \\ & \| & & \| & & \\ \to & \mathrm{Lie}(A') & \to & M(H)^{(p)}/VM(H) & \to & 0 \end{array}$$

Thus the type of $\mathrm{Lie}(A')$ is also $(2,1)$ if and only if $M(H)[V]$ and $M(H)^{(p)}/VM(H)$ have the same type. But from the exact sequence

$$0 \to M(H)[V] \to M(H) \xrightarrow{V} M(H)^{(p)} \to M(H)^{(p)}/VM(H) \to 0$$

we see that this is the case if and only if $M(H)$ is balanced. We conclude that H being Raynaud is in fact a *necessary and sufficient* condition for $A' = A/H$ to be of type $(2,1)$ as well.

We shall see later that in contrast to π, the map $\widetilde{\pi}$ *is* finite flat of degree $p+1$.

If we denote by $f : A \to A'$ the canonical homomorphism with kernel H, and identify A'^t with $A/H^\perp$, then $f'^t : A'^t \to A^t$ has kernel $A[p]/H^\perp$ and

$$p\phi = f^t \circ \psi \circ f.$$

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[2]In characteristic 0, or if H is étale, this is obvious, because the Lie algebra is not changed, but in characteristic p the type of the Lie algebra may well change under an isogeny.

[3]*A guide for the perplexed:* the covariant Dieudonné modules of a finite flat group scheme (resp. p-divisible group) is defined as the *contravariant* Dieudonné module of its Cartier (resp. Serre) dual. From the exact sequence $0 \to H^D \to A'^t \to A^t \to 0$ we get the top row of the diagram.

1.2.6. *The moduli problem* $(\widetilde{S}_0(p))$. There is a fourth moduli problem that one can define. It turns out to be equivalent to $(S_0(p))$, yet useful for later calculations and for the study of the moduli problem $\mathscr{T}$ mentioned in the introduction.

The moduli problem $(\widetilde{S}_0(p))$ assigns to every $\mathcal{O}_p$-algebra R isomorphism classes of tuples $(\underline{A}', J)$ where

- $\underline{A}'$ is as in $(\widetilde{S})$
- $J \subset A'[p]$ is a finite flat $\mathcal{O}_E$-subgroup scheme of rank p^4, containing $\ker(\psi)$, such that $J/\ker(\psi)$ is Raynaud, and which is maximal isotropic for the Mumford pairing $e_{p\psi}$.

Note that $\deg(p\psi) = p^8$.

Proposition 1.4. *The moduli problems* $(S_0(p))$ *and* $(\widetilde{S}_0(p))$ *are equivalent, hence* $(\widetilde{S}_0(p))$ *is also represented by* $\mathscr{S}_0(p)$.

Proof. To pass from $(\underline{A}, H)$ to $(\underline{A}', J)$ define

$$\underline{A}' = \underline{A}/H, \;\; J = A[p]/H,$$

and observe that $J/\ker(\psi) = A[p]/H^{\perp}$ is Raynaud, and that J is isotropic (hence, from degree considerations, maximal isotropic) for $e_{p\psi}$. To pass from $(\underline{A}', J)$ to $(\underline{A}, H)$ define $A = A'/J$, descend $p\psi$ to obtain a principal polarization ϕ on A, and let $H = A'[p]/J$. We leave to the reader the verification that we obtain a point of $(\widetilde{S}_0(p))$, as well as that these two constructions are inverse to each other. □

In terms of this new interpretation of $\mathscr{S}_0(p)$ the map $\widetilde{\pi}$ is simply "forget J".

Proposition 1.5. *The schemes* $\mathscr{S}, \mathscr{S}_0(p)$ *and* $\widetilde{\mathscr{S}}$ *are regular. They are flat over* $\mathcal{O}_p$ *and their special fibers are reduced. The maps* π *and* $\widetilde{\pi}$ *are surjective and proper.*

Proof. The "flat" and "reduced" assertions follow from the Main Result of [Gö], and from the fact that locally for the étale topology, a neighborhood of a point in the special fiber of $\mathscr{S}_0(p)$ or $\widetilde{\mathscr{S}}$ is isomorphic to an open neighborhood in the local model. Similarly, regularity follows from the determination of the completed local rings of the three schemes in [Bel] III.3.4.8. Although Bellaïche does not use the formalism of [Ra-Zi], he builds upon the earlier work of de Jong [dJ2], which except for the notation, yields identical results for the completed local rings as what one would get from the more general theory developed by Rapoport and Zink.

Properness and surjectivity of π and $\widetilde{\pi}$ are usually proved along with the proof of the representability of $\mathscr{S}_0(p)$. For the map π it is done in [Bel] III.3.2.3. For the map $\widetilde{\pi}$ the proof is similar, and we only sketch it. It is best described with the new interpretation of $\mathscr{S}_0(p)$ as representing the moduli problem $(\widetilde{S}_0(p))$. Consider first a larger moduli problem $(\widetilde{S}_0(p)')$ obtained from $(\widetilde{S}_0(p))$ by *relaxing* the Raynaud condition on $J/\ker(\psi)$. One proves, following de Jong, that this modified moduli problem is proper and surjective over $(\widetilde{S})$. Properness follows from the valuative criterion. The Raynaud condition is a closed condition, a fact which secures the properness of $\widetilde{\pi}$. Surjectivity clearly holds in the generic fiber. By [**?**], the generic fiber of $\widetilde{\mathscr{S}}$ is dense. Since $\widetilde{\pi}$ is already known to be proper, its image must be closed, hence is everything. □

1.2.7. *Diamond operators.* If $a \in (\mathcal{O}_E/N\mathcal{O}_E)^\times$ we denote by $\langle a \rangle$ the automorphism of $\mathscr{S}$, defined on the moduli problem by

$$\langle a \rangle (A,\phi,\iota,\eta) = (A,\phi,\iota,\eta\circ a) = (A,\phi,\iota,\iota(a)\circ\eta).$$

The same notation will be applied to the other moduli schemes.

1.3. **Translation into the language of Rapoport and Zink.** The moduli problems that we defined in the preceding sections are examples of the moduli problems defined in chapter 6 of [Ra-Zi], although the Raynaud condition is implicit there, as we shall now explain. It follows (from general results of Kottwitz) that, as has been claimed above, they are indeed representable by fine moduli schemes when $N \geq 3$. We remark that [Bel] gives an independent proof of the representability of $(S_0(p))$ by proving that it is *relatively representable* over (S).

Using the notation of [Ra-Zi] we take $B = E$, $\mathcal{O}_B = \mathcal{O}_E$, $V = E^3$ as before and $b^* = \bar{b}$. Let $\mathcal{L}, \widetilde{\mathcal{L}}$ and $\mathcal{L}_0(p)$ be the following *self-dual lattice chains* in V_p (see (1.1)):

$$\mathcal{L} = \{\cdots \subset p\Lambda_0 \subset \Lambda_0 \subset p^{-1}\Lambda_0 \subset \cdots\},$$

$$\widetilde{\mathcal{L}} = \{\cdots \subset p\Lambda_1 \subset \Lambda_2 \subset \Lambda_1 \subset p^{-1}\Lambda_2 \subset \cdots\},$$

$$\mathcal{L}_0(p) = \{\cdots \subset p\Lambda_0 \subset \Lambda_2 \subset \Lambda_1 \subset \Lambda_0 \subset p^{-1}\Lambda_2 \subset \cdots\}.$$

View the three lattice chains as categories, inclusions as morphisms. The moduli problem of type $(\mathcal{L})$, as defined in [Ra-Zi] Definition 6.9, is clearly our (S); just set $A = A_{\Lambda_0}$.

The moduli problem of type $(\widetilde{\mathcal{L}})$ is our $(\widetilde{S})$. Recall the definition of a "principally polarized $\widetilde{\mathcal{L}}$-set of abelian varieties of type $(2,1)$" over a base ring R as above ([Ra-Zi], Definition 6.6). First, one is given the $\widetilde{\mathcal{L}}$-set of abelian schemes $A_{\Lambda_\bullet}$ of type $(2,1)$. Then one gives the "principal polarization"[4] $\lambda : A_{\Lambda_\bullet} \simeq \widetilde{A}_{\Lambda_\bullet}$. Note that the $\widetilde{\mathcal{L}}$-set $\widetilde{A}_{\Lambda_\bullet}$ is of type $(1,2)$ because λ induces the Rosati involution on the endomorphism ring, hence switches types. We set

$$A' = A_{\Lambda_2}, \quad A'^t = A^t_{\Lambda_2} \simeq A^t_{p^{-1}\Lambda_2} = \widetilde{A}_{\Lambda_1}, \quad \psi = \lambda\circ\rho_{\Lambda_1,\Lambda_2}.$$

Then ψ is a polarization in the ordinary sense, of degree $p^2 = [\Lambda_1 : \Lambda_2]$. If $R = k$ is an algebraically closed field in characteristic p,

$$M(\ker(\psi)) = M(A_{\Lambda_1})/M(A_{\Lambda_2}) = \Lambda_1/\Lambda_2 \otimes k$$

([Ra-Zi] 6.10) is balanced, so $\ker(\psi)$ is Raynaud. Conversely, if we are given data as in (S), thanks to the fact that $\ker(\psi)$ is Raynaud the signature of $A'' = A'/\ker(\psi)$ (with $\mathcal{O}_E$-action induced by ι') is $(2,1)$ (as explained at the end of 1.2.4), so we can define

$$A_{\Lambda_2} = A', \quad A_{\Lambda_1} = A'', \quad \rho_{\Lambda_1,\Lambda_2} = \text{the canonical homomorphism},$$

and "polarize" the resulting $\widetilde{\mathcal{L}}$-set by letting λ be the unique type-reversing isomorphism of $\widetilde{\mathcal{L}}$-sets satisfying $\psi = \lambda\circ\rho_{\Lambda_1,\Lambda_2}$.

The proof that the moduli problem of type $(\mathcal{L}_0(p))$ is our $(S_0(p))$ is in principle identical, and we only sketch it. Once again, given the data $(S_0(p))$ we construct an

[4] We apologize for the unintentional double meaning attributed to tilde. We chose to denote the moduli problem $(\widetilde{S})$ with a tilde, hence it made sense to denote the corresponding lattice chain also $\widetilde{\mathcal{L}}$. In [Ra-Zi], passing to the *dual* $\widetilde{\mathcal{L}}$-set is also denoted by a tilde, hence the tilde on $\widetilde{A}_{\Lambda_\bullet}$.

$\mathcal{L}_0(p)$-set of abelian varieties by interlacing the previous two constructions. First, letting $A = A_{\Lambda_0} \simeq A_{p\Lambda_0}$ we use the Raynaud condition on H to ensure that $A' = A/H = A_{\Lambda_2}$ is of type $(2,1)$. Then we continue and define $A_{\Lambda_1} = A/H^{\perp}$ and the polarization λ as before.

1.4. **Modular curves on the Picard modular surface.**

1.4.1. *Embedding the modular curve.* Maps between Shimura data induce maps between Shimura varieties. Here we have unitary groups of signature $(1,1)$ at infinity mapping (in many ways) to our $\boldsymbol{G}$. These group homomorphisms give rise to morphisms of modular curves and Shimura curves to our Picard modular surface. Rather than go through the familiar yoga of Shimura data, we jump straight ahead to the moduli interpretation, thereby giving the morphism *on the level of integral structures*. We give only one example, which will be explored in connection with the geometry of the special fiber at p later on.

Let B_0 be a fixed elliptic curve defined over $\mathcal{O}_p$ with complex multiplication by $\mathcal{O}_E$ and CM type Σ. Such a curve exists because (p) splits completely in the Hilbert class field H of E, and if $\mathfrak{P}$ is a prime divisor of (p) in H, B_0 may be defined over $\mathcal{O}_{H,\mathfrak{P}} = \mathcal{O}_p$. The reduction of B_0 modulo p is a supersingular elliptic curve defined over κ. Let $\phi_0 : B_0 \simeq B_0^t$ be the canonical principal polarization of B_0, and $\iota_0 : \mathcal{O}_E \simeq \mathrm{End}(B_0)$.

Recall that $p \nmid N \geq 3$. Let $-D$ be the discriminant of E and $\delta = \sqrt{-D}$ a fixed square root of it in E. Assume for simplicity that D is odd and $(N, D) = 1$ (otherwise the construction below has to be modified slightly). Let $\mathscr{Z}_0$ be the scheme parametrizing $\mathcal{O}_E$-isomorphisms $\eta_0 : \mathcal{O}_E/N\mathcal{O}_E \simeq B_0[N]$. It is étale of relative dimension 0 over $\mathcal{O}_p$ and comes with a "multiplier morphism" ν_N to $\underline{\mathrm{Isom}}(\mathbb{Z}/N\mathbb{Z}, \mu_N)$. Write

$$\underline{B}_0 = (B_0, \phi_0, \iota_0, \eta_0) \in \mathscr{Z}_0(R)$$

for an R-valued point of $\mathscr{Z}_0$.

Let $\mathscr{X} = X_0(D; N)$ be the modular curve parametrizing elliptic curves B with a full level N structure $\nu : (\mathbb{Z}/N\mathbb{Z})^2 \simeq B[N]$ and a cyclic subgroup scheme M of order D. We view $\mathscr{X}$ as a scheme over $\mathcal{O}_p$. It too comes equipped with a "multiplier morpism" ν_N to $\underline{\mathrm{Isom}}(\mathbb{Z}/N\mathbb{Z}, \mu_N)$. If we identify $\det(B[N])$ with μ_N via the Weil pairing, then $\nu_N = \det \nu$. We remark that $\mathscr{X}$ is neither complete (the cusps are missing) nor connected ($\det \nu$ is not fixed), and that every subgroup scheme M as above is étale, since D is invertible.

Let R be an $\mathcal{O}_p$-algebra and $\underline{B} = (B, \nu, M) \in \mathscr{X}(R)$. Let $A_1(\underline{B})$ be the abelian surface $\mathcal{O}_E \otimes B/\delta \otimes M$. As D is odd, hence square-free, every class in $\mathcal{O}_E/\delta\mathcal{O}_E$ is represented by a rational integer. As δ kills $\delta \otimes M$, this subgroup is $\mathcal{O}_E$-stable. It is also maximal isotropic for the Mumford pairing induced by the canonical degree D^2 polarization

$$\phi_1' : \mathcal{O}_E \otimes B \to \delta^{-1}\mathcal{O}_E \otimes B = (\mathcal{O}_E \otimes B)^t.$$

The identification $\delta^{-1}\mathcal{O}_E \otimes B = (\mathcal{O}_E \otimes B)^t$ is such that the resulting Weil e_n-pairing between $\mathcal{O}_E \otimes B[n]$ and $\delta^{-1}\mathcal{O}_E \otimes B[n]$ is

$$e_n(\alpha \otimes u, \beta \otimes v) = e_n^B(u, v)^{Tr_{E/\mathbb{Q}}(\alpha\overline{\beta})},$$

where e_n^B is Weil's e_n-pairing on $B[n]$. We may therefore descend ϕ_1' to obtain a principal polarization ϕ_1 of $A_1(\underline{B})$. We let ι_1 be the natural action of $\mathcal{O}_E$ as endomorphisms of $A_1(\underline{B})$. It is of type $(\Sigma, \overline{\Sigma})$. Let

$$\eta_1 = \mathrm{id} \otimes \nu : (\mathcal{O}_E/N\mathcal{O}_E)^2 \simeq A_1[N],$$

a full level-N $\mathcal{O}_E$-structure.

Let $\underline{B}_0 \in \mathscr{Z}_0(R)$ and $\underline{B} \in \mathscr{X}(R)$ be such that $\nu_N(\underline{B}_0) = \nu_N(\underline{B})$. Define

$$A(\underline{B}_0, \underline{B}) = B_0 \times A_1, \;\; \phi = \phi_0 \times \phi_1, \;\; \iota = \iota_0 \times \iota_1, \;\; \eta = \eta_0 \times \eta_1.$$

The structure $\underline{A}(\underline{B}_0, \underline{B}) = (A, \phi, \iota, \eta) \in \mathscr{S}(R)$. Indeed, the assumption $\nu_N(\underline{B}_0) = \nu_N(\underline{B})$ allows us to define a multiplier for η so that it becomes compatible with ϕ, and the rest is obvious. This construction depends functorially on the input. In this way we have defined a morphism

$$\mathscr{Z}_0 \times_{\underline{\mathrm{Isom}}(\mathbb{Z}/N\mathbb{Z}, \mu_N)} \mathscr{X} \to \mathscr{S}.$$

A minor modification of this construction yields a morphism

$$\mathscr{Z}_0 \times_{\underline{\mathrm{Isom}}(\mathbb{Z}/N\mathbb{Z}, \mu_N)} \mathscr{X}_0(p) \to \mathscr{S}_0(p),$$

when we add a cyclic subgroup of order p to the level.

1.4.2. *Endomorphism rings of $\overline{\mathbb{F}}_p$ points of $\mathscr{S}$.* Let D be an indefinite quaternion algebra over $\mathbb{Q}$ equipped with a positive involution † and assume that E embeds in D as a †-stable subfield. Then

$$D = E \oplus E\xi$$

where $\xi^2 > 0$ is rational, $\xi a \xi^{-1} = \overline{a}$ for $a \in E$, $a^\dagger = \overline{a}$ and $\xi^\dagger = \xi$. Furthermore E is the unique quadratic imaginary †-stable subfield of D. Let $\mathcal{O}_D$ be a maximal order in D such that $\mathcal{O}_D \cap E = \mathcal{O}_E$. In this situation we may define the Shimura curve $\mathscr{X}_D$ parametrizing abelian surfaces A_1 with endomorphisms by $\mathcal{O}_D$, a principal polarization inducing † as the Rosati involution on D, and a full level N structure. Precisely as for the modular curve, we get a morphism from $\mathscr{Z}_0 \times_{\underline{\mathrm{Isom}}(\mathbb{Z}/N\mathbb{Z}, \mu_N)} \mathscr{X}_D$ to $\mathscr{S}$. Its image in $\mathscr{S}$ is called *an embedded Shimura curve.*

The points of $\mathscr{S}(\overline{\mathbb{F}}_p)$ lying on the embedded Shimura curves all represent non-simple abelian varieties. There are, however, points $\underline{A} \in \mathscr{S}(\overline{\mathbb{F}}_p)$ for which A is simple. We use the Honda-Tate theorem to construct them. More precisely, we construct A's with $\mathrm{End}^0(A) = M$ a CM field of degree 6.

Let L be a totally real non-Galois cubic field, in which p decomposes as $\mathfrak{p}\mathfrak{q}$, where $f(\mathfrak{p}/p) = 2$ and $f(\mathfrak{q}/p) = 1$. Then $M = LE$ is a degree 6 CM field and $\mathfrak{p} = P\overline{P}$ splits in M, while $\mathfrak{q} = Q$ remains inert. Let π be an element of M such that $(\pi) = P^{2h}Q^h$, where h kills the class of P^2Q in the class group of M. Then $\pi\overline{\pi} = \epsilon p^{2h}$ for a unit ϵ of L. Replacing π with $\epsilon^{-1}\pi^2$ and h with $2h$ we may assume that $\epsilon = 1$.

Let $q = p^{2h}$. Then π is a Weil q-number, and the Honda-Tate theorem implies that there exists a simple 3-dimensional abelian variety over $\mathbb{F}_q$ with $\mathrm{End}(A)$ equal to an order of M, and whose Frobenius of degree q is π. It is easily seen that A is absolutely simple. Changing A by an isogeny if necessary we may assume that $\mathrm{End}(A) \supset \mathcal{O}_E$, and that A carries a principal polarization. Of course, $End^0(A) = M$.

Since $\mathrm{End}^0(A)$, for any $\underline{A} \in \mathscr{S}(\overline{\mathbb{F}}_p)$, must contain a 6-dimensional semi-simple $\mathbb{Q}$-algebra, we see that the "most general" $\overline{\mathbb{F}}_p$-point of $\mathscr{S}$ carries an abelian variety with CM by a field of degree 6. Generic points of the special fiber of $\mathscr{S}$, by contrast, have no endomorphisms except for $\iota(\mathcal{O}_E)$.

2. The structure of the special fiber of $\mathscr{S}$

2.1. **Stratification.** Let k be a fixed algebraic closure of κ. Since we shall have no use for the generic fibers of our integral models any more, *we denote from now on by S, $\widetilde{S}$ and $S_0(p)$ their geometric special fibers, which are schemes defined over k.* We denote by $\mathcal{A}$ the universal abelian scheme over $\mathscr{S}$, and by $\mathcal{A}_x$ its fiber over a geometric point $x \in S(k)$.

Let $\mathfrak{G}$ be the unique (up to isomorphism) connected 1-dimensional p-divisible group over k of height 2. It is self-dual of slope 1/2, and isomorphic to the p-divisible group of any supersingular elliptic curve over k. Fix an embedding $\lambda : \mathcal{O}_p \hookrightarrow \mathrm{End}_k(\mathfrak{G})$ in which $a \in \mathcal{O}_p$ acts on $\mathrm{Lie}(\mathfrak{G})$ via the natural homomorphism $\Sigma : \mathcal{O}_p \twoheadrightarrow \kappa \hookrightarrow k$, and denote the pair $(\mathfrak{G}, \lambda)$ by $\mathfrak{G}_\Sigma$. Let $\mathfrak{G}_{\overline{\Sigma}}$ be the same p-divisible group with the embedding $\lambda \circ \sigma$, under which the action of $a \in \mathcal{O}_p$ on $\mathrm{Lie}(\mathfrak{G})$ is via $\overline{\Sigma} = \Sigma \circ \sigma$.

The following theorem is due to Vollaard [Vo], in particular §6. See also [dS-G1] Theorem 2.1.

Theorem 2.1. *(i) The special fiber S of $\mathscr{S}$ is the union of 3 locally closed strata defined over κ. The μ-*ordinary *stratum S_μ is open and dense, and $x \in S_\mu(k)$ if and only if*

$$\mathcal{A}_x[p^\infty] \simeq (\mathcal{O}_E \otimes \mu_{p^\infty}) \times \mathfrak{G}_\Sigma \times (\mathcal{O}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p)$$

as p-divisible groups with $\mathcal{O}_E$-action. Its complement, $S - S_\mu = S_{ss}$ is called the supersingular locus. It is a reduced (but reducible) complete curve, and if $x \in S_{ss}(k)$ then $\mathcal{A}_x[p^\infty]$ is supersingular, i.e. its Newton polygon is of constant slope 1/2. The superspecial *locus $S_{ssp} \subset S_{ss}$ is 0-dimensional and a point $x \in S_{ssp}(k)$ if and only if*

$$\mathcal{A}_x[p^\infty] \simeq \mathfrak{G}_\Sigma^2 \times \mathfrak{G}_{\overline{\Sigma}}$$

as p-divisible groups with $\mathcal{O}_E$-action. We let $S_{gss} = S_{ss} - S_{ssp}$ and call it the general supersingular *locus.*

Oort's a-number

$$a(\mathcal{A}_x) = \dim_k \mathit{Hom}(\alpha_p, \mathcal{A}_x[p])$$

is 1 if $x \in S_\mu(k)$ or $x \in S_{gss}(k)$ and 3 if $x \in S_{ssp}(k)$. Let $\alpha_p(\mathcal{A}_x)$ be the maximal α_p-subgroup of $\mathcal{A}_x[p]$. The action of κ on $\mathrm{Lie}(\alpha_p(\mathcal{A}_x))$ has signature Σ in the first two cases, and $(\Sigma, \Sigma, \overline{\Sigma})$ in the third case.

(ii) If S' is a connected component of S then $S' \cap S_{ss}$ is a connected component of S_{ss}. The non-singular locus of S_{ss} is precisely S_{gss}. The irreducible components of S_{ss} are Fermat curves, whose normalizations are isomorphic to the curve

$$\mathscr{C} : \; x^{p+1} + y^{p+1} + z^{p+1} = 0.$$

(iii) If $N \geq N_0(p)$ (an integer depending on p) the following also holds. The irreducible components of S_{ss} are already non-singular, and isomorphic to $\mathscr{C}$. Any two of them intersect at most at one point, and if they intersect, this point belongs to $S_{ssp}(k)$ and the intersection is transversal. There are $p^3 + 1$ superspecial points on each irreducible component of S_{ss}, and there are $p + 1$ irreducible components of S_{ss} intersecting transversally at each $x \in S_{ssp}(k)$.

Let X be the geometric special fiber of the modular curve $\mathscr{X}$ which was constructed[5] in §1.4. It is a non-singular curve in S. The following corollary is clear from the description of the strata of S.

Corollary 2.2. *The curve X does not intersect S_{gss}. If $\underline{B} \in X(k)$ is such that $\underline{A}(\underline{B}) \in S_{ssp}(k)$ then B is supersingular, and vice versa.*

2.2. **The tangent bundle of S.**

2.2.1. *The special line sub-bundle TS^+.* Outside S_{ssp}, one may define a natural line sub-bundle TS^+ of the tangent bundle TS of S. For this recall the following facts from [dS-G1]. Let $\Omega_{\mathcal{A}/S}$ be the sheaf of relative differentials of the universal abelian variety $\mathcal{A}$, and $\omega_{\mathcal{A}} = f_*\Omega_{\mathcal{A}/S}$ where $f : \mathcal{A} \to S$ is the structure morphism. Then $\omega_{\mathcal{A}}$ is a rank 3 vector bundle on S, can be identified with the cotangent space of $\mathcal{A}$ at the origin, and admits a decomposition

$$\omega_{\mathcal{A}} = \mathcal{P} \oplus \mathcal{L}$$

into a plane bundle $\mathcal{P}$ on which $\mathcal{O}_E$ acts via Σ and a line bundle $\mathcal{L}$ on which it acts via $\overline{\Sigma}$. Let $\Phi\colon S \to S$ be the absolute Frobenius morphism of degree p, and $\mathcal{A}^{(p)} = S \times_{\Phi,S} \mathcal{A}$ the base change of $\mathcal{A}$. Similar notation will be employed for the base change of the vector bundles $\mathcal{P}$ or $\mathcal{L}$. The Verschiebung homomorphism $\mathrm{Ver}_{\mathcal{A}/S}\colon \mathcal{A}^{(p)} \to \mathcal{A}$ induces maps

$$V_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{L}^{(p)},\ \ V_{\mathcal{L}}\colon \mathcal{L} \to \mathcal{P}^{(p)},$$

which, outside S_{ssp}, are both of rank 1. At the superspecial points these maps vanish. Let

$$\mathcal{P}_0 = \ker(V_{\mathcal{P}}).$$

Outside the superspecial points, $\mathcal{P}_0$ is a line sub-bundle of $\mathcal{P}$. Outside S_{ss}, the lines $\mathcal{P}_0^{(p)}$ and $V_{\mathcal{L}}(\mathcal{L})$ are distinct, but along S_{gss} they coincide. In fact,

$$V_{\mathcal{P}}^{(p)} \circ V_{\mathcal{L}}$$

(a global section of $\mathcal{L}^{p^2-1}$) is the Hasse invariant, and $V_{\mathcal{P}}^{(p)} \circ V_{\mathcal{L}} = 0$ is the equation defining S_{ss} as a subscheme of S.

The Kodaira-Spencer isomorphism is an isomorphism

$$\mathrm{KS} : \mathcal{P} \otimes \mathcal{L} \simeq \Omega_{S/k} = TS^{\vee}.$$

Definition. Outside S_{ssp}, we define TS^+ to be the annihilator of the line bundle $\mathrm{KS}(\mathcal{P}_0 \otimes \mathcal{L})$. We call TS^+ the *special sub-bundle* of TS. By an *integral curve* of TS^+ we mean a nonsingular curve $C \subset S - S_{ssp}$ for which $TS^+|_C = TC$, i.e. TS^+ is tangent to C.

Theorem 2.3. *(i) S_{gss} is an integral curve of TS^+.*

(ii) The modular curve $X_{ord} = X \cap S_\mu$ is an integral curve of TS^+.

Proof. Part (i), although not stated there in this form, was proved in [dS-G2] Proposition 3.11. For (ii) observe that if $x \in X_{ord}(k) \subset S_\mu(k)$ then we have the decomposition $\mathcal{A}_x = B_0 \times \mathcal{A}_{1,x}$ where $\mathcal{A}_1$ is the abelian surface constructed along X from the universal elliptic curve $\mathcal{B}$ (and the universal cyclic subgroup of rank D) as in §1.4. For the cotangent space we have accordingly

$$\omega_{\mathcal{A}}|_x = \omega_{\mathcal{A}_x} = \omega_{B_0} \oplus \omega_{\mathcal{A}_{1,x}},$$

[5]We abuse notation and call the curve $\mathscr{X}_0 \times_{\underline{\mathrm{Isom}}(\mathbb{Z}/N\mathbb{Z},\mu_N)} \mathscr{X}$ simply $\mathscr{X}$.

where the first summand is of type Σ and the second of type $(\Sigma, \overline{\Sigma})$. Thus

$$\mathcal{P}|_x = \omega_{B_0} \oplus \omega_{\mathcal{A}_{1,x}}(\Sigma).$$

As $\mathcal{A}_{1,x}$ is ordinary, V is injective on $\omega_{\mathcal{A}_{1,x}}(\Sigma)$ and

$$\mathcal{P}_0|_x = \ker(V : \mathcal{P}|_x \to \mathcal{L}^{(p)}|_x) = \omega_{B_0}.$$

As B_0 is constant along X, $\mathrm{KS}(\mathcal{P}_0 \otimes \mathcal{L}|_x) \subset \Omega_{S/k}|_x$ annihilates the line $T_xX \subset T_xS$. This proves that $T_xX = TS^+|_x$ as claimed. □

There are many modular curves and Shimura curves like X on S, and by similar arguments they are all integral curves of the special sub-bundle. It would be interesting to know if these are the only integral curves of TS^+ in S_μ. This is an "André-Oort type" question. It would imply, in particular, that there are no integral curves passing through the CM points constructed in §1.4.2. Note that in characteristic p there could be many integral curves tangent to a perfectly nice vector field. The curves $x - c + \lambda y^p = 0$, for varying c and λ, are all tangent to the vector field $\partial/\partial y$ in $\mathbb{A}^2$, and infinitely many of them pass through any given point. The correct formulation of the problem should probably ask for curves annihilated by a larger class of differential operators. Such a class should contain, besides the differential operators generated by TS^+, also "divided powers".

2.2.2. *A characterization in terms of generalized Serre-Tate coordinates.* We shall now give a *second* characterization of TS^+, which relates it to Moonen's work on generalized Serre-Tate coordinates in S_μ. For the following proposition see [Mo], Example 3.3.2 and 3.3.3(d) (case AU, $r = 3$, $m = 1$).

Proposition 2.4. *Let $x \in S_\mu$. Let $\widehat{\mathfrak{G}}$ be the formal group over k associated with the p-divisible group $\mathfrak{G}$ and let $\widehat{\mathbb{G}}_m$ be the formal multiplicative group over k. Then the formal neighborhood $Spf(\widehat{\mathcal{O}}_{S,x})$ of x has a natural structure of a $\widehat{\mathbb{G}}_m$-torsor over $\widehat{\mathfrak{G}}$. In particular, it contains a canonical copy of $\widehat{\mathbb{G}}_m$ sitting over the origin of $\widehat{\mathfrak{G}}$.*

Theorem 2.5. *Let $x \in S_\mu$. Then the line $TS^+|_x$ is tangent to the canonical copy of $\widehat{\mathbb{G}}_m$ in $Spf(\widehat{\mathcal{O}}_{S,x})$.*

At a point x lying on a modular curve X as above, the canonical copy of $\widehat{\mathbb{G}}_m$ is identified with the classical Serre-Tate coordinate on X, i.e. the formal completion of X at x coincides with $i(\widehat{\mathbb{G}}_m)$ as a closed formal subscheme of $Spf(\widehat{\mathcal{O}}_{S,x})$. In this case the theorem is a consequence of Theorem 2.3(ii). Our claim can therefore be viewed as an extension of Theorem 2.3(ii) to a general μ-ordinary point, at which the formal curve $i(\widehat{\mathbb{G}}_m)$ may no longer be "integrated".

Proof. Write $\widehat{\mathbb{G}}_m = Spf(k[[T-1]])$ with comultiplication $T \mapsto T \otimes T$, and let $i : \widehat{\mathbb{G}}_m \hookrightarrow Spf(\widehat{\mathcal{O}}_{S,x})$ be the embedding of formal schemes given by Proposition 2.4. It sends the closed point 1 of $\widehat{\mathbb{G}}_m$ to x. Let i_* be the induced map on tangent spaces

$$i_* : T\widehat{\mathbb{G}}_m|_1 \hookrightarrow TS|_x.$$

We have to show that $i_*(\partial/\partial T)$ annihilates $\mathrm{KS}(\mathcal{P}_0 \otimes \mathcal{L})|_x$. This is equivalent to saying that when we consider the pull back $i^*\mathcal{A}$ of the universal abelian scheme to $\widehat{\mathbb{G}}_m$, *its* Kodaira-Spencer map kills $\mathcal{P}_0 \otimes \mathcal{L}|_1$. For this recall the definition of $\mathrm{KS} = \mathrm{KS}(\Sigma)$ from [dS-G1], §1.4.2.

Let $\mathfrak{S} = \widehat{\mathbb{G}}_m$ and write for simplicity $\mathcal{A}$ for $i^*\mathcal{A}$. We then have the following commutative diagram

$$\begin{array}{ccc} \mathcal{P} = \omega_{\mathcal{A}/\mathfrak{S}}(\Sigma) & \hookrightarrow & H^1_{dR}(\mathcal{A}/\mathfrak{S})(\Sigma) \\ \downarrow \mathrm{KS} & & \downarrow \nabla \\ \mathcal{L}^\vee \otimes \Omega^1_{\mathfrak{S}} \simeq \omega^\vee_{\mathcal{A}^t/\mathfrak{S}}(\Sigma) \otimes \Omega^1_{\mathfrak{S}} & \longleftarrow & H^1_{dR}(\mathcal{A}/\mathfrak{S})(\Sigma) \otimes \Omega^1_{\mathfrak{S}} \end{array} \tag{2.1}$$

in which we identified $H^1(\mathcal{A}, \mathcal{O})$ with $H^0(\mathcal{A}^t, \Omega^1)^\vee$ and used the polarization to identify the latter with $\omega^\vee_{\mathcal{A}/\mathfrak{S}}$, reversing types. Here ∇ is the Gauss-Manin connection, and the tensor product is over $\widehat{\mathcal{O}}_{\mathfrak{S}} = k[[T-1]]$. Although ∇ is a derivation, KS is a homomorphism of vector bundles over $\widehat{\mathcal{O}}_{\mathfrak{S}}$. We shall show that $\mathrm{KS}(\mathcal{P}_0) = 0$, where $\mathcal{P}_0 = \ker(V : \omega_{\mathcal{A}/\mathfrak{S}} \to \omega^{(p)}_{\mathcal{A}/\mathfrak{S}}) \cap \mathcal{P}$.

At this point recall the filtration

$$0 \subset Fil^2 = \mathcal{A}[p^\infty]^m \subset Fil^1 = \mathcal{A}[p^\infty]^0 \subset Fil^0 = \mathcal{A}[p^\infty]$$

of the p-divisible group of $\mathcal{A}$ over $\mathfrak{S}$. The graded pieces are of height 2 and $\mathcal{O}_E$-stable. They are rigid (do not admit non-trivial deformations as p-divisible groups with $\mathcal{O}_E$ action) and given by

$$gr^2 = \mathcal{O}_E \otimes \mu_{p^\infty},\ \ gr^1 = \mathfrak{G},\ \ gr^0 = \mathcal{O}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

For any p-divisible group G over $\mathfrak{S}$ denote by $\mathbb{D}(G)$ the Dieudonné crystal associated to G, and let $D(G) = \mathbb{D}(G)_{\mathfrak{S}}$, *cf.* [Gro]. The $\widehat{\mathcal{O}}_{\mathfrak{S}}$-module $D(G)$ is endowed with an integrable connection ∇ and the pair $(D(G), \nabla)$ determines $\mathbb{D}(G)$.

In our case, we can identify $D(\mathcal{A}[p^\infty])$ with $H^1_{dR}(\mathcal{A}/\mathfrak{S})$, and the connection with the Gauss-Manin connection. The above filtration on $\mathcal{A}[p^\infty]$ induces therefore a filtration $Fil^\bullet$ on $H^1_{dR}(\mathcal{A}/\mathfrak{S})$ which is preserved by ∇. Since the functor $\mathbb{D}$ is contravariant, we write the filtration as

$$0 \subset Fil^1 H^1_{dR}(\mathcal{A}/\mathfrak{S}) \subset Fil^2 H^1_{dR}(\mathcal{A}/\mathfrak{S}) \subset Fil^3 = H^1_{dR}(\mathcal{A}/\mathfrak{S})$$

where

$$Fil^i H^1_{dR}(\mathcal{A}/\mathfrak{S}) = D(\mathcal{A}[p^\infty]/Fil^i\mathcal{A}[p^\infty]).$$

For example, $Fil^1 H^1_{dR}(\mathcal{A}/\mathfrak{S})$ is sometimes referred to as the "unit root subspace". As $Fil^2\mathcal{A}[p^\infty]$ is of multiplicative type, $\ker(V : H^1_{dR}(\mathcal{A}/\mathfrak{S}) \to H^1_{dR}(\mathcal{A}/\mathfrak{S})^{(p)})$ is contained in $Fil^2 H^1_{dR}(\mathcal{A}/\mathfrak{S})$. In particular,

$$\mathcal{P}_0 \subset Fil^2 H^1_{dR}(\mathcal{A}/\mathfrak{S}).$$

Let $G = \mathcal{A}[p^\infty]/\mathcal{A}[p^\infty]^m$, so that $Fil^2 H^1_{dR}(\mathcal{A}/\mathfrak{S}) = D(G)$. It follows that in computing KS on $\mathcal{P}_0$ we may use the following diagram instead of (2.1):

$$\begin{array}{ccc} \mathcal{P}_0 & \hookrightarrow & D(G)(\Sigma) \\ \downarrow \mathrm{KS} & & \downarrow \nabla \\ \mathcal{L}^\vee \otimes \Omega^1_{\mathfrak{S}} & \longleftarrow & D(G)(\Sigma) \otimes \Omega^1_{\mathfrak{S}} \end{array} \tag{2.2}$$

Finally, we have to use the description of the formal neighborhood of x as given in [Mo]. Since we are considering the pull-back of $\mathcal{A}$ to $\mathfrak{S}$ only, and not the full deformation over $Spf(\widehat{\mathcal{O}}_{S,x})$, the p-divisible groups $Fil^1\mathcal{A}[p^\infty]$, and dually $G = \mathcal{A}[p^\infty]/Fil^2$, are *constant* over $\mathfrak{S}$. Thus over $\mathfrak{S}$

$$G \simeq \mathfrak{G} \times (\mathcal{O}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p),$$

and ∇ maps $D(\mathfrak{G})$ to $D(\mathfrak{G})\otimes\Omega^1_{\mathfrak{S}}$. Since

$$\mathcal{P}_0=\omega_{\mathfrak{G}}=D(\mathfrak{G})(\Sigma)$$

as subspaces of $H^1_{dR}(\mathcal{A}/\mathfrak{S})$,

$$\nabla(\mathcal{P}_0)\subset\mathcal{P}_0\otimes\Omega^1_{\mathfrak{S}}.$$

The bottom arrow in (2.2) comes from the homomorphism

$$D(G)(\Sigma)\hookrightarrow H^1_{dR}(\mathcal{A}/\mathfrak{S})(\Sigma)\stackrel{pr}{\to}H^1(\mathcal{A},\mathcal{O})(\Sigma)\stackrel{\phi}{\simeq}H^1(\mathcal{A}^t,\mathcal{O})(\overline{\Sigma})=\mathcal{L}^\vee.$$

But the projection pr kills $\mathcal{P}_0\subset\omega_{\mathcal{A}/\mathfrak{S}}$. This concludes the proof. □

We shall later show that the line sub-bundle TS^+ has a *third* characterization, in connection with the ramification in the covering $\pi\colon S_0(p)\to S$. The definitions and the discussion of this section have obvious generalizations to higher dimensional unitary Shimura varieties. We intend to address them in a future work.

2.3. **The blow up of S at the superspecial points.** We denote by $S^\#$ the surface over k which is obtained by blowing up the superspecial points on S. The fiber of $S^\#\to S$ above a superspecial point x is a projective line which we denote by E_x. It is canonically identified with $\mathbb{P}(T_xS)$.

Since S has a canonical model over κ and the stratum S_{ssp} is defined over κ, $S^\#$ too has a canonical model over κ. In fact, it is the fine moduli space of a moduli problem $(S^\#)$ which is unique to characteristic p. For any κ-algebra R, $S^\#(R)$ classifies isomorphism classes of pairs $(\underline{A},\mathcal{P}_0)$ where

- $\underline{A}\in S(R)$
- $\mathcal{P}_0\subset\ker(V:\omega_{A/R}(\Sigma)\to\omega^{(p)}_{A/R}(\Sigma))$ is a line sub-bundle of $\mathcal{P}=\omega_{A/R}(\Sigma)$ which is annihilated by V.

If no geometric fiber of A/R is superspecial then $\mathcal{P}_0$ is unique. At superspecial points, however, V kills $\mathcal{P}$, so the additional data amounts to a choice of a line in the plane $\mathcal{P}$.

If $N=1$ then S is a stack defined over κ and the superspecial points are κ-rational. It follows that $\mathcal{P}$ is defined over κ too and we can equip each

$$E_x\simeq\mathbb{P}(\mathcal{P}|_x)=\mathbb{P}(\mathcal{P}\otimes\mathcal{L}|_x)\simeq\mathbb{P}(T_xS)$$

with a canonical κ-rational structure. If $N>1$ then level structure at N forces superspecial points to be defined over larger finite fields, but since $\mathcal{P}$ is independent of this extra level structure, the tangent space and the exceptional divisor E_x still carry a canonical κ structure.

In practice we use a coordinate ζ on E_x which is derived from a particular choice of a basis for the Dieudonné module of A_x at $x\in S_{ssp}$. This will be explained in Theorem 4.11 below.

3. Local structure of the three integral models

3.1. **Raynaud's classification.** Recall that k is our fixed algebraically closed field containing κ. In [Ray] Raynaud classifies the finite flat group schemes of rank p^2 over k, which admit an action of κ and satisfy the Raynaud condition discussed in 1.2.1. See also [Bel], III.2.3. They are given in the following table.

H	$(a_0,b_0;a_1,b_1)$	Lie(H)	Lie$(\alpha_p(H))$	α	β	γ	strata
$\kappa\otimes\mathbb{Z}/p\mathbb{Z}$	$(0,1;0,1)$	$\emptyset$	$\emptyset$	0	2	1	μ
$\kappa\otimes\mu_p$	$(1,0;1,0)$	$\Sigma,\overline{\Sigma}$	$\emptyset$	2	0	1	μ
$\kappa\otimes\alpha_p$	$(0,0;0,0)$	$\Sigma,\overline{\Sigma}$	$\Sigma,\overline{\Sigma}$	2	2	$1,2$	ssp
$\mathfrak{G}[p]_\Sigma$	$(0,1;1,0)$	Σ	Σ	1	1	1	gss/ssp
$\mathfrak{G}[p]_{\overline{\Sigma}}$	$(1,0;0,1)$	$\overline{\Sigma}$	$\overline{\Sigma}$	1	1	-	-
$\alpha_{p^2,\Sigma}$	$(0,1;0,0)$	Σ	Σ	1	2	2	gss
$\alpha_{p^2,\overline{\Sigma}}$	$(0,0;0,1)$	$\overline{\Sigma}$	$\overline{\Sigma}$	1	2	-	-
$\alpha^*_{p^2,\Sigma}$	$(0,0;1,0)$	$\Sigma,\overline{\Sigma}$	Σ	2	1	2	gss
$\alpha^*_{p^2,\overline{\Sigma}}$	$(1,0;0,0)$	$\Sigma,\overline{\Sigma}$	$\overline{\Sigma}$	2	1	-	-

Explanations

- Each group scheme is designated by a vector $(a_0,b_0;a_1,b_1)$ with entries from $\{0,1\}$ where $a_0b_0=a_1b_1=0$. There are 9 possibilities. As a scheme $H=Spec(A)$ where $A=k[X,Y]/(X^p-b_0Y,Y^p-b_1X)$. The group structure (Hopf algebra structure on A) involves the a_i. It is completely determined by the condition that the Cartier dual H^D is obtained by interchanging a_0 with b_0, a_1 with b_1. The twist $\kappa\otimes_{\sigma,\kappa}H$ of H is obtained by interchanging a_0 with a_1, and likewise b_0 with b_1.
- The column Lie(H) gives the signature of κ on Lie(H), with multiplicities.
- The column Lie$(\alpha_p(H))$ gives the signature of κ, with multiplicities, on the Lie algebra of the maximal α_p-subgroup of H (whose dimension is Oort's a-number).
- The invariants α,β are defined by
$$\alpha=\dim_k \mathrm{Lie}(H),\ \beta=\dim_k\mathrm{Lie}(H^D).$$
They satisfy
$$\alpha=2-b_0-b_1,\ \beta=2-a_0-a_1.$$
The third invariant, γ, is not an intrinsic invariant of H, but rather of the way it sits as an isotropic subgroup of $A[p]$. Recall that if $(\underline{A},H)$ is a point of $S_0(p)(k)$, we have a filtration
$$0\subset H\subset H^\perp\subset A[p]$$
with graded pieces $A[p]/H^\perp\simeq H^D$ and $H^\perp/H\simeq\ker\psi$ (see §1.2.5). We then set $\gamma=\dim_k\mathrm{Lie}(H^\perp/H)$.
- Finally, the last column indicates over which of the strata of S such points $(\underline{A},H)$ lie. A hyphen indicates that an H of the given type does not occur as an isotropic subgroup of $A[p]$ for $\underline{A}$ as in (S). This is the contents of the next lemma.

Lemma 3.1. *The subgroups $\mathfrak{G}[p]_{\overline{\Sigma}}$, $\alpha_{p^2,\overline{\Sigma}}$ and $\alpha^*_{p^2,\overline{\Sigma}}$ do not occur as isotropic subgroups of $A[p]$ for any $\underline{A}$ as in (S).*

Proof. We do the first example first. Let $M=M(A[p])$ be the covariant Dieudonné module of $A[p]$. It is a 6-dimensional vector space over k, with a κ action of signature $(3,3)$, and maps $F:M^{(p)}\to M$ and $V:M\to M^{(p)}$. The principal polarization

ϕ induces a non-degenerate alternating bilinear pairing $\langle , \rangle = \langle , \rangle_M : M \times M \to k$ satisfying, for $a \in \kappa,\ x \in M^{(p)},\ y, u, v \in M$

$$\langle \iota(a)u, v\rangle = \langle u, \iota(\overline{a})v\rangle$$

$$\langle Fx, y\rangle_M = \langle x, Vy\rangle_{M^{(p)}}.$$

By $\langle , \rangle_{M^{(p)}}$ we denote the base change of $\langle , \rangle_M$ to $M^{(p)} = k \otimes_{\sigma,k} M$. The first property shows that M_0 and M_1, the Σ- and $\overline{\Sigma}$-eigenspaces of κ, are maximal isotropic spaces for the pairing. The second property shows that

$$\mathrm{Lie}(A) = \mathrm{Lie}(A[p]) = M[V] = F(M^{(p)})$$

is another maximal isotropic subspace, which, according to our assumption on the signature of A, intersects M_0 in a 2-dimensional space, and M_1 in a line.

Now let $N = M(H) \subset M$ where H is assumed to be of type $\mathfrak{G}[p]_{\overline{\Sigma}}$ and isotropic. Decompose $N = N_0 \oplus N_1$ according to κ-type. Then $\mathrm{Lie}(H) = N_1$ is orthogonal to N_0 (because N is isotropic) but also to $\mathrm{Lie}(A)_0 = M_0[V]$ (because $\mathrm{Lie}(H) \subset \mathrm{Lie}(A)$ and $\mathrm{Lie}(A)$ is isotropic). Since N_0 is a line lying outside the two-dimensional $\mathrm{Lie}(A)_0$, we deduce that N_1 is orthogonal to all of M_0, contradicting the non-degeneracy of the pairing.

The argument for $H \simeq \alpha_{p^2,\overline{\Sigma}}$ is the same. To rule out $H \simeq \alpha^*_{p^2,\overline{\Sigma}}$ we need another argument, on the α_p-subgroup. $\mathrm{Lie}(H)$ alone does not distinguish it from $\alpha^*_{p^2,\Sigma}$, which, as we shall see later, does occur as a possible isotropic subgroup. If A is either μ-ordinary or general supersingular, then the α_p-subgroup of A is of rank p and type Σ, while the α_p-subgroup of $\alpha^*_{p^2,\overline{\Sigma}}$ is of rank p and type $\overline{\Sigma}$. Hence, $\alpha^*_{p^2,\overline{\Sigma}}$ is not isomorphic to a subgroup scheme of $A[p]$. If A is superspecial, then its p-divisible group is $\mathfrak{G}^3$, and does not admit a subgroup scheme of type $\alpha^*_{p^2}$ at all, because the kernels of Verschiebung and Frobenius on $A^{(p)}$ coincide, while $\alpha^*_{p^2}$ is killed by Frobenius but not by Verschiebung. □

3.2. **The completed local rings.**

3.2.1. *Generalities on local models.* The method of "local models" was introduced by de Jong [dJ2] and developed further by Rapoport and Zink in [Ra-Zi]. For a point x in the special fiber of a given Shimura variety these authors construct a generalized flag variety, and a point x' on it, so that suitable étale neighborhoods of x and x' become isomorphic. This allows them to compute the isomorphism type of the completed local rings of the original Shimura variety in terms of linear-algebra data. For the arithmetic schemes $\mathscr{S}$, $\mathscr{S}_0(p)$ and $\widetilde{\mathscr{S}}$ these computations were done in [Bel] III.4.3, and in this section we shall quote results from there, adhering as much as possible to the notation used by Bellaïche.

The method of local models is *flawed* when it comes to functoriality with respect to change of level at p. This is because Grothendieck's theory of the Dieudonné crystal, on which it is based, is functorial in divided power neighborhoods, but not beyond. This flaw appears already in the case of the modular curve $X_0(p)$ mapping to the j-line X. At a supersingular point $y \in X_0(p)(k)$ mapping to $x \in X(k)$ we get, for the relation between local models in characteristic p

$$k[[u]] \hookrightarrow k[[u,v]]/(uv),$$

while the correct model for the pair $\widehat{\mathcal{O}}_x \hookrightarrow \widehat{\mathcal{O}}_y$ is known to be, ever since Kronecker,

$$k[[u]] \hookrightarrow k[[u,v]]/((u^p - v)(v^p - u)).$$

Observe that modulo pth powers of the maximal ideal (where there is a canonical divided power structure) the two models are isomorphic, but over the whole formal neighborhood they are not. The second homomorphism is finite flat of degree $p+1$ while the first is neither finite nor flat.

Despite this flaw, relations between local models of Shimura varieties of PEL type with parahoric level structure suffice to tell us the relations between cotangent spaces, as well as the relations between the infinitesimal deformation theories when we vary the level.

3.2.2. *The standard model.* Fix $y = [\underline{A}, H] \in \mathscr{S}_0(p)(k)$. Let $x = \pi(y) \in \mathscr{S}(k)$ and $\widetilde{x} = \widetilde{\pi}(y) \in \widetilde{\mathscr{S}}(k)$. Then x is represented by the tuple $\underline{A} = (A, \phi, \iota, \eta)$ and $\widetilde{x}$ by $\underline{A}' = (A', \psi, \iota', \eta')$ where $A' = A/H$ and ψ is descended from $p\phi$, i.e. if $h : A \to A'$ is the canonical isogeny with $\ker(h) = H$ then

$$p\phi = h^t \circ \psi \circ h.$$

Similarly

$$\iota'(a) \circ h = h \circ \iota(a), \quad \eta' = h \circ \eta.$$

Associated with the data $(A, \phi, \iota, A', \psi, \iota', h)$ is the following linear-algebra data. Let

$$M_1 = \mathbb{D}(A)_{W(k)}, \ \ M_2 = \mathbb{D}(A')_{W(k)}$$

be the crystalline Dieudonné modules of the two abelian varieties. Here $\mathbb{D}(A)$ is the (contravariant) Dieudonné crystal associated to A, *cf.* [Gro]. In this section we use crystalline deformation theory as in [Bel]. The translation to *covariant* Cartier-Dieudonné theory, which will be employed in later sections, is standard (if painful), see the appendix to [C-C-O].

The modules M_i are free $W(k)$-modules of rank 6, and decompose under the action of $\mathcal{O}_E$ as a direct sum of two rank-3 submodules, denoted $M_i(\Sigma)$ and $M_i(\overline{\Sigma})$. The isogeny h induces an injective homomorphism

$$D(h) : M_2 \to M_1$$

respecting the $\mathcal{O}_E$-action, whose cokernel is a two-dimensional vector space over k of type $(1,1)$, as H is Raynaud. The polarizations result in type-reversing homomorphism

$$B : M_1^* \simeq M_1, \ \ B' : M_2^* \to M_2$$

where we have used the canonical identifications of $M_i^* = \mathrm{Hom}(M_i, W(k))$ with the crystalline Dieudonné modules of the dual abelian varieties. Clearly

$$D(h) \circ B' \circ D(h)^* = pB.$$

Denote by $\mathcal{M}_1$ the coherent sheaf on $\mathscr{S}$ which associates to a Zariski open U the module

$$\mathcal{M}_1(U) = \mathbb{D}(\mathcal{A})_U$$

($\mathcal{A}$ being the universal abelian variety over $\mathscr{S}$) and define $\mathcal{M}_2$ similarly on $\widetilde{\mathscr{S}}$. Denote by the same letters their pull-backs to $\mathscr{S}_0(p)$. Then the same sort of linear-algebra structure is induced on the sheaves $\mathcal{M}_i$, the map $D(h)$ resulting from the canonical isogeny

$$h : \mathcal{A} \to \mathcal{A}/\mathcal{H}$$

where $\mathcal{H}$ is the universal subgroup scheme of $\mathcal{A}$ over $\mathscr{S}_0(p)$. The following is Théorème III.4.2.5.3 of [Bel].

Theorem 3.2. *(i) There exist $W(k)$-bases $\{e_1,\dots,e_6\}$ of M_1 and $\{f_1,\dots,f_6\}$ of M_2 such that, if we denote by $\{e_i^*\}$ and $\{f_i^*\}$ the dual bases, the following properties hold.*

(a) $M_1(\Sigma)$ is spanned by $\{e_1,e_2,e_3\}$, $M_1(\overline{\Sigma})$ is spanned by $\{e_4,e_5,e_6\}$, and similarly for M_2.

(b) The matrices of the homomorphisms B, B' in these bases are given by

$$B=\begin{pmatrix} &&&&&1\\ &&&&1&\\ &&&1&&\\ &&-1&&&\\ &-1&&&&\\ -1&&&&& \end{pmatrix},\quad B'=\begin{pmatrix} &&&&&p\\ &&&&1&\\ &&&1&&\\ &&-1&&&\\ &-1&&&&\\ -p&&&&& \end{pmatrix},$$

i.e. $B(e_1^*)=-e_6$, $B'(f_1^*)=-pf_6$ etc.

(c) The matrix of $D(h)$ is given by

$$D(h)=\begin{pmatrix} 1&&&&&\\ &p&&&&\\ &&1&&&\\ &&&p&&\\ &&&&1&\\ &&&&&1 \end{pmatrix},$$

i.e. $D(h)(f_1)=e_1,\ D(h)(f_2)=pe_2$ etc.

(ii) The structure $(\mathcal{M}_1,\mathcal{M}_2,B,B',D(h))$ is locally Zariski isomorphic to

$$(M_1,M_2,B,B',D(h))\otimes_{W(k)}\mathcal{O}.$$

3.2.3. *The Hodge filtration.* Fix $y=[\underline{A},H]\in\mathscr{S}_0(p)(k)$ as above. The canonical isomorphism

$$M_1\otimes_{W(k)}k=\mathbb{D}(A)_k\simeq H^1_{dR}(A/k)$$

defines a 3-dimensional subspace

$$\omega_0\subset M_1\otimes_{W(k)}k$$

which maps isomorphically to $\omega_{A/k}$, and similarly a 3-dimensional subspace $\omega_0'\subset M_2\otimes_{W(k)}k$ which maps to $\omega_{A'/k}$. These subspaces are $\mathcal{O}_E$-invariant of type $(2,1)$. Furthermore, they are isotropic in the sense that if we denote by $\omega_0^\perp$ the annihilator of ω_0 in $M_1^*\otimes_{W(k)}k$, and similarly for ω_0', then

$$B(\omega_0^\perp)=\omega_0,\ \ B'(\omega_0'^\perp)\subset\omega_0'.$$

Equality (rather than inclusion) holds with B because ϕ, unlike ψ, is principal. Finally, the map $D(h)$ maps ω_0' to ω_0.

Lemma 3.3. *(i) The invariants (α,β,γ) at the point y are given by the formulae*

$$\alpha=\dim_k\omega_0/D(h)(\omega_0')$$
$$\beta=\dim_k M_1\otimes_{W(k)}k/\big(\omega_0+D(h)(M_2\otimes_{W(k)}k)\big)$$
$$\gamma=\dim_k\omega_0'/B'(\omega_0'^\perp).$$

(ii) (α,β,γ) *form a complete set of invariants of the structure*

$$(M_1\otimes_{W(k)} k, M_2\otimes_{W(k)} k, B, B', D(h), \omega_0, \omega_0').$$

Namely, any two structures (over k) of this form having the same set of invariants (α,β,γ) are isomorphic.

Proof. Part (ii) is an exercise in linear algebra which we leave out to the reader. In checking it observe that α determines the relative position of ω_0' and $\ker D(h)$, β determines the relative position of ω_0 and $\mathrm{Im}D(h)$, while γ is responsible for the relative position of $\ker B'$ and $\omega_0'^{\perp}$. To prove (i) consider the diagram

$$\begin{array}{ccccccccc} & & & & & & \omega_{H^D}^{\vee} & & \\ & & & & & & \cap & & \\ 0 & \to & \omega_{A'} & \to & H^1_{dR}(A'/k) & \to & \omega_{A'^t}^{\vee} & \to & 0 \\ & h^* & \downarrow & & \downarrow & & \downarrow & (h^t)^{*\vee} & \\ 0 & \to & \omega_A & \to & H^1_{dR}(A/k) & \to & \omega_{A^t}^{\vee} & \to & 0 \\ & & \downarrow & & & & & & \\ & & \omega_H & & & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

This gives the formulae for $\alpha=\dim_k\omega_H$ and $\beta=\dim_k\omega_{H^D}=\dim_k \operatorname{coker}(h^t)^{*\vee}$. The formula for γ comes from the fact that if $K=H^{\perp}/H=\ker\psi$ then $\omega_K=\omega_{A'}/B'(\omega_{A'^t})$. □

3.2.4. *Deformations.* The following is a consequence of the main theorem of [Gro], characterizing deformations of an abelian variety A (with extra structure) by means of linear-algebra data. See also [dJ2] and [Bel], Proposition III.4.3.6.

Let $\mathcal{C}_k$ be the category of local Artinian rings $(R,\mathfrak{m}_R)$ of residue field isomorphic to k, equipped with an isomorphism $R/\mathfrak{m}_R\simeq k$. Observe that every object of $\mathcal{C}_k$ comes with a canonical homomorphism $W(k)\to R$.

The local deformation problem $\mathcal{D}$ of the structure $(A,\phi,\iota,A',\psi,\iota',h)_{/k}$ associates to $R\in\mathcal{C}_k$ the set $\mathcal{D}(R)$ of isomorphism classes of similar structures over R, equipped with an isomorphism between their reduction modulo $\mathfrak{m}_R$ and the given structure over k. It is represented by the formal scheme $\mathrm{Spf}(\widehat{\mathcal{O}}_{\mathscr{S}_0(p),y})$. The *local model theorem* is the following.

Theorem 3.4. *The local deformation problem $\mathcal{D}$ is equivalent to the deformation problem $\widetilde{\mathcal{D}}$ which associates to every $(R,\mathfrak{m}_R)$ as above the set of structures*

$$(\omega\subset M_1\otimes_{W(k)} R,\ \omega'\subset M_2\otimes_{W(k)} R)$$

satisfying

(a) ω and ω' are rank-3 direct summands, $\mathcal{O}_E$-invariant of type $(2,1)$, reducing modulo $\mathfrak{m}_R$ to ω_0 and ω_0'.

(b) $B(\omega^{\perp})=\omega,\ \ B'(\omega'^{\perp})\subset\omega'$.

(c) $D(h)(\omega')\subset\omega$.

Similar results hold for the moduli problems represented by $\mathrm{Spf}(\widehat{\mathcal{O}}_{\mathscr{S},x})$ and $\mathrm{Spf}(\widehat{\mathcal{O}}_{\widetilde{\mathscr{S}},\widetilde{x}})$, obtained by forgetting part of the data.

The theorem allows us to compute, quite easily, the complete local rings $\mathbf{L}_y$, $\mathbf{L}_x$ and $\mathbf{L}_{\widetilde{x}}$ representing the deformation problem $\widetilde{\mathcal{D}}$, and deduce isomorphisms

$$\widehat{\mathcal{O}}_{\mathscr{S}_0(p),y}\simeq\mathbf{L}_y,\ \ \widehat{\mathcal{O}}_{\mathscr{S},x}\simeq\mathbf{L}_x,\ \ \widehat{\mathcal{O}}_{\widetilde{\mathscr{S}},\widetilde{x}}\simeq\mathbf{L}_{\widetilde{x}}.$$

Since the local deformation problems $\widetilde{\mathcal{D}}$ at x and $\widetilde{x}$ are obtained from the same problem at y by forgetting part of the data, we get canonical homomorphisms

$$\mathbf{L}_{\widetilde{x}} \to \mathbf{L}_y \leftarrow \mathbf{L}_x \tag{3.1}$$

between the local models. However, as remarked above, this diagram is *not* isomorphic to the corresponding diagram of homomorphisms between the completed local rings of the Picard modular schemes. The best one can get from the general theory is the following.

Theorem 3.5. *In the above situation the diagrams* $\mathbf{L}_{\widetilde{x}} \to \mathbf{L}_y \leftarrow \mathbf{L}_x$ *and* $\widehat{\mathcal{O}}_{\widetilde{\mathscr{S}},\widetilde{x}} \to \widehat{\mathcal{O}}_{\mathscr{S}_0(p),y} \leftarrow \widehat{\mathcal{O}}_{\mathscr{S},x}$ *become canonically isomorphic after one divides all the local rings by the pth powers of their maximal ideals. In particular, they induce isomorphic diagrams on cotangent spaces.*

3.3. Computations.

3.3.1. *Local model diagrams.* Let $W = W(k)$ be the ring of Witt vectors of k. The scheme $\mathscr{S}$ is smooth over W, so all its completed local rings are isomorphic to $\mathbf{L}_x = W[[r,s]]$. In the following table we catalog the diagrams (3.1) giving the local models at $x, \widetilde{x}$ and y, and the maps between them.

Proposition 3.6. *For a suitable choice of local parameters the local model diagram is given by the following table (where* $\mathbf{L}_x = W[[r,s]]$*)*

H at $y = [\underline{A}, H]$	$\mathbf{L}_y$	$\mathbf{L}_{\widetilde{x}}$	maps	in [Bel]
$\underline{\mu\text{-ord}}$:				
$\kappa \otimes \mu_p$	$W[[r,s]]$	$W[[a,b]]$	$a \mapsto r,\ b \mapsto ps$	II.1.c
$\kappa \otimes \mathbb{Z}/p\mathbb{Z}$	$W[[a,b]]$	$W[[a,b]]$	$r \mapsto pa,\ s \mapsto pb$	II.3
$\underline{\text{gss}}$:				
$\mathfrak{G}[p]$	$W[[a,c]]$	$W[[a,c]]$	$r \mapsto a,\ s \mapsto pc$	II.2
$\alpha^*_{p^2}$	$W[[r,s,c]]/(cs+p)$	$W[[a,b,c]]/(bc+p)$	$a \mapsto cr,\ b \mapsto s$	I.1.b
α_{p^2}	$W[[a,b,c]]/(bc+p)$	$W[[a,b,c]]/(bc+p)$	$r \mapsto pa,\ s \mapsto b$	I.2
$\underline{\text{ssp}}$:				
$\mathfrak{G}[p]$	$W[[a,c]]$	$W[[a,c]]$	$r \mapsto a,\ s \mapsto pc$	II.2
$\kappa \otimes \alpha_p$ (generic)	$W[[a,b,r]]/(ar+p)$	$W[[a,b]]$	$s \mapsto br$	II.1.a
$\kappa \otimes \alpha_p$ ($\sqrt[p+1]{-1}$)	$W[[a,b,r]]/(abr+p)$	$W[[a,b,c]]/(bc+p)$	$s \mapsto br,\ c \mapsto ar$	I.1.a

Explanations

- The first column indicates the stratum to which x belongs and the possible Raynaud types of the subgroup H in the fiber of π above x. The parentheses distinguishing the two cases where $H \simeq \kappa \otimes \alpha_p$ refer to the value of the coordinate ζ on the projective line $F_x \subset \pi^{-1}(x)$. This line maps isomorphically to $E_x \subset S^{\#}$ and we endow it with the coordinate ζ as in Section 2.3 and Theorem 4.11 below. The last entry in the table refers to points where $\zeta^{p+1} = -1$, "generic" refers to all the rest.
- The last column refers to the enumeration of the various cases in Bellaïche's thesis [Bel] III.4.3.8 (*cas.sous-cas.sous-sous-cas*).

The table implies that the special fiber $S_0(p)$ of $\mathscr{S}_0(p)$ is equidimensional of dimension 2. As we shall see in Theorems 4.1 and 4.5, it is the union of three smooth surfaces intersecting transversally. These surfaces are the closures of the strata denoted below by Y_m, Y_{et} and Y_{gss}. The first two are irreducible, but the third has several connected components. The non-singular points of $S_0(p)$, lying on only *one* of these surfaces, support an H of type $\kappa\otimes\mu_p$, $\kappa\otimes\mathbb{Z}/p\mathbb{Z}$ or $\mathfrak{G}[p]$. The points lying on the intersection of two of them support an H of type $\alpha^*_{p^2}, \alpha_{p^2}$ or $\kappa\otimes\alpha_p$ (generic). The remaining points, represented by the last row in the table, are those where all three surfaces meet.

The special fiber $\widetilde{S}$ of $\widetilde{\mathscr{S}}$ is the union of two smooth surfaces intersecting transversally. One of them, which is the closure of $\widetilde{\pi}(Y_m)=\widetilde{\pi}(Y_{et})$, is irreducible. The other one, which is the closure of $\widetilde{\pi}(Y_{gss})$, has several connected components. A point $\widetilde{x}=\widetilde{\pi}(y)$ lies on the intersection of these two surfaces if and only if y supports an H of type $\kappa\otimes\alpha_p$ ($\sqrt[p+1]{-1}$), $\alpha^*_{p^2}$ or α_{p^2}.

In the next subsections we work out two sample cases from the table, explaining how one arrives at the given description of the local model diagram.

3.3.2. *First example.* Assume that $x=\pi(y)$ is a gss point and $y\in S_0(p)(k)$ is such that $H\simeq\alpha_{p^2,\Sigma}$ (case I.2 in [Bel]). Here the invariants $(\alpha,\beta,\gamma)=(1,2,2)$. Using Lemma 3.3 one deduces that we may take, without loss of generality,

$$\omega_0=\langle e_1,e_3,e_5\rangle_k\,,\ \ \omega_0'=\langle f_2,f_3,f_5\rangle_k\,.$$

A little computation yields that the most general deformation satisfying (a) (b) and (c) of Theorem 3.4 is given by

$$\omega=\langle e_1-se_2,e_3-re_2,e_5+re_4+se_6\rangle_R$$

$$\omega'=\langle f_2+cf_1,f_3+acf_1,f_5+af_4+bf_6\rangle_R\,,$$

where $r,s,a,b,c\in\mathfrak{m}_R$ satisfy the relations

$$bc+p=0,\quad b=s,\quad pa=r.$$

It follows that

$$\mathbf{L}_{\widetilde{x}}=W(k)[[a,b,c]]/(bc+p)=\mathbf{L}_y\supset\mathbf{L}_x=W(k)[[r,s]].$$

In the special fiber we get

$$\mathbf{L}_{\widetilde{x}}\otimes_{W(k)}k=k[[a,b,c]]/(bc)=\mathbf{L}_y\otimes_{W(k)}k\leftarrow\mathbf{L}_x\otimes_{W(k)}k=k[[r,s]]$$

where $s\mapsto b$ and $r\mapsto 0$.

Corollary 3.7. *The map $\widehat{\mathcal{O}}_{\widetilde{S},\widetilde{x}}\to\widehat{\mathcal{O}}_{S_0(p),y}$ is an isomorphism. Identify $\widehat{\mathcal{O}}_{S_0(p),y}$ with $\mathbf{L}_y\otimes_{W(k)}k$. There are two analytic branches of $S_0(p)$ through y, given by $c=0$ and $b=0$, namely the closed embeddings of formal schemes*

$$\mathfrak{W}=Spf(k[[a,b]])\hookrightarrow\mathfrak{Y}=Spf\ (\widehat{\mathcal{O}}_{S_0(p),y})\hookleftarrow Spf(k[[a,c]])=\mathfrak{Z}.$$

The map $\Omega_{S/k}|_x\to\Omega_{\mathfrak{W}/k}|_y$ maps $ds\mapsto db,\ \ dr\mapsto 0$. The map $\Omega_{S/k}|_x\to\Omega_{\mathfrak{Z}/k}|_y$ is identically 0.

Proof. The map $\widehat{\mathcal{O}}_{\widetilde{\mathscr{S}},\widetilde{x}}\to\widehat{\mathcal{O}}_{\mathscr{S}_0(p),y}$ is an isomorphism even before we reduce these rings modulo p. Indeed, both are 3-dimensional complete regular local rings, and the map between them induces an isomorphism on the cotangent spaces $\mathfrak{m}/\mathfrak{m}^2$, hence is an isomorphism. Here we use the fact that the map between *cotangent spaces* coincides with the corresponding map on the local models, which happens to be an isomorphism.

The two branches of $\mathrm{Spf}(\widehat{\mathcal{O}}_{S_0(p),y})$ can be read off the reduction modulo p of the local model $\mathbf{L}_y$. As both branches are smooth over k, and so is the base S at x, the maps on cotangent spaces are easily calculated from the local models. □

3.3.3. *Second example.* For our second example assume that x is an ssp point and y is such that $H \simeq \kappa \otimes \alpha_p$ and $\zeta^{p+1} = -1$ (case I.1.a in [Bel]). In this case $(\alpha, \beta, \gamma) = (2, 2, 2)$ and we may assume that

$$\omega_0 = \langle e_1, e_3, e_5 \rangle_k \,, \;\; \omega_0' = \langle f_2, f_3, f_4 \rangle_k \,.$$

The most general deformation satisfying (a) (b) and (c) of Theorem 3.4 is given by

$$\omega = \langle e_1 - re_2, e_3 - se_2, e_5 + se_4 + re_6 \rangle_R$$

$$\omega' = \langle f_2 + abf_1, f_3 + bf_1, f_4 + af_5 + cf_6 \rangle_R \,,$$

where $r, s, a, b, c \in \mathfrak{m}_R$ satisfy the relations

$$bc + p = 0, \quad s = -rb, \quad c = ra.$$

The local models are therefore

$$\mathbf{L}_{\widetilde{x}} = W(k)[[a,b,c]]/(bc+p) \rightarrow \mathbf{L}_y = W(k)[[r,a,b]]/(rab+p) \leftarrow \mathbf{L}_x = W(k)[[r,s]]$$

and the maps between them are given by $c \mapsto ra$, $s \mapsto -rb$. Modulo pth powers of the maximal ideals these are also the maps between the completed local rings of the Picard modular surfaces at the corresponding points.

4. The global structure of $S_0(p)$

As before, fix an algebraic closure k of κ. In this section we concentrate on the structure of the geometric special fiber $S_0(p)$ over k.

4.1. The μ-ordinary strata.

4.1.1. *Lots of Frobenii.* Let $Y = S_0(p)$, and let

$$Y^\sigma = \Phi_k^* Y$$

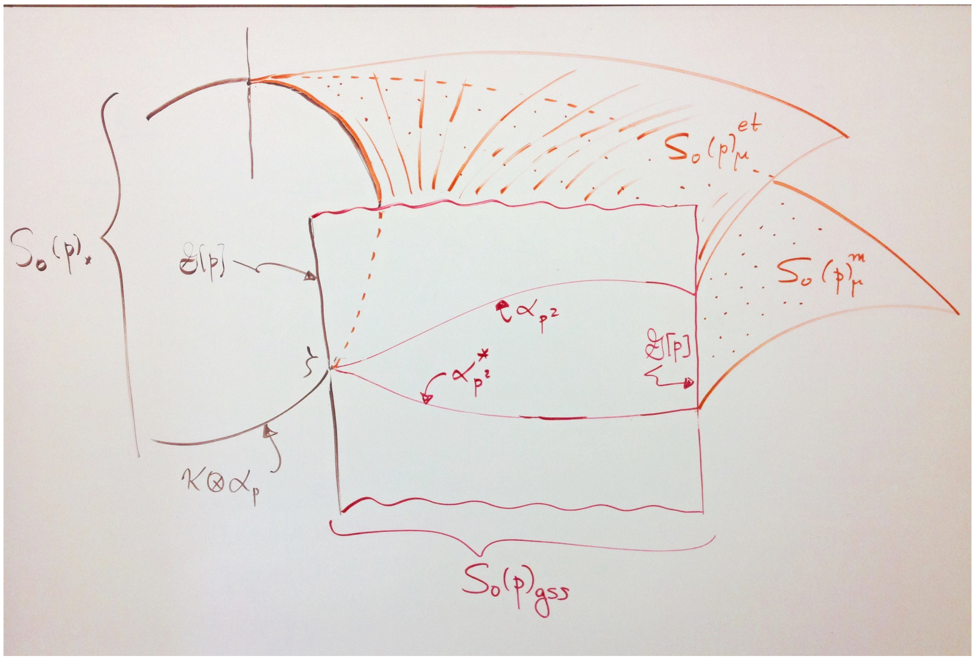
be its base change under the Frobenius of k. This is a fine moduli space for tuples $(\underline{A}_1, H_1)$ as in the moduli problem $(S_0(p))$ except that the signature of the $\mathcal{O}_E$-action on the Lie algebra of A_1 is now $(1,2)$ rather than $(2,1)$.

This Y^σ carries the universal abelian variety $\mathcal{A}_1 = \mathcal{A}^\sigma = \Phi_k^* \mathcal{A}$. It should be distinguished from $\mathcal{A}^{(p)} = \Phi_Y^* \mathcal{A}$, which lies over Y. The same remark and notation applies to the universal subgroup scheme H. The following diagram illustrates the situation.

$$\begin{array}{ccccccc}
\mathcal{A} & \xrightarrow{Fr_{\mathcal{A}/Y}} & \mathcal{A}^{(p)} & \longrightarrow & \mathcal{A}^\sigma & \longrightarrow & \mathcal{A} \\
 & \searrow & \downarrow & \square & \downarrow & \square & \downarrow \\
 & & Y & \xrightarrow{Fr_{Y/k}} & Y^\sigma & \longrightarrow & Y \\
 & & & \searrow & \downarrow & \square & \downarrow \\
 & & & & Spec(k) & \xrightarrow{\Phi_k} & Spec(k) \\
 & & & & & \searrow & \downarrow \\
 & & & & & & Spec(\mathbb{F}_p)
\end{array}$$

The three squares are Cartesian. The composition of the arrows in the three top rows are the maps $\Phi_{\mathcal{A}}$, Φ_Y and Φ_k.

FIGURE 4.1. The structure of $S_0(p)$



Consider now an R-valued point $\xi : Spec(R) \to Y$ and let $A = \xi^*\mathcal{A}$ be the abelian scheme over $Spec(R)$ represented by ξ (we suppress the role of H and the PEL structure). Consider

$$Fr_{Y/k}(\xi) = Fr_{Y/k} \circ \xi : Spec(R) \to Y^{\sigma}.$$

Then

$$A_1 = Fr_{Y/k}(\xi)^*\mathcal{A}_1 = \xi^* Fr_{Y/k}^* \Phi_k^* \mathcal{A} = \xi^* \Phi_Y^* \mathcal{A} = \Phi_R^* A = A^{(p)}.$$

In the moduli-problem language this means that for $(\underline{A}, H) \in Y(R)$

$$Fr_{Y/k}((\underline{A}, H)) = (\underline{A}^{(p)}, H^{(p)}).$$

The Frobenius $Fr_{A/R}$ is an isogeny $Fr_{A/R} : A \to A^{(p)}$. All of the above holds (forgetting the group H) also for S instead of $S_0(p)$.

4.1.2. *The μ-ordinary strata.* We study the part of $S_0(p)$ lying over S_μ, together with the map π. Recall that we work over the algebraically closed field k. We are motivated by the familiar diagram of maps of modular curves (which takes advantage of the fact that $X_0(p)$ is defined over $\mathbb{F}_p$)

$$\begin{array}{ccc} X_0(p)_{et} & \stackrel{Fr_{X/k}}{\to} & X_0(p)_{et} \\ \pi \downarrow & \rho \nearrow & \wr \downarrow \overline{\pi} \\ X_0(1) & = & X_0(1) \end{array}$$

where $\pi(A,H) = A$, $\overline{\pi}(A_1, H_1) = A_1/H_1$ and $\rho(A) = (A^{(p)}, A^{(p)}[\mathrm{Ver}])$.

Theorem 4.1. (i) *Let $Y_\mu = \pi^{-1}(S_\mu) \subset S_0(p)$. Then Y_μ is the disjoint union of two open sets Y_m and Y_{et}. A point $(\underline{A}, H) \in S_0(p)(k)$ lies on Y_m if and only if $H \simeq \kappa \otimes \mu_p$, and on Y_{et} if and only if $H \simeq \kappa \otimes \mathbb{Z}/p\mathbb{Z}$.*

(ii) *The map $\pi\colon Y_\mu \to S_\mu$ is finite flat of degree p^3+1. Restricted to Y_m it yields an isomorphism*

$$\pi_m\colon Y_m \simeq S_\mu.$$

Its inverse is the section

$$\sigma_m : S_\mu \to Y_m, \quad \sigma_m(\underline{A}) = (\underline{A}, A[p]^m),$$

cf. the proof below for the notation.

(iii) *Consider next Y_{et} and its base change Y_{et}^σ under the Frobenius of k. Let $(\underline{A}_1, H_1) \in Y_{et}^\sigma(R)$ for some k-algebra R. Then there exists a point $\underline{A} \in S_\mu(R)$ such that $\underline{A}_1 \simeq \underline{A}^{(p)} = \Phi_R^* \underline{A}$. In fact, let*

$$K_1 = H_1 + H_1^\perp[\mathrm{Fr}],$$

where $H_1^\perp$ is the annihilator of H_1 under the pairing $e_{p\phi_1}$ on $A_1[p]$. Then K_1 is a finite flat, maximal isotropic, $\mathcal{O}_E$-stable subgroup scheme of $A_1[p]$. Let $B = A_1/K_1$, and descend the polarization, endomorphisms, and level-N structure from A_1 to B. Then

$$\underline{B}^{(p)} \simeq \langle p \rangle \, \underline{A}_1$$

so we may take $\underline{A} = \langle p \rangle^{-1} \underline{B}$. Moreover, under the isomorphism $\underline{A}_1 \simeq \underline{A}^{(p)}$

$$K_1 \simeq A^{(p)}[\mathrm{Ver}].$$

(iv) *Restricted to Y_{et}, π yields a map π_{et}, which is of degree p^3 and totally ramified, i.e. $1-1$ on k-points. It factors as*

$$\pi_{et} = \overline{\pi}_{et} \circ Fr_{Y/k}$$

where $Fr_{Y/k}\colon Y_{et} \to Y_{et}^\sigma$ is the relative Frobenius morphism, and $\overline{\pi}_{et}\colon Y_{et}^\sigma \to S_\mu$ is totally ramified of degree p.

In fact, identify Y_{et}^σ with the moduli space for tuples $(\underline{A}_1, H_1)$ as before. Let K_1 and $\underline{A}$ be as in part (iii). Then the following holds:

$$\overline{\pi}_{et}((\underline{A}_1, H_1)) = \langle p \rangle^{-1} (\underline{A}_1/K_1) = \underline{A}. \tag{4.1}$$

In addition, if $(\underline{A}_1, H_1) = Fr_{Y/k}((\underline{A}, H)) = (\underline{A}^{(p)}, H^{(p)})$ for some $(\underline{A}, H) \in Y_\mu(R)$, then $K_1 = A^{(p)}[\mathrm{Ver}]$.

(v) *For any R-valued point $\underline{A}$ of S_μ, $H = \mathrm{Fr}(A^{(p)}[\mathrm{Ver}])$ is a finite flat, rank p^2, isotropic, Raynaud subgroup scheme of $A^{(p^2)}[p]$. Furthermore, it is étale. Define a map*

$$\rho_{et} : S_\mu \to Y_{et}^{\sigma^2} = Y_{et}$$

by setting

$$\rho_{et}(\underline{A}) = (\underline{A}^{(p^2)}, \mathrm{Fr}(A^{(p)}[\mathrm{Ver}])).$$

Then ρ_{et} is finite flat and totally ramified of degree p. We have

$$\rho_{et} \circ \pi_{et} = Fr_{Y/k}^2 : Y_{et} \to Y_{et}^{\sigma^2} = Y_{et}, \quad \rho_{et} \circ \overline{\pi}_{et} = Fr_{Y^\sigma/k}.$$

The following diagram summarizes what was said about the maps π_{et},$\overline{\pi}_{et}$, ρ_{et}.

$$\begin{array}{ccccccc} Y_{et}=Y_{et} & \overset{Fr_{Y/k}}{\longrightarrow} & Y_{et}^{\sigma} & \overset{Fr_{Y^\sigma/k}}{\longrightarrow} & Y_{et} & & \\ & \pi_{et}\searrow & \downarrow \overline{\pi}_{et} & \nearrow \rho_{et} & \downarrow \overline{\pi}_{et}^{\sigma} & \searrow \pi_{et} & \\ & & S_\mu & \overset{Fr_{S/k}}{\longrightarrow} & S_\mu^\sigma & \overset{Fr_{S^\sigma/k}}{\longrightarrow} & S_\mu. \end{array}$$

Proof. (i) Let $Y_\mu=\pi^{-1}(S_\mu)$. This is an open subset of $S_0(p)$. If R is any k-algebra and $\underline{A}\in S_\mu(R)$, then the group scheme $A[p]_{/R}$ admits a canonical filtration by finite flat $\mathcal{O}_E$-subgroup schemes

$$Fil^3A[p]=0\subset Fil^2A[p]=A[p]^m\subset Fil^1A[p]=A[p]^0\subset Fil^0A[p]=A[p].$$

Here Fil^1 is the maximal connected subgroup-scheme and is of rank p^4, while Fil^2 is the maximal subgroup scheme of multiplicative type (connected, with étale Cartier dual), and is of rank p^2. It is also equal to the annihilator of Fil^1 under the pairing $e_{p\phi}$. Moreover, the graded pieces are rigid in formal neighborhoods. This means that over any Artinian neighborhood $Spec(R)$ of a point, we have isomorphisms ($gr^i=Fil^i/Fil^{i+1}$)

$$gr^2A[p]\simeq\kappa\otimes\mu_p,\ \ gr^1A[p]\simeq\mathfrak{G}[p]_\Sigma,\ \ gr^0A[p]\simeq\kappa\otimes\mathbb{Z}/p\mathbb{Z},$$

as R-group schemes with $\mathcal{O}_E$-action. We remark that the filtration and the rigidity of its graded pieces hold for the whole p-divisible group. If $R=k$ (or any other perfect field), $A[p]$ splits canonically as the product of the three graded pieces. As these are pairwise non-isomorphic, the only rank-p^2 $\mathcal{O}_E$-subgroup schemes of $A[p]$ are then the unique copies of $\kappa\otimes\mu_p$, $\mathfrak{G}[p]_\Sigma$ or $\kappa\otimes\mathbb{Z}/p\mathbb{Z}$ in it. They are all Raynaud. Only the first and the last are isotropic for the Weil pairing. Thus, if $x\in S_\mu(k)$, there are only two points of $Y_\mu(k)$ above x. We call Y_m the component of Y_μ containing the k-points $(\underline{A},H)$ where $H\simeq\kappa\otimes\mu_p$, and Y_{et} the component containing the k-points where $H\simeq\kappa\otimes\mathbb{Z}/p\mathbb{Z}$. That these are indeed connected components follows from the above mentioned rigidity.

(ii) Let

$$\sigma_m:S_\mu\to Y_m$$

be the morphism defined on R-points (R any k-algebra) by $\underline{A}\mapsto(\underline{A},A[p]^m)$. It is a section of the map π, both $\pi\circ\sigma_m$ and $\sigma_m\circ\pi$ are the identity maps, hence π induces an isomorphism on Y_m.

This is not the case on Y_{et}, as we can not split the filtration of $A[p]$ functorially over arbitrary k-algebra, only over perfect fields. Let us prove that $\pi_{et}:Y_{et}\to S_\mu$ is finite flat and totally ramified of degree p^3. It follows from the computations of the completed local rings in §3.2 that Y_{et} is non-singular. The map π_{et} is quasi-finite and proper (see Proposition 1.5), hence finite. Any finite surjective morphism between non-singular varieties is automatically flat ([Eis] 18.17). In fact, the same argument, using regularity of the arithmetic schemes, proves that on the scheme $\mathscr{S}_0(p)'$ obtained by removing $Y_{ss}=\pi^{-1}(S_{ss})$ from the special fiber of $\mathscr{S}_0(p)$, the map π is finite flat to $\mathscr{S}'=\mathscr{S}-S_{ss}$. Since the degree in the generic fiber is p^3+1, so must be the degree in the special fiber. Since π was shown to be an isomorphism on Y_m, on Y_{et} it is finite flat of degree p^3, and of course, totally ramified ($1-1$ on geometric points). For another proof see [Bel] III.3.5.12.

(iii,iv) Since Y_{et}^σ is reduced, every R-point of Y_{et}^σ is a base-change of an R'-point under a homomorphism $R'\to R$, where R' is reduced. We may therefore assume in the proof of (iii) and (iv) that R is *reduced*.

We begin by showing that if $(\underline{A}_1, H_1)$ is an R-point of Y_{et}^{σ}, then $K_1 = H_1 + H_1^{\perp}[\mathrm{Fr}]$ is a finite flat subgroup-scheme of rank p^3 contained in $A_1[p]$. It is enough to prove this for the universal abelian scheme $\mathcal{A}_1$ over Y_{et}^{σ}, and its universal subgroup H_1. We use the criterion for flatness, saying that if $f : X' \to X$ is a finite morphism of schemes, X is reduced, and all the fibers of f have the same rank, then f is also flat ([Mu], p.432). By the open-ness of the flat locus of a morphism, if X is a variety over a field k, it is enough to check the constancy of the fiber rank at *closed* points of X. We shall use this criterion here for group schemes over Y_{et}^{σ}, noting that the base is a non-singular variety. First, $H_1^{\perp}$ is clearly finite flat of rank p^4 over Y_{et}^{σ} and $H_1^{\perp}[\mathrm{Fr}] = H_1^{\perp} \cap \mathcal{A}_1[\mathrm{Fr}]$ is a closed, hence finite, subgroup scheme. Its fiber rank (over the closed points of Y_{et}^{σ}!) is constantly p, so it is also flat. Next, $H_1 \cap H_1^{\perp}[\mathrm{Fr}] = H_1[\mathrm{Fr}] = 0$. Thus, as a subgroup functor of $\mathcal{A}_1[p]$,

$$H_1 + H_1^{\perp}[\mathrm{Fr}] \simeq (H_1 \times H_1^{\perp}[\mathrm{Fr}])/(H_1 \cap H_1^{\perp}[\mathrm{Fr}]) \simeq H_1 \times H_1^{\perp}[\mathrm{Fr}]$$

is a finite flat group scheme of rank p^3.

Define $\overline{\pi}_{et}$ to be the morphism sending $(\underline{A}_1, H_1) \in Y_{et}^{\sigma}(R)$ to $\langle p \rangle^{-1} \underline{B}$ where $B = A_1/K_1$. The type of $\mathrm{Lie}(B)$ will now be $(2,1)$, as can be easily checked. Since K_1 is a maximal isotropic subgroup scheme for the Weil pairing on $A_1[p]$, the polarization $p\phi_1$ on A_1 descends to a principal polarization of B. The tame level-N structure on A_1 gives rise to a tame level-N structure on B. This completes the definition of $\overline{\pi}_{et}$.

If $(\underline{A}_1, H_1) = (\underline{A}^{(p)}, H^{(p)})$ for $(\underline{A}, H) \in Y_{et}(R)$, and R is *reduced*, then K_1 is of rank p^3 and killed by Ver, as can be checked fiber-by-fiber. This shows that

$$K_1 = A^{(p)}[\mathrm{Ver}],$$

hence $A_1/K_1 \simeq A$ via $\mathrm{Ver} : A^{(p)} \to A$. The polarization $p\phi_1$ descends back to ϕ because $\phi_1 = \phi^{(p)}$. Finally, if

$$\eta : \Lambda/N\Lambda \simeq A[N]$$

is the level-N structure on A and $\eta_1 = \eta^{(p)}$, then

$$\mathrm{Ver} \circ \eta^{(p)} = \langle p \rangle \circ \eta,$$

concluding the proof that $\overline{\pi}_{et}(\underline{A}_1, H_1) = \underline{A}$. This holds in particular when $R = k$, which is enough to prove

$$\pi_{et} = \overline{\pi}_{et} \circ Fr_{Y/k}.$$

We remark that for a reduced R, to conclude that $K_1 = A^{(p)}[\mathrm{Ver}]$ we did not have to know that H_1 was of the form $H^{(p)}$, only that $A_1 = A^{(p)}$. Caution must be exercised when R is non-reduced though, because it is then possible to have $A^{(p)} \simeq B^{(p)}$ without $A \simeq B$. The isogeny Ver should be labeled by A or B, and the given isomorphism between $A^{(p)}$ and $B^{(p)}$ may not carry $\ker(\mathrm{Ver}_A)$ to $\ker(\mathrm{Ver}_B)$.

In general, applying the same argument to $(A_1^{(p)}, H_1^{(p)})$ implies that

$$H_1^{(p)} + H_1^{(p)\perp}[\mathrm{Fr}] = A_1^{(p)}[\mathrm{Ver}]$$

so

$$B^{(p)} = A_1^{(p)}/(H_1^{(p)} + H_1^{(p)\perp}[\mathrm{Fr}]) = A_1^{(p)}/A_1^{(p)}[\mathrm{Ver}] \simeq A_1.$$

By the remark above, $K_1 = B^{(p)}[\mathrm{Ver}]$. We emphasize, however, that the group H_1 need not be a Frobenius base change of a similar subgroup of B. To guarantee that the level-N structures also match we have to twist $\underline{B}$ by the diamond operator $\langle p \rangle^{-1}$ and set $\underline{A} = \langle p \rangle^{-1} \underline{B}$. Then $\underline{A}_1 \simeq \underline{A}^{(p)}$.

(v) The finite subgroup scheme $\mathcal{A}^{(p)}[\mathrm{Fr}] \cap \mathcal{A}^{(p)}[\mathrm{Ver}]$ is flat over S_μ, as it has constant fiber rank p and the base is reduced. The image

$$\mathrm{Fr}(\mathcal{A}^{(p)}[\mathrm{Ver}]) \subset \mathcal{A}^{(p^2)}[p],$$

is isomorphic to the quotient of $\mathcal{A}^{(p)}[\mathrm{Ver}]$ by $\mathcal{A}^{(p)}[\mathrm{Fr}] \cap \mathcal{A}^{(p)}[\mathrm{Ver}]$, hence is also finite and flat of rank p^2. It is isotropic, $\mathcal{O}_E$-stable and Raynaud. By base change from the universal case, for any R-valued point $\underline{A}$ of S_μ, $H = \mathrm{Fr}(A^{(p)}[\mathrm{Ver}])$ is a finite flat, rank p^2, isotropic, Raynaud subgroup scheme of $A^{(p^2)}[p]$. It is easily seen to be étale. Since ρ_{et} is defined functorially in terms of the moduli problem, it is a well-defined morphism.

It is enough to verify the equality $\rho_{et} \circ \pi_{et} = Fr^2_{Y/k}$ on k-valued points $(\underline{A}, H) \in Y_{et}(k)$, namely that

$$\mathrm{Fr}(A^{(p)}[\mathrm{Ver}]) = H^{(p^2)},$$

but if A is μ-ordinary this is clear. The relation $\rho_{et} \circ \overline{\pi}_{et} = Fr_{Y^\sigma/k}$ follows from $\rho_{et} \circ \pi_{et} = Fr^2_{Y/k}$ since $\pi_{et} = \overline{\pi}_{et} \circ Fr_{Y/k}$ and $Fr_{Y/k}$ is faithfully flat. The remaining assertions on ρ_{et} also follow from this relation. □

Corollary 4.2. *Over Y_{et} the universal abelian scheme $\mathcal{A} \simeq \mathcal{A}_1^{(p)} = Y_{et} \times_{\Phi_Y, Y_{et}} \mathcal{A}_1$ for another abelian scheme $\mathcal{A}_1$ of type* $(1,2)$.

Proof. In part (iii) of the theorem we showed the same for the universal abelian variety $\mathcal{A}_1$ over Y_{et}^σ. The corollary follows by base-changing back to Y_{et}, or by repeating the arguments throughout with type $(1,2)$ replacing type $(2,1)$. □

4.1.3. *A lemma on ramification.* Before we continue our study of Y_μ we need the following result.

Lemma 4.3. *Let $\pi : Y \to X$ be a finite flat totally ramified morphism of degree p between non-singular surfaces over k, an algebraically closed field of characteristic p. Let $\pi(y) = x$. Then there exist local parameters u, v at $y \in Y$ so that $\pi^* : \widehat{\mathcal{O}}_{X,x} \hookrightarrow \widehat{\mathcal{O}}_{Y,y}$ is*

$$k[[u^p, v]] \hookrightarrow k[[u, v]].$$

The class of u^p modulo $\widehat{\mathfrak{m}}^2_{X,x}$ spans $\ker(\pi^* : \Omega_{X/k}|_x \to \Omega_{Y/k}|_y)$, *and is therefore independent of any choice.*

Proof. See [Ru-Sh] Theorem 4, and the Corollary at the bottom of p. 1215 there. □

Definition. We call the line in T_xX which is the annihilator of $\ker(\pi^* : \Omega_{X/k}|_x \to \Omega_{Y/k}|_y)$ the *unramified direction* at x, and denote it by T_xX^{ur}. Then TX^{ur} is a line sub-bundle of TX.

If $C \subset X$ is a non-singular curve such that for every $x \in C$

$$T_xC = T_xX^{ur} \subset T_xX$$

(an *integral curve* for TX^{ur}), then $\pi : \pi^{-1}(C)^{red} \to C$ is indeed unramified, hence an isomorphism, because π^* is injective on

$$\Omega_{C/k} = \Omega_{X/k}/TC^\perp = \Omega_{X/k}/\ker(\pi^*).$$

4.1.4. *The unramified direction of* $\overline{\pi}_{et}$. The morphism π_{et} is "too ramified", and we study it via the factorization $\pi_{et} = \overline{\pi}_{et} \circ Fr_{Y/k}$. Since $\overline{\pi}_{et}$ is of degree p, it admits, as we have just seen, an "unramified direction". In §2.2 we have defined the special sub-bundle TS^+ in TS outside the superspecial locus. We shall now show that over S_μ it coincides with the sub-bundle of unramified directions for $\overline{\pi}_{et}$. Thus the latter can be defined intrinsically in terms of the automorphic vector bundles on S, without any reference to the covering π.

Theorem 4.4. *Let* $x = \pi_{et}(y) = \overline{\pi}_{et}(y^{(p)}) \in S_\mu$. *The unramified direction at* x *for the map* $\overline{\pi}_{et}$ *is* T_xS^+. *Equivalently, under the Kodaira-Spencer isomorphism*

$$\ker(\Omega_{S_\mu/k} \to \Omega_{Y^\sigma_{et}/k}) = \mathrm{KS}(\mathcal{P}_0 \otimes \mathcal{L}).$$

Proof. More precisely, we need to prove that over Y^σ_{et}

$$\ker(\overline{\pi}^*_{et}\Omega_{S_\mu/k} \to \Omega_{Y^\sigma_{et}/k}) = \overline{\pi}^*_{et}(\mathrm{KS}(\mathcal{P}_0 \otimes \mathcal{L})).$$

In parts (iii) and (iv) of Theorem 4.1 we have seen that if we denote by $\mathcal{A}_1$ the universal abelian scheme over Y^σ_{et} then $\mathcal{A}_1 = \mathcal{B}^{(p)}$, where $\mathcal{B} = \overline{\pi}^*_{et}\mathcal{A}$, and the morphism $\overline{\pi}_{et}$ is induced from $\mathrm{Ver} : \mathcal{A}_1 \to \mathcal{B}$, followed by $\langle p \rangle^{-1}$ on the level-N structure.

Along with the Kodaira-Spencer isomorphism $\mathrm{KS} : \mathcal{P} \otimes \mathcal{L} \simeq \Omega_{S_\mu/k}$ and its pull-back to Y^σ_{et} under $\overline{\pi}^*_{et}$, consider the Kodaira-Spencer *homomorphism* $\mathrm{KS}' : \overline{\pi}^*_{et}(\mathcal{P}^{(p)} \otimes \mathcal{L}^{(p)}) \to \Omega_{Y^\sigma_{et}/k}$ associated with the abelian scheme $\mathcal{A}_1$ over Y^σ_{et}. It is not an isomorphism (as at each point there is a direction where infinitesimally $\mathcal{A}_1$ is not varying, only its subgroup H_1), but is defined from the Gauss-Manin connection of $\mathcal{A}_1/Y^\sigma_{et}$ in the same way as KS was defined for $\mathcal{A}/S$. By the functoriality of the Gauss-Manin connection with respect to the Verschiebung isogeny

$$\mathrm{Ver} : \mathcal{A}_1 \to \mathcal{B}$$

we derive the following commutative diagram with exact rows ($s(u \otimes v) = v \otimes u$)

$$\begin{array}{ccccccc} 0 \to & \overline{\pi}^*_{et}(\mathcal{P}_0 \otimes \mathcal{L}) & \longrightarrow & \overline{\pi}^*_{et}(\mathcal{P} \otimes \mathcal{L}) & \overset{s\circ(V_{\mathcal{P}} \otimes V_{\mathcal{L}})}{\longrightarrow} & \overline{\pi}^*_{et}(\mathcal{P}^{(p)} \otimes \mathcal{L}^{(p)}) \\ & \downarrow & & \wr\downarrow \mathrm{KS} & & \downarrow \mathrm{KS}' \\ 0 \to & \ker(\overline{\pi}^*_{et}\Omega_{S_\mu/k} \to \Omega_{Y^\sigma_{et}/k}) & \longrightarrow & \overline{\pi}^*_{et}\Omega_{S_\mu/k} & \overset{\overline{\pi}^*_{et}}{\longrightarrow} & \Omega_{Y^\sigma_{et}/k} \end{array}.$$

This gives the isomorphism between the two line sub-bundles in the first column. Their annihilators in TS are the "special sub-bundle" TS^+ and the "line-bundle of unramified directions" TS^{ur}, hence these two are also equal. □

In the next section we shall see that the theorem extends to the gss locus. In fact, the same proof applies, once we extend the morphism $\overline{\pi}_{et}$ and the factorization $\pi_{et} = \overline{\pi}_{et} \circ Fr_{Y/k}$. See the proof of Theorem 4.5 (iii).

4.2. **The gss strata.** Recall that the supersingular locus $S_{ss} \subset S$ is the union of Fermat curves crossing transversally at the superspecial locus S_{ssp}. The complement of these crossing points was denoted S_{gss} and is therefore a disjoint union of open Fermat curves. In this section we study its pre-image under the morphism $S_0(p) \to S$ and show that it is a $\mathbb{P}^1$-bundle, intersecting transversally with the horizontal components of $S_0(p)$. Understanding the pre-image of S_{ssp} will be taken up in the next section.

4.2.1. *The $\mathbb{P}^1$-bundles.*

Theorem 4.5. (i) *Let $Y_{gss} = \pi^{-1}(S_{gss})^{red}$. Then Y_{gss} has the structure of a $\mathbb{P}^1$-bundle over the non-singular curve S_{gss}, with two distinguished non-intersecting non-singular curves*

$$Z_{et} \text{ and } Z_m.$$

A point $y = (\underline{A}, H) \in S_0(p)(k)$ lies on Z_{et} if and only if $H \simeq \alpha_{p^2,\Sigma}$ and on Z_m if and only if $H \simeq \alpha^*_{p^2,\Sigma}$. The fiber $\pi^{-1}(x)$ $(x \in S_{gss}(k))$ intersects each of the curves Z_{et} or Z_m at a unique point. At all other k-points $(\underline{A}, H)$ of Y_{gss}, the group $H \simeq \mathfrak{G}[p]_\Sigma$.

(ii) *The closure $\overline{Y}_m$ of Y_m intersects Y_{gss} transversally in Z_m. Let $Y_m^\dagger = Y_m \cup Z_m$, a locally closed subscheme of $S_0(p)$, and $S_\mu^\dagger = S_\mu \cup S_{gss}$. Then $Y_m^\dagger$ is a non-singular surface. The map $\pi_m \colon Y_m^\dagger \xrightarrow{\sim} S_\mu^\dagger$ is an isomorphism, and the section $\sigma_m : S_\mu \to Y_m$ extends to a section of π_m over $S_\mu^\dagger$.*

(iii) *The closure $\overline{Y}_{et}$ of Y_{et} intersects Y_{gss} transversally in Z_{et}. Let $Y_{et}^\dagger = Y_{et} \cup Z_{et}$, a locally closed subscheme of $S_0(p)$. Then $Y_{et}^\dagger$ is a non-singular surface. The morphism $\overline{\pi}_{et}$ of Theorem 4.1 extends to a morphism*

$$\overline{\pi}_{et} : Y_{et}^{\dagger\,\sigma} \to S_\mu^\dagger,$$

which is finite flat totally ramified of degree p. The factorization $\pi_{et} = \overline{\pi}_{et} \circ Fr_{Y/k}$ extends to $Y_{et}^\dagger$.

Restricted to Z_{et} the map π_{et} is totally ramified of degree p and $\overline{\pi}_Z = \overline{\pi}_{et}|_{Z_{et}^\sigma}$ is an isomorphism from Z_{et}^σ onto S_{gss}.

(iv) Setting

$$\rho_{et}(\underline{A}) = (\underline{A}^{(p^2)}, \mathrm{Fr}(A^{(p)}[\mathrm{Ver}]))$$

extends the map ρ_{et} to a finite flat totally ramified map of degree p from $S_\mu^\dagger$ to $Y_{et}^\dagger$. We have

$$\rho_{et} \circ \pi_{et} = Fr^2_{Y/k} : Y_{et}^\dagger \to Y_{et}^{\dagger\sigma^2} = Y_{et}^\dagger, \quad \rho_{et} \circ \overline{\pi}_{et} = Fr_{Y^\sigma/k}.$$

The proof of the theorem will be given in the next subsection. We caution the reader that the scheme-theoretic pre-image of S_{gss} under $\overline{\pi}_{et}$ is not reduced. It is rather a nilpotent thickening of degree p of the reduced curve Z_{et}^σ in $Y_{et}^{\dagger\sigma}$. Similarly the scheme-theoretic pre-image $\pi^{-1}(S_{gss})$ is non-reduced along Z_{et}, and only there.

We also caution that the formula (4.1) giving $\overline{\pi}_{et}$ on Y_{et}^σ is no longer valid for its continuous extension to Z_{et}^σ. The group functor $H_1 + H_1^\perp[\mathrm{Fr}]$ is represented by a finite flat group scheme on each of Y_{et}^σ and Z_{et}^σ separately, but even though the ranks of these group schemes are the same (p^3), they do not glue to give a group scheme over the whole of $Y_{et}^{\dagger\sigma}$. Indeed, at a closed point of Y_{et}^σ this group is the kernel of Ver, but this does not hold at closed points of Z_{et}^σ.[6]

The following diagram summarizes what the extensions of the maps $\pi_{et}, \overline{\pi}_{et}, \rho_{et}$ to the gss strata look like.

[6]If H_1 and H_2 are finite flat subgroup schemes of a finite flat group scheme G, then $H_1 \cap H_2$ is a finite subgroup scheme, but is not necessarily flat. If it is flat, then the sum $H_1 + H_2$, being isomorphic as a group functor to $H_1 \times H_2/(H_1 \cap H_2)$, is again represented by a finite flat group scheme. In general, however, the group-functor-quotient of a finite flat group scheme by a closed (hence finite) non-flat subgroup scheme, need not be represented by a group scheme at all, let alone by a finite flat group scheme. Thus the sum of two subgroup schemes need not be a group scheme!

$$\begin{array}{ccccc}
Z_{et} & \overset{Fr_{Z/k}}{\longrightarrow} & Z_{et}^{\sigma} & \overset{Fr_{Z^\sigma/k}}{\longrightarrow} & Z_{et}^{\sigma^2} = Z_{et} \\
 & \pi_{et} \searrow p & \simeq\downarrow \overline{\pi}_{et} & p \nearrow \rho_{et} & \\
 & & S_{gss} & &
\end{array}$$

Corollary 4.6. *(i) The maps $\overline{\pi}_{et}$ and σ_m induce an isomorphism*

$$\sigma_m \circ \overline{\pi}_{et} : Z_{et}^{\sigma} \overset{\sim}{\to} Z_m.$$

(ii) Setting $\theta = \rho_{et} \circ \pi_m : Y_m^\dagger \to Y_{et}^\dagger$ gives a commutative diagram of totally ramified finite flat morphisms between surfaces, and similarly between embedded curves (the diagonal arrows are embeddings):

$$\begin{array}{cccccc}
Z_m & & \overset{\theta}{\longrightarrow} & & Z_{et} & \\
| & \searrow & & & \vdots & \searrow \\
\simeq & & Y_m^\dagger & & \overset{\theta}{\longrightarrow} & Y_{et}^\dagger \\
\downarrow & & | & & \downarrow & | \\
S_{gss} & \cdots & \simeq & \overset{Fr^2}{\dashrightarrow} & S_{gss} & \pi_{et} \\
 & \searrow & \downarrow & & \searrow & \downarrow \\
 & & S_\mu^\dagger & & \overset{Fr^2}{\longrightarrow} & S_\mu^\dagger
\end{array}$$

The map θ is of degree p, and so is $\theta|_{Z_m}$. In particular, the latter factors through the Frobenius of the curve Z_m and yields an isomorphism $Z_m^\sigma \overset{\sim}{\to} Z_{et}$.

If Z_m' and Z_{et}' are two κ-components of Z_m and Z_{et} (i.e. defined and irreducible over κ) which map to the same κ-component S_{gss}' of S_{gss} then $\theta(Z_m') = Z_{et}'$.

Proof. The commutativity is easily checked in terms of the moduli problem. The degrees are calculated from the fact that π_m is an isomorphism, π_{et} has degree p^3 on $Y_{et}^\dagger$ and degree p on Z_{et}, while $Fr_{S/K}^2$ has degree p^4 on $S_\mu^\dagger$ and degree p^2 on S_{gss}. To summarize, in the front square we have $p^3 \times p = p^4 \times 1$, and in the back square we have $p \times p = p^2 \times 1$. The assertion about κ-components follows from the fact that $Fr_{S/k}^2$ preserves these components. □

Remark. We believe that if $N = 1$ (working with stacks) the geometrically irreducible components of S_{gss} are already defined over κ, hence θ exchanges the irreducible components of Z_m and Z_{et} within the same irreducible component of Y_{gss}. This is clearly not the case when $N > 1$. Compare with supersingular points on the modular curve $X(N)$.

4.2.2. *Proof of Theorem 4.5.* We first quote [Bu-We], Proposition 3.6. In the notation used there, the Dieudonné module of $A[p]$, for A supersingular but not superspecial, is the "Dieudonné space" $\underline{\overline{B}}(3)$. Our Dieudonné module M differs from the one appearing in [Bu-We], (3.2)(2) by a "Frobenius twist". This is because we use covariant Dieudonné theory, while [Bu-We] employs Cartier theory. See [C-C-O], Appendix B.3.10, where the first (used here) is denoted M_*, and the second (used in [Bu-We]) is denoted E_*.

Proposition 4.7. *Let $\underline{A} \in S_{gss}(k)$, and let $M = M(A[p])$ be the covariant Dieudonné module of $A[p]$. Then M has a basis over k denoted $\{e_1, e_2, e_3, f_1, f_2, f_3\}$ such that*

(i) $\mathcal{O}_E$ acts on the e_i via Σ and on the f_i via $\overline{\Sigma}$.

(ii) The antisymmetric pairing induced by the principal polarization ϕ is given by $\langle e_i, f_j\rangle = (-1)^j\delta_{ij}$, $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$.

(iii) F and V are given by the following table:

	e_1	e_2	e_3	f_1	f_2	f_3
F	$-f_3$	0	0	0	e_1	e_2
V	0	0	f_1	e_2	e_3	0

By this we mean that $Fe_1^{(p)} = -f_3$, $Ve_3 = f_1^{(p)}$, *etc. In particular,* $\mathrm{Lie}(A) = M[V] = \langle e_1, e_2, f_3\rangle$.

Let $\underline{A} \in S_\mu^\dagger(R)$, where R is an arbitrary k-algebra.

Lemma 4.8. *The R-subgroup scheme $\alpha_p(A^{(p)}) = A^{(p)}[\mathrm{Fr}] \cap A^{(p)}[\mathrm{Ver}]$ is finite flat of rank p, and $\mathcal{O}_E$-stable.*

Proof. We have already encountered the lemma when A was μ-ordinary. The extension to the gss stratum works the same. It is enough to prove the lemma for the universal abelian scheme $\mathcal{A}$ over $S_\mu^\dagger$. In this case $\alpha_p(\mathcal{A}^{(p)})$ is clearly finite and $\mathcal{O}_E$-stable, and its fibers all have the same rank p, as follows from Proposition 4.7. Let us make this point clear, because the proposition only deals with fibers over closed points. Let ξ be any point of $S_\mu^\dagger$ (not necessarily closed), and $\overline{\{\xi\}}$ its closure (a point, a curve, or an irreducible surface). By the open-ness of the flat locus there is a non-empty connected open subset $\xi \in U \subset \overline{\{\xi\}}$ such that $\alpha_p(\mathcal{A}^{(p)})|_U$ is finite and flat over U, hence all its fibers, at all the geometric points of U, have the same rank. But $U(k)$ is Zariski dense in U, and at a k-point the proposition tells us that the rank is p. Hence the rank is p at ξ as well. Since $S_\mu^\dagger$ is *reduced*, by [Mu], Corollary on p. 432, $\alpha_p(\mathcal{A}^{(p)})$ is also flat. □

Proposition 4.9. *The finite flat group scheme $A[p]_{/R}$ has a canonical filtration*

$$Fil^3A[p] = 0 \subset Fil^2A[p] \subset Fil^1A[p] \subset Fil^0A[p] = A[p]$$

by finite flat group schemes, which agrees with the canonical filtration over S_μ. The graded pieces are $\mathcal{O}_E$-stable, rank p^2 and Raynaud. Furthermore, $Fil^1A[p] = Fil^2A[p]^\perp$ (with respect to the Weil pairing). Over S_{gss} every geometric fiber of $gr^2A[p]$ is of type $\alpha^*_{p^2,\Sigma}$, $gr^1A[p]$ is of type $\kappa\otimes\alpha_p$, and $gr^0A[p]$ is of type $\alpha_{p^2,\Sigma}$. Let $R = k$ and assume that $\underline{A} \in S_{gss}(k)$. Then, with the notation of Proposition 4.7,

$$Fil^2M = \langle e_2, f_3\rangle\,, \quad Fil^1M = \langle e_1, e_2, f_1, f_3\rangle\,.$$

We remark that unlike μ-ordinary abelian varieties, the above filtration *does not split, even if $R = k$*. As we shall see, $A[p]$ does not admit a subgroup scheme of type $\kappa\otimes\alpha_p$ at all, and while it does admit a unique subgroup scheme of type $\alpha_{p^2,\Sigma}$, this subgroup scheme is contained in $Fil^1A[p]$, so does not lift $gr^0A[p]$.

Proof. Define

$$Fil^2A[p] = \mathrm{Ver}(A^{(p)}[\mathrm{Fr}]) \simeq A^{(p)}[\mathrm{Fr}]/A^{(p)}[\mathrm{Fr}] \cap A^{(p)}[\mathrm{Ver}].$$

This image exists because it is a quotient by a finite flat subgroup scheme. It is a closed subgroup scheme of $A[p]$. Since $A^{(p)}[\mathrm{Fr}]$ is finite flat of rank p^3, the Lemma implies that $Fil^2A[p]$ is finite flat of rank p^2. It is furthermore isotropic for the Weil pairing $e_{p\phi}$ on $A[p]$ associated with the principal polarization ϕ. By Cartier duality

$$Fil^1A[p] = Fil^2A[p]^{\perp}$$

is finite flat of rank p^4. These group schemes are clearly $\mathcal{O}_E$-stable.

The remaining assertions concern the geometric fibers of $A[p]$, so we assume that $R = k$. Over the μ-ordinary locus this is the same filtration that we encountered before. Assume that we are over S_{gss}, and use Proposition 4.7. Let $M = M(A[p])$. Since F is induced by Ver and V is induced by Fr, we have to compute $F(M^{(p)}[V])$. This turns out to be $\langle e_2, f_3\rangle$. A simple check of the table in §3.1 reveals that $gr^2 = Fil^2A[p]$ is of type $\alpha^*_{p^2,\Sigma}$. Similar computations apply to gr^1 and gr^0. □

We can now complete the proof of Part (i) of Theorem 4.5. From the analysis of the local models it follows that Y_{gss} is a non-singular surface, mapping under the map π to the non-singular curve S_{gss}. This is clear at points where $H \simeq \mathfrak{G}[p]$. At a point $y \in Y_{gss}$ where $H \simeq \alpha_{p^2,\Sigma}$ or $H \simeq \alpha^*_{p^2,\Sigma}$ the formal neighborhood of y in $S_0(p)$ has two non-singular analytic branches which intersect transversally. Since there are *at least* two irreducible components of $S_0(p)$ passing through y, the vertical component Y_{gss} and (at least) one horizontal component, we conclude that there are *precisely* two such components, and that they are non-singular at y. In particular, Y_{gss} is non-singular at y too.

By the Noether-Enriques Theorem ([Bea] Theorem III.4 and Proposition III.7) it is enough to prove that for any $x \in S_{gss}(k)$, the scheme-theoretic fiber

$$Y_x \subset Y_{gss}$$

of the map $\pi : Y_{gss} \to S_{gss}$ is isomorphic to $\mathbb{P}^1$. We rely on the computation of local models at points $y \in Y_x$ in [Bel] III.4.3.8. These show that for any $y \in Y_x$ the map

$$\pi^* : \Omega_{S_{gss},x} \to \Omega_{Y_{gss},y}$$

is injective, and $\pi : Y_{gss} \to S_{gss}$ is smooth at y. We do not reproduce these computations here, but remark that the most problematic points turn out to be the y that lie on Z_{et} (where $H \simeq \alpha_{p^2,\Sigma}$). At such points the claim follows from §3.3.2, as the analytic branch of $S_0(p)$ at y determined by Y_{gss} is the one denoted there $\mathfrak{W}$, while $S_{gss} \subset S$ is given infinitesimally by the equation $r = 0$. Y_x is therefore *a reduced non-singular* curve.

Let M be the covariant Dieudonné module of $A[p]$, where $A = \mathcal{A}_x$, see Proposition 4.7. The fiber Y_x represents the *relative* moduli problem, sending a k-algebra R to the set of finite flat rank p^2 isotropic Raynaud $\mathcal{O}_E$-subgroup schemes $H \subset A_R[p]$. Note that since A_R is a constant abelian scheme over $Spec(R)$ both Fr and Ver are defined on it, base-changing from k to R the corresponding isogenies of A. Let

$$\alpha_p(A_R) = A_R[\mathrm{Fr}] \cap A_R[\mathrm{Ver}].$$

This is a constant (finite flat) subgroup scheme of rank p, and if $R = k$, its Dieudonné submodule is $\langle e_2\rangle$. Let

$$\beta_p(A_R) = A_R[\mathrm{Fr}^2] \cap A_R[\mathrm{Ver}^2] \cap A_R[p],$$

another constant (finite flat) subgroup scheme, of rank p^3. If $R = k$, its Dieudonné submodule is $\langle e_2, f_1, f_3\rangle$. We claim that

$$\alpha_p(A_R) \subset H \subset \beta_p(A_R),$$

hence classifying $H_{/R}$ is the same as classifying finite flat rank p subgroups of $\beta_p(A_R)/\alpha_p(A_R)$. Since Y_x is a reduced non-singular curve, it is enough to check these inclusions when R is reduced and of finite type over k. Since the closed points of $Spec(R)$ are then dense, we may assume $R = k$. But over k, Ver and Fr are nilpotent on H, which is of rank p^2, so both Ver^2 and Fr^2 must kill it. On the other hand, H must contain an α_p-subgroup, because it is local with a local Cartier dual.

Now $\beta_p(A_R)/\alpha_p(A_R)$ is nothing but α_p^2 (of type $(\overline{\Sigma}, \overline{\Sigma})$) and it is well-known that the moduli problem of classifying its rank-p subgroups is represented by $\mathbb{P}^1_{/k}$. One checks that the isotropy and Raynaud conditions are automatically satisfied for such an H.

Let $R = k$. The subgroup scheme H is completely determined by its Dieudonné submodule

$$N_\lambda = \langle e_2, \lambda_1 f_1 + \lambda_3 f_3 \rangle$$

where $\lambda = (\lambda_1 : \lambda_3) \in \mathbb{P}^1(k)$. Here $N_0 = N_{(0:1)} = M(H)$ if $H = Fil^2(A[p]) \simeq \alpha_{p^2}^*$. Similarly, $N_\infty = N_{(1:0)} = M(H)$ where $H \simeq \alpha_{p^2}$ because $N_{(1:0)}$ is killed by F and V^2 but not by V. For all other values of $\lambda \neq 0, \infty$, $N_\lambda = M(H)$ where H is of type $\mathfrak{G}[p]_\Sigma$, because N_λ is killed by F^2 and V^2 but the kernels of F or V are only 1-dimensional.

Part (ii): Let us show that the totality of points $(\underline{A}, H) \in Y_{gss}(k)$ where $H \simeq \alpha_{p^2}^*$, makes up a *curve* Z_m, that π induces an isomorphism of this curve onto S_{gss}, and that the closure of Y_m intersects Y_{gss} transversally in this curve. For this purpose, consider the section

$$\sigma_m : S_\mu^\dagger \to S_0(p)$$

mapping an R-valued point $\underline{A}$ to $(\underline{A}, H)$, where $H = Fil^2 A[p] = \mathrm{Ver}(A^{(p)}[\mathrm{Fr}])$. The image of the section is a surface isomorphic to the base, intersecting Y_μ in its connected component Y_m and Y_{gss} in the curve Z_m. Finally, the transversality of the intersection of the closure of Y_m and Y_{gss} follows from the calculation of the completed local ring of $S_0(p)$ at a point $y \in Z_m$, see §3.2.

Part (iii): We turn our attention to the points $(\underline{A}, H) \in Y_{gss}(k)$ where $H \simeq \alpha_{p^2}$. The condition $\mathrm{Ver}(H^{(p)}) = 0$ is a *closed* condition on the moduli problem $S_0(p)$. It is satisfied throughout Y_{et} and on Y_{gss} it holds precisely at the given points where $H \simeq \alpha_{p^2}$. We claim that this set forms a curve Z_{et}, which is the intersection of the closure of Y_{et} and Y_{gss}. Indeed, π being proper, the closure of Y_{et} must meet *every* fiber Y_x for $x \in S_{gss}(k)$, and such a fiber has a unique point where $H \simeq \alpha_{p^2}$. That the intersection is transversal follows as before from §3.2.

Write $Y_{et}^\dagger = Y_{et} \cup Z_{et}$. The computations in §3.2 show that $Y_{et}^\dagger$ is non-singular. So is $Y_{et}^{\dagger\sigma}$.

We claim that since $\pi : Y_{et}^\dagger \to S$ factors through $Fr_{Y/k} : Y_{et}^\dagger \to Y_{et}^{\dagger\sigma}$ over the dense open set Y_{et}, it factors through $Fr_{Y/k}$ everywhere. Indeed, consider the local ring $\mathcal{O}_{S,x}$ at $x = \pi(y) \in S$, where $y \in Y_{et}^\dagger$ is a closed point. Let $y^{(p)} = Fr_{Y/k}(y) \in Y_{et}^{\dagger\sigma}$. For the function fields we have

$$k(S) \subset k(Y_{et}^{\dagger\sigma}) = k(Y_{et}^\dagger)^p \subset k(Y_{et}^\dagger).$$

Thus $\mathcal{O}_{S,x} \subset k(Y_{et}^\dagger)^p \cap \mathcal{O}_{Y_{et}^\dagger, y}$. But the ring on the right is just $\mathcal{O}_{Y_{et}^{\dagger\sigma}, y^{(p)}}$, because y is the *unique* point above $y^{(p)}$ in $Y_{et}^\dagger$ and $\mathcal{O}_{Y_{et}^{\dagger\sigma}, y^{(p)}}$ is normal. For every affine subset $U = Spec(R) \subset Y_{et}^{\dagger}$ the ring R is the intersection of all the $\mathcal{O}_{Y_{et}^{\dagger},y}$ for closed points $y \in U$, and similarly for $Fr_{Y/k}(U) \subset Y_{et}^{\dagger\sigma}$. This proves the claim.

Thus $\overline{\pi}_{et}$ extends to a morphism from $Y_{et}^{\dagger\sigma}$ to $S_{\mu}^{\dagger}$. It is a finite morphism, because $\pi_{et}: Y_{et}^{\dagger} \to S_{\mu}^{\dagger}$ is finite. Both source and target are non-singular surfaces, so by [Eis] 18.17 it is also flat, totally ramified of degree p. It therefore defines a line sub-bundle TS^{ur} of unramified directions in the tangent bundle there, as in Lemma 4.3, now over all of $S_{\mu}^{\dagger}$. Recall that the special sub-bundle TS^{+} was defined on the whole of $S_{\mu}^{\dagger}$ as well. The two line sub-bundles TS^{+} and TS^{ur} coincide over S_{μ} (Theorem 4.4), hence also over S_{gss}, by continuity.

As TS^{+} is tangent to S_{gss} along the general supersingular stratum, we get, from the discussion following Lemma 4.3, that $\overline{\pi}_{Z}: Z_{et}^{\sigma} \to S_{gss}$ is unramified. As it is also totally ramified (bijective on k-points), it is an isomorphism.

In retrospect, we can look at the factorization $\pi_{et} = \overline{\pi}_{et} \circ Fr_{Y/k}$ also from the moduli point of view as follows. Consider the abelian scheme $\mathcal{B} = \overline{\pi}_{et}^{*}\mathcal{A}$ which is the pull-back of the universal abelian scheme over $S_{\mu}^{\dagger}$ to $Y_{et}^{\dagger\sigma}$. Consider also the universal abelian scheme $\mathcal{A}_1$ over $Y_{et}^{\dagger\sigma}$. Over the dense open subset Y_{et}^{σ} $\mathcal{A}_1 \simeq \mathcal{B}^{(p)}$, as was shown in the proof of Theorem 4.1. It follows that this relation persists over Z_{et}^{σ}, and *a-fortiori* we may define $\overline{\pi}_{et}$ by sending $(\underline{A}_1, H_1) \in Y_{et}^{\dagger\sigma}(R)$ to $\langle p \rangle^{-1} \operatorname{Ver}(\underline{B}^{(p)}) \in S_{\mu}^{\dagger}(R)$.

Part (iv): By Lemma 4.8, and the arguments used before, $\mathrm{Fr}(A^{(p)}[\mathrm{Ver}])$ is a finite flat rank-p^2 isotropic Raynaud subgroup scheme of $A^{(p^2)}[p]$, for any $A \in S_{\mu}^{\dagger}(R)$, for any k-algebra R. Since ρ_{et} is now defined functorially in terms of the moduli problems, it is a well defined morphism. The argument is identical to the one used for the proof of Part (v) of Theorem 4.1.

Since the equality $\rho_{et} \circ \pi_{et} = Fr_{Y/k}^{2}$ has already been established on $Y_{et} = Y_{et}(k)$, it extends by continuity to $Y_{et}^{\dagger}$. The relation $\rho_{et} \circ \overline{\pi}_{et} = Fr_{Y^{\sigma}/k}$ follows from $\rho_{et} \circ \pi_{et} = Fr_{Y/k}^{2}$ since $\pi_{et} = \overline{\pi}_{et} \circ Fr_{Y/k}$ and $Fr_{Y/k}$ is faithfully flat. The remaining assertions on ρ_{et} also follow from this relation. This concludes the proof of Theorem 4.5.

4.2.3. *A closer look at Example 3.3.2.* It is instructive to look again at the diagram

$$\widehat{\mathcal{O}}_{S,x} \hookrightarrow \widehat{\mathcal{O}}_{S_0(p),y}$$

at a point $y \in Z_{et}(k)$. We have found the local models $\widehat{\mathcal{O}}_{S,x} \simeq k[[r,s]]$ and $\widehat{\mathcal{O}}_{S_0(p),y} \simeq k[[a,b,c]]/(bc)$. The map between the local models is

$$r \mapsto 0, \ \ s \mapsto b.$$

This is far from the correct map between the completed local rings, which should be injective. Let $\widehat{\mathcal{O}}_{\mathfrak{W}}$ and $\widehat{\mathcal{O}}_{\mathfrak{Z}}$ be the quotients of $\widehat{\mathcal{O}}_{\mathfrak{Y}} = \widehat{\mathcal{O}}_{S_0(p),y}$ which were introduced in §3.3.2. The first is obtained by modding out (c), and is the analytic branch determined by the inclusion $Y_{gss} \subset S_0(p)$. The second is obtained by modding out (b), and is the analytic branch determined by the inclusion $Y_{et}^{\dagger} \subset S_0(p)$.

Claim 4.10. The diagram $\widehat{\mathcal{O}}_{S,x} \to \widehat{\mathcal{O}}_{\mathfrak{W}}$ is isomorphic to the diagram

$$k[[r,s]] \to k[[a,b]], \ \ s \mapsto b + a^{p}, \ \ r \mapsto 0,$$

and the diagram $\widehat{\mathcal{O}}_{S,x} \to \widehat{\mathcal{O}}_{\mathfrak{Z}}$ is isomorphic to the diagram

$$k[[r,s]] \hookrightarrow k[[a,c]], \quad r \mapsto c^{p^2}, \ s \mapsto a^p.$$

This is more than could be deduced from the local models alone.

Proof. After a change of variable we may assume that $r = 0$ is the equation of S_{gss} in a formal neighborhood of x on S. Therefore r maps to 0 in $\widehat{\mathcal{O}}_{\mathfrak{W}}$. The local parameter s projects (modulo (r)) to a local parameter of the curve S_{gss}. We already know that it should map to b modulo pth powers. Since $b = c = 0$ is the formal equation of the curve Z_{et} (the intersection of the two analytic branches) on $\mathfrak{W}$, and since the map $Z_{et} \to S_{gss}$ is purely inseparable of degree p, we see that we may choose a so that $s \mod (b) = a^p$. A last change of variables allows us to assume that actually $s = b + a^p$.

The second diagram is treated similarly. Here the key point is to recall that the map π_{et} from $\mathfrak{Z}$ to $Spf(\widehat{\mathcal{O}}_{S,x})$ factors through Fr. The resulting map $\overline{\pi}_{et}$ on $\mathfrak{Z}^\sigma$ was shown to be of degree p and unramified in the direction of S_{gss}. □

Both diagrams are compatible with $\widehat{\mathcal{O}}_{S,x} \to \widehat{\mathcal{O}}_{\mathfrak{Y}} = \widehat{\mathcal{O}}_{S_0(p),y}$ being given by

$$r \mapsto c^{p^2}, \ s \mapsto b + a^p.$$

4.3. **The ssp strata.**

4.3.1. *The superspecial combs.* We now turn our attention to the superspecial strata of $S_0(p)$. Let $x \in S_{ssp}(k)$ and $Y_x = \pi^{-1}(x)$. We shall contend ourselves with the determination of the *reduced* scheme Y_x^{red}, of finite type over k. The scheme theoretic pre-image of x will not be reduced along the component denoted below F_x, see the discussion following the theorem.

Theorem 4.11. (i) *Y_x^{red} is the union of $p+2$ projective lines, arranged as follows. One irreducible component, which we call F_x, intersects the remaining $p+1$ projective lines transversally, each at a different point $\zeta \in F_x$. With a natural choice of a coordinate on F_x, this ζ can be taken to be*[7] *a $p+1$ root of -1. These $p+1$ projective lines, which we label as $G_x[\zeta]$, are disjoint from each other.*

A point $(\underline{A}, H) \in Y_x(k)$ lies on F_x if and only if $H \simeq \kappa \otimes \alpha_p$. If this is the case, the invariant $\gamma(\underline{A}, H) = 1$ if $(\underline{A}, H)$ lies on a non-singular point of Y_x, and is equal to 2 if it lies at the intersection of F_x and some $G_x[\zeta]$ (i.e. if it is the point ζ). Finally, if $(\underline{A}, H)$ lies on $G_x[\zeta]$ but not on F_x, the group $H \simeq \mathfrak{G}[p]_\Sigma$.

(ii) *The* closure *$\overline{Y}_\mu$ of $Y_\mu = Y_m \cup Y_{et}$ in $S_0(p)$ intersects Y_x^{red} in F_x.*

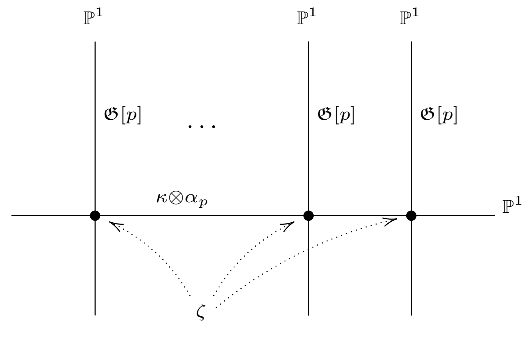
(iii) *Let W be the* closure *of an irreducible component of Y_{gss}. Then W is a $\mathbb{P}^1$-bundle over an irreducible component $\mathscr{C} = \pi(W)$ of S_{ss}. If $x \in S_{ssp}$ and $W_x = W \cap Y_x^{red}$ then W_x is one of the $G_x[\zeta]$. Precisely one such W passes through $G_x[\zeta]$ for a given x and ζ. Thus the closures of the irreducible components of Y_{gss} do not intersect each other.*

(iv) *The closures of the curves $W \cap Z_{et}$ and $W \cap Z_m$ intersect $G_x[\zeta]$ at the point $\zeta = G_x[\zeta] \cap F_x$.*

See Figures 4.1, 4.2. We refer to the irreducible components W of the closure of Y_{gss} as the *supersingular (ss) screens.* We refer to the Y_x for x superspecial as the

[7]This is a non-trivial statement, as it has consequences for the cross ratio of the intersection points, which is independent of the chosen coordinate on the basis of the comb.

FIGURE 4.2. The fiber of $S_0(p)$ above a superspecial point



superspecial (ssp) combs. The component F_x, which we draw horizontally, is called the *base* of the comb, and the vertical components $G_x[\zeta]$ are called its *teeth.* The points ζ are called the *roots* of the teeth.

Proof. (i) Let $A = \mathcal{A}_x$. We first analyze what happens on the level of Dieudonné modules. Fix a model of $\mathfrak{G}_\Sigma$ over k, let $\mathfrak{G}_{\overline{\Sigma}} = \mathfrak{G}_\Sigma^\sigma$ and fix the polarization

$$\lambda : \mathfrak{G}[p]_\Sigma \xrightarrow{\sim} \mathfrak{G}[p]_\Sigma^D = \mathfrak{G}[p]_{\overline{\Sigma}}$$

so that the resulting pairing on $\mathfrak{G}[p]_\Sigma$, $(x,y) \mapsto \langle x, \lambda(y)\rangle$ is alternating. The group scheme $A[p]$ is isomorphic to

$$\mathfrak{G}[p]_\Sigma^2 \times \mathfrak{G}[p]_{\overline{\Sigma}},$$

so that the polarization induced on it by ϕ_x is the product $\lambda^2 \times \lambda^\sigma$ of the polarizations of the three factors. Consequently [Bu-We], the polarized Dieudonné module $M = M(A[p])$ is given by $M = \langle e_1, e_2, e_3, f_1, f_2, f_3\rangle_k$, where the endomorphisms act on the e_i via Σ and on the f_i via $\overline{\Sigma}$, where $\langle e_i, f_j\rangle = \delta_{ij}$, $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$ and where the action of F and V is given by the table

	e_1	e_2	e_3	f_1	f_2	f_3
F	0	0	$-f_3$	e_1	e_2	0
V	0	0	f_3	$-e_1$	$-e_2$	0

By this we mean $Fe_3^{(p)} = -f_3,\ \ Ve_3 = f_3^{(p)}$ etc.

Let $H \subset A[p]$ be as in $(S_0(p))$. Since $M(H)$ is balanced we may write

$$M(H) = \langle \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3,\ \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3\rangle\,.$$

The conditions that have to be satisfied are $V(M(H)) \subset M(H)^{(p)},\ \ F(M(H)^{(p)}) \subset M(H)$, and the isotropy condition

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0.$$

Observe that $M(H)$ contains $\beta_1^p e_1 + \beta_2^p e_2$. If $\alpha_3 \neq 0$ this forces $\beta_1 = \beta_2 = 0$, and then the isotropy condition gives also $\beta_3 = 0$, an absurd. Therefore $\alpha_3 = 0$. We distinguish two cases.

Case I (the base of the comb): $\beta_1 = \beta_2 = 0$. This case is characterized by the fact that $M(H)$ is killed by both V and F, so that $H \simeq \kappa \otimes \alpha_p$. We may take $\beta_3 = 1$ and H is classified by

$$\zeta = (\alpha_1 : \alpha_2) \in \mathbb{P}^1(k).$$

Consider in this case the group $H^\perp/H$. Its Dieudonné module is given by

$$M(H^\perp/H) = \langle e_1, e_2, -\alpha_2 f_1 + \alpha_1 f_2, f_3 \rangle \mod \langle \alpha_1 e_1 + \alpha_2 e_2, f_3 \rangle .$$

An easy check shows that $H^\perp/H$ is of type $\mathfrak{G}[p]_\Sigma$, unless $\zeta^{p+1} = -1$, where it is of type $\kappa \otimes \alpha_p$. The invariant $\gamma(\underline{A}, H) = \dim_k \mathrm{Lie}(H^\perp/H)$ is thus 1 in the former case, and 2 in the latter.

Case II (the teeth of the comb): $(\beta_1 : \beta_2) \in \mathbb{P}^1(k)$. Then, $\zeta = (\alpha_1 : \alpha_2) = (\beta_1^p : \beta_2^p)$ and the isotropy condition forces

$$\alpha_1^{p+1} + \alpha_2^{p+1} = 0,$$

i.e. $\zeta^{p+1} = -1$. Fix ζ, hence the point $(\beta_1 : \beta_2)$. The H in question are classified by $\beta_3 \in \mathbb{A}^1(k)$. Their $M(H)$ is killed by V^2 and F^2 but neither by V nor by F, so H must be isomorphic to $\mathfrak{G}[p]_\Sigma$. We observe that when $\beta_3 = \infty$, i.e. $(\beta_1 : \beta_2 : \beta_3) = (0 : 0 : 1)$ we are back in Case I. This is the root of the tooth.

This analysis strongly *suggests* the picture outlined in Part (i), but does not quite *prove* it. To give a rigorous proof we proceed as follows. The fiber Y_x represents the relative moduli problem assigning to any k-algebra R the set of subgroup schemes $H \subset A_R[p]$ of type $(S_0(p))$. Observe that since A_R is constant, both Fr and Ver are defined on it, by base change from A. We let $\alpha_p(A_R) = A_R[\mathrm{Fr}] \cap A_R[\mathrm{Ver}]$ and

$$\alpha_p(H) = H \cap \alpha_p(A_R).$$

Case I. Consider first the closed locus $F_x \subset Y_x^{red}$ defined by

$$\mathrm{Fr}(H) = 0, \quad \mathrm{Ver}(H) = 0.$$

Over F_x we have $\alpha_p(H) = H$. Indeed, since F_x is a reduced curve it is enough to check the inclusion $H \subset \alpha_p(A_R)$ at geometric points, where it follows from the analysis of their Dieudonné modules as above. However, $\alpha_p(A_R) = \alpha_{p,\Sigma}^2 \times \alpha_{p,\overline{\Sigma}}$, so the problem becomes that of classifying $\mathcal{O}_E$-subgroup schemes of type $\kappa \otimes \alpha_p = \alpha_{p,\Sigma} \times \alpha_{p,\overline{\Sigma}}$ in it. As the factor of type $\alpha_{p,\overline{\Sigma}}$ is unique, this is the same as classifying subgroup schemes of rank p in $\alpha_{p,\Sigma}^2$, a problem that is represented by $\mathbb{P}^1_{/k}$. This gives us the base of the comb, whose k-points are described in terms of their Dieudonné submodules as before.

Case II. Let G_x be the open curve which is the complement of F_x in Y_x^{red}. Over G_x, the group $\alpha_p(H)$ is of rank p. Observe that $H \cap \mathfrak{G}[p]_\Sigma^2$ is non-zero, because otherwise, via projection to the third factor, H would be of type $\mathfrak{G}[p]_{\overline{\Sigma}}$, which is forbidden. It follows that $\alpha_p(H \cap \mathfrak{G}[p]_\Sigma^2)$ is also non-zero, so must coincide with $\alpha_p(H)$. The $\alpha_p \subset \mathfrak{G}[p]_\Sigma^2$ were classified before by $\mathbb{P}^1_{/k}$. Our $\alpha_p(H)$ is therefore classified by $\zeta = (\alpha_1 : \alpha_2) \in \mathbb{P}^1(R)$. The Dieudonné module computation above shows that ζ restricts, at every geometric point, to a $p+1$ root of -1. However, the equation $X^{p+1} + 1 = 0$ is separable, so if R is a local ring in characteristic p and $\zeta \in R$ satisfies this equation modulo $\mathfrak{m}_R$, it satisfies it in R. This means that $\alpha_p(H)$ is locally constant over $Spec(R)$. There remains the classification of $H/\alpha_p(H)$, which sits in general "diagonally" in $(\mathfrak{G}[p]_\Sigma^2/\alpha_p(H)) \times \mathfrak{G}[p]_{\overline{\Sigma}}$. The same argument that was used to show that $\alpha_p(H)$ is constant, shows now that the projection

K of $H/\alpha_p(H)$ to $(\mathfrak{G}[p]^2_\Sigma/\alpha_p(H))$ is constant, and in fact is given by the point $(\beta_1 : \beta_2) = (\alpha_1^p : \alpha_2^p) \in \mathbb{P}^1(R)$. The classification of $H/\alpha_p(H)$ is therefore the same as the classification of all the R-morphisms of this fixed K to $\alpha_p(\mathfrak{G}[p]_{\overline{\Sigma}})$. This moduli problem, of classifying morphisms from a fixed copy of α_p to another, is represented by $\mathbb{A}^1_{/k}$.This gives the tooth of the comb labeled $G_x[\zeta]$.

The two cases (I) and (II) cover Y_x^{red}. It remains to remark that the intersection of the closure of $G_x[\zeta]$ with F_x is transversal. This follows, as usual, from §3.2.

(ii) The condition $\mathrm{Fr}(H) = 0$ is a closed condition and holds throughout Y_m. It therefore holds also in the intersection of its closure $\overline{Y}_m$ with Y_x. As this condition is not satisfied on the teeth of the comb (outside their roots), the closure $\overline{Y}_m$ intersects Y_x in F_x. The same argument, applied to the condition $\mathrm{Ver}(H^{(p)}) = 0$ proves that the closure $\overline{Y}_{et}$ of Y_{et} also intersects Y_x in F_x. As we have previously shown that $Y_m^\dagger$ and $Y_{et}^\dagger$ are disjoint, $\overline{Y}_m$ and $\overline{Y}_{et}$ intersect only in the superspecial locus, and their intersection is the union of the F_x for $x \in S_{ssp}$. This intersection is transversal, as follows from the description of the completed local rings in §3.2.

(iii) The classification of the completed local rings of $S_0(p)$ shows that through a point $\zeta \in F_x$ which is not a root of a tooth (i.e. $\zeta^{p+1} \neq -1$) pass only 2 analytic branches. As $\overline{Y}_{et}$ and $\overline{Y}_m$ already account for these two analytic branches, the closure W of a connected component of Y_{gss} can only meet Y_x in one of the lines $G_x[\zeta]$. Since the points of $G_x[\zeta]$ are generically non-singular on $S_0(p)$, exactly one such W passes through every $G_x[\zeta]$. These W are non-singular surfaces projecting to a component $\mathscr{C}$ of S_{ss} and the fiber above each geometric point (including now the superspecial points) is $\mathbb{P}^1$. By the Noether-Enriques theorem quoted before, they are $\mathbb{P}^1$-bundles.

(iv) The condition $\mathrm{Fr}(H) = 0$ is a closed condition and holds throughout Z_m. It therefore holds also on its closure. It follows that this closure intersects a tooth $G_x[\zeta]$ at its root, because points other than the root support an H of type $\mathfrak{G}[p]_\Sigma$ which is not killed by Fr. A similar argument invoking the condition $\mathrm{Ver}(H^{(p)}) = 0$ proves that the closure of Z_{et} also meets the teeth of the combs in their roots. The two curves Z_{et} and Z_m, which are disjoint over the gss locus, intersect over every superspecial point.

This concludes the proof of the theorem. □

4.3.2. *The maps to $S^\#$.* Recall the construction of the blow-up $S^\#$ of S at the ssp points, given in §2.3. The exceptional divisor E_x at $x = [\underline{A}] \in S_{ssp}(k)$ classifies lines in $\mathcal{P} = \omega_{A/k}(\Sigma)$.

The isomorphism $\pi_m : Y_m^\dagger \simeq S_\mu^\dagger$ extends to an isomorphism

$$\pi_m^\# : \overline{Y}_m \simeq S^\#.$$

In terms of the moduli problems, it sends $(\underline{A}, H) \in \overline{Y}_m(R)$ to $(\underline{A}, \ker(\omega_{A/R}(\Sigma) \to \omega_{H/R}(\Sigma))$. If $R = k$, A is μ-ordinary and $H = A[p]^m$ then

$$\ker(\mathcal{P} = \omega_{A/k}(\Sigma) \to \omega_{H/k}(\Sigma)) = \mathcal{P}_0 = \mathcal{P}[V]$$

is uniquely determined by A. The same holds if A is gss and $H = Fil^2 A[p]$. On the other hand if $x = [\underline{A}]$ is ssp then $\mathcal{P}[V]$ is the whole of $\mathcal{P}$ and H "selects" a line in it. This establishes an isomorphism

$$F_x \simeq E_x.$$

From the universal property of blow-ups, the projection $\pi_{et} : \overline{Y}_{et} \to S$ also factors through a map

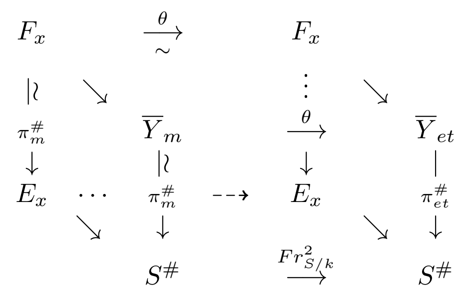
$$\pi_{et}^{\#} : \overline{Y}_{et} \to S^{\#}$$

mapping F_x to E_x. This map is now proper and quasi-finite, hence finite. The two surfaces are non-singular, so the map is also flat. Its degree is p^3. We have seen that on the open dense $Y_{et}^{\dagger}$ it factors through $Fr_{Y/k}$, i.e.

$$\pi_{et} = \overline{\pi}_{et} \circ Fr_{Y/k}$$

and this forces the map $\pi_{et}^{\#}$ to factor in the same way $\pi_{et}^{\#} = \overline{\pi}_{et}^{\#} \circ Fr_{Y/k}$ over the whole of $\overline{Y}_{et}$. The map $\overline{\pi}_{et}^{\#}$ is finite flat totally ramified of degree p, and it can be shown that it is ramified of degree p along the lines F_x^{σ}. Thus $\pi_{et}^{\#}$ is ramified of degree p^2 along F_x (and of an extra degree p in a normal direction).

We emphasize that $\pi_m^{\#}$ and $\pi_{et}^{\#}$ do not agree on F_x. Instead, the following diagram extends the one from Corollary 4.6.



The degrees of the maps in the front square (on surfaces) are $p^3 \times p = p^4 \times 1$. In the back square (on projective lines) they are $p^2 \times 1 = p^2 \times 1$.

4.3.3. *How embedded modular curves meet F_x.* Let X be the special fiber of the modular curve $\mathscr{X}$ which was constructed on $\mathscr{S}$ in §1.4. Consider the modular curve $\mathscr{X}_0(p)$ parametrizing, in addition to the triple $\underline{B} = (B, \nu, M)$, also a finite flat subgroup scheme $H_B \subset B[p]$ of rank p. Enhance the map $\mathscr{Z}_0 \times_{\underline{\mathrm{Isom}}(\mathbb{Z}/N\mathbb{Z}, \mu_N)} \mathscr{X} \to \mathscr{S}$ to a map

$$\mathscr{Z}_0 \times_{\underline{\mathrm{Isom}}(\mathbb{Z}/N\mathbb{Z}, \mu_N)} \mathscr{X}_0(p) \to \mathscr{S}_0(p), \quad (\underline{B}_0, \underline{B}, H_B) \mapsto (\underline{A}, H)$$

by setting H to be the image of $\mathcal{O}_E \otimes H_B$ in $A(\underline{B}_0, \underline{B})$. Note that since H_B is automatically isotropic, and the polarization on A is induced from the polarizations of B and B_0, this H is isotropic. It is also clearly Raynaud.

Proposition 4.12. *Let $X_0(p)$ be the special fiber of $\mathscr{X}_0(p)$. Let $x \in S_{ssp}(k)$. Then under the above morphism $X_0(p)$ meets the component $F_x \subset Y_x$ in a point ζ satisfying*

$$\zeta \in \kappa, \quad \zeta^{p+1} \neq -1.$$

Thus both the supersingular screens on $S_0(p)$ and the modular curves cross the superspecial strata F_x at $\mathbb{F}_{p^2}$-rational points, but while the supersingular screens cross at a ζ satisfying $\zeta^{p+1} = -1$, the modular curves cross at the remaining ones.

Proof. As we shall see in the next chapter, the κ-rational $\zeta \in F_x$ are characterized by the fact that $A' = A/H$ is superspecial. At other points of F_x this A' is supersingular of a-number 2, but not superspecial. For the pair (A, H) that is constructed from the "elliptic curve data" on $X_0(p)$, it is easily seen that A' is either μ-ordinary or superspecial, depending on whether B is ordinary or supersingular.

Among these κ-rational points the points with $\zeta^{p+1} = -1$ are characterized by $\gamma(\underline{A}, H) = 2$, i.e. the group $H^\perp/H = \ker(\psi)$ being isomorphic to $\kappa \otimes \alpha_p$. All the rest have $\gamma = 1$. In our case, $H = \mathcal{O}_E \otimes H_B$ is maximal isotropic in $A_1(\underline{B})[p]$, so its annihilator in $A[p] = A_1[p] \times B_0[p]$ is $H \times B_0[p]$. it follows that

$$H^\perp/H \simeq B_0[p] \simeq \mathfrak{G}[p]_\Sigma$$

and $\gamma = 1$. □

5. The structure of $\widetilde{S}$

5.1. **The global structure of $\widetilde{S}$.** The moduli space $\widetilde{\mathscr{S}}$ was defined in Section 1.2.3. Typically, moduli spaces involving parahoric level structure are "complicated", and may involve issues such as non-reduced components, complicated singularities etc. It is interesting, and important for our further applications, that $\widetilde{\mathscr{S}}$ turns out to be quite simple. In essence, its special fiber is a collection of smooth surfaces intersecting transversally at a reduced non-singular curve.

5.1.1. *Flatness of* $\widetilde{\pi}$. The following proposition stands in sharp contrast to the non-flatness of π. It is also key to understanding the geometry of the surface

$$\mathscr{T} = \mathscr{S}_0(p) \times_{\widetilde{\mathscr{S}}} \mathscr{S}_0(p).$$

This surface, which is generically of degree $(p+1)(p^3+1)$ over the Picard modular surface $\mathscr{S}$, "is" the geometrization of the Hecke operator T_p. We intend to study it in a future work.

Proposition 5.1. *The morphism* $\widetilde{\pi} : \mathscr{S}_0(p) \to \widetilde{\mathscr{S}}$ *is finite flat of degree* $p+1$.

Proof. Both arithmetic surfaces are regular. The map $\widetilde{\pi}$ is proper, and, as we shall see below, analyzing its geometric fibers one-by-one, also quasi-finite. It is therefore finite. By [Eis], 18.17, it is flat. The degree can be read off in characteristic 0. □

From now on we concentrate on the structure of the geometric special fiber $\widetilde{S}_k$ of $\widetilde{\mathscr{S}}$ over k, and omit the subscript k. We study $\widetilde{S}$ together with the map

$$\widetilde{\pi} : S_0(p) \to \widetilde{S}$$

and make strong use of the facts that we have already established for $S_0(p)$.

5.1.2. *The fibers of* $\widetilde{\pi}$. To study the geometric fibers of π we had to study, for a given $\underline{A}$, the subgroup schemes $H \subset A[p]$ for which $(\underline{A}, H) \in S_0(p)(k)$. This was achieved by analyzing $M(A[p])$ and its 2-dimensional, isotropic, balanced $\mathcal{O}_E$-stable Dieudonné submodules. To study the geometric fibers of $\widetilde{\pi}$ we have to look, for a given $\underline{A}'$, for all the possible $(\underline{A}, H)$ yielding $\underline{A}'$ upon the process of dividing by H and descending the polarization. Equivalently, by Proposition 1.4, we have to look for all the subgroup schemes J such that $(\underline{A}', J) \in \widetilde{S}_0(p)(k)$. This reduces the computation of the fibers of $\widetilde{\pi}$ to Dieudonné-module computations, as was the case with π. However, starting with one $(\underline{A}, H)$ mapping under $\widetilde{\pi}$ to $\underline{A}'$, finding all the others in the fiber above $\underline{A}'$ requires in general the knowledge of $M(A[p^2])$ and not only of $M(A[p])$. This makes the following sections technically more complicated than the previous ones.

5.1.3. *The stratification of* $\widetilde{S}$. We suppress (ι',η') from the notation and refer to R-points of $\widetilde{S}$ (R a k-algebra) as (A',ψ). Given $(A',\psi)\in\widetilde{S}(k)$ the subgroup scheme

$$\ker(\psi)\subset A'[p]$$

is of rank p^2, self-dual (i.e. isomorphic to its Cartier dual), stable under $\iota'(\mathcal{O}_E)$ and Raynaud. Its Lie algebra $\mathrm{Lie}(\ker(\psi))$ is 1 or 2-dimensional[8], and carries an action of κ. We call its type the *type (or signature) of* $\ker(\psi)$ and denote it by $\tau(\psi)$. Similarly the maximal α_p-subgroup of $A'[p]$ is of rank p,p^2 or p^3, and the κ-type of its Lie algebra is called the *a-type* of A', and denoted $a(A')$.

Theorem 5.2. (i) *The surface $\widetilde{S}$ is the union of 7 disjoint, locally closed, nonsingular strata $\widetilde{S}_*[**]$, as shown in the table. The name of each stratum indicates the type of $A'_{\widetilde{x}}$ for $\widetilde{x}$ in the stratum (μ-ordinary, gss or ssp), and, in brackets, the type of* $\ker(\psi)$. *The last column indicates what types of $(\underline{A},H)$ lie in $\widetilde{\pi}^{-1}(\widetilde{x})$. The first entry in the last column refers to the stratum of S in which $\underline{A}$ lies. The second refers to the type of H ($\mathfrak{G}$ stands for $\mathfrak{G}[p]$). If $\underline{A}$ is ssp there is a third entry, which we now explain.*

Recall that the ssp strata of $S_0(p)$ are unions of projective lines admitting a natural coordinate ζ. The third entry refers to ζ. Depending on whether $\zeta\in\mathbb{F}_{p^2}$ or not, and in the case of the components F_x, also on whether it is a $p+1$ root of -1, $\widetilde{\pi}(A,H)$ may land in different strata of $\widetilde{S}$.

	Stratum of $\widetilde{x}$	*dim.*	$\tau(\psi)$	$a(A')$	$\#\widetilde{\pi}^{-1}(\widetilde{x})$	$\widetilde{\pi}^{-1}(\widetilde{x})$
1	$\widetilde{S}_\mu$	2	Σ	Σ	2	$(\mu, et/m)$
2	$\widetilde{S}_{gss}[\overline{\Sigma}]$	2	$\overline{\Sigma}$	$\Sigma,\overline{\Sigma}$	$p+1$	$(gss,\mathfrak{G})/(ssp,\mathfrak{G},\neg\mathbb{F}_{p^2})$
3	$\widetilde{S}_{gss}[\Sigma,\overline{\Sigma}]$	1	$\Sigma,\overline{\Sigma}$	$\Sigma,\overline{\Sigma}$	2	$(gss,\alpha_{p^2}/\alpha^*_{p^2})$
4	$\widetilde{S}_{gss}[\Sigma]$	1	Σ	Σ,Σ	1	$(ssp,\kappa\otimes\alpha_p,\neg\mathbb{F}_{p^2})$
5	$\widetilde{S}_{ssp}[\overline{\Sigma}]$	0	$\overline{\Sigma}$	$\Sigma,\Sigma,\overline{\Sigma}$	$p+1$	$(ssp,\mathfrak{G},\mathbb{F}_{p^2})$
6	$\widetilde{S}_{ssp}[\Sigma,\overline{\Sigma}]$	0	$\Sigma,\overline{\Sigma}$	$\Sigma,\Sigma,\overline{\Sigma}$	1	$(ssp,\kappa\otimes\alpha_p,\ \sqrt[p+1]{-1})$
7	$\widetilde{S}_{ssp}[\Sigma]$	0	Σ	$\Sigma,\Sigma,\overline{\Sigma}$	1	$(ssp,\kappa\otimes\alpha_p,\mathbb{F}_{p^2}\neg\ \sqrt[p+1]{-1})$

(ii) *The closure relations between the various strata are described by the following diagram, where an arrow $X\to Y$ indicates specialization, i.e. that $Y\subset\overline{X}$.*

$$\begin{array}{ccccccc} & & & \widetilde{S}_\mu & & & \widetilde{S}_{gss}[\overline{\Sigma}] \\ & & \swarrow & & \searrow & \swarrow & | \\ & \widetilde{S}_{gss}[\Sigma] & & & & \widetilde{S}_{gss}[\Sigma,\overline{\Sigma}] & | \\ & \swarrow & & \searrow & \swarrow & & \downarrow \\ \widetilde{S}_{ssp}[\Sigma] & & & \widetilde{S}_{ssp}[\Sigma,\overline{\Sigma}] & & & \widetilde{S}_{ssp}[\overline{\Sigma}] \end{array}$$

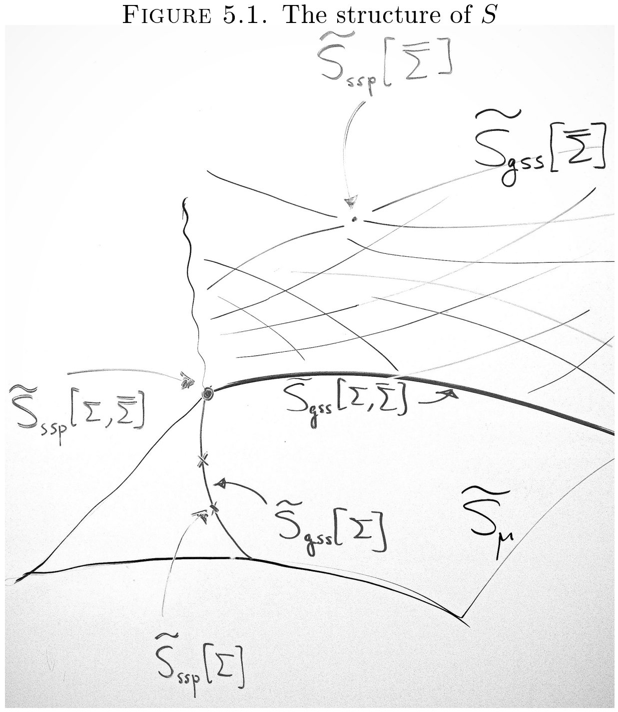
The strata $\widetilde{S}_{gss}[\Sigma,\overline{\Sigma}]$ and $\widetilde{S}_{ssp}[\Sigma,\overline{\Sigma}]$ are singular on $\widetilde{S}$, and the rest are nonsingular.

See Figure 5.1

Proof. The invariants $(\tau(\psi),a(A'))$ characterize the stratum in $\widetilde{S}$, and the seven cases in the last column are mutually exclusive and exhaustive. It is therefore enough to verify that starting with a point $(\underline{A},H)\in S_0(p)(k)$ in a prescribed

[8]If it were 0-dimensional, A' would be μ-ordinary and $\ker(\psi)\simeq\kappa\otimes\mathbb{Z}/p\mathbb{Z}$, but this group is not self-dual.

FIGURE 5.1. The structure of $\widetilde{S}$



stratum of $S_0(p)$, we end up with the right pair of invariants $(\tau(\psi), a(A'))$. For this we use the covariant Dieudonné module $M(A[p^\infty])$.

(1) If A is μ-ordinary, so is A', and vice versa. As in this case

$$A[p^\infty] \simeq (\mathcal{O}_E \otimes \mu_{p^\infty}) \oplus \mathfrak{G}_\Sigma \oplus (\mathcal{O}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p)$$

and H is either $\mathcal{O}_E \otimes \mu_p$ or $\mathcal{O}_E \otimes \mathbb{Z}/p\mathbb{Z}$, $H^\perp/H \simeq \mathfrak{G}[p]_\Sigma$ so $\tau(\psi) = \Sigma$. Since upon dividing by H we get $A'[p^\infty] \simeq A[p^\infty]$, $a(A') = \Sigma$. The map $Y_m \to \widetilde{S}_\mu$ is surjective, purely inseparable of degree p, while $Y_{et} \to \widetilde{S}_\mu$ is an isomorphism. This follows from the following two facts: (a) $Y_\mu \to \widetilde{S}_\mu$ is finite flat of degree $p+1$, (b) If $y \in Y_{et}(k)$ then $\widetilde{\pi}$ is étale at y, while if $y \in Y_m(k)$ it is ramified there (see §3.2). We conclude that if $\widetilde{x} \in \widetilde{S}_\mu(k)$ the fiber $\widetilde{\pi}^{-1}(\widetilde{x})$ contains precisely 2 points. Alternatively, we could have used the model $\widetilde{S}_0(p)$ (see §1.2.6) to show that there are precisely two possibilities for J to go with an $\underline{A}' \in \widetilde{S}_\mu(k)$.

(2) Assume next that A is gss and $H \simeq \mathfrak{G}[p]$. The analysis of $H^\perp/H$ is easy, since $H^\perp \subset A[p]$, so we can use Proposition 4.7. With the notation used there

$$M(H) = \langle e_2, \alpha_1 f_1 + \alpha_2 f_3 \rangle$$

for some $(\alpha_1 : \alpha_2) \neq 0, \infty$. It follows that $M(H^\perp/H) = \langle \alpha_2 \overline{e}_1 - \alpha_1 \overline{e}_3, \overline{f}_1 \rangle$ where the bar denotes the class modulo $M(H)$. Since this space is killed by V^2 and F^2 but neither by F nor by V, $H^\perp/H \simeq \mathfrak{G}[p]$. Since $M(H^\perp/H)[V] = \langle \overline{f}_1 \rangle$, $\mathrm{Lie}(H^\perp/H)$ is of type $\overline{\Sigma}$.

To analyze the α_p-subgroup of A' and conclude that it is of rank p^2 and type $(\Sigma, \overline{\Sigma})$, we need to know $M(A[p^2])$. This, unlike $M(A[p])$, depends on the particular

A, and not only on it being of type gss. The computations needed to verify this are deferred to the appendix.

(3) Assume that A is gss and $H \simeq \alpha_{p^2,\Sigma}$. Using the notation of Proposition 4.7

$$M(H) = \langle e_2, f_1 \rangle_k$$

so $M(H^\perp/H) = \langle \overline{e}_3, \overline{f}_3 \rangle$. This module is killed by both V and F so $H^\perp/H \simeq \kappa \otimes \alpha_p$, and its Lie algebra is of type $(\Sigma, \overline{\Sigma})$. The computation of $a(A')$ is again deferred to the appendix. The case A gss and $H \simeq \alpha^*_{p^2,\Sigma}$ is treated similarly.

(4) Assume that A is ssp. Then the covariant Dieudonné module $M = M(A[p^\infty])$ is freely spanned over $W(k)$ by a basis $e_1, e_2, e_3, f_1, f_2, f_3$ satisfying (i) $\mathcal{O}_E$ acts on the e_i via Σ and on the f_i via $\overline{\Sigma}$ (ii) $\langle e_i, f_j \rangle = \delta_{ij}$, $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$ (iii) the action of F and V is given by the table

	e_1	e_2	e_3	f_1	f_2	f_3
F	$-pf_1$	$-pf_2$	$-f_3$	e_1	e_2	pe_3
V	pf_1	pf_2	f_3	$-e_1$	$-e_2$	$-pe_3$

.

See [Bu-We], Lemma (4.1) and [Vo], Lemma 4.2. Note that Vollaard works over $W(\kappa)$ and uses a slightly different normalization, but over $W(k)$ her model and the one above become isomorphic. Let $\overline{M} = M/pM = M(A[p])$ (called in [Bu-We] the Dieudonné *space*) and denote by $\overline{e}_i$ and $\overline{f}_i$ the images of the basis elements. Using the notation of the proof of Theorem 4.11, we distinguish two cases.

Case I (the base of the comb): In this case H is of type $\kappa \otimes \alpha_p$ and

$$M(H) = \langle \alpha_1 \overline{e}_1 + \alpha_2 \overline{e}_2, \overline{f}_3 \rangle \subset \overline{M}.$$

As we have seen in the proof of Theorem 4.11, $H^\perp/H = \ker(\psi)$ is of type $\mathfrak{G}[p]_\Sigma$, unless $\zeta = (\alpha_1 : \alpha_2)$ satisfies $\zeta^{p+1} = -1$, where it is of type $\kappa \otimes \alpha_p$. This gives the entries for $\tau(\psi)$ in rows 4,6 and 7 of the table. We proceed to compute the a-number and a-type of A'. For this observe that $M' = M(A'[p^\infty])$ sits in an exact sequence

$$0 \to M \to M' \to M(H) \to 0,$$

hence inside the isocrystal $M_\mathbb{Q}$

$$M' = \langle e_i, p^{-1}(\widetilde{\alpha}_1 e_1 + \widetilde{\alpha}_2 e_2), f_1, f_2, p^{-1} f_3 \rangle.$$

Here we let $\widetilde{\alpha}_i$ denote any element of $W(k)$ mapping to α_i modulo p. To compute the Dieudonné module of the α_p-subgroup of A' we must compute

$$(M'/pM')[V] \cap (M'/pM')[F].$$

The kernel of V on M'/pM' is spanned over k by the images of the vectors $\{e_1, e_2, e_3, \widetilde{\alpha}_1^\sigma f_1 + \widetilde{\alpha}_2^\sigma f_2\}$ where σ is the Frobenius on $W(k)$. Similarly, the kernel of F is spanned by the images of $\{e_1, e_2, e_3, \widetilde{\alpha}_1^{\sigma^{-1}} f_1 + \widetilde{\alpha}_2^{\sigma^{-1}} f_2\}$. The span of $\{e_1, e_2, e_3\}$ in M'/pM' is two dimensional and of type Σ, Σ. We see that if $\zeta = (\alpha_1 : \alpha_2) \notin \mathbb{F}_{p^2}$ then $\alpha_p(A')$ is of rank p^2, hence A' is gss (supersingular but not superspecial), and $a(A') = \{\Sigma, \Sigma\}$. On the other hand if $\zeta \in \mathbb{F}_{p^2}$ then $\alpha_p(A')$ is of rank p^3, so A' is superspecial, and $a(A') = \{\Sigma, \Sigma, \overline{\Sigma}\}$. This completes the verification of $\tau(\psi)$ and $a(A')$ in rows 4,6 and 7 of the table.

Case II (the teeth of the comb): In this case H is of type $\mathfrak{G}[p]_\Sigma$,

$$M(H) = \langle \alpha_1 \overline{e}_1 + \alpha_2 \overline{e}_2, \beta_1 \overline{f}_1 + \beta_2 \overline{f}_2 + \beta_3 \overline{f}_3 \rangle \subset \overline{M}$$

where $\zeta = (\alpha_1 : \alpha_2) = (\beta_1^p : \beta_2^p)$ satisfies $\zeta^{p+1} = -1$ and $\beta_3 \in k$ is arbitrary. Now $M(H^\perp/H)$ is spanned by the images of $-\beta_3\bar{e}_1 + \beta_1\bar{e}_3$ and $\bar{f}_3$ modulo $M(H)$, so $H^\perp/H = \ker(\psi)$ is seen to be of type $\mathfrak{G}[p]_{\overline{\Sigma}}$. This confirms the invariant $\tau(\psi)$ in rows 2 and 5 of the table. Regarding $a(A')$ we compute, as in Case I, $M' = M(A'[p^\infty])$:

$$M' = \left\langle e_i, p^{-1}(\widetilde{\alpha}_1 e_1 + \widetilde{\alpha}_2 e_2), f_i, p^{-1}(\widetilde{\beta}_1 f_1 + \widetilde{\beta}_2 f_2 + \widetilde{\beta}_3 f_3)\right\rangle.$$

We find that $M'/pM'[V]$ is spanned over k by the images of

$$\{e_1, e_2, p^{-1}(\widetilde{\beta}_1^\sigma e_1 + \widetilde{\beta}_2^\sigma e_2) + \widetilde{\beta}_3^\sigma e_3, \widetilde{\alpha}_1^\sigma f_1 + \widetilde{\alpha}_2^\sigma f_2, f_3\}.$$

Note that $p^{-1}(\widetilde{\beta}_1^\sigma e_1 + \widetilde{\beta}_2^\sigma e_2) \in M'$ because of the relation $(\alpha_1 : \alpha_2) = (\beta_1^p : \beta_2^p)$. Likewise $M'/pM'[F]$ is spanned over k by the images of

$$\{e_1, e_2, p^{-1}(\widetilde{\beta}_1^{\sigma^{-1}} e_1 + \widetilde{\beta}_2^{\sigma^{-1}} e_2) + \widetilde{\beta}_3^{\sigma^{-1}} e_3, \widetilde{\alpha}_1^{\sigma^{-1}} f_1 + \widetilde{\alpha}_2^{\sigma^{-1}} f_2, f_3\}.$$

Now $\widetilde{\alpha}_1^{\sigma^{-1}} f_1 + \widetilde{\alpha}_2^{\sigma^{-1}} f_2$ and $\widetilde{\alpha}_1^\sigma f_1 + \widetilde{\alpha}_2^\sigma f_2$ both represent the class of $\widetilde{\beta}_1 f_1 + \widetilde{\beta}_2 f_2$ in M'/pM'. Similarly $p^{-1}(\widetilde{\beta}_1^\sigma e_1 + \widetilde{\beta}_2^\sigma e_2)$ and $p^{-1}(\widetilde{\beta}_1^{\sigma^{-1}} e_1 + \widetilde{\beta}_2^{\sigma^{-1}} e_2)$ both represent the class of $p^{-1}(\widetilde{\alpha}_1 e_1 + \widetilde{\alpha}_2 e_2)$ in M'/pM'. It follows that the span of f_3 and $\widetilde{\alpha}_1^\sigma f_1 + \widetilde{\alpha}_2^\sigma f_2$ in M'/pM' is 1-dimensional and of type $\overline{\Sigma}$. Regarding the Σ-component of $M'/pM'[V] \cap M'/pM'[F]$, e_1 and e_2 contribute a 1-dimensional piece there. If $\beta_3 \in \mathbb{F}_{p^2}$ then $p^{-1}(\widetilde{\beta}_1^\sigma e_1 + \widetilde{\beta}_2^\sigma e_2) + \widetilde{\beta}_3^\sigma e_3$ and $p^{-1}(\widetilde{\beta}_1^{\sigma^{-1}} e_1 + \widetilde{\beta}_2^{\sigma^{-1}} e_2) + \widetilde{\beta}_3^{\sigma^{-1}} e_3$ contribute another 1-dimensional piece, but otherwise they do not agree modulo pM'.

To sum up, if $\beta_3 \notin \mathbb{F}_{p^2}$ then A' is gss and $a(A') = \{\Sigma, \overline{\Sigma}\}$. If $\beta_3 \in \mathbb{F}_{p^2}$ then A' is ssp and $a(A') = \{\Sigma, \Sigma, \overline{\Sigma}\}$. This completes the verification of $\tau(\psi)$ and $a(A')$ in rows 2 and 5.

Since the morphism $\widetilde{\pi}$ is finite flat of degree $p+1$, the dimensions of the strata of $\widetilde{S}$ follow from the known dimensions of the strata of $S_0(p)$. Moreover, each geometric fiber has $p+1$ points if one counts multiplicities. We have already noted that the map $Y_m \to \widetilde{S}_\mu$ is surjective, purely inseparable of degree p, while $Y_{et} \to \widetilde{S}_\mu$ is an isomorphism. This proves that for $\widetilde{x} \in \widetilde{S}_\mu(k)$, $\#\widetilde{\pi}^{-1}(\widetilde{x}) = 2$, but it also proves that for $\widetilde{x} \in \widetilde{S}_{gss}[\Sigma, \overline{\Sigma}](k)$ we have $\#\widetilde{\pi}^{-1}(\widetilde{x}) = 2$. Indeed, such a point must have pre-images both in Z_{et} and in Z_m but the morphism $\widetilde{\pi} : Y_m \to \widetilde{S}$ being totally ramified and 1:1 on geometric points, must extend to a totally ramified morphism on Z_m, since the ramification locus is closed. Thus $\widetilde{\pi}$ is 1:1 on $Z_m(k)$. It is clearly 1:1 on $Z_{et}(k)$ because it is an isomorphism on Z_{et}.

Similar arguments show that $\widetilde{\pi}$ is totally ramified of degree $p+1$ on the base of the comb denoted F_x in Theorem 4.11, where A is ssp and H of type $\kappa \otimes \alpha_p$. This shows that $\#\widetilde{\pi}^{-1}(\widetilde{x}) = 1$ in rows 4,6 and 7 of the table.

Finally, at a generic point y lying on a tooth of a comb or on the gss screens (i.e. where A is ssp or gss but H is of type $\mathfrak{G}[p]_\Sigma$) then $\widetilde{\pi}$ induces an isomorphism on the completed local rings as can be seen from the table in Proposition 3.6, hence is étale. It follows that the image of such a point has $p+1$ distinct pre-images.

This concludes the proof of part (i) of the theorem. Part (ii) follows from the relations between the closures of the pre-images of the seven strata in $S_0(p)$. □

5.2. **Analysis of $\widetilde{\pi}$.**

5.2.1. *Analysis of $\widetilde{\pi}$ along the μ-ordinary strata.* We denote by $\widetilde{\pi}_{et}$ and $\widetilde{\pi}_m$ the restrictions of $\widetilde{\pi}$ to Y_{et} (or even $Y_{et}^{\dagger}$) and Y_m (or $Y_m^{\dagger}$).

Proposition 5.3. (i) *The map $\widetilde{\pi}_{et}\colon Y_{et} \xrightarrow{\sim} \widetilde{S}_\mu$ is an isomorphism. Denote by*

$$\widetilde{\sigma}_{et}\colon \widetilde{S}_\mu \xrightarrow{\sim} Y_{et}$$

the section which is its inverse. If $\underline{A}' \in \widetilde{S}_\mu(R)$ then $A'[\mathrm{Fr}] + \ker(\psi)$ is a finite flat subgroup J satisfying the conditions listed in Proposition 1.4, $p\psi$ descends to a principal polarization ϕ on A'/J and

$$\widetilde{\sigma}_{et}(\underline{A}') = (A'/A'[\mathrm{Fr}] + \ker(\psi), \phi, \iota', \langle p\rangle^{-1} \circ \eta', A'[p]/A'[\mathrm{Fr}] + \ker(\psi)).$$

(ii) The map $\widetilde{\pi}_m : Y_m \to \widetilde{S}_\mu$ is finite flat totally ramified of degree p.

Proof. We have already seen that $\widetilde{\pi}_{et}$ is an isomorphism and that $\widetilde{\pi}_m$ is a finite flat totally ramified map of degree p. It remains to check the assertion about $\widetilde{\sigma}_{et}$. Let us first check the claims made about J. As usual, by reduction to the universal object, we may assume that R is reduced. Then $A'[Fr] \cap \ker(\psi)$ is a finite group scheme over R, all of whose fibers have the same rank p, so is finite flat, and

$$J = A'[\mathrm{Fr}] + \ker(\psi) \simeq (A'[\mathrm{Fr}] \times \ker(\psi))/(A'[Fr] \cap \ker(\psi))$$

is finite flat of rank p^4. It is also maximal isotropic for $e_{p\psi}$, $\mathcal{O}_E$-stable and $J/\ker(\psi)$ is Raynaud. All these statements are checked fiber-by-fiber. We may therefore descend $p\psi$ to a principal polarization of A'/J and form the tuple $\widetilde{\sigma}_{et}(\underline{A}')$. It is now a simple matter to check that if $A' = A/H$ where $(\underline{A}, H) \in Y_{et}(k)$ then

$$A'/J = A/A[p] \overset{\times p}{\simeq} A$$

and $A'[p]/J = p^{-1}H/A[p]$ gets mapped back to H. When we add level-N structure twisted by the diamond operator $\langle p\rangle^{-1}$ to the definition of $\widetilde{\sigma}_{et}(\underline{A}')$ we ensure that $\widetilde{\sigma}_{et}$ is indeed the inverse of $\widetilde{\pi}_{et}$. □

The next corollary follows directly from the definitions of the various maps and we omit its proof.

Corollary 5.4. (i) *On R-points of the moduli problems the maps*

$$j_{et} = \pi_{et} \circ \langle p\rangle \circ \widetilde{\sigma}_{et}\colon \widetilde{S}_\mu \to S_\mu, \quad j_m = \widetilde{\pi}_m \circ \sigma_m\colon S_\mu \to \widetilde{S}_\mu$$

are given by

$$j_{et}(A', \psi, \iota', \eta') = (A'/A'[\mathrm{Fr}] + \ker(\psi), \phi, \iota', \eta') \quad j_m(A, \phi, \iota, \eta) = (A/A[p]^m, \psi, \iota, \eta).$$

Their compositions are the maps $Fr^2\colon S_\mu \to S_\mu^{(p^2)} = S_\mu$ or $Fr^2\colon \widetilde{S}_\mu \to \widetilde{S}_\mu^{(p^2)} = \widetilde{S}_\mu$ (here we use the fact that S and $\widetilde{S}$ are defined over κ).

(ii) *The maps*

$$w_m = \langle p\rangle \circ \widetilde{\sigma}_{et} \circ \widetilde{\pi}_m\colon S_0(p)^m \to S_0(p)^{et}, \quad w_{et} = \sigma_m \circ \pi_{et}\colon S_0(p)^{et} \to S_0(p)^m$$

are given by

$$w_m(\underline{A}, H) = (\underline{A}^{(p^2)}, \mathrm{Fr}(A^{(p)}[\mathrm{Ver}])), \quad w_{et}(\underline{A}, H) = (\underline{A}, A[p]^m).$$

5.2.2. *Analysis of* $\widetilde{\pi}$ *along the curves* Z_{et} *and* Z_m.

Proposition 5.5. *Let* $\widetilde{Z}$ *be the stratum* $\widetilde{S}_{gss}[\Sigma, \overline{\Sigma}]$. *The morphism* $\widetilde{\pi}_{et} : Z_{et} \to \widetilde{Z}$ *is an isomorphism. The morphism* $\widetilde{\pi}_m : Z_m \to \widetilde{Z}$ *is totally ramified of degree* p.

Proof. Let $Y_{et}^\dagger = Y_{et} \cup Z_{et}$ and $\widetilde{S}_\mu^\dagger = \widetilde{S}_\mu \cup \widetilde{Z}$. The map $\widetilde{\pi}_{et} : Y_{et}^\dagger \to \widetilde{S}_\mu^\dagger$ is finite, and induces an isomorphism between the open dense subsets $Y_{et} \simeq \widetilde{S}_\mu$. From the classification of the completed local rings in Proposition 3.6 it follows that $\widetilde{S}_\mu^\dagger$ is smooth, hence its local rings are integrally closed and $\widetilde{\pi}_{et}$ is an isomorphism. A similar argument shows that $\widetilde{\pi}_m : Y_m^\dagger \to \widetilde{S}_\mu^\dagger$ is finite flat totally ramified of degree p, where $Y_m^\dagger = Y_m \cup Z_m$.

In principle, the unramified direction (see Lemma 4.3) for $\widetilde{\pi}_m : Y_m^\dagger \to \widetilde{S}_\mu^\dagger$ at a point $\widetilde{x} \in \widetilde{Z}$ could be transversal to $\widetilde{Z}$ or tangential to it. We claim that it is everywhere transversal, i.e. the schematic pre-image of $\widetilde{Z}$ is Z_m (with its reduced structure) but $\widetilde{\pi}|_{Z_m}$ is totally ramified of degree p. This can be seen in a variety of ways.[9] We shall deduce it from Corollary 5.4. Observe first that the maps j_{et} and j_m extend to similarly denoted maps

$$j_{et} = \pi_{et} \circ \langle p \rangle \circ \widetilde{\sigma}_{et} \colon \widetilde{S}_\mu^\dagger \to S_\mu^\dagger, \quad j_m = \widetilde{\pi}_m \circ \sigma_m \colon S_\mu^\dagger \to \widetilde{S}_\mu^\dagger,$$

and may then be restricted to the gss curves $\widetilde{Z}$ and S_{gss}. The claim follows now from the following established facts: (a) $\sigma_m : S_{gss} \simeq Z_m$ and $\widetilde{\sigma}_{et} : \widetilde{Z} \simeq Z_{et}$ are isomorphisms, (b) $\pi_{et} : Z_{et} \to S_{gss}$ is totally ramified of degree p (equivalently, $\overline{\pi}_{et} : Z_{et}^{(p)} \simeq S_{gss}$ is an isomorphism) (c) $j_{et} \circ j_m = Fr^2$ hence, restricted to the curve S_{gss}, it is totally ramified of degree p^2. □

The same argument used to show that $\widetilde{\pi}_{et}$ extends to an isomorphism on $Y_{et}^\dagger$, and that $\widetilde{\pi}_m$ extends to a totally ramified map on $Y_m^\dagger$ gives the following.

Proposition 5.6. *Let* $\overline{Y}_{et}$ *and* $\overline{Y}_m$ *denote the closures of* Y_{et} *and* Y_m *in* $S_0(p)$. *Then* $\widetilde{\pi}_{et}$ *extends to an isomorphism from* $\overline{Y}_{et}$ *to the closure* $\overline{\widetilde{S}}_\mu$ *of* $\widetilde{S}_\mu$. *The map* $\widetilde{\pi}_m$ *extends to a totally ramified map of degree* p *from* $\overline{Y}_m$ *to* $\overline{\widetilde{S}}_\mu$.

A computation similar to the above, that we leave out, yields the following.

Corollary 5.7. *Let* $\theta : Y_m^\dagger \to Y_{et}^\dagger$ *be the map* $\theta = \rho_{et} \circ \pi_m$ *(see Corollary 4.6). Then*

$$\langle p \rangle \circ \widetilde{\pi}_{et} \circ \theta = \widetilde{\pi}_m.$$

5.2.3. *Analysis of* $\widetilde{\pi}$ *along the gss screens* Y_{gss}. Let W be an irreducible component of the closure $\overline{Y}_{gss}$ of Y_{gss}. As we have seen in Theorem 4.11, these irreducible components are smooth $\mathbb{P}^1$-bundles over Fermat curves, and do not intersect each other. Outside (the closure of) Z_{et} and Z_m the restriction of $\widetilde{\pi}$ to W, which we denote from now on $\widetilde{\pi}_W$, is étale. It is also étale at $y \in Z_{et}(k)$. This follows from §3.3.2.

[9] Were the unramified direction everywhere tangential to $\widetilde{Z}$, the schematic pre-image of $\widetilde{Z}$ would be a nilpotent thickening of order p of Z_m, but $\widetilde{\pi}$ would be an isomorphism on the reduced curve. In general, of course, there is also a "mixed option", where the unramified direction is generically transversal, but tangential to $\widetilde{Z}$ at finitely many points.

(1) We have

$$\widetilde{\pi}(\overline{Z}_m \cap W) = \widetilde{\pi}(\overline{Z}_{et} \cap W). \tag{5.1}$$

Proof: $\widetilde{\pi}(\overline{Z}_{et} \cap W)$ is an irreducible component of $\overline{\widetilde{Z}}$, the closure of the stratum $\widetilde{Z} = \widetilde{S}_{gss}[\Sigma, \overline{\Sigma}]$. So is $\widetilde{\pi}(\overline{Z}_m \cap W)$. The two intersect at the image of any point ζ which is "a base of a tooth of a comb", points where $\overline{Z}_{et}$ and $\overline{Z}_m$ meet. Since the irreducible components of $\overline{\widetilde{Z}}$ are disjoint, the two components coincide.

(2) We have

$$\widetilde{\pi}(W) \cap \overline{\widetilde{Z}} = \widetilde{\pi}(\overline{Z}_{et} \cap W).$$

Proof: this follows from (1) since $\widetilde{\pi}^{-1}(\overline{\widetilde{Z}}) = \overline{Z}_{et} \cup \overline{Z}_m$.

(3) Let W, W' be two components of $\overline{Y}_{gss}$. Then $\widetilde{\pi}(W) \cap \widetilde{\pi}(W') = \emptyset$.

Proof: Each $\widetilde{\pi}(W)$ is an irreducible component of $\widetilde{\pi}(\overline{Y}_{gss})$. But the irreducible components of $\widetilde{\pi}(\overline{Y}_{gss})$ are disjoint from each other and are uniquely determined by their intersection with $\overline{\widetilde{S}}_\mu$, i.e. with $\overline{\widetilde{Z}}$. The claim follows from (2), since $\widetilde{\pi}(\overline{Z}_{et} \cap W) \cap \widetilde{\pi}(\overline{Z}_{et} \cap W') = \emptyset$, as $\widetilde{\pi}_{et}$ is an isomorphism.

(4) We give another proof of (5.1). It is based on the following lemma, which is of independent interest. Recall that S is defined over $\kappa = \mathbb{F}_{p^2}$, although we consider it over $k = \overline{\mathbb{F}}_{p^2}$. It follows that $Gal(k/\kappa)$ permutes the irreducible components of S_{gss}. The diamond operators also act on these irreducible components.

Lemma 5.8. *Let Z be an irreducible component of S_{gss}. Then $Fr_{p^2}(Z) = \langle p \rangle (Z)$.*

Proof. For the proof of the lemma we may increase N. Indeed, if $N|N'$ and Z, Z' are as above for N and N', with Z' mapping to Z, then the validity of the lemma for Z' implies it for Z. Since the closure $\overline{Z}$ of every irreducible component of S_{gss} contains at least two superspecial points, and since when N is large enough, through any two superspecial points passes at most one such $\overline{Z}$ [Vo], it is enough to prove that for $x \in S_{ssp}(k)$

$$Fr_{p^2}(x) = \langle p \rangle (x).$$

Let $x = (A, \phi, \iota, \eta)$. Every supersingular elliptic curve B over k has a model B_0 over κ, whose Frobenius of degree p^2 satisfies

$$Fr_{p^2} = p.$$

By the Tate-Honda theorem [Ta], all the endomorphisms of B are already defined over κ. We may therefore assume that $A \simeq B^3$ and ι are defined over κ. Since A admits at least one principal polarization defined over κ, and its endomorphisms are all defined over κ, ϕ is defined over κ. Thus (A, ϕ, ι) is invariant under Fr_{p^2}. But the relation $Fr_{p^2} = p$ on $A[N]$ means that $Fr_{p^2}(\eta) = \langle p \rangle \circ \eta$, which concludes the proof. □

Now use the relation

$$\langle p \rangle^{-1} \circ Fr_p^2 = \langle p \rangle^{-1} \circ j_{et} \circ j_m = \pi_{et} \circ \widetilde{\sigma}_{et} \circ \widetilde{\pi}_m \circ \sigma_m$$

from Corollary 5.4, and its extension to $S_\mu^\dagger$ from the proof of Proposition 5.5. The left hand side fixes the irreducible components of S_{gss}, hence also the irreducible components W of $\overline{Y}_{gss}$. Let $y \in \overline{Z}_m \cap W$. Then $y' = \widetilde{\sigma}_{et} \circ \widetilde{\pi}_m(y) \in \overline{Z}_{et} \cap W$, or

$$\widetilde{\pi}_m(y) = \widetilde{\pi}_{et}(y'). \tag{5.2}$$

This shows that $\widetilde{\pi}(\overline{Z}_m \cap W) = \widetilde{\pi}(\overline{Z}_{et} \cap W)$ as was to be shown.

(5) The map $\widetilde{\pi}_W : W \to \widetilde{\pi}(W)$ is finite flat of degree $p+1$.
Proof: This follows from (3) since $\widetilde{\pi}$ in the large is finite flat of degree $p+1$.

We next want to analyze how $\widetilde{\pi}$ is behaved when restricted to a fiber $W_x = \pi^{-1}(x)$ of π above a gss point x. Recall that $W_x \simeq \mathbb{P}^1$.

(6) Let y_m and y_{et} be the unique points on $Z_m \cap W_x$ and $Z_{et} \cap W_x$ respectively. Then $\widetilde{\pi}(y_m) \neq \widetilde{\pi}(y_{et})$.

Proof: Equivalently, we have to show that the images under π of y and y' as in (5.2), which are in the same fiber for $\widetilde{\pi}$, are distinct. But $\pi(y') = \langle p \rangle^{-1} \pi(y)^{(p^2)}$. We claim that if $\pi(y) = (A, \phi, \iota, \eta)$ then already (A, ϕ, ι) is not defined over κ, so is not isomorphic to $(A^{(p^2)}, \phi^{(p^2)}, \iota^{(p^2)})$. This follows from the fact, established in [Vo], that when $N = 1$ any irreducible curve Z in the supersingular locus of the coarse moduli space associated with the algebraic stack S is defined over κ, and is birationally isomorphic to the Fermat curve

$$\mathscr{C} : x^{p+1} + y^{p+1} + z^{p+1} = 0.$$

Let $\mathscr{C} \to Z$ be the normalization of Z. This $\mathscr{C}$ has $p^3 + 1$ κ-rational points, which are precisely the points mapping to superspecial points on Z. Furthermore, all the self-intersections of Z are at κ-rational points. It follows that no $x \in Z(k)$ which is gss is fixed under Fr_p^2. Since the diamond operators do not affect (A, ϕ, ι), *a fortiori* $\pi(y') \neq \pi(y)$.

Starting with $x = x^{(1)} \in S_{gss}(k)$ we may now form a sequence of points $x^{(1)}, \ldots, x^{(r)}$ such that if $y_m^{(i)}$ and $y_{et}^{(i)}$ are the respective points on $W_{x^{(i)}}$ then

$$\widetilde{\pi}(y_m^{(i+1)}) = \widetilde{\pi}(y_{et}^{(i)}).$$

This sequence becomes periodic after d steps, where d is the minimal number so that $\langle p \rangle^{-d} \circ Fr_p^{2d}(x) = x$.

(7) The map $\widetilde{\pi} : W_x \to \widetilde{\pi}(W_x)$ is a birational isomorphism.
Proof: We have to show that the map is generically 1-1. For that it is enough to find a single point $y \in W_x$ so that $\widetilde{\pi}$ is étale at y and $\widetilde{\pi}^{-1}(\widetilde{\pi}(y)) = \{y\}$. In view of (6), the unique point on $Z_{et} \cap W_x$ is such a point.

We do not answer the question whether $\widetilde{\pi}$ is everywhere 1-1. We summarize the discussion of this section in the following theorem.

Theorem 5.9. *The map $\widetilde{\pi}$ induces a bijection between the vertical irreducible components of $\widetilde{S}$ and of $S_0(p)$. The map π induces a bijection between the vertical irreducible components of $S_0(p)$ and the irreducible components of the curve S_{ss}. The vertical irreducible components of $\widetilde{S}$ are mutually disjoint. Let W be a vertical irreducible component of $S_0(p)$. Then $\widetilde{\pi}_W$ is finite flat of degree $p+1$ and is étale outside $W \cap \overline{Z}_m$. The restriction of $\widetilde{\pi}_W$ to $W_x = \pi^{-1}(x)$ for $x \in S_{gss}$ is a birational isomorphism and maps the unique intersection points of W_x with Z_{et} and Z_m to distinct points.*

6. Appendix

6.1. **The classification of the gss Dieudonné modules.** In the appendix we perform some computations on the covariant Dieudonné module of a gss abelian variety. We first recall their classification, following Vollaard [Vo].

Fix $\delta \in \mu_{p^2-1} \subset W(\kappa) \subset W(\kappa)_{\mathbb{Q}} = E_p$ such that

$$\delta^\sigma = \delta^p = -\delta.$$

Let $\mathbf{M}$ be the free $W(\kappa)$-module on $e_1, e_2, e_3, f_1, f_2, f_3$ and let $\mathcal{O}_E$ act on the e_i via Σ (the canonical embedding of E in E_p) and on the f_i via $\overline{\Sigma}$. Let F be the σ-linear endomorphism[10] of $\mathbf{M}$ whose matrix w.r.t. the above basis is

$$\begin{pmatrix} & & & 1 & & \\ & & & & p & \\ & & & & & 1 \\ p & & & & & \\ & 1 & & & & \\ & & p & & & \end{pmatrix},$$

i.e. $F(e_1) = pf_1, F(e_2) = f_2, \ldots, F(f_3) = e_3$. Let V be the σ^{-1}-linear endomorphism with the same matrix. Note that $\tau = V^{-1}F$ is the identity on $\mathbf{M}$. Let $\mathbf{M}_k = W(k) \otimes_{W(\kappa)} \mathbf{M}$ and extend F, V semi-linearly as usual. Then τ becomes σ^2-linear.

Let $\langle , \rangle$ be the alternating pairing on $\mathbf{M}_k$ satisfying

$$\langle e_i, f_j \rangle = \delta \cdot \delta_{ij}, \quad \langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0.$$

This $\mathbf{M}_k$ is the Dieudonné module of $A_x[p^\infty]$ for any $x \in S_{ssp}(k)$. It is isomorphic[11] to the module used in part (4) of the proof of Theorem 5.2. The Lie algebra of A_x is identified with $\mathbf{M}_k/p\mathbf{M}_k[V] = V^{-1}p\mathbf{M}_k/p\mathbf{M}_k = F\mathbf{M}_k/p\mathbf{M}_k \simeq (\mathbf{M}_k/V\mathbf{M}_k)^{(p)}$ and is spanned over k by $\overline{e}_1, \overline{e}_3, \overline{f}_2$.

Following [Vo] we denote $\mathbf{M}(\Sigma) = \langle e_1, e_2, e_3 \rangle_{W(\kappa)}$ by $\mathbf{M}_0$ and $\mathbf{M}(\overline{\Sigma})$ by $\mathbf{M}_1$. We introduce on $\mathbf{M}_0$ the skew-hermitian form

$$\{x, y\} = \langle x, Fy \rangle.$$

We extend it to a bi-additive form on $\mathbf{M}_{0,k}$ which is linear in the first variable and σ-linear in the second. It satisfies

$$\{x, y\} = -\{y, \tau^{-1}(x)\}^\sigma, \quad \{\tau(x), \tau(y)\} = \{x, y\}^{\sigma^2}.$$

We denote the unitary isocrystal $\mathbb{Q} \otimes \mathbf{M}$ by $\mathbf{N} = \mathbf{N}_0 \oplus \mathbf{N}_1$ and write also C for $\mathbf{N}_0$. When we base-change to the field of fractions of $W(k)$ we shall add, as before, the subscript k. Note that the $\mathbb{Q}_p$-group $\boldsymbol{J} = GU(C, \{,\})$ is isomorphic, in our case, to $\boldsymbol{G}_{/\mathbb{Q}_p}$. (In general, it might be an inner form of it.)

If $\Lambda \subset C$ is a $W(\kappa)$-lattice we let

$$\Lambda^\vee = \{x \in C | \{x, \Lambda\} \subset W(\kappa)\}.$$

If $\mathbf{M}_k$ were the Dieudonné module of $A_x[p^\infty]$ for a superspecial point x, then the components of S_{ss} passing through x are classified, as we have seen before, by the set

$$\mathcal{J} = \{(1 : \zeta) \in \mathbb{P}^1(W(\kappa)) | \zeta^{p+1} + 1 = 0\}.$$

The vertices of the Bruhat-Tits tree of $\boldsymbol{J}$ are of two types. The *special* (s) lattices $\mathscr{L}^{(1)}$ are the lattices Λ' for which

$$\Lambda' \subset \Lambda'^\vee, \ \ length_{W(\kappa)}(\Lambda'^\vee/\Lambda') = 2.$$

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[10]In the appendix we depart from our habit of writing F as a *linear* map from $\mathbf{M}^{(p)}$ to $\mathbf{M}$.

[11]The change in notation is made to conform with [Vo]. Previously we tried to match [Bu-We].

For example, $\mathbf{M}_0 \in \mathscr{L}^{(1)}$. The *hyperspecial* (hs) lattices $\mathscr{L}^{(3)}$ are those satisfying $\Lambda = \Lambda^\vee$. Finally, the edges of the tree connect a lattice Λ' of type (s) to a vertex Λ of type (hs) if $\Lambda' \subset \Lambda \subset \Lambda'^\vee$. One computes that the $p+1$ vertices of type (hs) adjacent to $\mathbf{M}_0$ are the lattices

$$\Lambda_\zeta = \langle e_1, e_2, e_\zeta \rangle_{W(\kappa)}$$

where $\zeta \in \mathcal{J}$ and $e_\zeta = p^{-1}(e_1 + \zeta e_3)$.

Fix ζ and let $\Lambda = \Lambda_\zeta$, $V = \Lambda / p\Lambda$, a vector space over κ with basis $\overline{e}_1, \overline{e}_2, \overline{e}_\zeta$. The skew-hermitian pairing $(,) = \{,\} \mod p$ is given in this basis by the matrix

$$\delta \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

Theorems 2 and 3 of [Vo] imply the following. The k-points of the irreducible component of S_{ss} passing through the superspecial point x and labeled by ζ are in one-to-one correspondence with

$$Y_\Lambda(k) = \{U \subset V_k |\ \dim U = 2,\ U^\perp \subset U\}.$$

Here

$$U^\perp = \{x \in V_k | \, (x, U) = 0\}.$$

Caution has to be taken as we are over k and not κ : $(U^\perp)^\perp = \tau(U)$ and not U. The point x corresponds to $\mathbf{U} = \langle \overline{e}_1, \overline{e}_2 \rangle$. In general, let $a, b \in k$ and

$$U_{a,b} = \langle \overline{e}_1 + a\overline{e}_\zeta, \overline{e}_2 + b\overline{e}_\zeta \rangle.$$

Then

$$U_{a,b}^\perp = \langle \overline{e}_1 - b^p \overline{e}_2 - a^p \overline{e}_\zeta \rangle$$

is contained in $U_{a,b}$ if and only if

$$a^p + a - b^{p+1} = 0.$$

It follows ([Vo], Lemma 4.6) that the irreducible components of S_{ss} are isomorphic to the smooth projective curve whose equation is

$$x^p z + x z^p - y^{p+1} = 0.$$

This is just the Fermat curve $x^{p+1} + y^{p+1} + z^{p+1} = 0$ in disguise.

Moreover, the Dieudonné module of the abelian variety $A_{a,b}$ "sitting" at the point (a, b) is

$$M_{a,b} = M_{a,b}^0 \oplus M_{a,b}^1$$

where

$$M_{a,b}^0 = \langle e_1 + [a]e_\zeta, e_2 + [b]e_\zeta, pe_\zeta \rangle_{W(k)}$$

and

$$M_{a,b}^1 = \left\langle f_1 - [b]p^{-1}f_2 - [a]f_\zeta, f_2, pf_\zeta \right\rangle.$$

Here $[a]$ is the Teichmüller representative of a and $f_\zeta = p^{-1}(f_3 - \zeta f_1)$.

The matrices for F and V can now be computed. To simplify the notation let

$$\epsilon_1 = e_1 + [a]e_\zeta,\ \epsilon_2 = e_2 + [b]e_\zeta,\ \epsilon_3 = pe_\zeta,$$

$$\phi_1 = f_1 - [b]p^{-1}f_2 - [a]f_\zeta,\ \phi_2 = f_2,\ \phi_3 = pf_\zeta.$$

Then relative to the basis $\{\epsilon_1, \epsilon_2, \epsilon_3, \phi_1, \phi_2, \phi_3\}$

$$F = \begin{pmatrix} & & & 1 & & \\ & & & -[b^p] & p & \\ & & & \gamma & -[b] & 1 \\ p & & & & & \\ [b] & 1 & & & & \\ [a^p]+[a] & [b^p] & p & & & \end{pmatrix}$$

and

$$V = \begin{pmatrix} & & & 1 & & \\ & & & -[b^{1/p}] & p & \\ & & & \sigma^{-1}(\gamma) & -[b] & 1 \\ p & & & & & \\ [b] & 1 & & & & \\ [a^{1/p}]+[a] & [b^{1/p}] & p & & & \end{pmatrix}$$

with

$$\gamma = -p^{-1}([a^p] + [a] - [b^{p+1}]) \in W(k).$$

6.2. **The quotient A/H for gss A.** Let $(a,b) \in k^2$ but not in κ^2. This guarantees that $A_{a,b}$ is gss, and every gss A is of this sort, for an appropriate $x \in S_{ssp}(k)$ and an appropriate $\zeta \in \mathcal{J}$. Let $H \subset A[p]$ be an isotropic Raynaud subgroup scheme. Let $A' = A/H$.

We know that $M(H) \subset M_{a,b}/pM_{a,b}$ must contain $\ker V \cap \ker F = \langle \overline{\epsilon}_3 \rangle_k$. In addition, $M(H)$ should contain a vector $\overline{\eta}$ from $M^1_{a,b}/pM^1_{a,b}$ such that $F\overline{\eta} = \overline{\epsilon}_3$. We see that the most general form of such an η is

$$\eta = u(\phi_2 + [b^p]\phi_3) + v(\phi_2 + [b^{1/p}]\phi_3),$$

$(u:v) \in \mathbb{P}^1(W(k))$. Thus

$$M(H) = \langle \overline{\epsilon}_3, \overline{\eta} \rangle_k .$$

Note that by the assumption that (a,b) is not in κ^2, neither a nor b lies in κ. Thus H is uniquely classified by $(\overline{u} : \overline{v}) \in \mathbb{P}^1(k)$. The point $\overline{u} = 0$ corresponds to an H such that $M(H)$ is killed by F, or H is killed by Ver. This H will be of type $\alpha_{p^2,\Sigma}$ and $(\underline{A}, H)$ will lie then on Z_{et}. The point $\overline{v} = 0$ will correspond to an H such that $M(H)$ is killed by V, or H is killed by $Frob$. This H will be of type $\alpha^*_{p^2,\Sigma}$ and $(\underline{A}, H)$ will lie then on Z_m.

Assume from now on that we are not in these two special cases, so that H is of type $\mathfrak{G}_\Sigma[p]$. Then $M(A'[p^\infty]) = M'$ will sit in an exact sequence

$$0 \to M \to M' \to M(H) \to 0,$$

and inside $\mathbf{N}_k$, $M' = \left\langle \epsilon_1, \epsilon_2, p^{-1}\epsilon_3, \phi_1, p^{-1}\eta, \phi_3 \right\rangle_{W(k)}$, provided $\overline{u} \neq -\overline{v}$. If $\overline{u} = -\overline{v}$ the same basis works, if we replace ϕ_3 by ϕ_2. Assume from now on that $\overline{u} \neq -\overline{v}$. We calculate the matrices of F and V in this basis as we did for $M = M_{a,b}$ before. The matrix of F comes out to be

$$\begin{pmatrix} & & & 1 & & \\ & & & -[b^p] & u^\sigma + v^\sigma & \\ & & & p\gamma & u^\sigma([b^{p^2}] - [b]) & p \\ p & & & & & \\ p[b](u+v)^{-1} & p(u+v)^{-1} & & & & \\ [a^p]+[a]-[b]w & [b^p]-w & 1 & & & \end{pmatrix}$$

where we put $w = (u[b^p] + v[b^{1/p}])(u+v)^{-1}$, while the one of V is

$$\begin{pmatrix} & & & 1 & & \\ & & & -[b^{1/p}] & u^{\sigma^{-1}} + v^{\sigma^{-1}} & \\ & & & p\sigma^{-1}(\gamma) & v^{\sigma^{-1}}([b^{1/p^2}] - [b]) & p \\ p & & & & & \\ p[b](u+v)^{-1} & p(u+v)^{-1} & & & & \\ [a^{1/p}] + [a] - [b]w & [b^{1/p}] - w & 1 & & & \end{pmatrix}.$$

We see that $M'/pM'[V] \cap M'/pM'[F]$ is spanned by the images modulo pM' of ϕ_3 and of $x\epsilon_1 + y\epsilon_2 + zp^{-1}\epsilon_3$ provided $x, y, z \in W(k)$ are such that

$$x^\sigma([a^p] + [a] - [b]w) + y^\sigma([b^p] - w) + z^\sigma \equiv 0 \mod p$$

$$x^{\sigma^{-1}}([a^{1/p}] + [a] - [b]w) + y^{\sigma^{-1}}([b^{1/p}] - w) + z^{\sigma^{-1}} \equiv 0 \mod p.$$

These two equations are equivalent to

$$x([b^{1+1/p}] - [b^{1/p}]w^{\sigma^{-1}}) + y([b] - w^{\sigma^{-1}}) + z \equiv 0 \mod p,$$

$$x([b^{p+1}] - [b^p]w^\sigma) + y([b] - w^\sigma) + z \equiv 0 \mod p.$$

The solution set $(x, y, z) \mod p$ to these two equations is 1-dimensional, unless $\overline{w} \in \kappa$ and $\overline{b}^{p-1/p} = 1$, where it is 2-dimensional. This last condition however translates into $\overline{b} \in \kappa$, which we assumed not to be the case. We conclude that $M'/pM'[V] \cap M'/pM'[F]$ is always two-dimensional, of type $(\Sigma, \overline{\Sigma})$. This settles the a-type of A' in the cases that were deferred to the appendix in the proof of Theorem 5.2.

References

[Bel] J. Bellaïche: Congruences endoscopiques et représentations Galoisiennes, Thèse, Paris XI (Orsay), 2002.

[Bea] A. Beauville: Complex algebraic surfaces, second edition, London Mathematical Society Student Texts 34, Cambridge Univ. Press 1996.

[Bu-We] O. Bültel, T. Wedhorn: Congruence relations for Shimura varieties associated to some unitary groups, J. Instit. Math. Jussieu **5** (2006), pp. 229-261.

[C-C-O] C.-L. Chai, B. Conrad, F. Oort: Complex Multiplication and Lifting Problems, AMS, Providence, 2013.

[Cri] S. E. Crick: Local Moduli of Abelian Varieties, American Journal of Mathematics **97** (1975), pp. 851-861.

[dJ1] A.J. de Jong: The moduli spaces of polarized abelian varieties, Mathematische Annalen **295** (1993), pp. 485-503.

[dJ2] A.J. de Jong: The moduli spaces of principally polarized abelian varieties with $\Gamma_0(p)$-level structure, Journal of Algebraic Geometry **2** (1993), pp. 667-688.

[dS-G1] E. de Shalit, E. Z. Goren: Supersingular curves on Picard modular surfaces modulo an inert prime, Journal of Number Theory **171** (2017), pp. 391-421.

[dS-G2] E. de Shalit, E.Z. Goren: A theta operator on Picard modular forms modulo an inert prime, Res. Math. Sci. **3:**28 (2016).

[Eis] D. Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry, GTM 150, Springer-Verlag, New York, 1995.

[Gö] U. Görtz: On the flatness of models of certain Shimura varieties of PEL type, Math. Ann. **321** (2001), pp. 689-727.

[Gro] A. Grothendieck: Groupes de Barsotti-Tate et cristaux de Dieudonné, Univ. Montréal, Montréal, 1974.

[Kos] J.-S. Koskivirta: Congruence relations for Shimura varieties associated to unitary groups, Canad. J. Math. **66** (2014), pp. 1305-1326.

[Lan] K.-W. Lan: Arithmetic compactifications of PEL-type Shimura varieties, London Mathematical Society Monographs **36**, Princeton, 2013.

[La-Ra] R. Langlands, D. Ramakrishnan: The Zeta functions of Picard modular surfaces, Univ. Montréal, Montréal, 1992.

[Mo] B. Moonen: Serre-Tate theory for moduli spaces of PEL type, Ann. Scient. Éc. Norm. Sup., 4e série, t. **37** (2004), pp. 223-269.

[Mu] D. Mumford: The red book of varieties and schemes, LNM **1358**, Springer-Verlag, New-York, 1999.

[Nor] P. Norman: An Algorithm for Computing Local Moduli of Abelian Varieties, Annals of Mathematics **101** (1975), pp. 499-509.

[N-O] P. Norman, F. Oort: Moduli of Abelian Varieties, Annals of Mathematics **112** (1980), pp. 413-439.

[Ra-Zi] M. Rapoport, T. Zink: Period Spaces for p-divisible Groups, Annals of Mathematics Studies **141**, Princeton University Press, Princeton (1996).

[Ray] M. Raynaud: Schémas en groupes de type $(p,...,p)$, Bulletin de la Société Mathématique de France **102** (1974), pp. 241-280.

[Ru-Sh] A. N. Rudakov, I. R. Shafarevich: Inseparable morphisms of algebraic surfaces, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), pp. 1269–1307.

[Ta] J. Tate: Classes d'isogenie des varietes abeliennes sur un corps finis (d'apres T. Honda), Seminaire Bourbaki, Expose **352** (1968/69).

[Vo] I. Vollaard: The supersingular locus of the Shimura variety for $GU(1;s)$, Canad. J. Math. **62** (2010), pp. 668-720.

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