FOLIATIONS ON SHIMURA VARIETIES IN POSITIVE CHARACTERISTIC

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Abstract. This paper is a continuation of [G-dS1]. We study foliations of two types on Shimura varieties S in characteristic p. The first, which we call "tautological foliations", are defined on Hilbert modular varieties, and lift to characteristic 0. The second, the "V-foliations", are defined on unitary Shimura varieties in characteristic p only, and generalize the foliations studied by us before, when the CM field in question was quadratic imaginary. We determine when these foliations are p-closed, and the locus where they are smooth. Where not smooth, we construct a "successive blow up" of our Shimura variety to which they extend as smooth foliations. We discuss some integral varieties of the foliations. We relate the quotient of S by the foliation to a purely inseparable map from a certain component of another Shimura variety of the same type, with parahoric level structure at p, to S.

Contents

1. Introduction

Let S be a non-singular variety over a field k, and $\mathcal{T}$ its tangent bundle. A (smooth) foliation on S is a vector sub-bundle of $\mathcal{T}$ that is closed under the Lie bracket. If $\mathrm{char}(k) = p > 0$, a p-foliation is a foliation that is, in addition, closed under the operation $\xi \mapsto \xi^p$. As explained below, such p-foliations play an important role in studying purely inseparable morphisms $S \to S'$.

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Foliations have been studied in many contexts, and our purpose here is to explore certain p-foliations on Shimura varieties of PEL type, which bear a relation to the underlying moduli problem. The connection between the two topics can go either way. It can be seen as using the rich geometry of Shimura varieties to produce interesting examples of foliations, or, in the other direction, as harnessing a new tool to shed light on some geometrical aspects of these Shimura varieties, especially in characteristic p.

We study two types of foliations, that could well turn out to be particular cases of a more general theory. The first lie on Hilbert modular varieties. Let S be a Hilbert modular variety associated with a totally real field L, $[L:\mathbb{Q}]=g$ (see the text for details). Let Σ be any proper non-empty subset of $\mathscr{I}=\mathrm{Hom}(L,\mathbb{R})$. Fixing an embedding of $\bar{\mathbb{Q}}$ in $\bar{\mathbb{Q}}_p$ we view $\mathscr{I}$ also as the set of embeddings of L in $\bar{\mathbb{Q}}_p$. If p is unramified in L these embeddings end up in $\mathbb{Q}_p^{\mathrm{nr}}$ and the Frobenius automorphism $\phi \in Gal(\mathbb{Q}_p^{\mathrm{nr}}/\mathbb{Q}_p)$ permutes them.

Consider the uniformization $\Gamma \setminus \mathfrak{H}^{\mathscr{I}} \simeq S(\mathbb{C})$, describing the complex points of S as a quotient of the product of g copies of the upper half plane (indexed by $\mathscr{I}$) by an arithmetic subgroup of $\mathrm{SL}_2(L)$. The foliation $\mathscr{F}_\Sigma$, labelled by the subset Σ, is defined most easily complex analytically: at any point $x \in S(\mathbb{C})$ its fiber is spanned by $\partial/\partial z_i$ for $i \in \Sigma \subset \mathscr{I}$. Our main results about it are the following:

- Using the Kodaira-Spencer isomorphism, $\mathscr{F}_\Sigma$ can be defined algebraically, hence also in the characteristic p fiber of S, for any good unramified rational prime p. Its "reduction modulo p" is a smooth p-foliation if and only if the subset Σ is invariant under Frobenius.
- If the singleton $\{\sigma\}$ is not Frobenius-invariant (i.e. the corresponding prime of L is not of absolute degree 1), the obstruction to $\mathscr{F}_{\{\sigma\}}$ being p-closed can be identified with the square of a partial Hasse invariant.
- Let $\mathfrak{p}$ be a prime of L dividing p. Let $S_0(\mathfrak{p})$ be the special fiber of the integral model of the Hilbert modular variety with $\Gamma_0(\mathfrak{p})$-level structure studied in [Pa]. When Σ consists of all the embeddings *not* inducing $\mathfrak{p}$, the (purely inseparable) quotient of S by the foliation $\mathscr{F}_\Sigma$ can be identified with a certain irreducible component of $S_0(\mathfrak{p})$.
- It is easy to see that $\mathscr{F}_\Sigma$ does not have any integral varieties in characteristic 0. In contrast, as we show below, any p-foliation in characteristic p admits a plentiful supply of integral varieties. In our case, we show that certain Goren-Oort strata in the reduction modulo p of S are integral varieties of the foliation $\mathscr{F}_\Sigma$.

The second class of foliations studied in this paper live on unitary Shimura varieties M of arbitrary signature, associated with a CM field K. They generalize the foliations studied in [G-dS1] when K was quadratic imaginary, and are again labelled by subsets Σ of $\mathscr{I}^+=\mathrm{Hom}(L,\mathbb{R})$ where L is the totally real subfield of K. Unlike the foliations of the first type, they are particular to the characteristic p fibers, where p is again a good unramified prime, and are of different genesis. They are defined using the Verschiebung isogeny of the universal abelian scheme over the μ-ordinary locus M^{ord} of M. Their study relies to a great extent on the work of Wedhorn and Moonen cited in the bibliography. We refer to the text for the precise definition of the foliation denoted by $\mathscr{F}_\Sigma$, as it is a bit technical. A rough description is this: In the p-kernel of the universal abelian scheme over M^{ord} lives an important subgroup scheme, which played a special role in the work of Oort and his school, namely the maximal subgroup scheme of α_p-type. Its cotangent space serves to define the $\mathscr{F}_\Sigma$ via the Kodaira-Spencer isomorphism. The main results concerning $\mathscr{F}_\Sigma$ are the following:

- $\mathscr{F}_\Sigma$ is a smooth p-foliation on M^{ord}, regardless of what Σ is. Involutivity follows from the flatness of the Gauss-Manin connection, but being p-closed is more delicate, and is a consequence of a theorem of Cartier on the p-curvature of that connection.
- Although in general more complicated than the relation found in [G-dS1], one can work out explicitly the relation between the foliation $\mathscr{F}_\Sigma$ and the "cascade structure" defined by Moonen [Mo] on the formal completion $\widehat{M}_x$ of M at a μ-ordinary point x. While the cascade structure does not globalize, its "trace" on the tangent space, constructed from the foliations $\mathscr{F}_\Sigma$, does globalize neatly.
- There is a maximal open subset $M_\Sigma \subset M$ to which $\mathscr{F}_\Sigma$ extends as a smooth p-foliation with the same definition used to define it on M^{ord}. This M_Σ is a union of Ekedahl-Oort (EO) strata, and in fact consists of all the strata containing in their closure a smallest one, denoted M_Σ^{fol}. The description of which EO strata participate in M_Σ is given combinatorially in terms of "shuffles" in the Weyl group.
- Outside M_Σ the foliation $\mathscr{F}_\Sigma$ acquires singularities, but we construct a "successive blow up" $\beta : M^\Sigma \to M$, which is an isomorphism over M_Σ, to which the lifting of $\mathscr{F}_\Sigma$ extends as a smooth p-foliation. This M^Σ is an interesting (characteristic p) moduli problem in its own right. It is non-singular, and the extension of $\mathscr{F}_\Sigma$ to it is transversal to the fibers of β.
- When K is quadratic imaginary, the EO stratum M_Σ^{fol} was proved to be an integral variety of $\mathscr{F}_\Sigma$. A similar result is expected here when $\Sigma = \mathscr{I}^+$. This can be probably proved via elaborate Dieudonné module computations, as in [G-dS1], but in this paper we content ourselves with checking that the dimensions match.
- A natural interesting question is to identify the purely inseparable quotient of M^Σ by (the extended) $\mathscr{F}_\Sigma$ with a certain irreducible component of the special fiber of the Rapoport-Zink model of a unitary Shimura variety with parahoric level structure at p. This was done in our earlier paper when K was quadratic imaginary, and was used to obtain some new results on the geometry of that particular irreducible component. In the general case treated in this paper, we know of a natural candidate with which we would like to identify the quotient of M^Σ by $\mathscr{F}_\Sigma$. However, following the path set in [G-dS1] for a general CM field K would require a significant amount of work, and we leave this question for a future paper.

Section 2 is a brief review of general results on foliations, especially in characteristic p. The main two sections of the paper, Sections 3 and 4, are devoted to the two types of foliations respectively.

1.0.1. *Notation.*

- For any commutative $\mathbb{F}_p$-algebra R we let $\phi : R \to R$ be the homomorphism $\phi(x) = x^p$.

- If S is a scheme over $\mathbb{F}_p$, Φ_S denotes its absolute Frobenius morphism. It is given by the identity on the underlying topological space of S, and by the map ϕ on its structure sheaf. If $\mathcal{H}$ is an $\mathcal{O}_S$-module then we write $\mathcal{H}^{(p)}$ (or $\mathcal{H}^{(p)/S}$) for $\Phi_S^*\mathcal{H} = \mathcal{O}_S \otimes_{\phi,\mathcal{O}_S} \mathcal{H}$.
- If $T \to S$ is a morphism of schemes over $\mathbb{F}_p$, $T^{(p)}$ (or $T^{(p)/S}$) is $S \times_{\Phi_S,S} T$ and $\mathrm{Fr}_{T/S} : T \to T^{(p)}$ is the relative Frobenius morphism, characterized by the relation $pr_2 \circ \mathrm{Fr}_{T/S} = \Phi_T$.
- If $A \to S$ is an abelian scheme and $A^t \to S$ is its dual, then $\mathrm{Fr}_{A/S}$ is an isogeny and Verschiebung $\mathrm{Ver}_{A/S} : A^{(p)} \to A$ is the dual isogeny of $\mathrm{Fr}_{A^t/S}$.
- If $\mathcal{H}$ is an $\mathcal{O}_S$-module with $\mathcal{O}$ action (for some ring $\mathcal{O}$), and $\tau : \mathcal{O} \to \mathcal{O}_S$ is a homomorphism, then
$$\mathcal{H}[\tau] = \{\alpha \in \mathcal{H} |\, \forall a \in \mathcal{O}\ a.\alpha = \tau(a)\alpha\}.$$
If $T : \mathcal{H} \to \mathcal{G}$ is a homomorphism of sheaves of modules, we denote $\ker T = \mathcal{H}[T]$.
- If $x \in S$, the fiber of $\mathcal{H}$ at x is denoted $\mathcal{H}_x$. This is a vector space over the residue field $k(x)$. The same notation is used for the fiber $\mathcal{H}_x = x^*\mathcal{H}$ at a geometric point $x : \mathrm{Spec}(k) \to S$.
- If $\mathcal{H}^\vee$ is the dual of a locally free $\mathcal{O}_S$-module $\mathcal{H}$ we denote the pairing $\mathcal{H}^\vee \times \mathcal{H} \to \mathcal{O}_S$ by $\langle , \rangle$.
- By the Dieudonné module of a p-divisible group over a perfect field k in characteristic p, or of a finite commutative p-torsion group scheme over k, we understand its *contravariant* Dieudonné module.

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2. Generalities on Foliations

2.1. **Smooth foliations.** Let k be a perfect field and S a d-dimensional smooth k-variety. Let $\mathcal{T}$ denote the tangent bundle of S. If $U \subset S$ is Zariski open, then sections $\xi \in \mathcal{T}(U)$ are *vector fields* on U and act on $\mathcal{O}_S(U)$ as derivations. The space $\mathcal{T}(U)$ has a structure of a Lie algebra (of infinite dimension) over k, when we define
$$[\xi, \eta](f) = \xi(\eta(f)) - \eta(\xi(f)).$$
A *foliation* $\mathcal{F}$ on S is a saturated subsheaf $\mathcal{F} \subset \mathcal{T}$ closed under the Lie bracket. For every Zariski open set U, the vector fields on U along the foliation form a saturated $\mathcal{O}_S(U)$-submodule $\mathcal{F}(U) \subset \mathcal{T}(U)$ closed under the Lie bracket, i.e. if $f \in \mathcal{O}_S(U)$, $\xi \in \mathcal{T}(U)$ and $f\xi \in \mathcal{F}(U)$ then $\xi \in \mathcal{F}(U)$, and if $\xi, \eta \in \mathcal{F}(U)$ then $[\xi, \eta] \in \mathcal{F}(U)$.

The foliation $\mathcal{F}$ is called *smooth* if it is a vector sub-bundle of $\mathcal{T}$, namely if both $\mathcal{F}$ and $\mathcal{T}/\mathcal{F}$ are locally free sheaves. Since $\mathcal{F}$ is assumed to be saturated, and since a torsion-free finite module over a discrete valuation ring is free, the locus $Sing(\mathcal{F})$ where $\mathcal{F}$ is *not* smooth is a closed subset of S of codimension ≥ 2. As an example, the vector field $x\partial/\partial x + y\partial/\partial y$ generates a rank-1 foliation on $\mathbb{A}^2$, whose singular set is the origin.

If $k = \mathbb{C}$ then by a well known theorem of Frobenius every $x \in S - Sing(\mathcal{F})$ has a *classical* open neighborhood $V \subset S(\mathbb{C})$ which can be decomposed into a disjoint union of parallel leaves F of the foliation. Each F is a smooth complex submanifold of V and if $y \in F$ then the tangent space $T_yF \subset T_yS = \mathcal{T}_y$ is just the fiber of $\mathcal{F}$ at y. One says that F is an *integral subvariety* of the foliation. Whether a smooth foliation has an *algebraic* integral subvariety passing through a given point x, and the classification of all such integral subvarieties, is in general a hard problem, see [Bo].

There is a rich literature on foliations on algebraic varieties. See, for example, the book [Br].

2.2. **Foliations in positive characteristic.**

2.2.1. *p-foliations and purely inseparable morphisms of height 1.* If $\mathrm{char}(k) = p$ is positive then $\mathcal{T}$ is a p-Lie algebra, namely if $\xi \in \mathcal{T}(U)$ then $\xi^p = \xi \circ \cdots \circ \xi$ (composition p times) is also a derivation, hence lies in $\mathcal{T}(U)$. A foliation $\mathcal{F}$ is called a *p-foliation* if it is *p-closed*: whenever $\xi \in \mathcal{F}(U)$, then $\xi^p \in \mathcal{F}(U)$ as well.

The interest in p-foliations in characteristic p stems from their relation to purely inseperable morphisms of height 1. The following theorem has its origin in Jacobson's *inseparable Galois theory* for field extensions ([Jac]§8). We denote by $\phi : k \to k$ the p-power map, by $S^{(p)} = k \times_{\phi,k} S$ the p-transform of S, and by

$$\mathrm{Fr}_{S/k} : S \to S^{(p)}$$

the relative Frobenius morphism. We denote by $\Phi_S : S \to S$ the absolute Frobenius of S. Thus,

$$\Phi_S = pr_2 \circ \mathrm{Fr}_{S/k},$$

where $pr_2 : S^{(p)} = k \times_{\phi,k} S \to S$ is the projection onto the second factor.

Theorem 2.2.1. [Ek] *Let k be a perfect field,* $\mathrm{char}(k) = p$*. Let S be a smooth k-variety and denote by $\mathcal{T}$ its tangent bundle. There exists a one-to-one correspondence between smooth p-foliations $\mathcal{F} \subset \mathcal{T}$ and factorizations of the relative Frobenius morphism* $\mathrm{Fr}_{S/k} = g \circ f$,

$$S \xrightarrow{f} T \xrightarrow{g} S^{(p)},$$

where T is a smooth k-variety (equivalently, where f and g are finite and flat). We call T the quotient of S by the foliation $\mathcal{F}$.

Given $\mathcal{F}$, if (locally) $S = \mathrm{Spec}(A)$, then $T = \mathrm{Spec}(B)$ where $B = A^{\mathcal{F}=0}$ is the subring annihilated by $\mathcal{F}$, and f is induced by the inclusion $B \subset A$. Conversely, given a factorization as above, then $\mathcal{F} = \ker(df)$ where df is the map induced by f on the tangent bundle.

Furthermore, if $r = \mathrm{rk}(\mathcal{F})$ then $\deg(f) = p^r$.

As mentioned above, the birational version of this theorem is due to Jacobson. From this version one deduces rather easily a correspondence as in the theorem, when T is only assumed to be *normal,* and $\mathcal{F}$ is saturated, closed under the Lie bracket and p-closed, but not assumed to be smooth. The main difficulty is in showing that T is smooth if and only if $\mathcal{F}$ is smooth, i.e. locally a direct summand everywhere. The reference [Ek] only cites the work of Yuan [Yuan] and of Kimura and Nitsuma [Ki-Ni], but does not give the details. The proof in the book [Mi-Pe] seems to be wrong. A full account may be found in [Li].

2.2.2. *The obstruction to being p-closed.* If $\mathcal{F} \subset \mathcal{T}$ is a smooth foliation, the map $\xi \mapsto \xi^p \mod \mathcal{F}$ induces an $\mathcal{O}_S$-linear map of vector bundles

$$\kappa_{\mathcal{F}} : \Phi_S^* \mathcal{F} \to \mathcal{T}/\mathcal{F}$$

which is identically zero if and only if $\mathcal{F}$ is p-closed. See [Ek], Lemma 4.2(ii). We call the map $\kappa_{\mathcal{F}}$ the *obstruction to being p-closed*.

2.3. **Integral varieties in positive characteristic.** In contrast to the situation in characteristic 0, integral varieties of p-foliations in characteristic p abound, and are easily described. The goal of this section is to clarify their construction.[1] As in the previous section we let S be a smooth d-dimensional variety over a perfect field k of characteristic p, $\mathcal{T} = TS$ its tangent bundle, and $\mathcal{F}$ a smooth p-foliation of rank r. We denote by $S \xrightarrow{f} T$ the quotient of S by $\mathcal{F}$, as in Theorem 2.2.1.

Let $\mathcal{G} = \mathcal{T}/\mathcal{F} = \mathrm{Im}(df)$. This is a smooth p-foliation of rank $d - r$ on T and the quotient of T by $\mathcal{G}$ is $T \xrightarrow{g} S^{(p)}$. The factorizations of the relevant Frobenii are $Fr_{S/k} = g \circ f$ and $Fr_{T/k} = f^{(p)} \circ g$.

Definition 2.3.1. Let $\iota : W \hookrightarrow S$ be a closed subvariety of S and W^{sm} the (open dense) smooth locus in W. We say that W is an *integral variety* of $\mathcal{F}$ if at every $x \in W^{\mathrm{sm}}$ we have $T_xW = \iota^*\mathcal{F}_x$ (as subspaces of ι^*T_xS). In this case $\dim W = r$. We say that W is *transversal* to $\mathcal{F}$ at $x \in W^{\mathrm{sm}}$ if $T_xW \cap \iota^*\mathcal{F}_x = 0$, and that it is *generically transversal to* $\mathcal{F}$ if the set of points where it is transversal is a dense open set of W.

Remark 2.3.2. Unlike the case of characteristic 0, an integral subvariety of a smooth p-foliation need not be smooth. Consider, for example, the foliation generated by $\partial/\partial v$ on $\mathbb{A}^2 = Spec(k[u, v])$. The irreducible curve

$$u(u + v^p) + v^{2p} = 0$$

is an integral curve of the foliation, but is singular at $x = (0, 0)$. The curve $u - v^2 = 0$ is generically transversal to the same foliation, although it is not transversal to it at x.

If $S = Spec(A)$ and $W = Spec(A/I)$ for a prime ideal I, then regarding $\mathcal{F}$ as a submodule of the module of derivations of A over k, W is an integral variety of $\mathcal{F}$ if and only if $\mathcal{F}(I) \subset I$.

Proposition 2.3.3. *Let $\iota : W \hookrightarrow S$ be a closed r-dimensional subvariety of S and $Z = f(W) \hookrightarrow T$ the corresponding subvariety of T (also r-dimensional). Then the following are equivalent:*

(1) $f_W : W \to Z$ is purely inseparable of degree p^r;
(2) W is an integral variety of $\mathcal{F}$;
(3) $g_Z : Z \to W^{(p)}$ is a birational isomorphism;
(4) Z is generically transversal to $\mathcal{G}$.

Proof. Remark first that since $g_Z \circ f_W = Fr_{W/k}$ induces a purely inseparable field extension $k(W^{(p)}) \subset k(W)$ of degree p^r, (1) and (3) are equivalent, and in fact are equivalent to g_Z being separable (generically étale).

[1] This construction shows that the conjecture "of André-Oort type", suggested in §5.3 of [G-dS1], is far from being true.

Let $x \in W^{\mathrm{sm}}$ be such that $y = f(x) \in Z^{\mathrm{sm}}$. The commutative diagram

$$\begin{tikzcd} T_xW \arrow[r, hook] \arrow[d, "df_{W,x}"'] & T_xS \arrow[d, "df_x"] \\ T_yZ \arrow[r, hook] & T_yT \end{tikzcd}$$

and the fact that $\ker(df) = \mathcal{F}$ tell us that

$$\ker(df_W) = TW^{\mathrm{sm}} \cap \iota^*\mathcal{F}.$$

It follows that f_W is purely inseparable of degree p^r ($df_W = 0$) if and only if TW^{sm} and $\iota^*\mathcal{F}$, both rank-r vector sub-bundles of ι^*TS, coincide along W^{sm}. This shows the equivalence of (1) and (2). It also follows that f_W is separable (generically étale) if and only for generic x we have $T_xW^{\mathrm{sm}} \cap \iota^*\mathcal{F}_x = 0$. When applied to g and Z, instead of f and W, this gives the equivalence of (3) and (4). $\square$

Theorem 2.3.4. *Notation as above, any two points x_1, x_2 of S lie on an integral variety of $\mathcal{F}$.*

Proof. We have the diagram $S \xrightarrow{f} T \xrightarrow{g} S^{(p)}$, where the first arrow is dividing by the foliation $\mathcal{F}$ and the second arrow by the foliation $\mathcal{G}$. Let $y_i = f(x_i) \in T$. By Bertini's theorem there is a subvariety $Z \subset T$ of dimension r passing through the points y_i which is generically transversal to $\mathcal{G}$. Choosing W so that $g(Z) = W^{(p)}$, and hence $f(W) = Z$, we conclude from the previous Proposition that W is an integral variety of $\mathcal{F}$ passing through x_1 and x_2. $\square$

Thus integral varieties in characteristic p abound, and are easy to classify. Nevertheless, given a particular subvariety W, it is still interesting to decide whether it is an integral variety of $\mathcal{F}$ or not.

3. Tautological foliations on Hilbert modular varieties

3.1. **Hilbert modular schemes.** Let L be a totally real field, $[L : \mathbb{Q}] = g \geq 2$, $N \geq 4$ an integer, and $\mathfrak{c}$ a fractional ideal of L, relatively prime to N, called the *polarization module*. We denote by $\mathfrak{d}$ the different ideal of $L/\mathbb{Q}$. Let $D = \mathrm{disc}_{L/\mathbb{Q}}$ be the absolute discriminant of L.

Consider the moduli problem over $\mathbb{Z}[(ND)^{-1}]$, attaching to a scheme S over $\mathbb{Z}[(ND)^{-1}]$ the set $\mathscr{M}(S)$ of isomorphism classes of four-tuples

$$\underline{A} = (A, \iota, \lambda, \eta),$$

where

- A is an abelian scheme over S of relative dimension g.
- $\iota : \mathcal{O}_L \hookrightarrow \mathrm{End}(A/S)$ is an injective ring homomorphism making the tangent bundle TA a locally free sheaf of rank 1 over $\mathcal{O}_L \otimes \mathcal{O}_S$. We denote by A^t the dual abelian scheme, and by $\iota(a)^t$ the dual endomorphism induced by $\iota(a)$.
- $\lambda : \mathfrak{c} \otimes_{\mathcal{O}_L} A \simeq A^t$ is a $\mathfrak{c}$-polarization of A in the sense of [Ka] (1.0.7). This means that λ is an isomorphism of abelian schemes compatible with the

$\mathcal{O}_L$ action, where $a \in \mathcal{O}_L$ acts on the left hand side via $a \otimes 1 = 1 \otimes \iota(a)$ and on the right hand side via $\iota(a)^t$. Furthermore, under the identification

$$\mathrm{Hom}_{\mathcal{O}_L}(A, A^t) \simeq \mathrm{Hom}_{\mathcal{O}_L}(A, \mathfrak{c} \otimes_{\mathcal{O}_L} A)$$

induced by λ, the symmetric elements on the left hand side (the elements α satisfying $\alpha^t = \alpha$ after we canonically identify A^{tt} with A) correspond precisely to the elements of $\mathfrak{c}$, and those arising from an ample line bundle $\mathcal{L}$ on A via $\alpha_{\mathcal{L}}(x) = [\tau_x^* \mathcal{L} \otimes \mathcal{L}^{-1}]$ correspond to the totally positive cone in $\mathfrak{c}$. Note that the Rosati involution induced by λ on $\mathcal{O}_L$ is the identity.

- η is a $\Gamma_{00}(N)$-level structure on A in the sense of [Ka] (1.0.8), i.e. a closed immersion of S-group schemes $\eta : \mathfrak{d}^{-1} \otimes \mu_N \hookrightarrow A[N]$ compatible with ι.

The moduli problem $\mathscr{M}$ is representable by a smooth scheme over $\mathbb{Z}[(ND)^{-1}]$, of relative dimension g, which we denote by the same letter $\mathscr{M}$, and call the *Hilbert modular scheme*. If we want to remember the dependence on $\mathfrak{c}$ we use the notation $\mathscr{M}^{\mathfrak{c}}$ for $\mathscr{M}$. The Hilbert moduli scheme admits smooth toroidal compactifications, depending on some extra data. See [Lan].

The complex points $\mathscr{M}(\mathbb{C})$ of $\mathscr{M}$ may be identified, as a complex manifold, with $\Gamma \backslash \mathfrak{H}^g$, where $\mathfrak{H}$ is the upper half plane, and $\Gamma \subset \mathrm{SL}(\mathcal{O}_L \oplus \mathfrak{d}^{-1}\mathfrak{c}^{-1})$ is some congruence subgroup.

If $\mathfrak{c} = \gamma \mathfrak{c}'$ where $\gamma \gg 0$ then $\lambda \circ (\gamma \otimes 1) : \mathfrak{c}' \otimes_{\mathcal{O}_L} A \simeq A^t$ is a $\mathfrak{c}'$-polarization of A. Thus only the strict ideal class of $\mathfrak{c}$ in $Cl^+(L)$ matters: the moduli schemes $\mathscr{M}^{\mathfrak{c}}$ and $\mathscr{M}^{\mathfrak{c}'}$ are isomorphic, via an isomorphism depending on the choice of γ.

Fix a prime number p which is unramified in L and relatively prime to N. By the last remark, we may (and do) assume that p is also relatively prime to $\mathfrak{c}$. Write

$$p\mathcal{O}_L = \mathfrak{p}_1 \dots \mathfrak{p}_r, \;\; f_i = f(\mathfrak{p}_i/p)$$

the inertia degree of $\mathfrak{p}_i$, and let κ be a finite field into which all the $\kappa(\mathfrak{p}_i) = \mathcal{O}_L/\mathfrak{p}_i$ embed, i.e. $\kappa = \mathbb{F}_{p^n}$ where n is divisible by $\mathrm{lcm}\{f_1, \dots, f_r\}$. Let $W(\kappa)$ be the Witt vectors of κ. We have a decomposition

$$\mathbb{B} = \mathrm{Hom}(L, W(\kappa)[p^{-1}]) = \coprod_{\mathfrak{p}|p} \mathbb{B}_{\mathfrak{p}}$$

indexed by the r primes of $\mathcal{O}_L$ dividing p, where $\sigma : L \hookrightarrow W(\kappa)[1/p]$ belongs to $\mathbb{B}_{\mathfrak{p}}$ if $\sigma^{-1}(pW(\kappa)) \cap \mathcal{O}_L = \mathfrak{p}$. The Frobenius automorphism of $W(\kappa)$, denoted ϕ, operates on $\mathbb{B}$ from the left via $\sigma \mapsto \phi \circ \sigma$, and permutes each $\mathbb{B}_{\mathfrak{p}}$ cyclically.

We denote by

$$M = \mathscr{M} \times_{\mathbb{Z}[(ND)^{-1}]} \mathbb{F}_p$$

the special fiber of $\mathscr{M}$ at p. Note that M is smooth over $\mathbb{F}_p$.

3.2. **The tautological foliations.**

3.2.1. *The definition.* Complex analytically, the complex manifold $\Gamma \backslash \mathfrak{H}^g$ admits g tautological rank 1 smooth foliations, generated by the vector fields $\partial/\partial z_i$, where z_i $(1 \le i \le g)$ are the coordinate functions. As the derivations $\partial/\partial z_i$ commute with each other, any r of them generate a rank r analytic foliation on $\mathscr{M}(\mathbb{C})$. We shall now show that these foliations are in fact algebraic, and address the question which of them descends modulo p to a smooth p-foliation on M. We shall then relate the quotients of M by these foliations to Hilbert modular varieties with Iwahori level structure at p.

Convention. From now on we denote by $\mathscr{M}$ the base change of the Hilbert modular scheme from $\mathbb{Z}[(ND)^{-1}]$ to $W(\kappa)$ and by M its special fiber, a smooth variety over κ. Recall that κ is assumed to be large enough so that there are g distinct embeddings $\sigma : L \hookrightarrow W(\kappa)[1/p]$.

Let $\pi : A^{\mathrm{univ}} \to \mathscr{M}$ denote the universal abelian variety over $\mathscr{M}$ and

$$\underline{\omega} = \pi_* \Omega^1_{A^{\mathrm{univ}}/\mathscr{M}}$$

its *Hodge bundle.* Since $p \nmid \mathrm{disc}_{L/\mathbb{Q}}$, it decomposes under the action of $\mathcal{O}_L$ as a direct sum of g line bundles

$$\underline{\omega} = \oplus_{\sigma\in\mathbb{B}} \mathscr{L}_\sigma$$

where $\mathscr{L}_\sigma = \{\alpha \in \underline{\omega} |\, \iota(a)^*(\alpha) = \sigma(a)\alpha \ \forall a \in \mathcal{O}_L\}$.

Let $\underline{\mathrm{Lie}} = \underline{\mathrm{Lie}}(A^{\mathrm{univ}}/\mathscr{M}) = \underline{\omega}^\vee$ be the relative tangent space of the universal abelian variety. The Kodaira-Spencer isomorphism is an isomorphism of $\mathcal{O}_{\mathscr{M}}$-modules ([Ka], (1.0.19)-(1.0.20))

$$\begin{aligned}(3.1)\quad \mathrm{KS} : \mathcal{T}_{\mathscr{M}/W(\kappa)} &\simeq \mathrm{Hom}_{\mathcal{O}_L\otimes\mathcal{O}_{\mathscr{M}}}(\underline{\omega}, \underline{\mathrm{Lie}}((A^{\mathrm{univ}})^t/\mathscr{M})) \\ &\simeq \mathrm{Hom}_{\mathcal{O}_L\otimes\mathcal{O}_{\mathscr{M}}}(\underline{\omega}, \underline{\mathrm{Lie}} \otimes_{\mathcal{O}_L} \mathfrak{c}) = \underline{\mathrm{Lie}}^{\otimes 2} \otimes_{\mathcal{O}_L} \mathfrak{dc},\end{aligned}$$

where the second isomorphism results from the polarization $\lambda^{\mathrm{univ}} : \mathfrak{c} \otimes_{\mathcal{O}_L} A^{\mathrm{univ}} \simeq (A^{\mathrm{univ}})^t$. Since $(p, \mathfrak{dc}) = 1$ we have

$$\underline{\mathrm{Lie}} \otimes_{\mathcal{O}_L} \mathfrak{dc} = \underline{\mathrm{Lie}} \simeq \oplus_{\sigma\in\mathbb{B}} \mathscr{L}_\sigma^\vee.$$

We therefore get from KS a canonical decomposition of the tangent space of $\mathscr{M}$ into a direct sum of g line bundles

$$\mathcal{T}_{\mathscr{M}/W(\kappa)} \simeq \oplus_{\sigma\in\mathbb{B}} \mathscr{L}_\sigma^{-2}.$$

We denote by $\mathscr{F}_\sigma$ the line sub-bundle of $\mathcal{T}_{\mathscr{M}/W(\kappa)}$ corresponding to $\mathscr{L}_\sigma^{-2}$ under this isomorphism.

Lemma 3.2.1. *Let $\Sigma \subset \mathbb{B}$ be any set of embeddings of L into $W(\kappa)[1/p]$. Then $\mathscr{F}_\Sigma = \oplus_{\sigma\in\Sigma} \mathscr{F}_\sigma$ is involutive: if ξ, η are sections of $\mathscr{F}_\Sigma$, so is $[\xi, \eta]$.*

Proof. Recall that the Kodaira-Spencer isomorphism is derived from the Gauss-Manin connection

$$\nabla : H^1_{dR}(A^{\mathrm{univ}}/\mathscr{M}) \to H^1_{dR}(A^{\mathrm{univ}}/\mathscr{M}) \otimes_{\mathcal{O}_{\mathscr{M}}} \Omega^1_{\mathscr{M}/W(\kappa)}.$$

If ξ is a section of $\mathcal{T}_{\mathscr{M}/W(\kappa)}$ we denote by $\nabla_\xi : H^1_{dR}(A^{\mathrm{univ}}/\mathscr{M}) \to H^1_{dR}(A^{\mathrm{univ}}/\mathscr{M})$ the map obtained by contraction with ξ. The Gauss-Manin connection is well-known to be flat, namely

$$\nabla_{[\xi,\eta]} = \nabla_\xi \circ \nabla_\eta - \nabla_\eta \circ \nabla_\xi.$$

Now, ∇_ξ commutes with $\iota(a)^*$ for $a \in \mathcal{O}_L$, and therefore preserves the σ-isotypic component $H^1_{dR}(A^{\mathrm{univ}}/\mathscr{M})[\sigma]$ for each $\sigma \in \mathbb{B}$. *By definition,* $\xi \in \mathscr{F}_\sigma$ if for every $\tau \neq \sigma$ the operator ∇_ξ maps the subspace

$$\mathscr{L}_\tau = \underline{\omega}[\tau] \subset H^1_{dR}(A^{\mathrm{univ}}/\mathscr{M})[\tau]$$

to itself. Similarly, $\xi \in \mathscr{F}_\Sigma$ if the same holds for every $\tau \notin \Sigma$. It follows at once from the flatness of ∇ that if this condition holds for ξ and η, it holds for $[\xi, \eta]$. $\square$

We conclude that $\mathscr{F}_\Sigma$ is a smooth foliation. We call these foliations *tautological.*

3.2.2. *The main theorem.* We consider now the foliations $\mathscr{F}_\Sigma$ in the special fiber $M = \mathscr{M} \times_{W(\kappa)} \kappa$ only. The following theorem summarizes the main results in the Hilbert modular case. As we learned from [E-SB-T], point (i) was also observed there some years ago.

Theorem 3.2.2. *(i) The smooth foliation $\mathscr{F}_\Sigma$ is p-closed if and only if Σ is invariant under the action of Frobenius, namely $\phi \circ \Sigma = \Sigma$. In particular, $\mathscr{F}_\sigma$ is p-closed if and only if $f(\mathfrak{p}_\sigma/p) = 1$ where $\mathfrak{p}_\sigma$ is the prime induced by σ.*

(ii) Suppose that $f(\mathfrak{p}_\sigma/p) \neq 1$. Then, up to a unit, the obstruction $\kappa_{\mathscr{F}_\sigma}$ to $\mathscr{F}_\sigma$ being p-closed (2.2.2) is equal to the square of the $\phi\circ\sigma$-partial Hasse invariant $h_{\phi\circ\sigma}$ [Go].

(iii) Let $\mathfrak{p}$ be a prime of L above p and $\mathscr{F}_\mathfrak{p} = \mathscr{F}_{\mathbb{B}_\mathfrak{p}} = \oplus_{\sigma\in\mathbb{B}_\mathfrak{p}}\mathscr{F}_\sigma$ the corresponding p-foliation. The quotient of M by the p-foliation $\oplus_{\mathfrak{q}\neq\mathfrak{p}}\mathscr{F}_\mathfrak{q}$ may be identified with the étale component of the $\Gamma_0(\mathfrak{p})$-moduli scheme $M_0(\mathfrak{p})$ (see details in the proof).

3.3. **Preliminaries.**

3.3.1. *Tate objects.* We begin our proof of Theorem 3.2.2 by recalling a result of Katz [Ka], who computed the effect of the Kodaira-Spencer isomorphism on q-expansions.

As in [Ka] (1.1.4), let S be a set of g linearly independent $\mathbb{Q}$-linear forms $l_i : L \to \mathbb{Q}$ preserving (total) positivity. Let $\mathfrak{a}$ and $\mathfrak{b}$ be fractional ideals of L, relatively prime to N, such that $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$. Let $\mathcal{R} = W(\kappa) \otimes \mathbb{Z}((\mathfrak{a}\mathfrak{b}, S))$ be the ring defined in [Ka] (1.1.7), after base change to $W(\kappa)$. Let

$$\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q) = \mathbb{G}_m \otimes \mathfrak{d}^{-1}\mathfrak{a}^{-1}/q(\mathfrak{b})$$

be the abelian scheme over $\mathcal{R}$ constructed in [Ka] (1.1.13). For fixed $\mathfrak{a}$ and $\mathfrak{b}$ it is essentially independent of S. Since $\mathfrak{a}$ is relatively prime to N, $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ admits a *canonical* $\Gamma_{00}(N)$-level structure η_{can} (denoted in [Ka] (1.1.16) by i_{can}). It also admits a canonical $\mathfrak{c}$-polarization λ_{can} and a canonical action ι_{can} of $\mathcal{O}_L$. We thus obtain an object

$$\underline{\mathrm{Tate}}_{\mathfrak{a},\mathfrak{b}}(q) = (\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \iota_{can}, \lambda_{can}, \eta_{can})$$

over the ring $\mathcal{R}$ (for any choice of S). For the definition of q-expansions at the cusp labeled by the pair $(\mathfrak{a},\mathfrak{b})$, recalled below, we assume, in addition, that $\mathfrak{a}$ is relatively prime to p. This can always be achieved, since only the classes of $\mathfrak{a}$ and $\mathfrak{b}$ in the strict ideal class group $Cl^+(L)$ matter.

The Lie algebra of $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ is given by a canonical isomorphism ([Ka] (1.1.17))

$$\omega_\mathfrak{a} : \underline{\mathrm{Lie}} = \mathrm{Lie}(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)/\mathcal{R}) \simeq \mathrm{Lie}(\mathbb{G}_m \otimes \mathfrak{d}^{-1}\mathfrak{a}^{-1}/\mathcal{R}) = \mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes \mathcal{R},$$

hence the Kodaira-Spencer map KS (3.1) induces a map

$$\mathrm{KS} : \mathcal{T}_{\mathrm{Spec}(\mathcal{R})/W(\kappa)} \to \underline{\mathrm{Lie}}^{\otimes 2} \otimes_{\mathcal{O}_L} \mathfrak{d}\mathfrak{c} \simeq \mathfrak{d}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1} \otimes \mathcal{R}. \tag{3.2}$$

The tangent bundle $\mathcal{T}_{\mathrm{Spec}(\mathcal{R})/W(\kappa)}$ is the module of $W(\kappa)$-derivations of $\mathcal{R}$. For $\gamma \in \mathfrak{d}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1}$ we may consider the derivation $D(\gamma)$ of $\mathcal{R}$ (analogue of $q\frac{d}{dq}$) given by

$$D(\gamma)(\sum_{\alpha\in\mathfrak{a}\mathfrak{b},\ l_i(\alpha)\geq -n} a_\alpha q^\alpha) = \sum_{\alpha\in\mathfrak{a}\mathfrak{b},\ l_i(\alpha)\geq -n} \mathrm{Tr}_{L/\mathbb{Q}}(\alpha\gamma) a_\alpha q^\alpha.$$

We then have the following elegant result.

Lemma 3.3.1. *(*[Ka] *(1.1.20)) The image of $D(\gamma)$ under the Kodaira-Spencer map* KS *in* (3.2) *is* $\gamma \otimes 1$.

3.3.2. *Hilbert modular forms and partial Hasse invariants.* In this subsection we set up notation for Hilbert modular forms, and recall some results due to the first author and to Diamond and Kassaei on the q-expansions of such forms over κ.

By our assumption on κ, for any $W(\kappa)$-algebra R we have

$$\mathcal{O}_L \otimes R \simeq \oplus_{\sigma\in\mathbb{B}} R_\sigma$$

where R_σ is the ring R equipped with the action of $\mathcal{O}_L$ via $\sigma : \mathcal{O}_L \hookrightarrow W(\kappa)$. A *weight* is a tuple $k = (k_\sigma)_{\sigma\in\mathbb{B}}$ with $k_\sigma \in \mathbb{Z}$. We shall often write k also as the element

$$\sum_{\sigma\in\mathbb{B}} k_\sigma[\sigma] \in \mathbb{Z}[\mathbb{B}].$$

The weight k defines a homomorphism of algebraic groups over $W(\kappa)$, $\chi = \chi_k : \mathrm{Res}_{W(\kappa)}^{\mathcal{O}_L\otimes W(\kappa)}(\mathbb{G}_m) \to \mathbb{G}_m$, given on R-points by

$$\chi : (\mathcal{O}_L \otimes R)^\times \to R^\times,\ \ \chi(a\otimes x) = \prod_{\sigma\in\mathbb{B}} (\sigma(a)x)^{k_\sigma}.$$

A weight k, level N Hilbert modular form (HMF) f over $W(\kappa)$ "à la Katz" is a rule associating to any $W(\kappa)$-algebra R, any four-tuple $\underline{A}$ over R as above, and any $\mathcal{O}_L \otimes R$-basis ω of $\underline{\omega}_{A/R}$ (such a basis always exists locally on R) an element $f(\underline{A}, \omega) \in R$, which depends only on the isomorphism type of the pair $(\underline{A}, \omega)$, is compatible with base change $R \to R'$ (over $W(\kappa)$), and satisfies

$$f(\underline{A}, \alpha\omega) = \chi(\alpha)^{-1} f(\underline{A}, \omega)$$

for $\alpha \in (\mathcal{O}_L \otimes R)^\times$. The $(\mathfrak{a}, \mathfrak{b})$-$q$-expansion of f is the element

$$f(\underline{\mathrm{Tate}}_{\mathfrak{a},\mathfrak{b}}(q), (\mathrm{Tr}_{L/\mathbb{Q}} \otimes 1) \circ \omega_{\mathfrak{a}}) \in \mathcal{R},$$

where $\mathcal{R}$, $\mathfrak{a}$ and $\mathfrak{b}$ are as above. Note that by our assumption that $\mathfrak{a}$ is relatively prime to p, $\mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes \mathcal{R} = \mathfrak{d}^{-1} \otimes \mathcal{R}$. Since $\mathrm{Tr}_{L/\mathbb{Q}} : \mathfrak{d}^{-1} \to \mathbb{Z}$ is an $\mathcal{O}_L$-basis of $\mathrm{Hom}(\mathfrak{d}^{-1}, \mathbb{Z})$,

$$(\mathrm{Tr}_{L/\mathbb{Q}} \otimes 1) \circ \omega_{\mathfrak{a}} : \underline{\mathrm{Lie}} \to \mathcal{R}$$

is indeed an $\mathcal{O}_L \otimes \mathcal{R}$-basis of $\underline{\omega}_{A/\mathcal{R}}$ for $A = \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$.

For a weight k denote by $\mathscr{L}_\chi$ the line bundle

$$\mathscr{L}_\chi = \bigotimes_{\sigma\in\mathbb{B}} \mathscr{L}_\sigma^{\otimes k_\sigma}$$

on $\mathscr{M}$. Let R be a $W(\kappa)$-algebra and $\underline{A}$ a four-tuple over R as above, corresponding to a morphism $h : \mathrm{Spec}(R) \to \mathscr{M}$ over $W(\kappa)$. An $\mathcal{O}_L \otimes R$-basis ω of $\omega_{A/R}$ yields R-bases ω_σ of the line bundles $\omega_{A/R}[\sigma] = h^*\mathscr{L}_\sigma$ for every $\sigma \in \mathbb{B}$, hence a basis ω_χ of $\bigotimes_{\sigma\in\mathbb{B}} \omega[\sigma]^{\otimes k_\sigma} = h^*\mathscr{L}_\chi$. If f is a weight k HMF then $f(\underline{A}, \omega)\cdot\omega_\chi$ is independent of ω. We may therefore regard f as a global section of $\mathscr{L}_\chi$, and vice versa, any global section of $\mathscr{L}_\chi$ over $\mathscr{M}$ is a HMF of weight k and level N.

Since we assume that $g \geq 2$, by the Köcher principle any HMF f is automatically holomorphic at the cusps, and the q-expansions of f lie in

$$\mathcal{R} \cap \{a_0 + \sum_{\alpha\gg 0} a_\alpha q^\alpha \,|\, a_0, a_\alpha \in W(\kappa)\}.$$

The same analysis holds if we restrict to κ-algebras R rather than $W(\kappa)$-algebras, and yields a definition of HMF's of weight k and level N over κ, as well as an interpretation of such modular forms as global sections of $\mathscr{L}_\chi$ over M, the special fiber of $\mathscr{M}$. We also get the mod-p $(\mathfrak{a},\mathfrak{b})$-$q$-expansion of a modular form over κ as an element of $\mathcal{R}/p\mathcal{R}$ by the same recipe. However, in general, not every HMF over κ lifts to a HMF over $W(\kappa)$. The exact sequence

$$0 \to \mathscr{L}_\chi \stackrel{\times p}{\to} \mathscr{L}_\chi \to \mathscr{L}_\chi/p\mathscr{L}_\chi \to 0$$

shows that the obstruction to lifting a HMF over κ lies in $H^1(\mathscr{M},\mathscr{L}_\chi)$.

Let $M_k(N,W(\kappa)) = H^0(\mathscr{M},\mathscr{L}_{\chi_k})$ denote the space of weight k, level N HMF's over $W(\kappa)$ and similarly $M_k(N,\kappa) = H^0(M,\mathscr{L}_{\chi_k})$ the space of weight k, level N HMF's over κ. The *q-expansion principle* says that a modular form over $W(\kappa)$ (or over κ), all of whose q-expansions, for all choices of $(\mathfrak{a},\mathfrak{b})$ - corresponding to the various cusps of the Hilbert modular scheme - vanish, is 0.

The space

$$M_*(N,W(\kappa)) = \bigoplus_{k\in\mathbb{Z}[\mathbb{B}]} M_k(N,W(\kappa))$$

carries a natural ring structure, and is called the *ring of modular forms of level N* over $W(\kappa)$. Similar terminology applies to the ring $M_*(N,\kappa)$ of modular forms of level N over κ. The q-expansion homomorphisms extend naturally to ring homomorphisms from these rings to the rings $\mathcal{R}$ or $\mathcal{R}/p\mathcal{R}$ (depending on the choice of $\mathfrak{a}$ and $\mathfrak{b}$). However, different HMF's over κ (of different weights) may now have the same q-expansions.

An important role in the study of q-expansions in characteristic p is played by the *g partial Hasse invariants* h_σ ($\sigma \in \mathbb{B}$). These are modular forms over κ, of weights

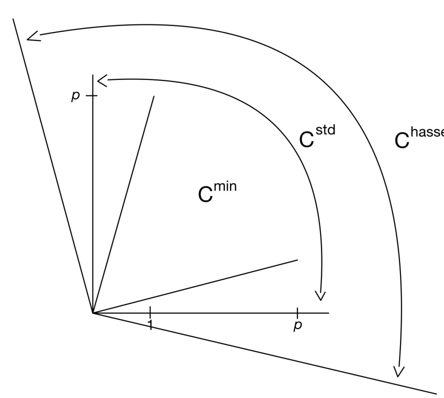
$$k_\sigma = p[\phi^{-1}\circ\sigma] - [\sigma].$$

Their q-expansion at every cusp is 1. See Theorem 2.1 of [Go].

Following [DK] we define the following cones $C^{\min} \subset C^{\mathrm{std}} \subset C^{\mathrm{hasse}}$ in $\mathbb{R}[\mathbb{B}]$:

- $C^{\min} = \{\sum a_\sigma[\sigma] |\ pa_\sigma \geq a_{\phi^{-1}\circ\sigma}\ \forall\sigma\}$ (the *minimal cone*)
- $C^{\mathrm{std}} = \{\sum a_\sigma[\sigma] |\ a_\sigma \geq 0\ \forall\sigma\}$ (the *standard cone*)
- $C^{\mathrm{hasse}} = \{\sum_\sigma a_\sigma(p[\phi^{-1}\circ\sigma]-[\sigma]) |\ a_\sigma \geq 0\}$ (the *Hasse cone*).

For example, when $g=2$ and p is split in L all three cones coincide. In contrast, when p is inert in L they look as follows:



Let $f \in M_k(N,\kappa)$. According to [A-G] Proposition 8.9, any other $g \in M_{k'}(N,\kappa)$ having the same q-expansions as f is a product of f and partial Hasse invariants raised to integral (possibly negative) powers. Furthermore, there exists a weight $\Phi(f)$, called the *filtration* of f, and $g \in M_{\Phi(f)}(N,\kappa)$ having the same q-expansions as f, such that any other $g_1 \in M_{k'}(N,\kappa)$ with the same q-expansions is of the form

$$g_1 = g\prod_{\sigma\in\mathbb{B}} h_\sigma^{n_\sigma}$$

for some integers $n_\sigma \geq 0$. These results generalize well-known results of Serre for elliptic modular forms in characteristic p.

Theorem 3.3.2. *(*[DK]*, Corollary 1.2) The filtration $\Phi(f) \in C^{\min}$.*

3.3.3. *Hilbert modular varieties with Iwahori level structure at p.* The last piece of input needed for the proof of Theorem 3.2.2 concerns $\Gamma_0(\mathfrak{p})$-moduli problems, for the primes $\mathfrak{p}$ of L dividing p. References for the results quoted below are [G-K, Pa, St].

Let $\mathscr{M}_0(\mathfrak{p})$ be the moduli problem over $W(\kappa)$ classifying isomorphism types of tuples $(\underline{A}, H)$ where $\underline{A} \in \mathscr{M}$ and H is a finite flat $\mathcal{O}_L$-invariant isotropic subgroup scheme of $A[\mathfrak{p}]$ of rank p^f, where $f = f(\mathfrak{p}/p)$. The meaning of "isotropic" is the following.

Since we assumed that $\mathfrak{c}$ is relatively prime to p there is a canonical isomorphism of group schemes $\mathfrak{c}\otimes_{\mathcal{O}_L} A[p] \simeq A[p]$ (sending $\alpha\otimes u$ to $\iota(\alpha)u$). The canonical e_p-pairing $A[p]\otimes A^t[p] \to \mu_p$ therefore induces, via λ, a perfect Weil pairing

$$\langle .,.\rangle_\lambda : A[p]\otimes A[p] \to \mu_p.$$

It restricts to a perfect pairing on $A[\mathfrak{p}]$ (since the Rosati involution on $\mathcal{O}_L$ is the identity), and H is a maximal isotropic subgroup scheme of $A[\mathfrak{p}]$.

If $\mathfrak{c}$ is not relatively prime to p, we may change it, in the definition of $\mathscr{M}$, to $\mathfrak{c}' = \gamma\mathfrak{c}$ where $\gamma \gg 0$, so that $\mathfrak{c}'$ is now relatively prime to p, and get an isomorphic moduli problem (the isomorphism depending on γ). By "isotropic" we then mean that H is isotropic with respect to the Weil pairing induced by the isomorphism $\mathfrak{c}'\otimes_{\mathcal{O}_L} A[p] \simeq A[p]$ as above.

As before, the moduli problem $\mathscr{M}_0(\mathfrak{p})$ is represented by a scheme, flat over $W(\kappa)$, which we denote by the same letter, and the forgetful morphism is a proper morphism $\mathscr{M}_0(\mathfrak{p}) \to \mathscr{M}$. The scheme $\mathscr{M}_0(\mathfrak{p})$ is normal and Cohen-Macaulay. We let

$$M_0(\mathfrak{p}) \to M$$

be the characteristic p fiber of this morphism, over the field κ. This morphism has been studied in detail in [G-K]. Away from the ordinary locus, it is neither finite nor flat.

Let M^{ord} be the ordinary locus of M, the open dense subset where none of the partial Hasse invariants h_σ vanishes. If k is an algebraically closed field containing κ and $x : \mathrm{Spec}(k) \to M$ a k-valued point of M, then x lies on M^{ord} if and only if the corresponding abelian variety $A_x = x^*(A^{\mathrm{univ}})$ is ordinary.

Let $M_0(\mathfrak{p})^{\mathrm{ord}}$ be the open subset of $M_0(\mathfrak{p})$ which lies over M^{ord}. Then $M_0(\mathfrak{p})^{\mathrm{ord}}$ is the disjoint union of two smooth varieties. The component $M_0(\mathfrak{p})^{\mathrm{ord,m}}$ (the *multiplicative* component) classifies tuples $(\underline{A}, H)$ where H is multiplicative (its Cartier dual is étale). The forgetful morphism is an isomorphism $M_0(\mathfrak{p})^{\mathrm{ord,m}} \simeq M^{\mathrm{ord}}$, its inverse given by the section $\underline{A} \mapsto (\underline{A}, A[\mathrm{Fr}]\cap A[\mathfrak{p}])$. Here $\mathrm{Fr} : A \to A^{(p)}$ is the relative Frobenius morphism (everything over a base scheme S lying over κ).

The second component $M_0(\mathfrak{p})^{\mathrm{ord,et}}$ (the *étale* component) classifies pairs $(\underline{A}, H)$ where H is étale. The forgetful morphism $M_0(\mathfrak{p})^{\mathrm{ord,et}} \to M^{\mathrm{ord}}$ is finite flat, purely inseparable of height 1 and degree p^f.

To discuss the Atkin-Lehner map w we have to bring the polarization module $\mathfrak{c}$ back into the picture, because w will in general change it. We therefore recall that all our constructions depended on an auxiliary ideal $\mathfrak{c}$ (at least on its narrow ideal class in $Cl^+(L)$), and use the notation $M^\mathfrak{c}$, $M_0^\mathfrak{c}(\mathfrak{p})$ etc. to emphasize this dependency. We now define

$$w : M_0^\mathfrak{c}(\mathfrak{p}) \to M_0^{\mathfrak{cp}}(\mathfrak{p}), \;\; w(\underline{A}, H) = (\underline{A}/H, A[\mathfrak{p}]/H).$$

Here $\underline{A}/H$ is the tuple $(A/H, \iota', \lambda', \eta')$ where ι' and η' are induced by ι and η. The $\mathfrak{cp}$-polarization λ' of A/H is obtained as follows. Giving λ is equivalent to giving a homomorphism

$$\psi_1 : \mathfrak{c} \to \mathrm{Hom}_{\mathcal{O}_L}(A, A^t)_{sym} = \mathcal{P}_A$$

such that $\psi_1(\alpha)$ is in the cone of $\mathcal{O}_L$-polarizations $\mathcal{P}_A^+$ if and only if $\alpha \gg 0$, and such that ψ_1 induces an isomorphism

$$\mathfrak{c} \otimes_{\mathcal{O}_L} A \simeq A^t.$$

Denoting A/H by B and letting $f : A \to B$ be the canonical homomorphism, the polarization λ' is determined by a similar homomorphism

$$\psi_2 : \mathfrak{cp} \to \mathrm{Hom}_{\mathcal{O}_L}(B, B^t)_{sym} = \mathcal{P}_B,$$

defined by the relation

$$f^t \circ \psi_2(\alpha) \circ f = \psi_1(\alpha)$$

for all $\alpha \in \mathfrak{pc} \subset \mathfrak{c}$. The *existence* of $\psi_2(\alpha)$ stems from the fact that H is isotropic for the pairing $A[p] \times A[p] \to \mu_p$ induced by $\psi_1(\alpha)$. Its *uniqueness* is obvious, and because of this uniqueness $\psi_2(\cdot)$ is a homomorphism. See [Pa], 2.2. The subgroup scheme $A[\mathfrak{p}]/H \subset (A/H)[\mathfrak{p}] = \mathfrak{p}^{-1}H/H$ is finite and flat, $\mathcal{O}_L$-invariant, and isotropic with respect to λ', of rank p^f. The tuple $(\underline{A}/H, A[\mathfrak{p}]/H)$ therefore lies on $M_0^{\mathfrak{cp}}(\mathfrak{p})$, and w is well defined.

In the definition of w we may replace $\mathfrak{cp}$ by a polarization module $\mathfrak{c}'$ which is relatively prime to p, as was assumed for $\mathfrak{c}$, by multiplying by an appropriate $\gamma \gg 0$, keeping the strict ideal class of $\mathfrak{cp}$ unchanged.

The Atkin-Lehner map is not an involution if the class of $\mathfrak{p}$ in $Cl^+(L)$ is not trivial. Indeed,

$$w^2(\underline{A}, H) = (\underline{A}/A[\mathfrak{p}], \mathfrak{p}^{-1}H/A[\mathfrak{p}]) \in M_0^{\mathfrak{cp}^2}(\mathfrak{p}).$$

Nevertheless, as in the case of modular curves, it preserves the ordinary locus and exchanges the ordinary étale and ordinary multiplicative components:

$$w : M_0^\mathfrak{c}(\mathfrak{p})^{\mathrm{ord,m}} \simeq M_0^{\mathfrak{cp}}(\mathfrak{p})^{\mathrm{ord,et}}, \;\; w : M_0^\mathfrak{c}(\mathfrak{p})^{\mathrm{ord,et}} \simeq M_0^{\mathfrak{cp}}(\mathfrak{p})^{\mathrm{ord,m}}.$$

We now define $M_0^\mathfrak{c}(\mathfrak{p})^{\mathrm{m}}$ to be the *Zariski closure* of $M_0^\mathfrak{c}(\mathfrak{p})^{\mathrm{ord,m}}$ in $M_0^\mathfrak{c}(\mathfrak{p})$, and $M_0^\mathfrak{c}(\mathfrak{p})^{\mathrm{et}}$ the Zariski closure of $M_0^\mathfrak{c}(\mathfrak{p})^{\mathrm{ord,et}}$. The following proposition summarizes the situation.

Proposition 3.3.3. *Both $M_0^\mathfrak{c}(\mathfrak{p})^{\mathrm{m}}$ and $M_0^\mathfrak{c}(\mathfrak{p})^{\mathrm{et}}$ are finite flat over $M^\mathfrak{c}$. The forgetful morphism is an isomorphism $M_0^\mathfrak{c}(\mathfrak{p})^{\mathrm{m}} \simeq M^\mathfrak{c}$, and $M_0^\mathfrak{c}(\mathfrak{p})^{\mathrm{et}}$ is purely inseparable of height 1 and degree p^f over $M^\mathfrak{c}$. The map w induces isomorphisms*

$$w : M_0^\mathfrak{c}(\mathfrak{p})^{\mathrm{m}} \simeq M_0^{\mathfrak{cp}}(\mathfrak{p})^{\mathrm{et}}, \;\; w : M_0^\mathfrak{c}(\mathfrak{p})^{\mathrm{et}} \simeq M_0^{\mathfrak{cp}}(\mathfrak{p})^{\mathrm{m}}.$$

Both $M_0^\mathfrak{c}(\mathfrak{p})^{\mathrm{m}}$ and $M_0^\mathfrak{c}(\mathfrak{p})^{\mathrm{et}}$ are therefore smooth over κ.

Proof. Taking Zariski closures, it is clear that w maps $M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{m}}$ to $M_0^{\mathfrak{c}\mathfrak{p}}(\mathfrak{p})^{\mathrm{et}}$ and $M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{et}}$ to $M_0^{\mathfrak{c}\mathfrak{p}}(\mathfrak{p})^{\mathrm{m}}$. Let h be the order of $[\mathfrak{p}]$ in $Cl^+(L)$ and let γ be a totally positive element of L such that $(\gamma) = \mathfrak{p}^h$. Applying w successively $2h$ times we get the map

$$M_0^{\mathfrak{c}}(\mathfrak{p}) \to M_0^{\gamma^2\mathfrak{c}}(\mathfrak{p}), \ \ (\underline{A}, H) \mapsto (\underline{A}/A[\gamma], \gamma^{-1}H/A[\gamma]).$$

Denote by $\gamma_* : M_0^{\gamma^2\mathfrak{c}}(\mathfrak{p}) \to M_0^{\mathfrak{c}}(\mathfrak{p})$ the isomorphism

$$(A', \iota', \lambda', \eta'; H') \mapsto (A', \iota', \lambda' \circ (\gamma^2 \otimes 1), \eta' \circ (\gamma \otimes 1); H').$$

Identifying $A/A[\gamma]$ with A under γ (carrying $\gamma^{-1}H/A[\gamma]$ to H) it is easy to check that γ_* maps the tuple $(\underline{A}/A[\gamma], \gamma^{-1}H/A[\gamma]) \in M_0^{\gamma^2\mathfrak{c}}(\mathfrak{p})$ back to $(\underline{A}, H) \in M_0^{\mathfrak{c}}(\mathfrak{p})$. We see that

$$\gamma_* \circ w^{2h}$$

is the identity, hence $w : M_0^{\mathfrak{c}}(\mathfrak{p}) \to M_0^{\mathfrak{c}\mathfrak{p}}(\mathfrak{p})$ is an isomorphism.

The multiplicative component maps isomorphically to $M^{\mathfrak{c}}$ since the section $\underline{A} \mapsto (\underline{A}, A[\mathrm{Fr}] \cap A[\mathfrak{p}])$ extends from the ordinary part to all of $M^{\mathfrak{c}}$ and must map it to $M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{m}}$ by continuity. It follows that $M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{m}}$ is smooth. Since $M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{et}}$ is isomorphic to $M_0^{\mathfrak{c}\mathfrak{p}}(\mathfrak{p})^{\mathrm{m}}$, it is also smooth.

It remains to prove that the morphism $\pi : M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{et}} \to M^{\mathfrak{c}}$ is finite flat, purely inseparable of height 1 and degree p^f. Consider $(M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{et}})^{(p)} = \kappa \times_{\phi,\kappa} M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{et}}$, which we canonically identify with $M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{et}}$, since it is actually defined over $\mathbb{F}_p$, and the relative Frobenius morphism (we use Y as a shorthand for $M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{et}}$)

$$\mathrm{Fr}_{Y/\kappa} : M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{et}} \to M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{et}}.$$

As $M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{et}}$ is smooth, $\mathrm{Fr}_{Y/\kappa}$ is finite and flat of degree p^g. We claim that there is a morphism $\theta : M^{\mathfrak{c}} \to M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{et}}$ such that $\mathrm{Fr}_{Y/\kappa} = \theta \circ \pi$. This will force both θ and π to be finite flat and purely inseparable of height 1, as finite morphisms between regular schemes are flat ("miracle flatness", [Stacks], Lemma 10.128.1). As the degree of π over the ordinary locus is p^f, this is its degree everywhere, and as a by-product we get that the degree of θ is p^{g-f}.

We define θ in the language of moduli problems. Let $\sigma : M^{\mathfrak{c}} \simeq M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{m}}$ be the section $\underline{A} \mapsto (\underline{A}, A[\mathrm{Fr}] \cap A[\mathfrak{p}])$ described before. Then $w \circ \sigma : M^{\mathfrak{c}} \simeq M_0^{\mathfrak{c}\mathfrak{p}}(\mathfrak{p})^{\mathrm{et}}$ is an isomorphism. Let $\mathfrak{p}' = \prod_{\mathfrak{q} \neq \mathfrak{p}} \mathfrak{q}$ be the product of the primes of L dividing p that are different from $\mathfrak{p}$. Let

$$\theta' : M_0^{\mathfrak{c}\mathfrak{p}}(\mathfrak{p}) \to M_0^{\mathfrak{c}\mathfrak{p}\mathfrak{p}'}(\mathfrak{p}) = M_0^{\mathfrak{c}p}(\mathfrak{p})^{\mathrm{et}}$$

be the map

$$\theta' : (\underline{A}, H) \mapsto (\underline{A}/A[\mathrm{Fr}] \cap A[\mathfrak{p}'], H \mod A[\mathrm{Fr}] \cap A[\mathfrak{p}']).$$

As $H \subset A[\mathfrak{p}]$, and $\mathfrak{p}$ and $\mathfrak{p}'$ are relatively prime, this map is well-defined and in fact sends $M_0^{\mathfrak{c}\mathfrak{p}}(\mathfrak{p})^{\mathrm{et}}$ to $M_0^{\mathfrak{c}p}(\mathfrak{p})^{\mathrm{et}}$ (it is enough to check this on the ordinary locus). We let

$$\theta = \theta' \circ w \circ \sigma : M^{\mathfrak{c}} \to M_0^{\mathfrak{c}p}(\mathfrak{p})^{\mathrm{et}} \simeq M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{et}}.$$

In the last step, we have used the fact that $\mathfrak{p}\mathfrak{p}' = (p)$ is principal to identify $M_0^{\mathfrak{c}p}(\mathfrak{p})^{\mathrm{et}} \simeq M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{et}}$ sending $(A, \iota, \lambda, \eta; H)$ to $(A, \iota, \tilde{\lambda}, \eta; H)$ where if $\lambda : \mathfrak{c}p \otimes_{\mathcal{O}_L} A \simeq A^t$ is a $\mathfrak{c}p$-polarization, $\tilde{\lambda} : \mathfrak{c} \otimes_{\mathcal{O}_L} A \simeq A^t$ is the $\mathfrak{c}$-polarization given by $\tilde{\lambda} = \lambda \circ (p \otimes 1)$. It follows that

$$\theta(\underline{A}, H) = (\underline{A}/A[\mathrm{Fr}], A[\mathfrak{p}] \mod A[\mathrm{Fr}]).$$

To conclude the proof we have to show that for $(\underline{A}, H) \in M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{et}}$

$$(\underline{A}/A[\mathrm{Fr}], A[\mathfrak{p}] \mod A[\mathrm{Fr}]) = \mathrm{Fr}_{Y/\kappa}(\underline{A}, H).$$

It is enough to verify this for $(\underline{A}, H) \in M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{ord,et}}$, as two morphisms that coincide on a dense open set, are equal. Assume that $(\underline{A}, H) \in M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{ord,et}}(k)$ is a k-valued point, for a κ-algebra k. Then $\mathrm{Fr} = \mathrm{Fr}_{A/k}$ is the relative Frobenius of A over k (not to be confused with $\mathrm{Fr}_{Y/\kappa}$ on the right hand side!). But $\underline{A}/A[\mathrm{Fr}] \simeq \underline{A}^{(p)/k}$ (base change with respect to the absolute Frobenius of k), and since H is the unique étale subgroup of $A[\mathfrak{p}]$ of order p^f, and $A[\mathfrak{p}] \mod A[\mathrm{Fr}]$ is the unique étale subgroup of $A^{(p)/k}[\mathfrak{p}]$ of order p^f, we must also have $A[\mathfrak{p}] \mod A[\mathrm{Fr}] \simeq H^{(p)/k}$.

Finally, let us explain the equality $(\underline{A}^{(p)/k}, H^{(p)/k}) = \mathrm{Fr}_{Y/\kappa}(\underline{A}, H)$. Intuitively, "the moduli of the object obtained by Frobenius base change is the Frobenius base change of the original moduli". However, as there are two *different* relative Frobenii involved, care must be taken. Let $(\underline{A}, H) \in M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{ord,et}}(k)$ correspond to the point $x : \mathrm{Spec}(k) \to M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{ord,et}}$ (over κ). By the functoriality of Frobenius, we have the following commutative diagram, where we have substituted $Y = M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{ord,et}}$:

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[r, "x"] \arrow[d, "\mathrm{Fr}_{k/\kappa}"'] & Y \arrow[d, "\mathrm{Fr}_{Y/\kappa}"] \\
\mathrm{Spec}(k^{(p)/\kappa}) \arrow[r, "x^{(p)/\kappa}"] \arrow[d] & Y^{(p)/\kappa} \arrow[d] \\
\mathrm{Spec}(k) \arrow[r, "x"] & Y
\end{tikzcd}$$

Here the vertical unmarked arrows are the base change maps with respect to the absolute Frobenius of κ. The composition of the two vertical arrows on the right is the absolute Frobenius Φ_Y of Y, and the composition of the two arrows on the left is the absolute Frobenius Φ_k of $\mathrm{Spec}(k)$. We therefore have

$$(\underline{A}^{(p)/k}, H^{(p)/k}) = \Phi_k^*(x^*(\underline{A}^{\mathrm{univ}}, H^{\mathrm{univ}})) = (\Phi_Y \circ x)^*(\underline{A}^{\mathrm{univ}}, H^{\mathrm{univ}}) = \mathrm{Fr}_{Y/\kappa}(\underline{A}, H).$$

(In the last step, we identified $Y^{(p)/\kappa}$ with Y, as it is defined over $\mathbb{F}_p$, hence we may identify $\mathrm{Fr}_{Y/\kappa}(\underline{A}, H)$ with $\Phi_Y(\underline{A}, H)$.) □

Corollary 3.3.4. *The smooth κ-variety $M_0^{\mathfrak{c}}(\mathfrak{p})^{\mathrm{et}}$ is the quotient of $M^{\mathfrak{c}}$ by a smooth p-foliation of rank $g - f$.*

Proof. The finite flat morphism θ defined in the proof of the previous proposition is purely inseparable of height 1 and degree p^{g-f}. The corollary now follows easily from Theorem 2.2.1. □

3.4. **Proof of Theorem 3.2.2.**

3.4.1. *Proof of parts (i) and (ii).* Let $\Sigma \subset \mathbb{B}$. We have seen that the foliation $\mathscr{F}_\Sigma$ is involutive. The obstruction $\kappa_{\mathscr{F}_\Sigma}$ to it being p-closed lies in

$$\mathrm{Hom}(\Phi_M^* \mathscr{F}_\Sigma, \mathcal{T}_{M/\kappa}/\mathscr{F}_\Sigma) \simeq \bigoplus_{\sigma \in \Sigma} \bigoplus_{\tau \notin \Sigma} \mathrm{Hom}(\mathscr{L}_\sigma^{-2p}, \mathscr{L}_\tau^{-2})$$

$$= \bigoplus_{\sigma \in \Sigma} \bigoplus_{\tau \notin \Sigma} \Gamma(M, \mathscr{L}_\sigma^{2p} \otimes \mathscr{L}_\tau^{-2}).$$

Here we used the fact that the pull back under the absolute Frobenius $\Phi_M : M \to M$ of a *line* bundle $\mathscr{L}$ is isomorphic to $\mathscr{L}^p$. Indeed, in characteristic p the map $f \otimes s \mapsto fs^{\otimes p}$ is an isomorphism $\mathcal{O}_M \otimes_{\phi,\mathcal{O}_M} \mathscr{L} \simeq \mathscr{L}^p$.

Lemma 3.4.1. *Let $\sigma \neq \tau$. We have*

$$\Gamma(M, \mathscr{L}_\sigma^{2p} \otimes \mathscr{L}_\tau^{-2}) = \begin{cases} 0 & \tau \neq \phi \circ \sigma; \\ \kappa \cdot h_\tau^2 & \tau = \phi \circ \sigma. \end{cases}$$

Proof. Let h be a non-zero Hilbert modular form on M of weight $2p[\sigma] - 2[\tau]$. According to Theorem 3.3.2 there exist integers $a_\beta \geq 0$ such that

$$\Phi(h) = 2p[\sigma] - 2[\tau] - \sum_{\beta \in \mathbb{B}} a_\beta(p[\beta] - [\phi \circ \beta]) \in C^{\min} \subset C^{\mathrm{std}}$$

(we end up using only the weaker result that the left hand side lies in C^{std}). If τ is not in the ϕ-orbit of σ and $\tau \in \mathbb{B}_\mathfrak{p}$ we get, upon summing the coefficients of $\beta \in \mathbb{B}_\mathfrak{p}$ in $\Phi(h)$ that

$$-2 - (p-1) \sum_{\beta \in \mathbb{B}_\mathfrak{p}} a_\beta \geq 0,$$

a clear contradiction. It follows that $\tau = \phi^i \sigma$ for some $1 \leq i \leq f-1$ where $f \geq 2$ is the length of the ϕ-orbit of σ (if $\sigma \in \mathbb{B}_\mathfrak{p}$, then $f = f(\mathfrak{p}/p)$).

Labelling the $\beta \in \mathbb{B}_\mathfrak{p}$ by $0, 1, \ldots, f-1$ so that $\phi \circ [i] = [i+1 \mod f]$, and assuming, without loss of generality, that $[\sigma] = [0]$ and $[\tau] = [i]$ for some $1 \leq i \leq f-1$ we have

$$2p[0] - 2[i] - \sum_{j=0}^{f-1} a_j(p[j] - [j+1 \mod f]) = \sum_{j=0}^{f-1} k_j[j],$$

where $a_j \geq 0$ and $k_j \geq 0$.

Summing over the coefficients we get $2p - 2 - (p-1)\sum a_j = \sum k_j \geq 0$, hence $\sum a_j = 0, 1$ or 2. We can not have $a_j = 0$ for all j, since $\Phi(h) \in C^{\mathrm{std}}$. If $a_m = 1$ and all the other $a_j = 0$ we again reach a contradiction since the coefficient of $[i]$ comes out negative, no matter what m is. There remains the case where $\sum a_j = 2$. In this case all the $k_j = 0$. Looking at the coefficient of $[0]$ we must have $2p - pa_0 + a_{f-1} = 0$. This forces $a_0 = 2$ hence all the other $a_j = 0$ and $i = 1$.

This means that $\tau = \phi \circ \sigma$ and hh_τ^{-2} has weight 0, i.e. is a constant in κ, proving the lemma. ∎

The lemma implies that if Σ is invariant under ϕ (i.e. is a union of certain $\mathbb{B}_\mathfrak{p}$'s for the primes $\mathfrak{p}$ above p) then $\kappa_{\mathscr{F}_\Sigma}$ vanishes, and $\mathscr{F}_\Sigma$ is therefore p-closed. To prove the converse, completing the proof of part (i) of Theorem 3.2.2, it is enough to show that when $\Sigma = \{\sigma\}$ the obstruction $\kappa_{\mathscr{F}_\sigma}$ is a *non-zero* multiple of $h_{\phi\circ\sigma}^2$. This will establish, at the same time, claim (ii) of the Theorem.

To this end we use q-expansions. Let $\mathcal{R}$ be one of the rings associated with the cusps as in 3.3.1, and $R = \mathcal{R}/p\mathcal{R}$. The pull back of the tangent bundle of M to $\mathrm{Spec}(R)$ is identified with the Lie algebra $\mathrm{Der}(R/\kappa)$ and the Kodaira-Spencer isomorphism yields the isomorphism (3.2)

$$\mathrm{KS} : \mathcal{T}_{\mathrm{Spec}(R)/\kappa} = \mathrm{Der}(R/\kappa) \simeq \mathcal{O}_L \otimes R = \bigoplus_{\sigma \in \mathbb{B}} R_\sigma.$$

Let $\{e_\sigma\}$ be the idempotents of $\mathcal{O}_L \otimes R$ corresponding to this decomposition. Then $\mathrm{KS} : \mathscr{L}_\sigma^{-2}|_{\mathrm{Spec}(R)} \simeq Re_\sigma = R_\sigma$. The ring $\mathcal{O}_L \otimes R$ has an endomorphism

$$\varphi(\alpha \otimes r) = \alpha \otimes r^p$$

and $\varphi(e_\sigma) = e_{\phi\circ\sigma}$. Indeed,

$$\alpha\otimes 1\cdot\varphi(e_\sigma) = \varphi(\alpha\otimes 1\cdot e_\sigma) = \varphi(1\otimes\sigma(\alpha)\cdot e_\sigma) = 1\otimes\sigma(\alpha)^p\cdot\varphi(e_\sigma) = 1\otimes(\phi\circ\sigma)(\alpha)\cdot\varphi(e_\sigma)$$

so $\varphi(e_\sigma)$, being an idempotent in $R_{\phi\circ\sigma}$, must equal $e_{\phi\circ\sigma}$.

Let $\xi_\sigma \in \mathscr{L}_\sigma^{-2}$ be the derivation mapping to e_σ under KS. If $e_\sigma = \sum_j \gamma_j \otimes r_j$ ($\gamma_j \in \mathcal{O}_L$, $r_j \in R$) then by Katz' formula (Lemma 3.3.1)

$$\xi_\sigma(\sum_\alpha a_\alpha q^\alpha) = \sum_\alpha a_\alpha(\sum_j r_j \mathrm{Tr}_{L/\mathbb{Q}}(\alpha\gamma_j))q^\alpha.$$

It follows that when we iterate ξ_σ p times we get the derivation

$$\xi_\sigma^p(\sum_\alpha a_\alpha q^\alpha) = \sum_\alpha a_\alpha(\sum_j r_j^p \mathrm{Tr}_{L/\mathbb{Q}}(\alpha\gamma_j))q^\alpha,$$

i.e. the derivation corresponding to $\varphi(e_\sigma) = e_{\phi\circ\sigma}$. We conclude that $\xi_\sigma^p = \xi_{\phi\circ\sigma} \neq 0$, hence $\kappa_{\mathscr{F}_\sigma}$ is a non-zero section of $\mathscr{L}_{\phi\circ\sigma}^{-2} \otimes \mathscr{L}_\sigma^{2p}$.

Remark. The reader might have noticed that the same q-expansion computation can be used to give an alternative proof of all of (i) and (ii) in Theorem 3.2.2. However, we found Lemma 3.4.1 and the relation to the main result of [DK] of independent interest, especially considering the extension of our results to other PEL Shimura varieties.

3.4.2. *Proof of (iii).* We now turn to the last part of the theorem, identifying the quotient of M by the smooth p-foliation

$$\mathscr{G} = \bigoplus_{\mathfrak{q}\neq\mathfrak{p}} \mathscr{F}_\mathfrak{q}$$

with the purely inseparable, finite flat map $\theta : M \to M_0(\mathfrak{p})^{\mathrm{et}}$ constructed in Proposition 3.3.3. Note that the rank of $\mathscr{G}$ is $g - f$, while $\deg(\theta) = p^{g-f}$. Since $M_0(\mathfrak{p})^{\mathrm{et}}$ is the quotient of M by the smooth p-foliation $\ker(d\theta)$, and the latter is also of rank $g - f$, it is enough to prove that $d\theta$ annihilates $\mathscr{G}$ to conclude that

$$\ker(d\theta) = \mathscr{G},$$

thus proving part (iii) of Theorem 3.2.2.

To simplify the notation write $N = M_0(\mathfrak{p})^{\mathrm{et}}$, let k be an algebraically closed field containing κ, $x \in M(k)$ and $y = \theta(x) \in N(k)$. Let $\underline{A}$ be the tuple parametrized by x. Then y parametrizes the tuple $(\underline{A}^{(p)}, \mathrm{Fr}(A[\mathfrak{p}]))$ where $\mathrm{Fr} = \mathrm{Fr}_{A/k}$.

Write TM for the tangent bundle $\mathcal{T}_{M/\kappa}$ and T_xM for its fiber at x, the tangent space to M at x. Similar meanings are attached to the symbols TN and T_yN. Let $k[\epsilon]$ be the ring of dual numbers, $\epsilon^2 = 0$. In terms of the moduli problem,

$$T_xM = \{\underline{\widetilde{A}} \in M(k[\epsilon]) | \, \underline{\widetilde{A}} \mod \epsilon = \underline{A}\}/\simeq$$

and its origin is the "constant" tuple $\mathrm{Spec}(k[\epsilon]) \times_{\mathrm{Spec}(k)} \underline{A}$. Similarly,

$$T_yN = \{(\underline{\widetilde{B}}, \widetilde{H}) \in N(k[\epsilon]) | \, (\underline{\widetilde{B}}, \widetilde{H}) \mod \epsilon = (\underline{A}^{(p)}, \mathrm{Fr}(A[\mathfrak{p}]))\}/\simeq$$

and its origin is the "constant" tuple $\mathrm{Spec}(k[\epsilon]) \times_{\mathrm{Spec}(k)} (\underline{A}^{(p)}, \mathrm{Fr}(A[\mathfrak{p}]))$.

Let $\widetilde{x} = \underline{\widetilde{A}} \in T_xM$ be a tangent vector at x. In terms of moduli problems

$$d\theta(\widetilde{x}) = \theta(\underline{\widetilde{A}}) = (\underline{\widetilde{A}}^{(p)}, \mathrm{Fr}(\widetilde{A}[\mathfrak{p}])) \in N(k[\epsilon])$$

where now $\widetilde{A}^{(p)} = \widetilde{A}^{(p)/k[\epsilon]}$ is the base change of $\widetilde{A}$ with respect to the raising-to-power p homomorphism $\phi_{k[\epsilon]} : k[\epsilon] \to k[\epsilon]$, the PEL structure ι, λ, η accompanying $\widetilde{A}$ in the definition of $\underline{\widetilde{A}}$ undergoes the same base change, and $\mathrm{Fr} = \mathrm{Fr}_{\widetilde{A}/k[\epsilon]} : \widetilde{A} \to \widetilde{A}^{(p)}$ is the relative Frobenius of $\widetilde{A}$ over $k[\epsilon]$.

We have to show that if $\widetilde{x} \in \mathscr{G}_x \subset T_xM$ then $d\theta(\widetilde{x}) = 0$, namely that the tuple $(\underline{\widetilde{A}}^{(p)}, \mathrm{Fr}(\widetilde{A}[\mathfrak{p}]))$ is *constant* along $\mathrm{Spec}(k[\epsilon])$. That $\underline{\widetilde{A}}^{(p)}$ is constant along $\mathrm{Spec}(k[\epsilon])$ is always true, regardless of whether $\widetilde{x} \in \mathscr{G}_x$ or not, simply becasue $\phi_{k[\epsilon]}$ factors as the projection modulo ϵ, $k[\epsilon] \twoheadrightarrow k$, followed by ϕ_k, and then by the inclusion $k \hookrightarrow k[\epsilon]$:

$$\phi_{k[\epsilon]} : k[\epsilon] \twoheadrightarrow k \stackrel{\phi_k}{\to} k \hookrightarrow k[\epsilon].$$

It all boils down to the identity $(a + b\epsilon)^p = a^p$.

Suppose $\widetilde{x} \in \mathscr{G}_x$ and let us show that $\mathrm{Fr}(\widetilde{A}[\mathfrak{p}])$ is also constant. Recall the local models M^{loc}, N^{loc} of M and N constructed in [Pa], 3.3. Let R be any κ-algebra. Let $W = (\mathcal{O}_L \otimes R)^2$ with the induced $\mathcal{O}_L$ action and the perfect alternate pairing (e_1, e_2 is the standard basis)

$$\langle a \otimes t \cdot e_1, b \otimes s \cdot e_2 \rangle = - \langle b \otimes s \cdot e_2, a \otimes t \cdot e_1 \rangle = \mathrm{Tr}_{L/\mathbb{Q}}(ab)ts \in R,$$

$$\langle a \otimes t \cdot e_i, b \otimes s \cdot e_i \rangle = 0 \quad (i = 1, 2).$$

Then $M^{\mathrm{loc}}(R)$ is the set of rank-1 $\mathcal{O}_L \otimes R$ local direct summands $\omega \subset W$ which are totally isotropic (equal to their own annihilator) under $\langle , \rangle$.

Similarly, the R-points of N^{loc} are given by the following data. Fix $a \in \mathcal{O}_L$ such that $\mathfrak{p} = (a, p)$ but $a \equiv 1 \mod \mathfrak{q}$ for every prime $\mathfrak{q} \neq \mathfrak{p}$ above p. Let $u : W \to W$ be the $\mathcal{O}_L \otimes R$-linear map sending e_1 to e_1 and e_2 to $a \otimes 1 \cdot e_2$. Equivalently, if we decompose $W = \oplus_{\mathfrak{q}|p} W[\mathfrak{q}]$, $w = \sum_{\mathfrak{q}|p} w(\mathfrak{q})$, then u sends $e_2(\mathfrak{p})$ to 0, but every $e_2(\mathfrak{q})$ for $\mathfrak{q} \neq \mathfrak{p}$ to itself. Then $N^{\mathrm{loc}}(R)$ is the set of pairs (ω, ω') of rank-1 $\mathcal{O}_L \otimes R$ local direct summands $\omega, \omega' \subset W$ which are totally isotropic such that $u(\omega) \subset \omega'$.

The scheme N^{loc} is a closed subscheme of a product of two Grassmannians, and its projection to the first factor is M^{loc}.

We shall use the R-points of the local models with $R = k$ or $k[\epsilon]$ to study the map $d\theta$ between T_xM and T_yN. Let $W = (\mathcal{O}_L \otimes k)^2$, $\widetilde{W} = (\mathcal{O}_L \otimes k[\epsilon])^2$, and suppose $x = \underline{A}$ corresponds to $\xi = (\omega \subset W) \in M^{\mathrm{loc}}(k)$. We may fix an identification $W = H^1_{dR}(A/k)$ so that $\omega = \omega_A = H^0(A, \Omega^1_{A/k})$.

Let $\widetilde{x} = \underline{\widetilde{A}} \in M(k[\epsilon])$ map to x modulo ϵ. Let $\widetilde{\xi} = (\widetilde{\omega} \subset \widetilde{W}) \in M^{\mathrm{loc}}(k[\epsilon])$ correspond to $\widetilde{x}$ under the isomorphism $T_xM \simeq T_\xi M^{\mathrm{loc}}$, i.e. $\widetilde{\omega} \mod \epsilon = \omega$. Fix $\alpha \in \omega$ and suppose that $\widetilde{\alpha} = \alpha + \epsilon\beta$ ($\beta \in W$) is an element of $\widetilde{\omega}$ mapping to α modulo ϵ. Since $\widetilde{\alpha}$ is uniquely determined modulo $\epsilon\widetilde{\omega} = \epsilon\omega$, the image $\overline{\beta}$ of β in

$$W/\omega = H^1(A, \mathcal{O}) = \mathrm{Lie}(A^t) = \omega^\vee_{A^t}$$

is well-defined. We have thus associated to $\widetilde{x}$ a map from $\omega = \omega_A$ to $\omega^\vee_{A^t}$. Using the polarization λ we view this as a map $\mathrm{KS}(\widetilde{x})$ from ω_A to $\omega^\vee_A$. It is straightforward to prove that $\widetilde{\omega}$ is totally isotropic if and only if $\mathrm{KS}(\widetilde{x})$ is symmetric, i.e. $\mathrm{KS}(\widetilde{x})^\vee = \mathrm{KS}(\widetilde{x})$. The following is a reformulation of the Kodaira-Spencer isomorphism:

Proposition. *The map* KS *is an isomorphism from* T_xM *onto the space of symmetric* $\mathcal{O}_L \otimes k$*-homomorphisms from* ω_A *to* $\omega_A^\vee$*. The fiber* $\mathscr{G}_x$ *of the foliation* $\mathscr{G}$ *consists of those tangent vectors* $\widetilde{x}$ *for which* $\mathrm{KS}(\widetilde{x})$ *annihilates* $\omega_A[\mathfrak{p}]$.

In the decomposition $\mathcal{O}_L \otimes k = \bigoplus_{\sigma\in\mathbb{B}} k_\sigma$, $\mathfrak{p}\otimes k$ is the ideal of all (α_σ) for which $\alpha_\sigma = 0$ whenever $\sigma \in \mathbb{B}_\mathfrak{p}$. We therefore have

$$\omega[\mathfrak{p}] = \bigoplus_{\sigma\in\mathbb{B}_\mathfrak{p}} \omega[\sigma]$$

($\omega[\mathfrak{p}]$ denotes the kernel of $\mathfrak{p}$, $\omega[\sigma]$ denotes the σ-isotypical component of ω), and recall that each $\omega[\sigma]$ is one-dimensional over k. We conclude that if $\widetilde{x} \in \mathscr{G}_x$ then whenever $\alpha \in \omega$ and $\widetilde{\alpha} = \alpha + \epsilon\beta \in \widetilde{\omega}$ maps to it modulo ϵ, then for any $\sigma \in \mathbb{B}_\mathfrak{p}$ the σ-component of β is proportional to the σ-component of α. In other words, $\widetilde{\omega}[\mathfrak{p}] = k[\epsilon]\otimes_k \omega[\mathfrak{p}]$.

By the crystalline deformation theory of Grothendieck (equivalently, by the local model), the group scheme $\widetilde{A}[p]$ over $k[\epsilon]$ is completely determined by the lifting $\widetilde{\omega}$ of ω to $k[\epsilon]$. In the notation used above $\widetilde{\omega}$ determines the point $\widetilde{\xi} = (\widetilde{\omega} \subset \widetilde{W}) \in M^{\mathrm{loc}}(k[\epsilon])$, hence the deformation $\widetilde{x} = \underline{\widetilde{A}} \in M(k[\epsilon])$, and therefore $\widetilde{A}[p]$. Furthermore, the subgroup scheme $\widetilde{A}[\mathfrak{p}]$ is constant along $k[\epsilon]$ if and only if $\widetilde{\omega}[\mathfrak{p}] = k[\epsilon]\otimes_k \omega[\mathfrak{p}]$. We have therefore proved that if $\widetilde{x} \in \mathscr{G}_x$, $\widetilde{A}[\mathfrak{p}]$ is constant along $k[\epsilon]$. With it, so are $\widetilde{A}[\mathfrak{p}][\mathrm{Fr}]$ and the quotient $\mathrm{Fr}(\widetilde{A}[\mathfrak{p}])$.

3.5. **Integral varieties.** Our goal in this section is to prove the following theorem.

Theorem 3.5.1. *(i) Assume that* $\emptyset \subsetneqq \Sigma \subsetneqq \mathbb{B}$*. Then the foliation* $\mathscr{F}_\Sigma$ *does not have any (algebraic) integral variety in the generic fiber* $\mathscr{M}_\mathbb{Q}$ *of the Hilbert modular variety.*

(ii) Assume that Σ *is invariant under* ϕ*. Let* Σ^c *be its complement. Then the Goren-Oort stratum* M_Σ *(see definition (3.3) below) is an integral variety of* $\mathscr{F}_{\Sigma^c}$ *in the characteristic* p *fiber* M *of the Hilbert modular variety.*

Proof. The proof of part (i) is transcendental. Had there been an integral variety to $\mathscr{F}_\Sigma$ in characteristic 0, it would provide an algebraic integral variety over $\mathbb{C}$. But over the universal covering $\mathfrak{H}^g$ the *analytic* leaves of the foliation are easily determined. If $z_0 = (z_{0,\sigma})_{\sigma\in\mathbb{B}}$ is a point of $\mathfrak{H}^g$, the leaf through it is the coordinate "plane"

$$H_\Sigma(z_0) = \{z \in \mathfrak{H}^g \,|\, z_\tau = z_{0,\tau}\ \forall\tau \notin \Sigma\}.$$

Unless Σ is empty or the whole of $\mathbb{B}$, these coordinate "planes" do not descend to algebraic varieties in $\Gamma \setminus \mathfrak{H}^g$ because the map $H_\Sigma(z_0) \to \Gamma\setminus\mathfrak{H}^g$ has a dense image. In fact, [R-T] Proposition 3.4 shows that the analytic leaves of these foliations do not even contain any algebraic curves.

(ii) Let $\mathcal{H} = H^1_{dR}(A^{\mathrm{univ}}/M)$ be the relative de Rham cohomology of the universal abelian variety over M and $\nabla : \mathcal{H} \to \mathcal{H}\otimes_{\mathcal{O}_M} \Omega^1_{M/\kappa}$ the Gauss-Manin connection. For a vector field $\xi \in \mathcal{T}_{M/\kappa}(U)$ over a Zariski open set $U \subset M$ we denote by

$$\nabla_\xi : \mathcal{H}(U) \to \mathcal{H}(U)$$

the ξ-derivation obtained by contracting $\Omega^1_{M/\kappa}$ with ξ. It satisfies ($a \in \mathcal{O}_M(U)$)

$$\nabla_\xi(at) = \xi(a)t + a\nabla_\xi(t).$$

If $\sigma, \tau \in \mathbb{B}$ and τ is not in the ϕ-orbit of σ, and if $\xi \in \mathscr{F}_\tau(U) \subset \mathcal{T}_{M/\kappa}(U)$ then, since $\mathrm{KS}(\xi)$ annihilates

$$\mathscr{L}_\sigma = \underline{\omega}[\sigma] \subset \underline{\omega} \subset \mathcal{H},$$

∇_ξ induces an $\mathcal{O}_M$-derivation of the line bundle $\mathscr{L}_\sigma$ over U. The same holds with $\mathscr{L}_{\phi\circ\sigma}$ since $\tau \neq \phi \circ \sigma$. By the usual rules of derivations, we obtain a derivation ∇_ξ of the line bundle $\mathrm{Hom}(\mathscr{L}_{\phi\circ\sigma}, \mathscr{L}_\sigma^p)$, of which the partial Hasse invariant $h_{\phi\circ\sigma}$ is a global section.

Let us elaborate on the last statement. If t is a section of $\mathscr{L}_\sigma$ then the induced derivation of

$$\mathscr{L}_\sigma^p = \mathscr{L}_\sigma^{\otimes p} \simeq \mathscr{L}_\sigma{}^{(p)}$$

(the first is the p-th tensor product, the second the base-change by the absolute Frobenius of M) is given by

$$\nabla_\xi^{(p)}(at^p) = \xi(a)t^p$$

(equivalently, on Frobenius base-change, $\nabla_\xi^{(p)}(a \otimes t) = \xi(a) \otimes t$; note that this is a *canonical* derivation, independent of the original ∇_ξ). If $\mathcal{H}$ is the relative de Rham cohomology of A^{univ}, and $\mathcal{H}^{(p)}$ the relative de Rham cohomology of $A^{\mathrm{univ}(p)}$, then the same formula applied to the Gauss-Manin connection of A^{univ} gives the Gauss-Manin connection of $A^{\mathrm{univ}(p)}$. Once again, the latter is the *canonical* connection which exists on the Frobenius base change of any vector bundle. *Any* section of the form $1 \otimes t$ is flat, just as any function which is a p-th power is annihilated by all the derivations.

Finally, if $h : \mathscr{L}_{\phi\circ\sigma} \to \mathscr{L}_\sigma^p$ is a homomorphism of line bundles, then$\nabla_\xi h$ is the homomorphism

$$(\nabla_\xi h)(t) = \nabla_\xi^{(p)}(h(t)) - h(\nabla_\xi(t)).$$

Lemma 3.5.2. *The partial Hasse invariant $h_{\phi\circ\sigma}$ is horizontal for $\xi \in \mathscr{F}_\tau$, i.e. $\nabla_\xi(h_{\phi\circ\sigma}) = 0$.*

Proof. Identify $\mathscr{L}_\sigma^p$ with $\mathscr{L}_\sigma^{(p)} = \Phi_M^* \mathscr{L}_\sigma = \Phi_M^*(\underline{\omega}[\sigma]) = (\Phi_M^* \underline{\omega})[\phi\circ\sigma]$. By definition, $h_{\phi\circ\sigma}$ is the $\phi \circ \sigma$ component of the $\mathcal{O}_L$-homomorphism

$$V : \underline{\omega} \to \underline{\omega}^{(p)} = \Phi_M^* \underline{\omega},$$

induced by the Verschiebung isogeny $\mathrm{Ver}_{A^{\mathrm{univ}}/M} : A^{\mathrm{univ}(p)} \to A^{\mathrm{univ}}$. It is horizontal since the Gauss-Manin connection commutes, in general, with any map on $\mathcal{H} = H^1_{dR}(A^{\mathrm{univ}}/M)$ induced by an isogeny, and in particular

$$\nabla_\xi(h_{\phi\circ\sigma}) = \nabla_\xi^{(p)} \circ h_{\phi\circ\sigma} - h_{\phi\circ\sigma} \circ \nabla_\xi = 0.$$

□

Let $H_{\phi\circ\sigma}$ be the hypersurface defined by the vanishing of $h_{\phi\circ\sigma}$ in M. By the results of [G-O], it is smooth, and $h_{\phi\circ\sigma}$ vanishes on it to first order. Furthermore, for different σ's these hypersurfaces intersect transversally. Let $x \in H_{\phi\circ\sigma}$ and choose a Zariski open neighborhood U of x on which $\mathrm{Hom}(\mathscr{L}_{\phi\circ\sigma}, \mathscr{L}_\sigma^p)$ is trivial. Let e be a basis of $\mathrm{Hom}(\mathscr{L}_{\phi\circ\sigma}, \mathscr{L}_\sigma^p)$ over U and write $h_{\phi\circ\sigma} = he$ for some $h \in \mathcal{O}_M(U)$. Then $H_{\phi\circ\sigma} \cap U$ is given by the equation $h = 0$ and h vanishes on it to first order. Furthermore, if $\xi \in \mathscr{F}_\tau(U)$, by the Lemma we have

$$0 = \nabla_\xi(h_{\phi\circ\sigma}) = \xi(h) \cdot e + h\nabla_\xi(e),$$

so along $H_{\phi\circ\sigma} = \{h = 0\}$ we also have $\xi(h) = 0$. This proves that ξ is parallel to $H_{\phi\circ\sigma}$, i.e. $\xi_x \in T_x H_{\phi\circ\sigma} \subset T_x M$. Let Σ be a ϕ-invariant subset of $\mathbb{B}$. Since the same analysis holds for every $\sigma \in \Sigma$ and every $\tau \notin \Sigma$ we get that at every point x of

$$M_\Sigma = \{x|\, h_\sigma(x) = 0 \ \forall \sigma \in \Sigma\} \tag{3.3}$$

the p-foliation

$$\mathscr{F}_{\Sigma^c} = \oplus_{\tau\notin\Sigma}\mathscr{F}_\tau$$

is contained in $T_x M_\Sigma \subset T_x M$. As both $\mathscr{F}_{\Sigma^c}$ and TM_Σ are vector bundles of the same rank $g - \#(\Sigma)$, and both are local direct summands of TM, we have shown that the Goren-Oort stratum M_Σ is an integral variety of $\mathscr{F}_{\Sigma^c}$. □

4. V-foliations on unitary Shimura varieties

4.1. Notation and preliminary results on unitary Shimura varieties.

4.1.1. *The moduli scheme.* We now turn to the second type of foliations considered in this paper, on unitary Shimura varieties in characteristic p. Let K be a CM field, $[K : \mathbb{Q}] = 2g$ and $L = K^+$ its totally real subfield. Let $\rho \in Gal(K/L)$ denote complex conjugation. Let $E \subset \mathbb{C}$, *the field of definition,* be a number field containing all the conjugates[2] of K. For

$$\tau \in \mathscr{I} := \mathrm{Hom}(K, E) = \mathrm{Hom}(K, \mathbb{C})$$

we write $\bar\tau = \tau \circ \rho$. We let $\mathscr{I}^+ = \mathscr{I} / \langle\rho\rangle = \mathrm{Hom}(L, E)$ and write elements of it as unordered pairs $\{\tau, \bar\tau\}$.

Let $d \geq 1$ and fix a *PEL-type $\mathcal{O}_K$-lattice* $(\Lambda, \langle,\rangle, h)$ of rank d over $\mathcal{O}_K$ ([Lan], 1.2.1.3). Thus Λ is a projective $\mathcal{O}_K$-module of rank d (regarded, if we forget the $\mathcal{O}_K$-action, as a lattice of rank $2gd$), $\langle,\rangle$ is a non-degenerate alternating bilinear form $\Lambda \times \Lambda \to 2\pi i\mathbb{Z}$, satisfying $\langle ax, y\rangle = \langle x, \bar a y\rangle$ for $a \in \mathcal{O}_K$, and

$$h : \mathbb{C} \to \mathrm{End}_{\mathcal{O}_K}(\Lambda \otimes \mathbb{R})$$

is an $\mathbb{R}$-linear ring homomorphism satisfying (i) $\langle h(z)x, y\rangle = \langle x, h(\bar z)y\rangle$ (ii) $(x, y) = (2\pi i)^{-1}\langle x, h(i)y\rangle$ is an inner product (symmetric and positive definite) on the real vector space $\Lambda \otimes \mathbb{R}$.

The $2gd$-dimensional complex vector space $V = \Lambda \otimes \mathbb{C}$ breaks up as a direct sum

$$V = V_0 \oplus V_0^c$$

of two $\langle,\rangle$-isotropic subspaces, where $V_0 = \{v \in V|\, h(z)v = 1 \otimes z \cdot v\}$ and $V_0^c = \{v \in V|\, h(z)v = 1 \otimes \bar z \cdot v\}$. The inclusion $\Lambda \otimes \mathbb{R} \subset \Lambda \otimes \mathbb{C} = V$ allows us to identify V_0 with the real vector space $\Lambda \otimes \mathbb{R}$, and then its complex structure is given by $J = h(i)$.

As representations of $\mathcal{O}_K$

$$V_0 \simeq \sum_{\tau\in\mathscr{I}} r_\tau \tau, \ \ V_0^c \simeq \sum_{\tau\in\mathscr{I}} r_\tau \bar\tau,$$

where the r_τ are non-negative integers satisfying $r_\tau + r_{\bar\tau} = d$ for each τ. We call the collection $\{r_\tau\}$ (or the formal sum $\sum_{\tau\in\mathscr{I}} r_\tau\tau$) the *signature* of $(\Lambda, \langle,\rangle, h)$ ([Lan] 1.2.5.2), or the *CM type.*

Let $N \geq 3$ (the *tame level*) be an integer which is relatively prime to the discriminant of the lattice $(\Lambda, \langle,\rangle)$. Let S be the set of *bad primes*, defined to be the

[2]We do not insist on the field of definition being the minimal possible one, i.e. the reflex field of the CM type.

rational primes that ramify in K, divide N, or divide the discriminant of Λ. The primes $p \notin S$ are called *good*, and we fix once and for all such a prime p.

Consider the following moduli problem $\mathscr{M}$ over $\mathcal{O}_E[1/S]$. For an $\mathcal{O}_E[1/S]$-algebra R, the set $\mathscr{M}(R)$ is the set of isomorphism classes of tuples $\underline{A} = (A, \iota, \lambda, \eta)$ where:

- A is an abelian scheme of relative dimension gd over R.
- $\iota : \mathcal{O}_K \hookrightarrow \mathrm{End}(A/R)$ is an embedding of rings, rendering $\mathrm{Lie}(A/R)$ an $\mathcal{O}_K$-module of type $\sum_{\tau \in \mathscr{I}} r_\tau \tau$.
- $\lambda : A \to A^t$ is a $\mathbb{Z}_{(p)}^\times$-polarization whose Rosati involution preserves $\iota(\mathcal{O}_K)$ and induces on it complex conjugation.
- η is a full level-N structure compatible via λ with $(\Lambda \otimes \widehat{\mathbb{Z}}^{(p)}, \langle , \rangle)$.

See [Lan], 1.4.1.2 for more details, in particular pertaining to the level-N structure.

The moduli problem $\mathscr{M}$ is representable by a smooth scheme over $\mathcal{O}_E[1/S]$, which we denote by the same letter. Its complex points form a finite disjoint union of Shimura varieties associated with the unitary group of signature $\{r_\tau\}$. Denote by

$$\underline{A}^{\mathrm{univ}} = (A^{\mathrm{univ}}, \iota^{\mathrm{univ}}, \lambda^{\mathrm{univ}}, \eta^{\mathrm{univ}})$$

the universal tuple over $\mathscr{M}$.

We let κ be a finite field, large enough to contain all the residue fields of the primes of E above p. Fix, once and for all, an embedding $E \hookrightarrow W(\kappa)[1/p]$, and consider

$$M = \kappa \times_{\mathcal{O}_E[1/S]} \mathscr{M},$$

the special fiber at the chosen prime of $\mathcal{O}_E[1/S]$, base-changed to κ. It is a smooth variety over κ of dimension $\sum_{\{\tau,\bar\tau\} \in \mathscr{I}^+} r_\tau r_{\bar\tau}$. We let $\mathcal{T}$ be its tangent bundle.

Via the fixed embedding of $\mathcal{O}_E$ in $W(\kappa)$ we regard $\mathscr{I}$ also as the set of homomorphisms of $\mathcal{O}_K$ to κ. For a prime $\mathfrak{P}$ of $\mathcal{O}_K$ above p we let $\mathscr{I}_{\mathfrak{P}}$ be those homomorphisms that factor through $\kappa(\mathfrak{P}) = \mathcal{O}_K/\mathfrak{P}$,

$$\mathscr{I} = \coprod_{\mathfrak{P}|p} \mathscr{I}_{\mathfrak{P}}, \quad \mathscr{I}_{\mathfrak{P}} = \mathrm{Hom}(\kappa(\mathfrak{P}), \kappa) = \mathrm{Hom}(\mathcal{O}_{K,\mathfrak{P}}, W(\kappa)).$$

The Frobenius $\phi(x) = x^p$ acts on $\mathscr{I}$ on the left via $\tau \mapsto \phi \circ \tau$ and the $\mathscr{I}_{\mathfrak{P}}$ are its orbits, each of them permuted cyclically by ϕ. Following Moonen's convention [Mo], when we use $\mathscr{I}_{\mathfrak{P}}$ as an indexing set, we shall also write i for τ and $i+1$ for $\phi \circ \tau$. This will be done without further notice to avoid the heavy notation $\tau_{i+1} = \phi \circ \tau_i$.

4.1.2. *The Kodaira-Spencer isomorphism.* Let $\pi : A^{\mathrm{univ}} \to \mathscr{M}$ be the structure morphism of the universal abelian variety, and

$$\underline{\omega} = \pi_*(\Omega^1_{A^{\mathrm{univ}}/\mathscr{M}}) \subset \mathcal{H} = \mathbb{R}^1\pi_*(\Omega^\bullet_{A^{\mathrm{univ}}/\mathscr{M}})$$

its relative de-Rham cohomology $\mathcal{H}$ and Hodge bundle $\underline{\omega}$. These are vector bundles on $\mathscr{M}$ of ranks $2gd$ and gd respectively. The Hodge bundle $\underline{\omega}$ is the dual bundle to the relative Lie algebra $\underline{\mathrm{Lie}} = \mathrm{Lie}(A^{\mathrm{univ}}/\mathscr{M})$. Since E contains all the conjugates of K, S contains all the ramified primes in K, and $\mathcal{O}_{\mathscr{M}}$ is an $\mathcal{O}_E[1/S]$-algebra, these vector bundles decompose under the action of $\mathcal{O}_K$ into isotypical parts

$$\underline{\omega} = \bigoplus_{\tau \in \mathscr{I}} \underline{\omega}[\tau] \subset \bigoplus_{\tau \in \mathscr{I}} \mathcal{H}[\tau] = \mathcal{H}.$$

For each τ the rank of $\underline{\omega}[\tau]$ is r_τ, and we have a short exact sequence (the Hodge filtration)

$$0 \to \underline{\omega}[\tau] \to \mathcal{H}[\tau] \to R^1\pi_*(\mathcal{O}_{A^{\mathrm{univ}}})[\tau] \to 0.$$

Since λ^{univ} is a prime-to-p quasi-isogeny, and the Rosati involution on $\iota^{\mathrm{univ}}(\mathcal{O}_K)$ is complex conjugation, *after* we base-change from $\mathcal{O}_E[1/S]$ to $W(\kappa)$ the polarization induces an isomorphism

$$R^1\pi_*(\mathcal{O}_{A^{\mathrm{univ}}})[\tau] \simeq R^1\pi_*(\mathcal{O}_{(A^{\mathrm{univ}})^t})[\bar{\tau}] = \underline{\mathrm{Lie}}[\bar{\tau}] = \underline{\omega}[\bar{\tau}]^\vee.$$

Since $\mathrm{rk}(\underline{\omega}[\tau]) + \mathrm{rk}(\underline{\omega}[\bar{\tau}]) = d$, each $\mathcal{H}[\tau]$ is of rank d.

We introduce the shorthand notation

$$\mathcal{P}_\tau = \underline{\omega}[\tau].$$

The Hodge filtration exact sequence can be written therefore, over $W(\kappa)$, as

$$0 \to \mathcal{P}_\tau \to \mathcal{H}[\tau] \to \mathcal{P}_{\bar{\tau}}^\vee \to 0. \tag{4.1}$$

For any abelian scheme A/R, there is a canonical perfect pairing

$$\{,\}_{dR} : H^1_{dR}(A/R) \times H^1_{dR}(A^t/R) \to R.$$

In our case, using the prime-to-p quasi-isogeny λ to identify $H^1_{dR}(A/R)[\bar{\tau}]$ with $H^1_{dR}(A^t/R)[\tau]$, we get a pairing

$$\{,\}_{dR} : \mathcal{H}[\tau] \times \mathcal{H}[\bar{\tau}] \to \mathcal{O}_{\mathscr{M}}.$$

Under this pairing $\mathcal{P}_\tau$ and $\mathcal{P}_{\bar{\tau}}$ are exact annihilators of each other, and the induced pairing between $\mathcal{P}_\tau$ and $\mathcal{P}_\tau^\vee$ is the natural one.

The Gauss-Manin connection is a flat connection

$$\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_{\mathscr{M}}.$$

If $\xi \in \mathcal{T}$ is a vector field (on an open set in $\mathscr{M}$, omitted from the notation) we denote by $\nabla_\xi : \mathcal{H} \to \mathcal{H}$ the ξ-derivation of $\mathcal{H}$ obtained by contracting $\Omega^1_{\mathscr{M}}$ with ξ. Since ∇ commutes with the endomorphisms, ∇_ξ preserves the τ-isotypical parts $\mathcal{H}[\tau]$ for every $\tau \in \mathscr{I}$. When ∇_ξ is applied to $\mathcal{P}_\tau$ and the result is projected to $\mathcal{P}_{\bar{\tau}}^\vee$, we get an $\mathcal{O}_{\mathscr{M}}$*-linear* homomorphism

$$\mathrm{KS}^\vee(\xi)_\tau \in \mathrm{Hom}(\mathcal{P}_\tau, \mathcal{P}_{\bar{\tau}}^\vee) \simeq \mathcal{P}_\tau^\vee \otimes \mathcal{P}_{\bar{\tau}}^\vee.$$

Using the formalism of the Gauss-Manin connection and the symmetry of the polarization, it is easy to check that when we identify $\mathcal{P}_\tau^\vee \otimes \mathcal{P}_{\bar{\tau}}^\vee$ with $\mathcal{P}_{\bar{\tau}}^\vee \otimes \mathcal{P}_\tau^\vee$,

$$\mathrm{KS}^\vee(\xi)_\tau = \mathrm{KS}^\vee(\xi)_{\bar{\tau}}.$$

Thus $\mathrm{KS}^\vee(\xi)_\tau$ depends only on the pair $\{\tau, \bar{\tau}\} \in \mathscr{I}^+$, i.e. on $\tau|_L$. When we combine these maps, we get an $\mathcal{O}_{\mathscr{M}}$*-linear* homomorphism

$$\mathrm{KS}^\vee(\xi) \in \bigoplus_{\{\tau,\bar{\tau}\}\in\mathscr{I}^+} \mathcal{P}_\tau^\vee \otimes \mathcal{P}_{\bar{\tau}}^\vee.$$

Proposition 4.1.1 (The Kodaira-Spencer isomorphism)**.** *The map*

$$\mathrm{KS}^\vee : \mathcal{T} \simeq \bigoplus_{\{\tau,\bar{\tau}\}\in\mathscr{I}^+} \mathcal{P}_\tau^\vee \otimes \mathcal{P}_{\bar{\tau}}^\vee$$

sending ξ to $\mathrm{KS}^\vee(\xi)$ is an isomorphism.

We let KS be the isomorphism dual to $\mathrm{KS}^\vee$, namely

$$\mathrm{KS}: \bigoplus_{\{\tau,\bar\tau\}\in\mathscr{I}^+} \mathcal{P}_\tau\otimes\mathcal{P}_{\bar\tau}\simeq\Omega^1_{\mathscr{M}}. \tag{4.2}$$

4.1.3. *The μ-ordinary locus of M.* For a general signature, the abelian varieties parametrized by the mod p fiber M of our moduli space are never ordinary. There is, however, a dense open set $M^{\mathrm{ord}}\subset M$ at whose geometric points A^{univ} is "as ordinary as possible". To make this precise we introduce, following [Mo] 1.2.3, certain standard Dieudonné modules and their associated p-divisible groups.

Let k be an algebraically closed field containing κ. If A is an abelian variety over the field k, with endomorphisms by $\mathcal{O}_K$, its p-divisible group $A[p^\infty]$ breaks up as a product

$$A[p^\infty]=\prod_{\mathfrak{P}|p}A[\mathfrak{P}^\infty]$$

over the primes of $\mathcal{O}_K$ above p, and $A[\mathfrak{P}^\infty]$ becomes a p-divisible group with $\mathcal{O}_{K,\mathfrak{P}}$-action.

Fix a prime $\mathfrak{P}|p$ and write $\mathcal{O}=\mathcal{O}_{K,\mathfrak{P}}$. Let

$$\mathfrak{f}:\mathscr{I}_\mathfrak{P}=\mathrm{Hom}(\mathcal{O},W(k))\to[0,d]$$

be an integer-valued function. Let $M(d,\mathfrak{f})$ be the following Dieudonné module with $\mathcal{O}$-action over $W(k)$. First,

$$M(d,\mathfrak{f})=\oplus_{i\in\mathscr{I}_\mathfrak{P}}M_i$$

where $M_i=\oplus_{j=1}^d W(k)e_{i,j}$ is a free $W(k)$-module of rank d. We let $\mathcal{O}$ act on M_i via the homomorphism $i:\mathcal{O}\to W(k)$. We let F (resp. V) be the ϕ-semilinear (resp. ϕ^{-1}-semilinear) endomorphism of $M(d,\mathfrak{f})$ satisfying (recall the convention that if i refers to the embedding τ then $i+1$ refers to $\phi\circ\tau$)

$$F(e_{i,j})=\begin{cases} e_{i+1,j} & 1\le j\le d-\mathfrak{f}(i)\\ pe_{i+1,j} & d-\mathfrak{f}(i)<j\le d\end{cases}$$

and

$$V(e_{i+1,j})=\begin{cases} pe_{i,j} & 1\le j\le d-\mathfrak{f}(i)\\ e_{i,j} & d-\mathfrak{f}(i)<j\le d\end{cases}.$$

Then $M(d,\mathfrak{f})$ is a Dieudonné module with $\mathcal{O}$-action, of rank $[\mathcal{O}:\mathbb{Z}_p]d$ over $W(k)$. We let $X(d,\mathfrak{f})$ be the unique p-divisible group with $\mathcal{O}$-action over k whose contravariant Dieudonné module is $M(d,\mathfrak{f})$.

Let $N(d,\mathfrak{f})=M(d,\mathfrak{f})/pM(d,\mathfrak{f})$. This is the Dieudonné module of the finite group scheme $Y(d,\mathfrak{f})=X(d,\mathfrak{f})[p]$. The cotangent space of $X(d,\mathfrak{f})$ is canonically isomorphic to $N(d,\mathfrak{f})[F]=\bigoplus_{i\in\mathscr{I}_\mathfrak{P}}\bigoplus_{j=d-\mathfrak{f}(i)+1}^d ke_{i,j}$. It inherits a $\kappa(\mathfrak{P})$-action and its i-isotypic subspace $\bigoplus_{j=d-\mathfrak{f}(i)+1}^d ke_{i,j}$ is $\mathfrak{f}(i)$-dimensional.

Let X be a p-divisible group with $\mathcal{O}$-action over k. In [Mo], Theorem 1.3.7, it is proved that if either X is isogenous to $X(d,\mathfrak{f})$, or $X[p]$ is isomorphic to $Y(d,\mathfrak{f})$, then X is already isomorphic to $X(d,\mathfrak{f})$.

Definition 4.1.2. Let A be an abelian variety over k with $\mathcal{O}_K$-action, of dimension $2gd$. Then A is called *μ-ordinary* if every $A[\mathfrak{P}^\infty]$ with its $\mathcal{O}_{K,\mathfrak{P}}$-action is isomorphic to some $X(d,\mathfrak{f})$.

Let A be a μ-ordinary abelian variety over k with $\mathcal{O}_K$-action and CM type $\{r_\tau\}$ as before. The cotangent space of A may be identified with that of $A[p^\infty]$. From the relation between the cotangent space of $A[p^\infty]$ and its Dieudonné module, it follows that if $\tau = i \in \mathscr{I}_{\mathfrak{P}}$, $\mathfrak{f}(i) = r_\tau$.

It follows that the function $\mathfrak{f}$ is determined by the signature, hence all the μ-ordinary A/k parametrized by geometric points of M have isomorphic p-divisible groups. Wedhorn proved the following fundamental theorem.

Theorem. [Wed, Mo] *There is a dense open set $M^{\mathrm{ord}} \subset M$ such that for any geometric point $x \in M(k)$ the abelian variety A_x^{univ} is μ-ordinary if and only if $x \in M^{\mathrm{ord}}(k)$.*

Using the slope decomposition explained below, it is possible to attach a Newton polygon to a p-divisible group with $\mathcal{O}_K$-action, and the points of M^{ord} are characterized also as those whose Newton polygon lies *below* every other Newton polygon (Newton polygons *go up under specialization*).

4.1.4. *Slope decomposition over M^{ord}.* We next review the slope decomposition of the $\mathfrak{P}$-divisible group of a μ-ordinary abelian variety A with $\mathcal{O}_K$-action over an algebraically closed field k containing κ. Recall that $A[\mathfrak{P}^\infty] \simeq X(d, \mathfrak{f})$. For each $1 \le j \le d$ the submodule

$$M(d, \mathfrak{f})^j = \bigoplus_{i \in \mathscr{I}_{\mathfrak{P}}} W(k) e_{i,j}$$

of $M(d, \mathfrak{f})$ is a sub-Dieudonné module of $\mathcal{O}$-height 1. Its *slope* is the rational number

$$\frac{|\{i |\, j > d - \mathfrak{f}(i)\}|}{|\mathscr{I}_{\mathfrak{P}}|}.$$

Note that the slope of $M(d, \mathfrak{f})^{j+1}$ is greater or equal than the slope of $M(d, \mathfrak{f})^j$, and if they are equal, $M(d, \mathfrak{f})^{j+1} \simeq M(d, \mathfrak{f})^j$.

Let $0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_r \le 1$ be the distinct slopes obtained in this way, and for $1 \le \nu \le r$ let d^ν be the number of j's with $\mathrm{slope}(M(d, \mathfrak{f})^j) = \lambda_\nu$. Define a function $\mathfrak{f}^\nu : \mathscr{I}_{\mathfrak{P}} \to \{0, d^\nu\}$ by

$$\mathfrak{f}^\nu(i) = \begin{cases} 0 & \text{if } \sum_{\ell=1}^{\nu-1} d^\ell < d - \mathfrak{f}(i) \\ d^\nu & \text{if } \sum_{\ell=1}^{\nu-1} d^\ell \ge d - \mathfrak{f}(i). \end{cases}$$

Then grouping together the $M(d, \mathfrak{f})^j$ of slope λ_ν we get an isoclinic Dieudonné module isomorphic to $M(d^\nu, \mathfrak{f}^\nu)$. We arrive at the *slope decomposition*

$$M(d, \mathfrak{f}) = \bigoplus_{\nu=1}^{r} M(d^\nu, \mathfrak{f}^\nu),$$

and similarly for the p-divisible group

$$X(d, \mathfrak{f}) = \prod_{\nu=1}^{r} X(d^\nu, \mathfrak{f}^\nu).$$

This description is valid for μ-ordinary p-divisible groups (with $\mathcal{O}$-action) over algebraically closed fields only. Its significance stems from the fact that when we study deformations, the isoclinic p-divisible groups with $\mathcal{O}$-action deform uniquely (are rigid), and the deformations arise only from non-trivial extensions of one isoclinic subquotient by another one, of a higher slope. Over an artinian ring with residue field k the slope decomposition is replaced by a *slope filtration*. The study of the universal deformation space via these extensions lead Moonen to introduce his *cascade* structures, which are the main topic of [Mo]. Finally, we remind the reader that by the Serre-Tate theorem, deformations of a tuple $\underline{A} \in M(k)$ correspond to deformations of $A[p^\infty]$ with its $\mathcal{O}_K$-structure and polarization. Moonen's theory of cascades supplies therefore "coordinates" at a μ-ordinary point $x \in M^{\mathrm{ord}}(k)$, reminiscent of the Serre-Tate coordinates at an ordinary point of the usual modular curve.

4.1.5. *Duality.* Finally, let us examine duality. Quite generally, the Cartier-Serre dual of $A[p^\infty]$ is $A^t[p^\infty]$. A $\mathbb{Z}_{(p)}^\times$-polarization λ therefore makes $A[p^\infty]$ self-dual. In the presence of an $\mathcal{O}_K$-action as above, the duality induced by λ sets $A[\mathfrak{P}^\infty]$ in duality with $A[\bar{\mathfrak{P}}^\infty]$. Let $\mathfrak{p}$ be the prime of L underlying $\mathfrak{P}$. We distinguish two cases.

(a) If $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}\bar{\mathfrak{P}}$ is split, there are no further restrictions on $A[\mathfrak{P}^\infty]$, but $A[\bar{\mathfrak{P}}^\infty]$ is completely determined by $A[\mathfrak{P}^\infty]$.

(b) If $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}$ is inert, $A[\mathfrak{P}^\infty]$ is self-dual. In this case let $m = [\kappa(\mathfrak{p}) : \mathbb{F}_p]$ be the inertia degree of $\mathfrak{p}$, so that $[\kappa(\mathfrak{P}) : \mathbb{F}_p] = 2m$. Complex conjugation $\rho \in Gal(K/L)$ fixes $\mathfrak{P}$, so induces an automorphism of $\kappa(\mathfrak{P})$, and for $\tau \in \mathscr{I}_{\mathfrak{P}}$ we have $\tau \circ \rho = \phi^m \circ \tau$. Recall that we denoted τ by i and $\phi^m \circ \tau$ by $i + m$. If A is μ-ordinary, the self-duality of $A[\mathfrak{P}^\infty]$ is manifested ([Mo] 3.1.1, Moonen's $\varepsilon = +1$ in our case) in a *perfect symmetric* $W(k)$-linear pairing

$$\varphi : M(d, \mathfrak{f}) \times M(d, \mathfrak{f}) \to W(k)$$

such that

$$\varphi(Fx, y) = \varphi(x, Vy)^\phi$$
$$\varphi(ax, y) = \varphi(x, \bar{a}y)$$

($a \in \mathcal{O}_K$). This (or the relation $r_\tau + r_{\bar{\tau}} = d$) implies that $\mathfrak{f}(i) + \mathfrak{f}(i + m) = d$. In fact, M_i is orthogonal to $M_{i'}$ unless $i' = i + m$ and we can choose the basis $\{e_{i,j}\}$ in such a way that

$$\varphi(e_{i,j}, e_{i+m,j'}) = c_{i,j}\delta_{j',d+1-j}$$

with some $c_{i,j} \in W(k)^\times$ ($\delta_{a,b}$ is Kronecker's delta). This means that the Dieudonné modules $M(d, \mathfrak{f})^j$ and $M(d, \mathfrak{f})^{d+1-j}$ are dual under this pairing. See [Mo] 3.2.3, case (AU).

4.2. **The V-foliations on the μ-ordinary locus.**

4.2.1. *Construction.* Consider the universal abelian variety A^{univ} over M, which we now denote for brevity A, and its Verschiebung isogeny

$$\mathrm{Ver} : A^{(p)} = M \times_{\Phi_M, M} A \to A.$$

The relative de Rham cohomology of $A^{(p)}$, denoted $\mathcal{H}^{(p)}$, may be identified with $\Phi_M^*\mathcal{H}$, and its Hodge bundle $\underline{\omega}^{(p)}$ with $\Phi_M^*\underline{\omega}$. Letting $a \in \mathcal{O}_K$ act on $A^{(p)}$ as $\iota^{(p)}(a) = 1 \times \iota(a)$ we get an induced action $\iota^{(p)}$ of $\mathcal{O}_K$ on $\mathcal{H}^{(p)}$ and on $\underline{\omega}^{(p)}$. However, for $\tau \in \mathscr{I}$

$$\mathcal{H}[\tau]^{(p)} := \Phi_M^*(\mathcal{H}[\tau]) = \mathcal{H}^{(p)}[\phi \circ \tau],$$

because if $x \in \mathcal{H}[\tau]$ and $1 \otimes x \in \mathcal{O}_M \otimes_{\phi, \mathcal{O}_M} \mathcal{H}[\tau] = \Phi_M^*(\mathcal{H}[\tau])$, then

$$\iota^{(p)}(a)(1 \otimes x) = 1 \otimes \tau(a)x = \tau(a)^p \otimes x = \phi \circ \tau(a) \cdot (1 \otimes x).$$

The isogeny Ver commutes with the endomorphisms,

$$\mathrm{Ver} \circ \iota^{(p)}(a) = \iota(a) \circ \mathrm{Ver},$$

and therefore induces a homomorphism of vector bundles

$$V : \mathcal{H}[\tau] \to \mathcal{H}^{(p)}[\tau] = \mathcal{H}[\phi^{-1} \circ \tau]^{(p)},$$

and similarly on $\underline{\omega}[\tau] = \mathcal{P}_\tau$

$$V : \mathcal{P}_\tau \to (\mathcal{P}^{(p)})_\tau = (\mathcal{P}_{\phi^{-1}\circ\tau})^{(p)}.$$

We shall use the notation $\mathcal{P}^{(p)}_{\phi^{-1}\circ\tau}$ for the right hand side.

At a μ-ordinary geometric point $x \in M^{\mathrm{ord}}(k)$ we may identify the fiber $\mathcal{H}_x = H^1_{dR}(A_x/k)$ with the (contravariant) Dieudonné module of $A_x[p]$, and the linear map $V : \mathcal{H}_x \to \mathcal{H}^{(p)}_x$ with the ϕ^{-1}-semilinear endomorphism V of the Dieudonné module. Let $\tau = i \in \mathscr{I}_{\mathfrak{P}}$. Recalling the shape of the Dieudonné module $M(d, \mathfrak{f})$ of $A_x[\mathfrak{P}^\infty]$ we conclude that $\mathcal{P}_{\tau,x}[V] = \underline{\omega}_x[\tau][V] = 0$ if $\mathfrak{f}(i-1) \geq \mathfrak{f}(i)$, and

$$\mathcal{P}_{\tau,x}[V] = \sum_{j=d-\mathfrak{f}(i)+1}^{d-\mathfrak{f}(i-1)} k e_{i,j}$$

if $\mathfrak{f}(i-1) < \mathfrak{f}(i)$. We recall that $\mathfrak{f}(i) = r_\tau$ and $\mathfrak{f}(i-1) = r_{\phi^{-1}\circ\tau}$.

The following is the main definition of the second part of our paper.

Definition 4.2.1. Let $\Sigma \subset \mathscr{I}^+$. Let[3] $\mathscr{F}_\Sigma \subset \mathcal{T}$ be the subsheaf on M^{ord} which is the *annihilator*, under the pairing between $\mathcal{T}$ and Ω^1_M, of

$$\mathrm{KS}\left(\sum_{\{\tau,\bar\tau\}\in\Sigma} (\mathcal{P}_\tau \otimes \mathcal{P}_{\bar\tau})[V \otimes V]\right).$$

Our first goal is to prove that $\mathscr{F}_\Sigma$ is a p-foliation. Note that at every $x \in M(k)$

$$\mathcal{P}_{\tau,x} \otimes \mathcal{P}_{\bar\tau,x}[V \otimes V] = \mathcal{P}_{\tau,x}[V] \otimes \mathcal{P}_{\bar\tau,x} + \mathcal{P}_{\tau,x} \otimes \mathcal{P}_{\bar\tau,x}[V]. \tag{4.3}$$

By the discussion above, if $x \in M^{\mathrm{ord}}(k)$, the first term is a subspace whose dimension is

$$\max\{0, r_\tau - r_{\phi^{-1}\circ\tau}\} \cdot (d - r_\tau)$$

and the second is of dimension $r_\tau \cdot \max\{0, r_{\phi^{-1}\circ\tau} - r_\tau\}$. Here we used the relations $\overline{\phi^{-1}\circ\tau} = \phi^{-1}\circ\tau\circ\rho = \phi^{-1}\circ\bar\tau$ and $r_{\bar\tau} = d - r_\tau$. At most one of the terms is non-zero, and both are zero if and only if either $r_\tau = r_{\phi^{-1}\circ\tau}$,$r_\tau = 0$ or $r_\tau = d$. In particular,

$$\dim_k \mathcal{P}_{\tau,x} \otimes \mathcal{P}_{\bar\tau,x}[V \otimes V]$$

is the same for all $x \in M^{\mathrm{ord}}(k)$.

[3] The reader might have noticed a twist in our notation. While the foliations denoted $\mathscr{F}_\Sigma$ on Hilbert modular varieties, in the first part of our paper, grow with Σ, our current $\mathscr{F}_\Sigma$ become smaller when Σ grows. This could be solved by labelling our $\mathscr{F}_\Sigma$ by the *complement* of Σ, but as the two types of foliations are distinct and of a different nature, we did not find it necessary to reconcile the two conventions.

Lemma 4.2.2. *For $\{\tau,\bar\tau\}\in\mathscr{I}^+$ let*

$$r_V^{\mathrm{ord}}\{\tau,\bar\tau\}=\max\{0,r_\tau-r_{\phi^{-1}\circ\tau}\}\cdot(d-r_\tau)+r_\tau\cdot\max\{0,r_{\phi^{-1}\circ\tau}-r_\tau\}$$

(this quantity is symmetric in τ and $\bar\tau$). Over M^{ord}, the subsheaf $\mathscr{F}_\Sigma$ is a vector sub-bundle of $\mathcal{T}$ of corank $r_V(\Sigma)=\sum_{\{\tau,\bar\tau\}\in\Sigma}r_V^{\mathrm{ord}}\{\tau,\bar\tau\}$. Its rank is given by the formula

$$\mathrm{rk}(\mathscr{F}_\Sigma)=\sum_{\{\tau,\bar\tau\}\notin\Sigma}r_\tau r_{\bar\tau}+\sum_{\{\tau,\bar\tau\}\in\Sigma}\min\{r_\tau,r_{\phi^{-1}\circ\tau}\}\cdot\min\{r_{\bar\tau},r_{\phi^{-1}\circ\bar\tau}\}.$$

Proof. The sheaf $\mathscr{G}_\tau=(\mathcal{P}_\tau\otimes\mathcal{P}_{\bar\tau})[V\otimes V]$ is a subsheaf of $\mathcal{P}_\tau\otimes\mathcal{P}_{\bar\tau}$, and $\mathscr{F}_\Sigma$, the annihilator of $\mathrm{KS}(\sum_{\{\tau,\bar\tau\}\in\Sigma}\mathscr{G}_\tau)$ in $\mathcal{T}$, is *saturated*. This is because $\mathcal{O}_M$ has no zero divisors: if $f\in\mathcal{O}_M$, $\xi\in\mathcal{T}$ and $f\xi\in\mathscr{F}_\Sigma$, then for every $\omega\in\mathrm{KS}(\sum_{\{\tau,\bar\tau\}\in\Sigma}\mathscr{G}_\tau)$ we have $f\langle\omega,\xi\rangle=\langle\omega,f\xi\rangle=0$, so $\langle\omega,\xi\rangle=0$.

Quite generally, if M is a variety over a field k, $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves of rank n, and $T\in\mathrm{Hom}_{\mathcal{O}_M}(\mathcal{F},\mathcal{G})$ is such that the fiber rank $\mathrm{rk}(T_x)=m$ is constant on M, then $\ker(T)$ is locally a direct summand (i.e. a vector sub-bundle) of rank $n-m$. Compare [Mu], II.5 Lemma 1, p.51. It is in fact enough to verify the constancy of $\mathrm{rk}(T_x)$ at closed points, because every other point $y\in M$ contains closed points in its closure, and the fiber rank can only go up under specialization. Applying this with $\mathcal{F}=\mathcal{P}_\tau\otimes\mathcal{P}_{\bar\tau}$, $\mathcal{G}=\mathcal{F}^{(p)}$ and $T=V\otimes V$ we deduce that $\mathscr{G}_\tau$, hence also $\mathscr{F}_\Sigma$, are vector sub-bundles.

The rank of $\mathscr{F}_\Sigma$ is

$$\sum_{\{\tau,\bar\tau\}\in\mathscr{I}^+}r_\tau r_{\bar\tau}-\sum_{\{\tau,\bar\tau\}\in\Sigma}r_V^{\mathrm{ord}}\{\tau,\bar\tau\}.$$

For $\{\tau,\bar\tau\}\in\Sigma$ we first assume that $r_{\phi^{-1}\circ\tau}\le r_\tau$. We then have

$$r_\tau r_{\bar\tau}-r_V^{\mathrm{ord}}\{\tau,\bar\tau\}=r_\tau r_{\bar\tau}-(r_\tau-r_{\phi^{-1}\circ\tau})r_{\bar\tau}=r_{\phi^{-1}\circ\tau}r_{\bar\tau}.$$

But by our assumption $r_{\phi^{-1}\circ\tau}=\min\{r_\tau,r_{\phi^{-1}\circ\tau}\}$ and $r_{\bar\tau}=\min\{r_{\bar\tau},r_{\phi^{-1}\circ\bar\tau}\}$. The case $r_{\phi^{-1}\circ\tau}\ge r_\tau$ is treated similarly. □

4.2.2. *Closure under Lie brackets and p-power.*

Lemma 4.2.3. *The vector bundle $\mathscr{F}_\Sigma$ is involutive: if ξ,η are sections of $\mathscr{F}_\Sigma$, so is $[\xi,\eta]$.*

Proof. The proof is essentially the same as the proof of Proposition 3 in [G-dS1]. For $\alpha\in\mathcal{P}_\tau$ and $\beta\in\mathcal{P}_{\bar\tau}$ we have the formula

$$\langle\mathrm{KS}(\alpha\otimes\beta),\xi\rangle=\{\nabla_\xi(\alpha),\beta\}_{dR}\in\mathcal{O}_M$$

(loc. cit. Lemma 4). Thus, $\xi\in\mathscr{F}_\Sigma$ if and only if for every τ such that $\{\tau,\bar\tau\}\in\Sigma$ we have

$$\nabla_\xi(\mathcal{P}_\tau[V])\perp\mathcal{P}_{\bar\tau}$$

under the pairing $\{,\}_{dR}:\mathcal{H}[\tau]\times\mathcal{H}[\bar\tau]\to\mathcal{O}_M$. But the left annihilator of $\mathcal{P}_{\bar\tau}$ is $\mathcal{P}_\tau$. The Gauss-Manin connection commutes with isogenies, so in particular carries $\mathcal{H}[V]$ to itself. It follows that $\xi\in\mathscr{F}_\Sigma$ if and only if

$$\nabla_\xi(\mathcal{P}_\tau[V])\subset\mathcal{P}_\tau[V]\qquad(4.4)$$

for any τ such that $\{\tau,\bar\tau\}\in\Sigma$. The Gauss-Manin connection is well-known to be flat, i.e.

$$\nabla_{[\xi,\eta]}=\nabla_\xi\circ\nabla_\eta-\nabla_\eta\circ\nabla_\xi,$$

so if both ξ and η lie in $\mathscr{F}_\Sigma$, (4.4) implies that so does $[\xi,\eta]$. □

Lemma 4.2.4. *The vector bundle $\mathscr{F}_\Sigma$ is p-closed: if ξ is a section of $\mathscr{F}_\Sigma$, so is ξ^p.*

Proof. Again we follow the proof of Proposition 3 in [G-dS1]. The p-curvature

$$\psi(\xi) = \nabla_\xi^p - \nabla_{\xi^p}$$

does not vanish identically, but is only a nilpotent endomorphism of $\mathcal{H}$ ([Ka-Tur], §5). However, denoting by $\mathcal{H}^{(p)} = \Phi_M^*\mathcal{H} = H^1_{dR}(A^{(p)}/M)$ $(A = A^{\mathrm{univ}})$ the relative de Rham cohomology of $A^{(p)}$, and by F the map induced by the relative Frobenius $\mathrm{Fr} : A \to A^{(p)}$ on cohomology, we have

$$\mathcal{H}[V] = F^*\mathcal{H}^{(p)}.$$

Furthermore, since the Gauss-Manin connection commutes with any isogeny, the following diagram commutes

$$\begin{tikzcd} \mathcal{H}^{(p)} \arrow[r, "F^*"] \arrow[d, "\nabla^{(p)}=\nabla_{\mathrm{can}}"'] & \mathcal{H} \arrow[d, "\nabla"] \\ \mathcal{H}^{(p)} \otimes \Omega_M \arrow[r, "F^*\otimes 1"] & \mathcal{H} \otimes \Omega_M. \end{tikzcd} \tag{4.5}$$

Here $\nabla^{(p)}$, the Gauss-Manin connection of $A^{(p)}$, turns out to be the canonical connection ∇_{can} that exists on *any* vector bundle of the form $\Phi_M^*\mathcal{H}$, namely if $f \otimes \alpha \in \mathcal{O}_M \otimes_{\phi,\mathcal{O}_M} \mathcal{H}$,

$$\nabla_{can}(f \otimes \alpha) = df \otimes \alpha$$

(that this is well-defined follows from the rule $d(g^pf) = g^pdf$). The commutativity of (4.5) implies that the restriction of ∇, the Gauss-Manin connection of A, to $F^*\mathcal{H}^{(p)} = \mathcal{H}[V]$ is the connection denoted ∇_{can} in [Ka-Tur]. It follows from Cartier's theorem (loc. cit. Theorem 5.1) that $\psi(\xi)$ vanishes when restricted to $\mathcal{H}[V]$. We conclude the proof as in the previous lemma, using the criterion (4.4) for ξ to lie in $\mathscr{F}_\Sigma$. □

Altogether we proved the following.

Theorem 4.2.5. *The sheaf $\mathscr{F}_\Sigma$ is a smooth p-foliation on M^{ord}.*

We stress the difference between the tautological foliations on Hilbert modular varieties, which were p-closed only if Σ was invariant under the action of ϕ, and the V-foliations on unitary Shimura varieties that are always p-closed. The reason lies in the last Lemma, in the delicate relation between the p-curvature of the Gauss-Manin connection and the kernel of Verschiebung.

4.3. **Relation with Moonen's cascade structure.**

4.3.1. *Moonen's cascade structure.* In [Mo] Moonen generalized the notion of Serre-Tate coordinates to any Shimura variety of PEL type in characteristic p. Here we recall his results in the case of our unitary Shimura variety M.

Fix a μ-ordinary geometric point $x : \mathrm{Spec}(k) \to M^{\mathrm{ord}}$. Let $W = W(k)$ be the ring of Witt vectors over k, and consider the category $\mathbf{C}_W$ of local artinian W-algebras with residue field k, morphisms being local homomorphisms inducing the identity on k. Let $\mathbf{FS}_W$ be the category of affine formal schemes $\mathfrak{X}$ over W with the property that $\Gamma(\mathfrak{X},\mathcal{O}_\mathfrak{X})$ is a profinite W-algebra (the last condition regarded as a "smallness" condition). By a theorem of Grothendieck, associating to $\mathfrak{X} \in \mathbf{FS}_W$ the functor of points $R \mapsto \mathfrak{X}(R)$ ($R \in \mathbf{C}_W$) identifies $\mathbf{FS}_W$ with the category of left-exact functors from $\mathbf{C}_W$ to sets.

Equip $\mathbf{FS}_W$ with the flat topology and let $\mathfrak{T} = \widehat{\mathbf{FS}}_W$ be the topos of sheaves of sets on it. Since the flat topology is coarser than the canonical topology, $\mathbf{FS}_W$ embeds in $\widehat{\mathbf{FS}}_W$ (Yoneda's lemma). In particular, we may consider $\mathrm{Spf}(\widehat{\mathcal{O}}_{\mathscr{M},x})$ as a sheaf of sets on $\mathbf{FS}_W$.

Let $\mathbb{D} = \mathrm{Def}(\underline{X}')$ be the universal deformation space of the pair

$$\underline{X}' = (A_x^{\mathrm{univ}}[p^\infty], \iota_x^{\mathrm{univ}}).$$

For every $\mathfrak{X} \in \mathbf{FS}_W$ the deformations of $\underline{X}'$ over $\mathfrak{X}$ make up the set $\mathbb{D}(\mathfrak{X})$, and $\mathbb{D} \in \widehat{\mathbf{FS}}_W$. In fact, it is representable by a formal scheme in $\mathbf{FS}_W$.

If $\underline{X}'^{,D}$ is the Cartier dual of X, with the $\mathcal{O}_K$-action

$$\iota^D(a) := \iota(\bar{a})^D,$$

then $\mathrm{Def}(\underline{X}'^{,D}) = \mathbb{D}$ as well, since any deformation of $\underline{X}'$ yields a deformation of $\underline{X}'^{,D}$ and vice versa. (The cascade structures defined below will be dual, though.) The polarization $\lambda_x^{\mathrm{univ}} : X \simeq X^D$ (intertwining the actions ι and ι^D) induces an automorphism

$$\gamma : \mathbb{D} = \mathrm{Def}(\underline{X}') \simeq \mathrm{Def}(\underline{X}'^{,D}) = \mathbb{D},$$

and the subsheaf $\mathbb{D}^\lambda = \{x \in \mathbb{D} |\, \gamma(x) = x\}$ of symmetric elements in $\mathbb{D}$ is the universal deformation space of

$$\underline{X} = (A_x^{\mathrm{univ}}[p^\infty], \iota_x^{\mathrm{univ}}, \lambda_x^{\mathrm{univ}}).$$

See [Mo], 3.3.1. Like $\mathbb{D}$, its sub-functor $\mathbb{D}^\lambda \in \widehat{\mathbf{FS}}_W$, and is representable. Indeed, $\mathbb{D}^\lambda$ is represented by the formal scheme $\mathrm{Spf}(\widehat{\mathcal{O}}_{\mathscr{M},x})$, and its characteristic p fiber by $\mathrm{Spf}(\widehat{\mathcal{O}}_{M,x})$.

We refer to [Mo], 2.2.1, for the precise definition of an *r-cascade* in a topos $\mathfrak{T}$. A 1-cascade is a point, a 2-cascade is a sheaf of commutative groups, and a 3-cascade is a biextension. The general structure of an r-cascade in $\mathfrak{T}$ generalizes these notions (see below).

In [Mo], 2.3.6, Moonen defines a structure of an r-cascade on $\mathbb{D}$, where r is the number of slopes of X (see §4.1.4). Thus $\mathrm{Spf}(\widehat{\mathcal{O}}_{\mathscr{M},x})$ (or rather, the sheaf that it represents) is endowed with the structure of symmetric elements in the self-dual r-cascade $\mathbb{D}$.

4.3.2. *Previous results.* When K is quadratic imaginary and p is inert we showed in [G-dS1], Theorem 13, that the foliation we have constructed (denoted there $\mathcal{TS}^+$, and here $\mathscr{F}_\Sigma$) is compatible with Moonen's cascade structure on M^{ord} in the following sense. Let (n, m) be the signature, $n \geq m$, so that $\mathrm{rk}(\mathscr{F}_\Sigma) = m^2$. Let $x \in M^{\mathrm{ord}}(k)$. The number of slopes of $A_x^{\mathrm{univ}}[p^\infty]$ turns out to be three, and $\mathrm{Spf}(\widehat{\mathcal{O}}_{M,x})$ acquires from the cascade structure a structure of a $\widehat{\mathbb{G}}_m^{m^2}$-torsor. The formal torus $\widehat{\mathbb{G}}_m^{m^2}$ gives rise to an m^2-dimensional subspace of the tangent space at x, and in loc. cit. we proved that this subspace coincided with the foliation.

We find that while the cascade structure "lives" only on the formal neighborhood of x, and does not globalize, its "trace" on the tangent space globalizes to the foliation that we constructed in the tangent bundle of M^{ord}.

4.3.3. *The general case: subspaces of the tangent bundle defined by the cascade.* We shall now describe how this result generalizes to the setting of our paper. For simplicity let us assume that p is inert in L, $p\mathcal{O}_L = \mathfrak{p}$. Both the foliation and the cascade structure break up as products of corresponding structures over the primes $\mathfrak{p}$ of L dividing p, so with a little more book-keeping the general case follows the same pattern as the inert case.

We shall also assume that $\mathfrak{p}$ splits in K, and write as before $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}\bar{\mathfrak{P}}$. The other case ($\mathfrak{p}$ inert in K) is a little more complicated, but can be handled in a similar way. The assumption that $\mathfrak{p}$ splits in K allows us to concentrate on the deformation space of $\underline{X} = A_x^{\mathrm{univ}}[\mathfrak{P}^\infty]$, as a p-divisible group with $\mathcal{O}_K$-action, and ignore the polarization. The resulting deformation space is just an r-cascade $\mathbb{D}_{\mathfrak{P}}$. Then, $\mathbb{D} = \mathbb{D}_{\mathfrak{P}} \times \mathbb{D}_{\mathfrak{P}}^\vee$, and the polarization induces an isomorphism $\mathbb{D}_{\mathfrak{P}} \simeq \mathbb{D}_{\mathfrak{P}}^\vee = \mathbb{D}_{\bar{\mathfrak{P}}}$ that we denote $x \mapsto \gamma(x)$; $\mathbb{D}^\lambda$ is the set of pairs $(x, \gamma(x)) \in \mathbb{D}$, and is therefore isomorphic to $\mathbb{D}_{\mathfrak{P}}$.

Because of the relation $\mathscr{F}_{\Sigma_1 \cup \Sigma_2} = \mathscr{F}_{\Sigma_1} \cap \mathscr{F}_{\Sigma_2}$ it is enough to determine the relation of $\mathscr{F}_{\{\tau,\bar{\tau}\}}$ to the cascade structure, and the general case will follow from it. As we have seen in (4.2), the Kodaira-Spencer map yields an isomorphism

$$\mathrm{KS}^\vee : \mathcal{T} \simeq \bigoplus_{\{\sigma,\bar{\sigma}\}\in\mathscr{I}^+} \mathcal{P}_\sigma^\vee \otimes \mathcal{P}_{\bar{\sigma}}^\vee,$$

and we may write

$$\mathcal{T}_{\{\sigma,\bar{\sigma}\}} = (\mathrm{KS}^\vee)^{-1}(\mathcal{P}_\sigma^\vee \otimes \mathcal{P}_{\bar{\sigma}}^\vee).$$

We then have

$$\mathscr{F}_{\{\tau,\bar{\tau}\}} = \bigoplus_{\{\sigma,\bar{\sigma}\}\neq\{\tau,\bar{\tau}\}} \mathcal{T}_{\{\sigma,\bar{\sigma}\}} \oplus \mathscr{E}_{\{\tau,\bar{\tau}\}} \tag{4.6}$$

where $\mathscr{E}_{\{\tau,\bar{\tau}\}} \subset \mathcal{T}_{\{\tau,\bar{\tau}\}}$ is the annihilator of $\mathrm{KS}(\mathcal{P}_\tau \otimes \mathcal{P}_{\bar{\tau}}[V \otimes V])$. If we choose the labelling so that $r_{\phi^{-1}\circ\tau} \leq r_\tau$ then over M^{ord} we have $\mathcal{P}_\tau \otimes \mathcal{P}_{\bar{\tau}}[V \otimes V] = \mathcal{P}_\tau[V] \otimes \mathcal{P}_{\bar{\tau}}$ and

$$\mathrm{rk}\ \mathscr{E}_{\{\tau,\bar{\tau}\}} = r_{\phi^{-1}\tau} \cdot (d - r_\tau).$$

Although our p-foliation is $\mathscr{F}_{\{\tau,\bar{\tau}\}}$ and not $\mathscr{E}_{\{\tau,\bar{\tau}\}}$ (the latter is an involutive sub-bundle but need not be p-closed!), for our purpose it will be enough to relate $\mathscr{E}_{\{\tau,\bar{\tau}\}}$ to the cascade structure.

Choose the notation $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}\bar{\mathfrak{P}}$ so that $\tau \in \mathscr{I}_{\mathfrak{P}}$ and $\bar{\tau} \in \mathscr{I}_{\bar{\mathfrak{P}}}$. As in §4.1.3 let $X(d, \mathfrak{f})$ (now for $\mathfrak{f} : \mathscr{I}_{\mathfrak{P}} \to [0, d]$) be the standard p-divisible group with $\mathcal{O}_K$-structure over k whose Dieudonné module is $M(d, \mathfrak{f})$. As before, $\mathfrak{f}(i) = r_\sigma$ if $i = \sigma \in \mathscr{I}_{\mathfrak{P}}$. Let r be the number of distinct slopes of $X(d, \mathfrak{f})$ (not to be confused with the CM type $\{r_\sigma\}$).

Fix $x \in M^{\mathrm{ord}}(k)$. We recall some notation related to the cascade structure at x. There is a canonical r-cascade structure

$$\mathscr{C} = \{\Gamma^{(a,b)}, G^{(a,b)} |\, 1 \leq a < b \leq r\}$$

on the deformation space

$$\mathrm{Spf}(\widehat{\mathcal{O}}_{M,x}) = \mathbb{D}_{\mathfrak{P}}.$$

The $\Gamma^{(a,b)}$ are formal schemes supported at x, and $\Gamma^{(1,r)} = \mathbb{D}_{\mathfrak{P}}$. They are equipped with (left and right) morphisms

$$\lambda : \Gamma^{(a,b)} \to \Gamma^{(a,b-1)}, \ \ \rho : \Gamma^{(a,b)} \to \Gamma^{(a+1,b)},$$

satisfying $\lambda \circ \rho = \rho \circ \lambda$ (where applicable). Each $\Gamma^{(a,b)}$ is endowed with a structure of a relative bi-extension of

$$\Gamma^{(a,b-1)} \times_{\Gamma^{(a+1,b-1)}} \Gamma^{(a+1,b)}$$

by a formal group that we denote by $G^{(a,b)}$ (in the category of formal schemes *over* $\Gamma^{(a+1,b-1)}$). See the following diagram.

$$\begin{tikzcd}
& \Gamma^{(a,b)} \arrow[ddl, "\lambda"'] \arrow[ddr, "\rho"] \arrow[d, dotted, "G^{(a,b)}"'] & \\
& \Gamma^{(a,b-1)} \underset{\Gamma^{(a+1,b-1)}}{\times} \Gamma^{(a+1,b)} \arrow[dl, dotted] \arrow[dr, dotted] & \\
\Gamma^{(a,b-1)} \arrow[dr, "\rho"] & & \Gamma^{(a+1,b)} \arrow[dl, "\lambda"'] \\
& \Gamma^{(a+1,b-1)} &
\end{tikzcd}$$

In fact, $G^{(a,b)} = \mathrm{Ext}(X^{(a)}, X^{(b)})$ where $X^{(\nu)}$ is the ν-th isoclinic component of $A_x^{\mathrm{univ}}[\mathfrak{P}^\infty] \simeq X(d, \mathfrak{f})$. This identification should be interpreted as an identity between fppf sheaves of $\mathcal{O}_K$-modules; each $X^{(\nu)}$, with its $\mathcal{O}_K$-structure, is rigid, so admits a unique canonical lifting $X_R^{(\nu)}$ to any local artinian ring R with residue field k, and

$$G^{(a,b)}(R) = \mathrm{Ext}_R(X_R^{(a)}, X_R^{(b)}).$$

Let

$$\mathcal{T}_x^{(a,b)} = \ker d(\lambda, \rho)$$

where $d(\lambda, \rho)$ is the differential of the map

$$(\lambda, \rho) : \Gamma^{(a,b)} \to \Gamma^{(a,b-1)} \times_{\Gamma^{(a+1,b-1)}} \Gamma^{(a+1,b)}.$$

This is the subspace of the tangent space of $\Gamma^{(a,b)}$ "in the direction" of $G^{(a,b)}$. Thus,

$$\mathcal{T}_x^{(a,b)} = \mathrm{Ext}_{k[\varepsilon]}(X_{k[\varepsilon]}^{(a)}, X_{k[\varepsilon]}^{(b)})$$

is the k-vector space of all the extensions, over the ring of dual numbers $k[\varepsilon]$, of the formal $\mathcal{O}_K$-module $X^{(a)}$ by the formal $\mathcal{O}_K$-module $X^{(b)}$.

Using these spaces we define subspaces

$$\mathcal{T}_x^{[a,b]} \subset \mathcal{T}_x$$

by a descending induction on $b - a$ for $1 \leq a < b \leq r$. First, $\mathcal{T}_x^{[1,r]} = \mathcal{T}_x^{(1,r)}$. Suppose $\mathcal{T}_x^{[a,b]}$ have been defined when $b - a > s$, let $b - a = s$ and consider $U = \mathcal{T}_x / \sum_{[a,b] \subsetneq [a',b']} \mathcal{T}_x^{[a',b']}$. Then $\mathcal{T}_x^{(a,b)}$ is a subspace of U and we let $\mathcal{T}_x^{[a,b]}$ be its preimage in $\mathcal{T}_x$, so that

$$\mathcal{T}_x^{(a,b)} = \mathcal{T}_x^{[a,b]} / \sum_{[a,b] \subsetneq [a',b']} \mathcal{T}_x^{[a',b']}.$$

For example, if $r = 3$ then $\mathrm{Spf}(\widehat{\mathcal{O}}_{M,x}) = \Gamma^{(1,3)}$ has the structure of a bi-extension of $\Gamma^{(1,2)} \times \Gamma^{(2,3)}$ by $G^{(1,3)}$, $\mathcal{T}_x^{[1,3]}$ is the tangent space to $G^{(1,3)}$, $\mathcal{T}_x^{[1,2]}/\mathcal{T}_x^{[1,3]} = \mathcal{T}_x^{(1,2)}$ is the tangent space to $\Gamma^{(1,2)}$ and, likewise, $\mathcal{T}_x^{[2,3]}/\mathcal{T}_x^{[1,3]} = \mathcal{T}_x^{(2,3)}$ is the tangent space to $\Gamma^{(2,3)}$ (all tangent spaces are at the origin). In general, we have defined a collection of subspaces of $\mathcal{T}_x$ indexed by closed intervals $[a,b]$ with $1 \le a < b \le r$, so that $\mathcal{T}_x^I \supset \mathcal{T}_x^J$ whenever $I \subset J$. The filtration of $\mathcal{T}_x$ by the $\mathcal{T}_x^{[a,b]}$ is not linearly ordered, but its graded pieces are the $\mathcal{T}_x^{(a,b)}$, the tangent spaces to the $G^{(a,b)}$.

4.3.4. *The relation between the cascade and $\mathscr{E}_{\{\tau,\bar\tau\}}$.* Depending on our τ, we define two integers $0 \le p_\tau \le q_\tau \le r$. Recall that the ν-th isoclinic group $X^{(\nu)}$ is of the form

$$X^{(\nu)} = X(d^\nu, \mathfrak{f}^{(\nu)}) = X(1, \mathfrak{g}^{(\nu)})^{d^\nu}$$

with $\mathfrak{g}^{(\nu)} : \mathscr{I}_{\mathfrak{P}} \to \{0,1\}$, $\mathfrak{f}^{(\nu)} = d^\nu \mathfrak{g}^{(\nu)}$, and $\mathfrak{g}^{(\nu+1)} \ge \mathfrak{g}^{(\nu)}$. If $\tau = i \in \mathscr{I}_{\mathfrak{P}}$ we let

$$p_\tau = \#\{\nu |\, \mathfrak{g}^{(\nu)}(i) = 0\}, \;\; q_\tau = p_{\phi^{-1}\circ\tau} = \#\{\nu |\, \mathfrak{g}^{(\nu)}(i-1) = 0\},$$

so that $\mathfrak{f}(i) = \sum_{\nu=p_\tau+1}^r d^\nu$ and similarly $\mathfrak{f}(i-1) = \sum_{\nu=q_\tau+1}^r d^\nu$.

Proposition 4.3.1. *The fiber of $\mathscr{E}_{\{\tau,\bar\tau\}}$ at x coincides with the subspace $\mathcal{T}_x^{[p_\tau, q_\tau+1]} \cap \mathcal{T}_{\{\tau,\bar\tau\},x}$. If $p_\tau = 0$ or $q_\tau = r$ this is 0.*

We can summarize the situation as follows. The Kodaira-Spencer isomorphism induces a "vertical" decomposition of the tangent bundle into the direct sum of the $\mathcal{T}_{\{\sigma,\bar\sigma\}}$. The foliation $\mathscr{F}_{\{\tau,\bar\tau\}}$ is "vertical" in the sense that it combines the subspace $\mathscr{E}_{\{\tau,\bar\tau\}} \subset \mathcal{T}_{\{\tau,\bar\tau\}}$ with the full $\mathcal{T}_{\{\sigma,\bar\sigma\}}$ for all $\{\sigma,\bar\sigma\} \ne \{\tau,\bar\tau\}$. The cascade structure, on the other hand, results from the slope decomposition of $X(d,\mathfrak{f})$, and induces a "horizontal" filtration by the $\mathcal{T}^{[a,b]}$ on the tangent bundle. The proposition describes the way these vertical decomposition and horizontal filtration interact.

Proof. We first compute the dimension of $\mathcal{T}_x^{[p_\tau,q_\tau+1]} \cap \mathcal{T}_{\{\tau,\bar\tau\},x}$. The graded pieces of $\mathcal{T}_x^{[p_\tau,q_\tau+1]}$ are the $\mathcal{T}_x^{(a,b)}$ for $1 \le a \le p_\tau$ and $q_\tau + 1 \le b \le r$. The dimension of $\mathcal{T}_x^{(a,b)}$ is the dimension of the formal group

$$G^{a,b} = \mathrm{Ext}(X^{(a)}, X^{(b)}) = \mathrm{Ext}(X(1,\mathfrak{g}^{(a)}), X(1,\mathfrak{g}^{(b)}))^{d^a d^b},$$

which is $d^a d^b \sum_{i\in\mathscr{I}_{\mathfrak{P}}} (\mathfrak{g}^{(b)}(i) - \mathfrak{g}^{(a)}(i))$. The dimension of $\mathcal{T}_x^{[p_\tau,q_\tau+1]}$ is therefore

$$\sum_{1\le a\le p_\tau} \sum_{q_\tau+1\le b\le r} d^a d^b \sum_{i\in\mathscr{I}_{\mathfrak{P}}} (\mathfrak{g}^{(b)}(i) - \mathfrak{g}^{(a)}(i)),$$

and the dimension of $\mathcal{T}_x^{[p_\tau,q_\tau+1]} \cap \mathcal{T}_{\{\tau,\bar\tau\},x}$ is the contribution of $i = \tau$ to this sum. But from the way p_τ and q_τ were defined it follows that for the particular index i corresponding to τ we have, for all a and b in the above range, $\mathfrak{g}^{(b)}(i) = 1$ and $\mathfrak{g}^{(a)}(i) = 0$. It follows that

$$\dim \mathcal{T}_x^{[p_\tau,q_\tau+1]} \cap \mathcal{T}_{\{\tau,\bar\tau\},x} = \sum_{1\le a\le p_\tau} \sum_{q_\tau+1\le b\le r} d^a d^b = r_{\phi^{-1}\circ\tau} \cdot (d - r_\tau) = \mathrm{rk}\, \mathscr{E}_{\{\tau,\bar\tau\}}.$$

To conclude the proof of the Proposition it is therefore enough to show the inclusion

$$W_x := \mathcal{T}_x^{[p_\tau,q_\tau+1]} \cap \mathcal{T}_{\{\tau,\bar\tau\},x} \subset \mathscr{E}_{\{\tau,\bar\tau\}},$$

i.e. that W_x annihilates $\mathrm{KS}(\mathcal{P}_\tau[V] \otimes \mathcal{P}_{\bar\tau})$. Let

$$i : \mathfrak{S} \hookrightarrow \mathrm{Spf}(\widehat{\mathcal{O}}_{M,x})$$

be the infinitesimal neighborhood of x in the direction of W_x. More precisely,

$$\mathfrak{S} = \mathrm{Spf}((\mathcal{O}_{M,x}/\mathfrak{m}_x^2)/((\mathfrak{m}_x/\mathfrak{m}_x^2)[W_x]))$$

where $(\mathfrak{m}_x/\mathfrak{m}_x^2)[W_x]$ is the subspace of the cotangent space at x annihilated by $W_x \subset \mathcal{T}_x$. To conform with [G-dS1] we introduce the following notation. Let $\mathcal{A} = i^*A^{\mathrm{univ}}$,

$$\mathcal{H} = H^1_{dR}(\mathcal{A}/\mathfrak{S}) = i^*H^1_{dR}(A^{\mathrm{univ}}/\widehat{\mathcal{O}}_{M,x}),\ \ \mathcal{P} = i^*\mathcal{P}_\tau,\ \ \mathcal{P}_0 = \mathcal{P}[V],\ \ \mathcal{Q} = i^*\mathcal{P}_{\bar{\tau}}.$$

Over M^{ord}, the p-divisible group $A^{\mathrm{univ}}[\mathfrak{P}^\infty]$ has an $\mathcal{O}_K$-stable slope filtration by p-divisible groups

$$\mathscr{X}^{(r)} \subset \mathscr{X}^{(r-1,r)} \subset \cdots \subset \mathscr{X}^{(1,r)} = A^{\mathrm{univ}}[\mathfrak{P}^\infty],$$

characterized by the fact that at each geometric point $x \in M^{\mathrm{ord}}$

$$\mathscr{X}_x^{(a,r)} = X^{(a)} \times \cdots \times X^{(r)},$$

where $X^{(\nu)}$ is the ν-th isoclinic factor of $A_x^{\mathrm{univ}}[\mathfrak{P}^\infty]$ (the slopes increasing with ν).

Let

$$0 \subset \mathrm{Fil}^2 = i^*(\mathscr{X}^{(q_\tau+1,r)}) \subset \mathrm{Fil}^1 = i^*(\mathscr{X}^{(p_\tau+1,r)}) \subset \mathrm{Fil}^0 = \mathcal{A}[\mathfrak{P}^\infty].$$

It follows from the construction of the cascade $\mathscr{C} = \{\Gamma^{(a,b)}\}$ that while the full $\mathcal{A}[\mathfrak{P}^\infty]$ does deform over $\mathfrak{S}$, its subquotient p-divisible groups Fil^1 and $\mathrm{Fil}^0/\mathrm{Fil}^2$ are constant there: they are obtained (with their $\mathcal{O}_K$-structure) by base change from the fiber at x. Indeed, as the only non-zero graded pieces of $\mathcal{T}_x^{[p_\tau,q_\tau+1]}$ are the $\mathcal{T}_x^{(a,b)}$ for $a \le p_\tau$ and $q_\tau + 1 \le b$, only extensions of $X^{(a)}$ by $X^{(b)}$ for a an b in these ranges contribute to the deformation of $\mathcal{A}[\mathfrak{P}^\infty]$ over $\mathfrak{S}$. But such an $X^{(a)}$ does not participate in Fil^1, and $X^{(b)}$ does not participate in $\mathrm{Fil}^0/\mathrm{Fil}^2$.

It also follows from our choice of p_τ and q_τ that $\mathcal{P}_0$ pairs trivially with the tangent space to Fil^2, so can be considered a subspace of the cotangent space of the p-divisible group $G = \mathrm{Fil}^0/\mathrm{Fil}^2$. Let $D(G) = \mathbb{D}(G)_{\mathfrak{S}}$ be the evaluation of the Dieudonné crystal associated to G on $\mathfrak{S}$. This is an $\mathcal{O}_{\mathfrak{S}}$-module with $\mathcal{O}_K$-action, and it is equipped with an integrable connection ∇. We have $D(G) \subset D(\mathcal{A}[p^\infty]) = H^1_{dR}(\mathcal{A}/\mathfrak{S}) = \mathcal{H}$ and

$$\nabla : D(G) \to D(G) \otimes \Omega^1_{\mathfrak{S}}$$

is induced by the Gauss-Manin connection on $\mathcal{H}$. Therefore, the diagram

$$\begin{array}{ccc} \mathcal{P} = \omega_{\mathcal{A}/\mathfrak{S}}[\tau] & \hookrightarrow & \mathcal{H}[\tau] \\ \Big\downarrow {\scriptstyle \mathrm{KS}_{\mathcal{A}/\mathfrak{S}}} & & \Big\downarrow {\scriptstyle \nabla} \\ \mathcal{Q}^\vee \otimes \Omega^1_{\mathfrak{S}} \xrightarrow[\lambda]{\sim} \omega^\vee_{\mathcal{A}^t/\mathfrak{S}}[\tau] \otimes \Omega^1_{\mathfrak{S}} & \twoheadleftarrow & \mathcal{H}[\tau] \otimes \Omega^1_{\mathfrak{S}} \end{array}$$

used to compute $\mathrm{KS}_{\mathcal{A}/\mathfrak{S}}$, can be replaced, when we restrict to $\mathcal{P}_0 = \mathcal{P}[V]$, by the diagram

$$\begin{array}{ccc} \mathcal{P}_0 & \hookrightarrow & D(G)[\tau] \\ \Big\downarrow {\scriptstyle \mathrm{KS}_{\mathcal{A}/\mathfrak{S}}} & & \Big\downarrow {\scriptstyle \nabla} \\ \mathcal{Q}^\vee \otimes \Omega^1_{\mathfrak{S}} & \longleftarrow & D(G)[\tau] \otimes \Omega^1_{\mathfrak{S}}. \end{array}$$

However, by the constancy of $G = \mathrm{Fil}^0/\mathrm{Fil}^2$ over $\mathfrak{S}$, the left arrow vanishes. We must therefore have $\mathrm{KS}_{\mathcal{A}/\mathfrak{S}}(\mathcal{P}_0 \otimes \mathcal{Q}) = 0$. By the functoriality of Kodaira-Spencer homomorphisms with respect to base change we conclude that W_x, the tangent space to $\mathfrak{S}$, annihilates $\mathrm{KS}_{A^{\mathrm{univ}}/M}(\mathcal{P}_\tau[V] \otimes \mathcal{P}_{\bar\tau}) \subset \Omega^1_{M/k}$, i.e. is contained in $\mathscr{E}_{\{\tau,\bar\tau\}}$, and this completes the proof. $\square$

4.4. **The extension of the foliations to inner Ekedahl-Oort strata.**

4.4.1. *The problem.* This section is largely combinatorial. The variety M has a stratification by locally closed subsets M_w, of which M^{ord} is the largest, called the Ekedahl-Oort (EO) strata of M. They are labelled by certain Weyl group cosets w (or by their distinguished representatives of shortest length), recalled below. They are characterized by the fact that the isomorphism class of $A_x^{\mathrm{univ}}[p]$, with its $\mathcal{O}_K$-structure and polarization, is the same for all geometric points x in a given stratum, and the strata are maximal with respect to this property. See [Mo2, Wed2].

Consider a (not necessarily closed) point $x \in M$. Let $k(x)$ be its residue field. By (4.3) and the fact that

$$\mathcal{P}_{\tau,x}[V] \otimes \mathcal{P}_{\bar\tau,x} \cap \mathcal{P}_{\tau,x} \otimes \mathcal{P}_{\bar\tau,x}[V] = \mathcal{P}_{\tau,x}[V] \otimes \mathcal{P}_{\bar\tau,x}[V],$$

the dimension of $\ker(V \otimes V : (\mathcal{P}_\tau \otimes \mathcal{P}_{\bar\tau})_x \to (\mathcal{P}^{(p)}_{\phi^{-1}\circ\tau} \otimes \mathcal{P}^{(p)}_{\phi^{-1}\circ\bar\tau})_x)$ is given by

$$r_V\{\tau,\bar\tau\}(x) = \dim \mathcal{P}_{\tau,x}[V] \cdot r_{\bar\tau} + r_\tau \cdot \dim \mathcal{P}_{\bar\tau,x}[V] - \dim \mathcal{P}_{\tau,x}[V] \cdot \dim \mathcal{P}_{\bar\tau,x}[V]. \tag{4.7}$$

This quantity depends only on $A_x^{\mathrm{univ}}[p]$, and is therefore *constant along each EO strata.* At $x \in M^{\mathrm{ord}}$ it was expressed in terms of the CM type by the formula

$$r_V^{\mathrm{ord}}\{\tau,\bar\tau\} = \max\{0, r_\tau - r_{\phi^{-1}\circ\tau}\} \cdot (d - r_\tau) + r_\tau \cdot \max\{0, r_{\phi^{-1}\circ\tau} - r_\tau\}.$$

(We did it at geometric points, but the same works at every schematic point of M^{ord}, not necessarily closed.) Since $r_V\{\tau,\bar\tau\}(x)$ can only increase under specialization, $r_V^{\mathrm{ord}}\{\tau,\bar\tau\}$ is the minimal value attained by it, and

$$M_\Sigma = \{x \in M \,|\, r_V\{\tau,\bar\tau\}(x) = r_V^{\mathrm{ord}}\{\tau,\bar\tau\} \ \forall\{\tau,\bar\tau\} \in \Sigma\} \tag{4.8}$$

is a Zariski open set, which is a union of EO strata. Clearly,

$$M_{\Sigma_1 \cup \Sigma_2} = M_{\Sigma_1} \cap M_{\Sigma_2}.$$

In our earlier work [G-dS1], where there was only one Σ to consider, this set was denoted by $M_\sharp$.

The proof of Lemma 4.2.2 shows that over M_Σ the sheaf

$$\mathscr{F}_\Sigma = \mathrm{KS}\left(\sum_{\{\tau,\bar\tau\}\in\Sigma} (\mathcal{P}_\tau \otimes \mathcal{P}_{\bar\tau})[V \otimes V]\right)^{\perp} \tag{4.9}$$

remains a vector sub-bundle of $\mathcal{T}$ of corank $r_V(\Sigma) = \sum_{\{\tau,\bar\tau\}\in\Sigma} r_V^{\mathrm{ord}}\{\tau,\bar\tau\}$. The properties of being involutive and p-closed extend by continuity from the open dense μ-ordinary stratum. We conclude:

Theorem 4.4.1. *The vector bundle $\mathscr{F}_\Sigma$ extends as a* smooth p-foliation to the Zariski open set M_Σ.

Our task is therefore to calculate $r_V\{\tau,\bar\tau\}(x)$ at $x \in M_w$ for the different EO strata M_w, and see for which of them it equals $r_V^{\mathrm{ord}}\{\tau,\bar\tau\}$. This will give us an explicit description of M_Σ.

4.4.2. *Previous results.* When $L=\mathbb{Q}$ and K is quadratic imaginary, and when $p\mathcal{O}_K=\mathfrak{P}$ is inert (the split case being trivial for quadratic imaginary K), we showed in [G-dS1] that there exists a smallest EO stratum M^{fol} in $M_\sharp$. Any other stratum lies in $M_\sharp$ if and only if it contains M^{fol} in its closure. In this case $\Sigma=\{\tau,\bar\tau\}$, and writing $(r_\tau,r_{\bar\tau})=(n,m)$ we may assume, without loss of generality, that $n\geq m$. We find that $r_V(\Sigma)=(n-m)m$, hence

$$\mathrm{rk}\,\mathscr{F}_\Sigma=m^2.$$

The dimension of M^{fol} is also m^2, while the dimension of M itself is nm. For example, for Shimura varieties attached to the group $U(n,1)$, the foliation extends everywhere except to the lowest, 0-dimensional, EO stratum, consisting of the so-called superspecial points.

In case K is quadratic imaginary, the labelling of the EO strata of M is by (n,m)-*shuffles.* We review it to motivate the type of combinatorics that will show up in the general case. A permutation w of $\{1,\dots,n+m\}$ is called an (n,m)-shuffle if

$$w^{-1}(1)<\cdots<w^{-1}(n),\;\; w^{-1}(n+1)<\cdots<w^{-1}(n+m),$$

i.e. w interlaces the intervals $[1,n]$ and $[n+1,n+m]$ but keeps the order within each interval. If w is an (n,m)-shuffle and M_w is the corresponding EO stratum, then its dimension is equal to the length of w

$$\dim(M_w)=\ell(w)=\sum_{i=1}^{n}(w^{-1}(i)-i).$$

The unique w for which this gets the value nm is

$$w^{\mathrm{ord}}=\begin{pmatrix}1&\cdots&m&m+1&\cdots&n+m\\ n+1&\cdots&n+m&1&\cdots&n\end{pmatrix} \tag{4.10}$$

and the corresponding M_w is M^{ord}. The formula for $r_V\{\tau,\bar\tau\}(x)$ at $x\in M_w(k)$ reads

$$r_V\{\tau,\bar\tau\}(x)=a(w)\cdot m=|\{1\leq i\leq n|\,1\leq w^{-1}(i)\leq n\}|\cdot m$$

(compare [G-dS1] (2.3)). For example, for w^{ord} this is $(n-m)m$. It is readily seen that there is a unique (n,m)-shuffle w^{fol} for which $a(w)$ is still $n-m$, but for which $\dim(M_w)=m^2$ is the smallest possible. This is the permutation

$$w^{\mathrm{fol}}=\begin{pmatrix}1&\cdots&n-m&n-m+1&\cdots&n&n+1&\cdots&n+m\\ 1&\cdots&n-m&n+1&\cdots&n+m&n-m+1&\cdots&n\end{pmatrix},$$

whose corresponding EO stratum is M^{fol}.

4.4.3. *Weyl group cosets and EO strata.* We return to a general CM field K. Denote by $\Pi_{e,d-e}$ the set of $(e,d-e)$-shuffles in the symmetric group $\mathfrak{S}_d$. They serve as representatives (of minimal length) for

$$\mathfrak{S}_e\times\mathfrak{S}_{d-e}\setminus\mathfrak{S}_d.$$

For an $(e,d-e)$-shuffle π let

$$\check\pi=w_0\circ\pi\circ w_0$$

where $w_0(\nu)=d+1-\nu$ is the element of maximal length in $\mathfrak{S}_d$. This $\check\pi$ is a $(d-e,e)$-shuffle, and $\pi\mapsto\check\pi$ is a bijection between $\Pi_{e,d-e}$ and $\Pi_{d-e,e}$. Explicitly,

$$\check\pi^{-1}(d+1-\nu)=d+1-\pi^{-1}(\nu).$$

Let $w = (w_\tau)_{\tau\in\mathscr{I}}$ where $w_\tau \in \Pi_{r_\tau, d-r_\tau}$ and $w_{\bar\tau} = \check{w}_\tau$. Note that w_τ and $w_{\bar\tau}$, being conjugate by w_0, have the same length.

Let k be, as usual, an algebraically closed field containing κ. Consider the following Dieudonné module with $\mathcal{O}_K$-structure N_w attached to w:

- $N_w = \bigoplus_{i\in\mathscr{I}} \bigoplus_{j=1}^d ke_{i,j}$, $\mathcal{O}_K$ acting on $\bigoplus_{j=1}^d ke_{i,j}$ via $i : \mathcal{O}_K \to k$.
- $F(e_{i,j}) = \begin{cases} 0 & \text{if } w_i(j) \le \mathfrak{f}(i) \\ e_{i+1,m} & \text{if } w_i(j) = \mathfrak{f}(i) + m. \end{cases}$
- $V(e_{i+1,j}) = \begin{cases} 0 & \text{if } j \le d - \mathfrak{f}(i) \\ e_{i,n} & \text{if } j = d - \mathfrak{f}(i) + w_i(n). \end{cases}$

This N_w is endowed with a pairing, setting its τ and $\bar\tau$-components in duality, but we suppress it from the notation, as it is irrelevant to the computation that we have to make.

Proposition 4.4.2. *(*[Mo2]*, Theorem 6.7) The EO strata M_w of M are in one-to-one correspondence with the w's as above. If $x \in M_w(k)$ is a geometric point, the Dieudonné module of $A_x^{\mathrm{univ}}[p]$ is isomorphic, with its $\mathcal{O}_K$-structure, to N_w. The dimension of M_w is given by*

$$\dim(M_w) = \ell(w) := \sum_{\{\tau,\bar\tau\}\in\mathscr{I}^+} \ell(w_\tau). \tag{4.11}$$

As an example, the μ-ordinary stratum M^{ord} corresponds to $w = w^{\mathrm{ord}} = (w_i^{\mathrm{ord}})_{i\in\mathscr{I}}$ where w_i^{ord} is given by (4.10) with $(n, m) = (\mathfrak{f}(i), d - \mathfrak{f}(i))$.

We now consider $r_V\{\tau, \bar\tau\}(x)$, as in (4.7), for $x \in M_w(k)$. As usual we write $\tau = i$ and $r_\tau = \mathfrak{f}(i)$. We have

$$\begin{aligned} \dim \mathcal{P}_{\tau,x}[V] &= |\{j|\, j \le r_{\phi^{-1}\circ\bar\tau},\, w_\tau(j) \le r_\tau\}| \\ &= |\{j|\, j \le d - \mathfrak{f}(i-1),\, w_i(j) \le \mathfrak{f}(i)\}| \\ &\ge \max\{0, \mathfrak{f}(i) - \mathfrak{f}(i-1)\}, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} \dim \mathcal{P}_{\bar\tau,x}[V] &= |\{j|\, j \le r_{\phi^{-1}\circ\tau},\, w_{\bar\tau}(j) \le r_{\bar\tau}\}| \\ &= |\{\ell|\, \ell \le r_{\bar\tau},\, w_{\bar\tau}^{-1}(\ell) \le r_{\phi^{-1}\circ\tau}\}| \\ &= |\{\ell|\, \ell \le r_{\bar\tau},\, \check{w}_\tau^{-1}(\ell) \le r_{\phi^{-1}\circ\tau}\}| \\ &= |\{m|\, d+1-m \le r_{\bar\tau},\, d+1-w_\tau^{-1}(m) \le r_{\phi^{-1}\circ\tau}\}| \\ &= |\{j|\, r_{\phi^{-1}\circ\bar\tau} + 1 \le j,\, r_\tau + 1 \le w_\tau(j)\}| \\ &= |\{j|\, d - \mathfrak{f}(i-1) + 1 \le j,\, \mathfrak{f}(i) + 1 \le w_i(j)\}| \\ &\ge \max\{0, \mathfrak{f}(i-1) - \mathfrak{f}(i)\}. \end{aligned} \tag{4.13}$$

Lemma 4.4.3. *We have $r_V\{\tau, \bar\tau\}(x) = r_V^{\mathrm{ord}}\{\tau, \bar\tau\}$ precisely when the following conditions are satisfied:*

- If $\mathfrak{f}(i-1) \le \mathfrak{f}(i)$,

$$|\{j|\, j \le d - \mathfrak{f}(i-1),\, w_i(j) \le \mathfrak{f}(i)\}| = \mathfrak{f}(i) - \mathfrak{f}(i-1), \tag{4.14}$$

- If $\mathfrak{f}(i) \le \mathfrak{f}(i-1)$,

$$|\{j|\, d - \mathfrak{f}(i-1) + 1 \le j,\, \mathfrak{f}(i) + 1 \le w_i(j)\}| = \mathfrak{f}(i-1) - \mathfrak{f}(i). \tag{4.15}$$

Proof. We begin by noting that when $\mathfrak{f}(i-1)=\mathfrak{f}(i)$ the two conditions agree with each other, since each of them is equivalent, in this case, to $w_i = w_i^{\mathrm{ord}}$, so the Lemma is consistent.

Let us dispose first of the cases where $\mathfrak{f}(i) \in \{0, d\}$. In these extreme cases $\mathcal{P}_\tau$ or $\mathcal{P}_{\bar\tau}$ are zero, so

$$r_V\{\tau,\bar\tau\}(x) = r_V^{\mathrm{ord}}\{\tau,\bar\tau\} = 0$$

for all x. The conditions of the Lemma also hold then, trivially, everywhere. We therefore assume from now on that $0<\mathfrak{f}(i)<d$.

Suppose $\mathfrak{f}(i-1)\le \mathfrak{f}(i)$. If (4.14) holds then

$$\dim \mathcal{P}_{\tau,x}[V]\otimes \mathcal{P}_{\bar\tau,x} = (\mathfrak{f}(i)-\mathfrak{f}(i-1))\cdot(d-\mathfrak{f}(i)) = r_V^{\mathrm{ord}}\{\tau,\bar\tau\}.$$

We claim that $|\{j|\, d-\mathfrak{f}(i-1)+1 \le j,\, \mathfrak{f}(i)+1\le w_i(j)\}| = 0$, and therefore $\mathcal{P}_{\bar\tau,x}[V]=0$ and $r_V\{\tau,\bar\tau\}(x) = r_V^{\mathrm{ord}}\{\tau,\bar\tau\}$ as desired. Suppose, to the contrary, that for some j we had $d-\mathfrak{f}(i-1)<j$ and $\mathfrak{f}(i)<w_i(j)$. Then there would be *fewer* than $\mathfrak{f}(i-1)$ values of j such that $d-\mathfrak{f}(i-1)<j$ and $w_i(j)\le \mathfrak{f}(i)$, hence *more than* $\mathfrak{f}(i)-\mathfrak{f}(i-1)$ values of j such that $j\le d-\mathfrak{f}(i-1)$ and $w_i(j)\le\mathfrak{f}(i)$. This would violate condition (4.14).

Conversely, still under $\mathfrak{f}(i-1)\le\mathfrak{f}(i)$, assume that $r_V\{\tau,\bar\tau\}(x) = r_V^{\mathrm{ord}}\{\tau,\bar\tau\} = (\mathfrak{f}(i)-\mathfrak{f}(i-1))\cdot(d-\mathfrak{f}(i))$. Since we assumed that $d-\mathfrak{f}(i)\neq 0$ we can not have $\dim\mathcal{P}_{\tau,x}[V] > \mathfrak{f}(i)-\mathfrak{f}(i-1)$ and condition (4.14) must hold.

The arguments when $\mathfrak{f}(i)\le\mathfrak{f}(i-1)$ are analogous. □

It is easy to see that if $\mathfrak{f}(i-1)\le\mathfrak{f}(i)$ the $(\mathfrak{f}(i), d-\mathfrak{f}(i))$-shuffle w_i^{fol} given by

$$\begin{pmatrix} 1 & \cdots & \mathfrak{f}(i)-\mathfrak{f}(i-1) & \mathfrak{f}(i)-\mathfrak{f}(i-1)+1 & \cdots & d-\mathfrak{f}(i-1) & d-\mathfrak{f}(i-1)+1 & \cdots & d \\ 1 & \cdots & \mathfrak{f}(i)-\mathfrak{f}(i-1) & \mathfrak{f}(i)+1 & \cdots & d & \mathfrak{f}(i)-\mathfrak{f}(i-1)+1 & \cdots & \mathfrak{f}(i) \end{pmatrix}$$

satisfies (4.14), and this is the $(\mathfrak{f}(i), d-\mathfrak{f}(i))$-shuffle of smallest length satisfying it. Its length is then

$$\ell(w_i^{\mathrm{fol}}) = (d-\mathfrak{f}(i))\mathfrak{f}(i-1).$$

Similarly if $\mathfrak{f}(i)\le\mathfrak{f}(i-1)$ letting $\mathfrak{g}(i)=d-\mathfrak{f}(i)$ the same holds with w_i^{fol} given by

$$\begin{pmatrix} 1 & \cdots & \mathfrak{g}(i-1) & \mathfrak{g}(i-1)+1 & \cdots & \mathfrak{g}(i-1)+\mathfrak{f}(i) & \mathfrak{g}(i-1)+\mathfrak{f}(i)+1 & \cdots & d \\ \mathfrak{f}(i)+1 & \cdots & \mathfrak{g}(i-1)+\mathfrak{f}(i) & 1 & \cdots & \mathfrak{f}(i) & \mathfrak{g}(i-1)+\mathfrak{f}(i)+1 & \cdots & d \end{pmatrix}.$$

In this case the length is

$$\ell(w_i^{\mathrm{fol}}) = \mathfrak{f}(i)(d-\mathfrak{f}(i-1)).$$

We arrive at the following result.

Theorem 4.4.4. *The EO stratum $M_w\subset M_\Sigma$ if and only if for every $\{\tau,\bar\tau\}\in\Sigma$, writing $\tau=i$, $\phi^{-1}\circ\tau = i-1$ as usual,* (4.14) *or* (4.15) *hold.*

There exists a unique EO stratum $M_w\subset M_\Sigma$ of smallest dimension. It is given by the following recipe: $w_i = w_i^{\mathrm{fol}}$ if $\{\tau,\bar\tau\}\in\Sigma$ and $w_i = id.$ otherwise. Denote this M_w by M_Σ^{fol}. Its dimension is given by

$$\dim M_\Sigma^{\mathrm{fol}} = \sum_{\{\tau,\bar\tau\}\in\Sigma} \min(r_\tau, r_{\phi^{-1}\circ\tau})\cdot\min(r_{\bar\tau}, r_{\phi^{-1}\circ\bar\tau}).$$

Any other EO stratum M_w lies in M_Σ if and only if M_Σ^{fol} lies in its closure.

Proof. By the computations above, the unique M_w of smallest dimension which is still contained in M_Σ, i.e. for which $r_V\{\tau,\bar\tau\}(x) = r_V^{\rm ord}\{\tau,\bar\tau\}$ for all $\{\tau,\bar\tau\}\in\Sigma$, is obtained when $w_i = w_i^{\rm fol}$ whenever $\{\tau,\bar\tau\}\in\Sigma$ (this condition is symmetric in τ and $\bar\tau$), and $w_i = id.$ otherwise. Its dimension follows from (4.11), and the computation of the lengths of the $w_i^{\rm fol}$.

If $M_\Sigma^{\rm fol}$ lies in the closure of M_w then for $x\in M_\Sigma^{\rm fol}$ and $y\in M_w$ and $\{\tau,\bar\tau\}\in\Sigma$ we have

$$r_V^{\rm ord}\{\tau,\bar\tau\}\le r_V\{\tau,\bar\tau\}(y)\le r_V\{\tau,\bar\tau\}(x) = r_V^{\rm ord}\{\tau,\bar\tau\}$$

so equality holds and $M_w\subset M_\Sigma$. Conversely, suppose that $M_w\subset M_\Sigma$. Assume, without loss of generality, that $r_{\phi^{-1}\circ\tau}\le r_\tau$. Then condition (4.14) holds, so writing $\tau = i$, we must have that w_i is the permutation

$$\begin{pmatrix} 1 & \cdots & \mathfrak{f}(i)-\mathfrak{f}(i-1) & \mathfrak{f}(i)-\mathfrak{f}(i-1)+1 & \cdots & d-\mathfrak{f}(i-1) & d-\mathfrak{f}(i-1)+1 & \cdots & d \\ * & \cdots & * & * & \cdots & * & \mathfrak{f}(i)-\mathfrak{f}(i-1)+1 & \cdots & \mathfrak{f}(i) \end{pmatrix}.$$

Since the blocks $[1,\mathfrak{f}(i)-\mathfrak{f}(i-1)]$ and $[\mathfrak{f}(i)+1,d]$ must appear in the bottom row in increasing order (but interlaced), it is easy to check that the permutation $w_i^{\rm fol}$ is smaller than or equal to w_i in the Bruhat order on the Weyl group of GL_d. This is enough (although in general, not equivalent) for $M_\Sigma^{\rm fol}$ to lie in the closure of M_w. (For the closure relation between EO strata, see [V-W].) □

4.4.4. *Integral varieties.* The results of §2.3 imply that integral varieties of the foliation $\mathscr{F}_\Sigma$ abound. Nonetheless, it's interesting to identify specific examples. By Lemma 4.2.2, when $\Sigma = \mathscr{I}^+$, $\dim M_\Sigma^{\rm fol} = \mathrm{rk}\,\mathscr{F}_\Sigma$. We expect $M_\Sigma^{\rm fol}$ to be an integral variety of $\mathscr{F}_\Sigma$ in this case. This has been proved when K is quadratic imaginary in [G-dS1], Theorem 25, and would be analogous to Theorem 3.5.1 above. The proof of Theorem 25 in [G-dS1] was not conceptual, and involved tedious computations with Dieudonné modules.

4.5. **The moduli problem M^Σ and the extension of the foliation to it.**

4.5.1. *The moduli scheme M^Σ.* As we have seen in the previous section, $\mathscr{F}_\Sigma$, defined by (4.9), is a p-foliation on all of M, but is smooth only on the Zariski open M_Σ. Our goal is to define a "successive blow-up" $\beta: M^\Sigma\to M$, which is an isomorphism over M_Σ, and a natural extension of $\mathscr{F}_\Sigma$ to a *smooth* p-foliation on M^Σ. See [G-dS1] §4.1 for K quadratic imaginary, where M^Σ was denoted $M^\sharp$.

Fix $\{\tau,\bar\tau\}\in\Sigma$, and assume that $r_{\phi^{-1}\circ\tau}\le r_\tau$ (otherwise switch notation between τ and $\bar\tau$). Consider the moduli problem $M^{\tau,\bar\tau}$ on κ-algebras R given by

$$M^{\tau,\bar\tau}(R) = \{(\underline{A},\mathcal{N})|\,\underline{A}\in M(R),\,\mathcal{N}\subset\mathcal{P}_\tau[V]\,\text{a subbundle of rank}\,r_\tau - r_{\phi^{-1}\circ\tau}\}/\simeq.$$

The forgetful map $\beta: M^{\tau,\bar\tau}\to M$ is bijective above M_Σ, since

$$\mathrm{rk}(\mathcal{P}_\tau[V]) = r_\tau - r_{\phi^{-1}\circ\tau}$$

holds there.

Let $(n,m) = (r_\tau, r_{\phi^{-1}\circ\tau})$ and consider the relative Grassmannian $Gr(n-m,\mathcal{P}_\tau)$ over M, classifying sub-bundles $\mathcal{N}$ of rank $n-m$ in the rank n bundle $\mathcal{P}_\tau$. This is a smooth scheme over M, of relative dimension $(n-m)m$. As the condition $V(\mathcal{N}) = 0$ is closed, the moduli problem $M^{\tau,\bar\tau}$ is representable by a closed subscheme of $Gr(n-m,\mathcal{P}_\tau)$. The fiber $M_x^{\tau,\bar\tau} = \beta^{-1}(x)$ is the Grassmannian of $(n-m)$-dimensional subspaces in $\mathcal{P}_{\tau,x}[V]$.

Suppose $x \in M_w$ where $w = (w_\sigma)_{\sigma\in\mathscr{I}}$, $w_\sigma \in \Pi_{r_\sigma, d-r_\sigma}$ and $w_{\bar\sigma} = \check{w}_\sigma$. As we have computed in (4.12)

$$\dim \mathcal{P}_{\tau,x}[V] = a_\tau(w) := |\{j | j \le d - r_{\phi^{-1}\circ\tau}, w_\tau(j) \le r_\tau\}| \ge n - m,$$

and consequently

$$\dim M_x^{\tau,\bar\tau} = (n-m)(a_\tau(w) - n + m).$$

Denote by $M_w^{\tau,\bar\tau} = \beta^{-1}(M_w)$. We have shown that it is smooth of relative dimension $(n-m)(a_\tau(w)-n+m)$ over M_w. When the EO strata undergo specialization, this dimension jumps.

The main result concerning $M^{\tau,\bar\tau}$ is the following.

Theorem 4.5.1. *The scheme $M^{\tau,\bar\tau}$ is a non-singular variety, and β induces a bijection between its irreducible (= connected) components and those of M. In particular, $M^{\tau,\bar\tau,\mathrm{ord}}$ is dense in $M^{\tau,\bar\tau}$.*

Proof. This is, mutatis mutandis, the proof of Theorem 15 in [G-dS1] §4.1.3, and we refer to our earlier paper for details. □

Corollary 4.5.2. *The moduli problem which is the fiber product of the $M^{\tau,\bar\tau}$ over M, for all $\{\tau,\bar\tau\} \in \Sigma$, is represented by a smooth κ-variety M^Σ. The map $\beta : M^\Sigma \to M$ induces a bijection on irreducible components, and is an isomorphism over M_Σ. In particular, $M^{\Sigma,\mathrm{ord}}$ is dense in M^Σ. Over an EO stratum M_w that is not contained in M_Σ the map β is no longer an isomorphism, but it is smooth of relative dimension*

$$\sum_{\{\tau,\bar\tau\}\in\Sigma} (r_\tau - r_{\phi^{-1}\circ\tau})(a_\tau(w) - r_\tau + r_{\phi^{-1}\circ\tau}).$$

Here for each pair $\{\tau,\bar\tau\}$ we choose τ so that $r_\tau \ge r_{\phi^{-1}\circ\tau}$, and

$$a_\tau(w) = |\{j | j \le d - r_{\phi^{-1}\circ\tau}, w_\tau(j) \le r_\tau\}|.$$

We can also draw the following corollary.

Corollary 4.5.3. *The open set $M_\Sigma \subset M$ is the largest open set in M to which the foliation $\mathscr{F}_\Sigma$ extends as a smooth foliation.*

Proof. We have already noted that $\mathscr{F}_\Sigma$ is a saturated foliation everywhere on M. Outside M_Σ the dimension of its fibers is strictly larger than their dimension over M^{ord}, so $\mathscr{F}_\Sigma$ can not be a vector sub-bundle on any open set larger than M_Σ. □

Remark 4.5.4. Since $\beta : M^\Sigma \to M$ is a birational projective morphism between non-singular varieties, it follows from the general theory ([Stacks] 29.43 and [H] Theorem 7.17 and exercise 7.11(c)) that β is a blow up at an ideal sheaf supported on $M - M_\Sigma$.

4.5.2. *Extending the foliation to M^Σ.* Above M_Σ, the map β is an isomorphism

$$M^\Sigma \supset \beta^{-1}(M_\Sigma) \simeq M_\Sigma,$$

so the foliation $\mathscr{F}_\Sigma$ induces a foliation on $\beta^{-1}(M_\Sigma)$. We now explain how to extend it to a smooth foliation on all of M^Σ.

Let k be, as usual, an algebraically closed field containing κ, and $y \in M^\Sigma(k)$ a geometric point, with image $x = \beta(y) \in M(k)$. The point y "is" a pair $(\underline{A}_x^{\mathrm{univ}}, \mathcal{N}_y)$ where $\mathcal{N}_y = (\mathcal{N}_{y,\tau})$. Here

- τ range over representatives of the pairs $\{\tau, \bar{\tau}\} \in \Sigma$, chosen in such a way that $r_{\phi^{-1}\circ\tau} \leq r_\tau$,
- $\mathcal{N}_{y,\tau} \subset \mathcal{P}_{x,\tau}[V]$ is an $r_\tau - r_{\phi^{-1}\circ\tau}$-dimensional subspace.

Let $\mathcal{H}_x = H^1_{dR}(A_x^{\mathrm{univ}}/k)$ and $\underline{\omega}_x = H^0(A_x^{\mathrm{univ}}, \Omega^1)$. The polarization λ_x on A_x^{univ} induces a perfect pairing

$$\{,\}_\lambda : \underline{\omega}_x \times \mathcal{H}_x/\underline{\omega}_x \to k$$

satisfying $\{\iota(a)u, v\}_\lambda = \{u, \iota(\bar{a})v\}_\lambda$ for $a \in \mathcal{O}_K$. Adapting the proof of Theorem 15 in [G-dS1] §4.1.3 to our situation, we see that the tangent space $\mathcal{T}M_y^\Sigma$ at the point y can be described as the set of pairs (φ, ψ) where

- $\varphi \in \mathrm{Hom}_{\mathcal{O}_K}(\underline{\omega}_x, \mathcal{H}_x/\underline{\omega}_x)^{sym}$ is an $\mathcal{O}_K$-linear homomorphism from $\underline{\omega}_x$ to $\mathcal{H}_x/\underline{\omega}_x$ which is symmetric with respect to $\{,\}_\lambda$, i.e. satisfies $\{u, \varphi(v)\}_\lambda = \{v, \varphi(u)\}_\lambda$ for $u, v \in \underline{\omega}_x$. By Kodaira-Spencer, such a φ represents a tangent vector to M at x, and the projection $(\varphi, \psi) \mapsto \varphi$ corresponds to the map $d\beta : \mathcal{T}M_y^\Sigma \to \mathcal{T}M_x$. We can write φ as a tuple (φ_τ) where $\tau \in \mathscr{I}$ and $\varphi_\tau \in \mathrm{Hom}(\mathcal{P}_{x,\tau}, \mathcal{H}_{x,\tau}/\mathcal{P}_{x,\tau})$. The symmetry condition and the fact that $\{,\}_\lambda$ induces a perfect pairing between $\mathcal{P}_{x,\bar{\tau}}$ and $\mathcal{H}_{x,\tau}/\mathcal{P}_{x,\tau}$ for every $\tau \in \mathscr{I}$, imply that $\varphi_{\bar{\tau}}$ is determined by φ_τ, and that for a given choice of representatives τ for the pairs $\{\tau, \bar{\tau}\} \in \mathscr{I}^+$ the φ_τ may be chosen arbitrarily.
- $\psi \in \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{N}_y, \mathcal{H}_x[V]/\mathcal{N}_y)$ satisfies $\varphi|_{\mathcal{N}_y} = \psi \mod \underline{\omega}_x$.

The second condition means that $\psi = (\psi_\tau)$ where the τ range over the same set of representatives for $\{\tau, \bar{\tau}\} \in \Sigma$ as above, $\psi_\tau \in \mathrm{Hom}(\mathcal{N}_{y,\tau}, \mathcal{H}_{x,\tau}[V]/\mathcal{N}_{y,\tau})$ and $\varphi_\tau|_{\mathcal{N}_{y,\tau}} = \psi_\tau \mod \mathcal{P}_{x,\tau}$.

The tangent space to the fiber $\beta^{-1}(x)$ is the space of (φ,ψ) with $\varphi = 0$, or, alternatively, the space

$$\{(\psi_\tau) |\, \psi_\tau \in \mathrm{Hom}(\mathcal{N}_{y,\tau}, \mathcal{P}_{x,\tau}[V]/\mathcal{N}_{y,\tau})\}.$$

Theorem 4.5.5. *There exists a unique smooth p-foliation $\mathscr{F}^\Sigma$ on M^Σ , characterized by the fact that at any geometric point $y \in M^\Sigma(k)$ as above, $\mathscr{F}_y^\Sigma$ is the subspace*

$$\mathscr{F}_y^\Sigma = \{(\varphi, \psi) |\, \psi = 0\}.$$

The foliation $\mathscr{F}^\Sigma$ agrees with $\mathscr{F}_\Sigma$ on $\beta^{-1}(M_\Sigma) \simeq M_\Sigma$ and, in general, is transversal to the fibers of β.

Proof. A straightforward adaptation of the proof of Proposition 20 in [G-dS1] §4.3. □

Intuitively, $\mathscr{F}_y^\Sigma$ are the directions in $\mathcal{T}M_y^\Sigma$ in which $\mathcal{N}_y$ does not undergo any infinitesimal deformation. By the alluded transversality, its projection to $\mathcal{T}M_x$ is injective, and identifies $\mathscr{F}_y^\Sigma$ with the subspace of $\varphi = (\varphi_\tau)_{\tau\in\mathscr{I}}$ such that for every $\{\tau, \bar{\tau}\} \in \Sigma$ (and τ a representative as above) $\varphi_\tau(\mathcal{N}_{y,\tau}) = 0$. This subspace, however, varies with $y \in \beta^{-1}(x)$.

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