

ON THE STRUCTURE OF CERTAIN Γ -DIFFERENCE MODULES

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ABSTRACT. This is a largely expository paper, providing a self-contained account on the results of [Sch-Si1, Sch-Si2], in the cases denoted there 2Q and 2M. These papers of Schäfke and Singer supplied new proofs to the main theorems of [Bez-Bou, Ad-Be], on the rationality of power series satisfying a pair of independent q -difference, or Mahler, equations.

We emphasize the language of Γ -difference modules, instead of difference equations or systems. Although in the two cases mentioned above this is only a semantic change, we also treat a new case, which may be labeled 1M1Q. Here the group Γ is generalized dihedral rather than abelian, and the language of equations is inadequate.

In the last section we explain how to generalize the main theorems in case 2Q to finite characteristic.

INTRODUCTION

Adamczewski and Bell proved in 2017 the following theorem, conjectured some 30 years earlier by Loxton and van der Poorten [vdPo].

Theorem 1. [Ad-Be] *Let p and q be multiplicatively independent natural numbers. Consider the endomorphisms*

$$\sigma f(x) = f(x^p), \quad \tau f(x) = f(x^q)$$

of the field of rational functions $K = \mathbb{C}(x)$ and of its completion at 0, the field of Laurent series $k = \mathbb{C}((x))$. If $f \in k$ satisfies the two Mahler equations

$$\sum_{i=0}^n a_i \sigma^{n-i}(f) = 0, \quad \sum_{i=0}^m b_i \tau^{m-i}(f) = 0,$$

with $a_i, b_i \in K$, then $f \in K$.

For an account on Mahler's equations and their role in transcendence theory, see the survey paper [Ad]. A similar theorem has been proved by Bézivin and Boutabaa in 1992.

Theorem 2. [Bez-Bou] *Let p and q be multiplicatively independent complex numbers. Consider the automorphisms*

$$\sigma f(x) = f(px), \quad \tau f(x) = f(qx)$$

of the fields $K = \mathbb{C}(x)$ and $k = \mathbb{C}((x))$. If $f \in k$ satisfies the two q -difference equations

$$\sum_{i=0}^n a_i \sigma^{n-i}(f) = 0, \quad \sum_{i=0}^m b_i \tau^{m-i}(f) = 0$$

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with $a_i, b_i \in K$, then $f \in K$.

The proofs of these two theorems used a variety of techniques. In the case of Theorem 1 it relied on Cobham's theorem [Co] in the theory of automata. Theorem 2 was proved by p -adic techniques, for an auxiliary prime¹ p , and involved, in its original formulation, some unnecessary restrictions. Recently, Schäfke and Singer [Sch-Si1, Sch-Si2] provided a uniform treatment of the two theorems, as well as of other similar results. Besides emphasizing common features, they eliminated the dependence, in the work of Adamczewski and Bell, on Cobham's theorem. In fact, the latter could now be deduced from Theorem 1. In Theorem 2 they still required p or q to be of absolute value different than 1 (at least after some automorphism of the complex numbers), but this restriction can be removed.

The goal of this largely expository paper is to provide yet another look at the same theorems, leading to a third, new example. We give a self-contained treatment, based on the notion of a Γ -difference module, which is introduced in Section 1, and we shift the focus from *equations* to *modules*. This is similar to studying linear partial differential equations via D -modules. The letter Γ signifies the group of automorphisms (of K or of some extension field \tilde{K}) generated by the operators σ and τ , which, in the two examples cited above, is free abelian of rank 2. This approach allows us to isolate, in Section 2, the formal aspects of the theory. Once we globalize, in Section 3, our proof of Theorem 1 follows the line of [Sch-Si1, Sch-Si2]. In Theorem 2 we remove the unnecessary restriction that $|p| \neq 1$ or $|q| \neq 1$ (see Step VI of §3.2).

We treat the above two theorems, corresponding to the cases 2M and 2Q in [Sch-Si1]. In Section 4 we give a third example that might be denoted 1M1Q. In this case the group Γ is no longer abelian, but rather *generalized dihedral*. As a result, the main theorem does not lend itself to a simple-minded formulation in terms of equations as above, but its formulation (and proof) in the language of difference modules is completely analogous to the first two cases.

In the last section we explain how to generalize Theorem 2, as well as case 2Q of our Main Theorem (Theorem 7), if an arbitrary field of constants, possibly of finite characteristic, is substituted for the field of complex numbers.

Finally we remark that in [dS1, dS2] a similar situation, not discussed in the present paper, is studied, where the field of rational functions is replaced by a field of elliptic functions. New issues arise there. One is the issue of periodicity. Another one is the existence of non-trivial Γ -invariant vector bundles on the elliptic curve. Nevertheless, these issues can be analyzed, and a theorem analogous to the two theorems cited above, where the operators σ and τ are induced by isogenies of the elliptic curve, and the coefficients a_i and b_i are elliptic functions, is proved there. We stress that in the elliptic case, a power series f satisfying two elliptic p - and q -difference equations need not be elliptic. Instead, it belongs to a slightly larger ring of functions, generated over the field of elliptic functions (in the variable z) by z, z^{-1} and the Weierstrass zeta function of z .

Recent work of Adamczewski, Dreyfus, Hardouin and Wibmer established a far-reaching strengthening of the above mentioned theorems. In [ADHW] they show that if $f, g \in \mathbb{C}((x))$ do not belong to $\mathbb{C}(x)$, f satisfies a σ -difference equation and g a τ -difference equation, then f and g are algebraically independent over $\mathbb{C}(x)$.

¹Having nothing to do with the complex number denoted by p in the statement of the theorem.

Special cases of this result have been proved by various authors before. Our survey raises two immediate questions: (a) Can a generalization of this type be phrased (and proved) in the context of *difference modules*, that will apply for example in the case $1M1Q$, not amenable to a formulation in terms of two difference equations? (b) Can the prerequisites for a such a theorem be axiomatized (and checked) to include, for example, ground fields of elliptic functions?

1. Γ -DIFFERENCE MODULES

1.1. Definitions and examples. Let K be a field and Γ a group, acting on K by automorphisms. We make no assumption whatsoever on the nature of Γ , nor do we require the action to be faithful. In fact, the case of a trivial action is not excluded. The fixed field $C = K^\Gamma$ is called the *field of constants*.

Definition 3. A Γ -*difference module* over K is a finite dimensional K -vector space M equipped with a semi-linear action of Γ . In other words, for every $\gamma \in \Gamma$ there is a $\Phi_\gamma \in GL_C(M)$ satisfying

$$\Phi_\gamma(av) = \gamma(a)\Phi_\gamma(v) \quad (a \in K, v \in M)$$

and

$$\Phi_{\gamma\delta} = \Phi_\gamma \circ \Phi_\delta \quad (\gamma, \delta \in \Gamma).$$

If the action of Γ on K is trivial, this notion is nothing but a linear representation of Γ over K . If $K_0 \subset K$ is a Γ -invariant subfield then we say that M *descends to* K_0 , or has an underlying K_0 -*structure*, if there exists a Γ -difference module M_0 over K_0 such that $M \simeq K \otimes_{K_0} M_0$, the Γ -action extended semi-linearly. This may apply in particular to $K_0 = C$. In general, if M descends to K_0 , M_0 need not be unique, not even up to an isomorphism over K_0 . If M descends to the trivial module over C , i.e. if $M \simeq K^r$, Γ acting in the coordinates, we call M *trivial*.

Example 4. If Γ is a finite group acting faithfully then K/C is a Galois extension with $Gal(K/C) = \Gamma$, and Hilbert's theorem 90 says that every Γ -difference module over K is trivial.

Denoting by $K\langle\Gamma\rangle$ the twisted group ring of Γ over K , a Γ -difference module is nothing but a $K\langle\Gamma\rangle$ -module, finite dimensional over K . The category of Γ -difference modules over K will be denoted ΓDiff_K . The tensor product $M \otimes N$ of two Γ -difference modules is defined as the tensor product over K , with the usual Γ -action, $\Phi_\gamma(u \otimes v) = \Phi_\gamma(u) \otimes \Phi_\gamma(v)$. The dual M^\vee is defined as the space of K -linear functionals $\lambda : M \rightarrow K$ with the action

$$\Phi_\gamma(\lambda)(v) = \gamma(\lambda(\Phi_\gamma^{-1}v)),$$

and the internal hom is $\underline{Hom}(M, N) = M^\vee \otimes N$.

It is easily checked that with these definitions ΓDiff_K becomes a rigid abelian tensor category ([De-Mi], Definition 1.15). The object $\underline{1}$ is the trivial Γ -module K and $\underline{End}(\underline{1}) = C$. The Tannakian formalism applies to this category, as described for example in the last section of the first chapter of [vdP-Si], but we shall not dwell on this aspect here.

The following easy Proposition should be viewed as a generalization of Hilbert's theorem 90.

Proposition 5. *Suppose that*

$$1 \rightarrow \Delta \rightarrow \Gamma \rightarrow \bar{\Gamma} \rightarrow 1$$

is a short exact sequence of groups, Δ is finite, and the action of Δ on K is faithful. Let $K_0 = K^\Delta$. Then the categories ΓDiff_K and $\bar{\Gamma}\text{Diff}_{K_0}$ are equivalent.

Proof. Consider the two functors $\alpha : \Gamma\text{Diff}_K \rightarrow \bar{\Gamma}\text{Diff}_{K_0}$ and $\beta : \bar{\Gamma}\text{Diff}_{K_0} \rightarrow \Gamma\text{Diff}_K$ defined by

$$\alpha(M) = M^\Delta, \quad \beta(M_0) = K \otimes_{K_0} M_0.$$

Since Δ is normal in Γ , the action of Γ on K induces an action of $\bar{\Gamma}$ on K_0 and $\alpha(M)$ becomes a $\bar{\Gamma}$ -difference module. Likewise, $\beta(M_0)$ becomes a Γ -difference module if the action of Γ on M_0 (which factors through $\bar{\Gamma}$) is extended semi-linearly to $K \otimes_{K_0} M_0$. Hilbert's Theorem 90 says that when we restrict the action from Γ to Δ these two functors give an equivalence between ΔDiff_K and $\bar{\Delta}\text{Diff}_{K_0}$, the latter being the category of finite dimensional vector spaces over K_0 . In particular $\dim_{K_0} \alpha(M) = \dim_K(M)$ and $\dim_K \beta(M_0) = \dim_{K_0}(M_0)$. Galois theory gives $\alpha \circ \beta(M_0) = M_0$. On the other hand there is an injective map $K \otimes_{K_0} \alpha(M) \rightarrow M$ respecting the action of Γ , so by dimension counting it must be an isomorphism and we also have $\beta \circ \alpha(M) = M$. \square

Thus, when studying Γ -difference modules over a field K , we may always factor out finite normal subgroups of Γ , if they act faithfully on K . For example, if Γ is a semisimple algebraic group acting faithfully on K , we may assume, without loss of generality, that it is of adjoint type.

1.2. Matrices and classification. If we choose a basis e_1, \dots, e_r of M over K we may associate to any $\gamma \in \Gamma$ its matrix (a_{ij}) , defined by

$$\Phi_\gamma(e_j) = \sum_{i=1}^r a_{ij} e_i.$$

It is customary to denote by A_γ the *inverse* of this matrix, namely $A_\gamma^{-1} = (a_{ij})$. The condition $\Phi_\gamma \circ \Phi_\delta = \Phi_{\gamma\delta}$ gets translated to the *consistency condition*

$$A_{\gamma\delta} = \gamma(A_\delta) \cdot A_\gamma,$$

which must hold for every $\gamma, \delta \in \Gamma$. Conversely, a collection of matrices $\{A_\gamma\}$ satisfying the above conditions, termed a *consistent collection of matrices*, defines a Γ -difference module structure on K^r by letting

$$\Phi_\gamma(v) = A_\gamma^{-1} \gamma(v).$$

If e'_1, \dots, e'_r is another basis and $C = (c_{ij})$ is the transition matrix, i.e.

$$e'_j = \sum_{i=1}^r c_{ij} e_i,$$

then the matrix A'_γ corresponding to γ in the new basis is

$$(1.1) \quad A'_\gamma = \gamma(C)^{-1} A_\gamma C.$$

The equivalence relation defined by (1.1) is called *gauge equivalence*. It follows that Γ -difference modules of rank r over K are classified by gauge equivalence classes

of consistent collections of matrices $A_\gamma \in GL_r(K)$, or what is the same, by the non-abelian cohomology

$$H^1(\Gamma, GL_r(K)).$$

Example 6. (i) $\Gamma = \langle \sigma \rangle$ is infinite cyclic. In this case a consistent collection is determined uniquely by A_σ , which may be chosen arbitrarily, and the gauge equivalence classes correspond to the twisted conjugacy classes

$$B(GL_r(K)) = GL_r(K) / \sim$$

where $A' \sim A$ if there exists a $C \in GL_r(K)$ with $A' = \sigma(C)^{-1}AC$.

(ii) Let k be a perfect field of characteristic p and $K = W(k)[1/p]$ where $W(k)$ is the ring of Witt vectors of k . Let σ denote the Frobenius automorphism of K and $\Gamma = \langle \sigma \rangle$. A Γ -difference module over K is called also an *F-isocrystal*. This notion is central to p -adic Hodge theory.

(iii) Replacing the group $GL_r(K)$ by $G(K)$ for an arbitrary linear algebraic group G over K , one arrives at the notion of a Γ -*difference module with G-structure*. These objects are classified by $H^1(\Gamma, G(K))$, and when $\Gamma = \langle \sigma \rangle$ by $B(G(K))$, defined as above. In example (ii) they have been analyzed in [Kot].

(iv) $\Gamma = \langle \sigma, \tau \rangle \simeq \mathbb{Z}^2$ (i.e. σ and τ commute and are multiplicatively independent: $\sigma^a \tau^b = 1$ if and only if $a = b = 0$). In this case a Γ -difference module is defined by the pair (A_σ, A_τ) , subject to the consistency condition

$$(1.2) \quad \sigma(A_\tau)A_\sigma = \tau(A_\sigma)A_\tau,$$

up to gauge equivalence. This is the example underlying the two theorems cited in the introduction.

1.3. Difference modules and difference equations. From now on let, as in the introduction,

$$K = \mathbb{C}(x), \quad k = \mathbb{C}((x)).$$

To give a uniform treatment of Theorem 1 (case 2M) and Theorem 2 (case 2Q) we introduce also the fields

$$\tilde{K} = \bigcup_{s=1}^{\infty} \mathbb{C}(x^{1/s})$$

and

$$\tilde{k} = \bigcup_{s=1}^{\infty} \mathbb{C}((x^{1/s})).$$

The field \tilde{k} is the field of Puiseux series, and is the algebraic closure of k .

In both theorems, σ and τ are endomorphisms of the algebraic group $\mathbb{G} = \mathbb{G}_{m, \mathbb{C}}$ or $\mathbb{G}_{a, \mathbb{C}}$, and can be extended to *automorphisms* of its universal covering $\tilde{\mathbb{G}}$. In the q -difference case (2Q) the additive group is simply connected, so $\mathbb{G} = \tilde{\mathbb{G}}$. In the Mahler case (2M) the extension of σ or τ to an automorphism of $\tilde{\mathbb{G}}$ depends on the choice of a compatible sequence of s th roots of the function x , namely $\sigma(x^{1/s}) = x^{p/s}$ and $\tau(x^{1/s}) = x^{q/s}$. We fix such a choice once and for all. Replacing $x^{1/s}$ by $\zeta_s x^{1/s}$ where ζ_s is an s th root of 1 (and, to maintain the compatibility, $\zeta_{st}^t = \zeta_s$) results in twisting the action of σ on $x^{1/s}$ by ζ_s^{p-1} and the action of τ on the same element by ζ_s^{q-1} . The field \tilde{K} is the function field of $\tilde{\mathbb{G}}$, and σ and τ induce automorphisms of \tilde{K} and of \tilde{k} .

In both cases we therefore let

$$\Gamma = \langle \sigma, \tau \rangle,$$

acting via automorphisms on the fields K, k in case 2Q, and on the fields \tilde{K}, \tilde{k} in case 2M. The significance of the assumption on the multiplicative independence of p and q is that $\Gamma \simeq \mathbb{Z}^2$.

Theorem 7 (Main Theorem). *In either case 2Q or case 2M, any Γ -difference module M over K (in case 2Q) or \tilde{K} (in case 2M) has an underlying \mathbb{C} -structure M_0 , i.e. there exists a Γ -invariant \mathbb{C} -submodule $M_0 \subset M$, such that $M = K \otimes_{\mathbb{C}} M_0$ (case 2Q), or $M = \tilde{K} \otimes_{\mathbb{C}} M_0$ (case 2M).*

Remark. (i) An equivalent formulation is that any pair (A_σ, A_τ) of matrices from $GL_r(K)$ (resp. $GL_r(\tilde{K})$) satisfying the consistency equation (1.2) is gauge-equivalent to a pair (A_σ^0, A_τ^0) of constant matrices from $GL_r(\mathbb{C})$.

(ii) Equivalently, the natural map $H^1(\Gamma, GL_r(\mathbb{C})) \rightarrow H^1(\Gamma, GL_r(K))$ (resp. $H^1(\Gamma, GL_r(\tilde{K}))$) is surjective.

(iii) In case 2M the underlying complex structure is *unique*, equiv. the pair (A_σ^0, A_τ^0) is unique up to conjugation in $GL_r(\mathbb{C})$, equiv. the map $H^1(\Gamma, GL_r(\mathbb{C})) \rightarrow H^1(\Gamma, GL_r(K))$ is bijective. In the case 2Q this is false, already in rank 1. See remark 16.

(iv) Note that in the formulation of the last theorem the field k or \tilde{k} plays no role. It will, however, reappear in its proof. Note also that the formulation of the theorem is purely algebraic. By this we mean that if ι is an arbitrary automorphism of \mathbb{C} and M is a Γ -difference module, then so is the module $M^\iota = \mathbb{C} \otimes_{\iota, \mathbb{C}} M$ obtained from it by transport of structure, and M descends to \mathbb{C} if and only if M^ι descends to \mathbb{C} . The topological or dynamical nature of M^ι may nevertheless be completely different, as ι is, in general, non-continuous.

Proposition 8. *Theorem 7 implies Theorem 1 and Theorem 2.*

Proof. Observe first that in the case 2M, to prove Theorem 1 it is enough to prove the analogous theorem with K and k replaced by \tilde{K} and \tilde{k} , where (the extended) σ and τ become automorphisms. This is because $k \cap \tilde{K} = K$. To unify the notation, in this proof only, we let the symbols k and K stand, in case 2M, for the fields \tilde{k} and \tilde{K} .

Let $M \subset k$ be the $K \langle \Gamma \rangle$ -span of f , and let $\Phi_\sigma = \sigma$ and $\Phi_\tau = \tau$. The condition imposed on f , that it simultaneously satisfies the two functional equations with coefficients from K , is equivalent to the condition

$$\dim_K M < \infty.$$

Indeed, thanks to the commutativity of Γ , M is spanned by $\sigma^i \tau^j f$ for $0 \leq i < n$ and $0 \leq j < m$.

Let e_1, \dots, e_r be a basis over \mathbb{C} of the submodule M_0 , whose existence is guaranteed by Theorem 7. Then

$$f \in M = \sum_{i=1}^r K e_i \subset k.$$

Replacing x by $x^{1/s}$ for some s in the case 2M, we may assume that all the e_i are in $\mathbb{C}((x))$. The column vector $\underline{e} = {}^t(e_1, \dots, e_r)$ satisfies

$$\underline{e}(\sigma x) = B\underline{e}(x)$$

for an invertible constant matrix $B \in GL_r(\mathbb{C})$. (In the notation introduced above, $B = ({}^tA_\sigma^0)^{-1}$). Write $\underline{e} = \sum_{n=n_0}^{\infty} v_n x^n$ with $v_n \in \mathbb{C}^r$. In case 2M this gives

$$\sum_{n=n_0}^{\infty} v_n x^{pn} = \sum_{n=n_0}^{\infty} Bv_n x^n,$$

from where we deduce that $v_n = 0$ for $n \neq 0$, so each $e_i \in \mathbb{C}$. In case 2Q the same equation gives

$$\sum_{n=n_0}^{\infty} v_n p^n x^n = \sum_{n=n_0}^{\infty} Bv_n x^n,$$

from where we deduce that $v_n = 0$ for n sufficiently large, since the matrix B can have only finitely many eigenvalues. Thus in this case, too, all the $e_i \in K$. As the e_i are linearly independent over K , in both cases we must have $r = 1$ and $f \in Ke_1 = K \subset k$. \square

2. THE STRUCTURE OF FORMAL Γ -DIFFERENCE MODULES

The results of this part appear in various variations in the literature, sometimes over the fields of Hahn series or of convergent power series replacing the fields of Puiseux or Laurent series. The ideas date back to works of Manin and Dieudonné on formal groups. We prove all that we shall need later on in the global theory from first principles. The reader may consult [Roq, Sau, vdP-Re] and the references therein for the historical development of the subject, and for further results.

2.1. Formal (p, q) -difference modules.

2.1.1. Rank 1 formal q -difference modules. In this section we prove an analogue of the Main Theorem over k instead of K , in the case 2Q of two q -difference operators. The case of two Mahler operators will be discussed in the next section. One starts by examining the structure of a Γ -difference module M , for $\Gamma = \langle \tau \rangle$ infinite cyclic. Adding a second multiplicatively independent and commuting operator σ imposes a serious restriction on the structure of M , and forces it to descend to \mathbb{C} .

Let $q \in \mathbb{C}^\times$ and assume that q is not a root of unity. Fix once and for all a compatible sequence of roots $q^{1/s}$. Let $\Gamma = \langle \tau \rangle$ act on the field $\tilde{k} = \bigcup_{s=1}^{\infty} \mathbb{C}((x^{1/s}))$ via $\tau f(x^{1/s}) = f(q^{1/s} x^{1/s})$. (To get the results below we have to work over \tilde{k} , although τ is already an automorphism of k .) Write $k_s = \mathbb{C}((x^{1/s}))$.

A Γ -difference module over \tilde{k} is the same as a $\tilde{k} \langle \Phi, \Phi^{-1} \rangle$ -module which is finite dimensional over \tilde{k} . Here the twisted Laurent polynomials ring $\tilde{k} \langle \Phi, \Phi^{-1} \rangle$ satisfies the relation

$$\Phi a = \tau(a) \Phi$$

for $a \in \tilde{k}$. We shall call a Γ -difference module over \tilde{k} or over k also a (formal) q -difference module.

Rank-1 q -difference modules over \tilde{k} are classified by $\tilde{k}^\times / (\tilde{k}^\times)^{\tau-1}$, see (i) from example 6 with $r = 1$. Every element $f \in \tilde{k}^\times$ can be written uniquely as $cx^\lambda f_1$ where $c \in \mathbb{C}^\times$, $f_1 \in U_1(k_s) = 1 + x^{1/s} \mathbb{C}[[x^{1/s}]]$ (the principal units of k_s) for some $s \in \mathbb{N}$, and $\lambda \in \mathbb{Q}$. Since $f_1 = g_1^{\tau-1}$ for $g_1 \in U_1(k_s)$ (solve successively

for the coefficients of g_1 , using the fact that q is not a root of unity), and since $(x^\mu)^{\tau-1} = q^\mu$, we see that classes in $\tilde{k}^\times / (\tilde{k}^\times)^{\tau-1}$ are represented by cx^λ , where λ (the *slope*) is uniquely determined, and $c \in \mathbb{C}^\times$ is determined up to multiplication by q^α for some $\alpha \in \mathbb{Q}$. We therefore have the following easy Proposition.

Proposition 9. *Let $c \in \mathbb{C}^\times$ and $\lambda \in \mathbb{Q}$. Let $I_{\lambda,c} = \tilde{k}e$ with $\Phi(e) = cx^\lambda e$. Then every rank-1 q -difference module over \tilde{k} is isomorphic to some $I_{\lambda,c}$ and $I_{\lambda,c} \simeq I_{\mu,d}$ if and only if $\lambda = \mu$ and $cd^{-1} = q^\alpha$ for some $\alpha \in \mathbb{Q}$.*

2.1.2. *Newton polygons.* We review well-known facts about Newton polygons and slopes. Let $\nu : \tilde{k}^\times \rightarrow \mathbb{Q}$ be the valuation of \tilde{k} , normalized by $\nu(x) = 1$. If $P \in \tilde{k}[T]$,

$$P(T) = a_0 T^r + \cdots + a_{r-1} T + a_r$$

$a_0, a_r \neq 0$, we consider the points $(i, \nu(a_i)) \in \mathbb{R}^2$ ($0 \leq i \leq r$). The highest piecewise linear convex graph lying on or below these points is called the Newton polygon \mathcal{N}_P of P . It has two vertical edges, connecting $(0, \infty)$ to $(0, \nu(a_0))$, and $(r, \nu(a_r))$ to (r, ∞) . The other edges have rational slopes $\lambda_1 < \lambda_2 < \cdots < \lambda_s$ and integral horizontal lengths r_1, r_2, \dots, r_s with $\sum r_i = r$. The polynomial P has precisely r_i roots α in \tilde{k}^\times of valuation $\nu(\alpha) = \lambda_i$. If we make a change of variable $Q(T) = P(a^{-1}T)$ then \mathcal{N}_Q has slopes $\lambda_i + \nu(a)$. After such a change of variables, we may therefore assume, when dealing with Newton polygons, that the smallest slope of P is 0. The definition of \mathcal{N}_P may be extended to an arbitrary non-zero $P \in \tilde{k}[T, T^{-1}]$ so that \mathcal{N}_{PT} is obtained from \mathcal{N}_P by a horizontal shift one unit to the right. It has the same slopes and the same horizontal lengths.

If $P \in \tilde{k}[T]$ is written as above we let $P(\Phi) = \sum_{i=0}^r a_i \Phi^{r-i} \in \tilde{k}\langle \Phi, \Phi^{-1} \rangle$. Note however that $P(T) \mapsto P(\Phi)$ is not a homomorphism, as Φ does not commute with \tilde{k} .

Let M be a cyclic q -difference module, generated by the vector v . We shall later see (Birkhoff's cyclicity lemma) that every q -difference module is cyclic, but at this stage we do not know it yet. Let $P(T)$ be a monic polynomial of minimal degree such that $P(\Phi)v = 0$. Such a polynomial exists since the $\Phi^i v$ are linearly dependent over \tilde{k} . Write

$$(2.1) \quad P(T) = T^r + a_1 T^{r-1} + \cdots + a_{r-1} T + a_r.$$

Then $a_r \neq 0$, since otherwise, as $\Phi^{-1}P(\Phi)v = 0$, the polynomial $Q = \tau^{-1}(P)/T$ has degree $r-1$ and still satisfies $Q(\Phi)v = 0$. Let $\mathcal{D} = \tilde{k}\langle \Phi, \Phi^{-1} \rangle$ and consider the left ideal $\mathcal{D}P(\Phi)$. The homomorphism of \mathcal{D} -modules

$$\mathcal{D}/\mathcal{D}P(\Phi) \twoheadrightarrow M,$$

sending $Q(\Phi) \in \mathcal{D}$ to $Q(\Phi)v$ is surjective. As the module on the left is generated over \tilde{k} by $1, \Phi, \dots, \Phi^{r-1}$ and M contains the linearly independent vectors $v, \Phi v, \dots, \Phi^{r-1}v$, both sides have dimension r and this map is an isomorphism.

If we replace M by $M \otimes I_{\lambda,1}$ ($\lambda \in \mathbb{Q}$) and the cyclic vector v by $v \otimes e$ where e is the basis of $I_{\lambda,1}$, then $\Phi^i(v \otimes e) = q^{\lambda \binom{i}{2}} x^{i\lambda} \Phi^i(v) \otimes e$ so the polynomial P is replaced (up to a scalar multiple) by

$$Q(T) = q^{-\lambda \binom{r}{2}} T^r + q^{-\lambda \binom{r-1}{2}} x^\lambda a_1 T^{r-1} + \cdots + x^{(r-1)\lambda} a_{r-1} T + x^{r\lambda} a_r$$

and the points $(i, \nu(a_i))$ by $(i, \nu(a_i) + i\lambda)$. After such a twist of M we may assume that the slopes of \mathcal{N}_P are ≥ 0 and that the first (smallest) slope is 0.

Replacing the variable x by some $x^{1/s}$ we may therefore assume that all the roots of P are in $k = \mathbb{C}(\!(x)\!)$, that the slopes are integral, and that the smallest slope is 0. In particular, all the $a_i \in \mathbb{C}[[x]]$.

2.1.3. Factorization in $k\langle\Phi\rangle$.

Lemma 10. *Assume that in (2.1) the $a_i \in \mathbb{C}[[x]]$ ($i \geq 1$), and at least one of them is a unit (these conditions are equivalent to the smallest slope of P being 0). Then there exists a unit $b_0 \in \mathbb{C}[[x]]^\times, b_1, \dots, b_{r-1} \in \mathbb{C}[[x]]$ and $c \in \mathbb{C}^\times$ such that in $k\langle\Phi\rangle$ we have*

$$P(\Phi) = \Phi^r + a_1\Phi^{r-1} + \dots + a_r = \tau(b_0)^{-1}(\Phi - c)(b_0\Phi^{r-1} + b_1\Phi^{r-2} + \dots + b_{r-1}).$$

Proof. We write $a_0 = 1$, $u = b_0$ and solve successively for the coefficient of Φ^{r-i} . Define

$$b_i = \sum_{j=0}^i c^j \tau^{-j}(u) \tau^{-j-1}(a_{i-j})$$

($0 \leq i \leq r-1$), where $u \in \mathbb{C}[[x]]^\times$ and $c \in \mathbb{C}^\times$ are still to be determined. These equations guarantee that

$$\tau(u)^{-1}(\tau(b_{i+1}) - cb_i) = a_{i+1}$$

for $0 \leq i \leq r-2$. To get the last coefficient we need

$$-cb_{r-1} = \tau(u)a_r$$

or, with $\tau(u) = v$, all that remains is to find $v \in \mathbb{C}[[x]]^\times$ and $c \in \mathbb{C}^\times$ such that

$$\sum_{j=0}^r c^j \tau^{-j}(v) \tau^{-j}(a_{r-j}) = 0.$$

Write $v = 1 + t_1x + t_2x^2 + \dots$ and $\tau^{-j}(a_{r-j}) = a_{j,0} + a_{j,1}x + a_{j,2}x^2 + \dots$. For c we take a non-zero solution of

$$\sum_{j=0}^r c^j a_{j,0} = 0.$$

Here we use the fact that since the smallest slope is 0, there is $j < r$ with $a_{j,0} \neq 0$, and of course $a_{r,0} = 1$. We also insist that for $i \geq 1$ $q^{-i}c$ is not a root of the same polynomial. This can be achieved because q is not a root of unity, so we may replace c by the last element in the sequence $c, q^{-1}c, q^{-2}c, \dots$ solving the equation.

We then solve successively for the t_i . We get

$$t_i \left(\sum_{j=0}^r c^j q^{-ji} a_{j,0} \right) = r_i$$

where r_i is an expression involving the $a_{j,m}$, c and t_ℓ for $\ell < i$. By our assumption on c the term in parenthesis does not vanish, so we can solve for t_i . \square

Corollary 11. *Assume that, after replacing x by $x^{1/s}$ for some s , the smallest slope of P is an integer μ . Then*

$$P(\Phi) = \tau(u)^{-1}(\Phi - cx^\mu)uP_1(\Phi)$$

where P_1 is a monic polynomial of degree $r-1$ and $u \in \mathbb{C}[[x]]^\times$.

Proof. Let μ be the smallest slope of P and consider the module $M \otimes I_{-\mu,1}$ with the cyclic vector $v \otimes e$. Since

$$(x^\mu \Phi)^i (v \otimes e) = \Phi^i v \otimes e$$

we deduce that if $Q(T) = \sum_{i=0}^r a_i T^i$ is the monic minimal polynomial of $v \otimes e$ then

$$Q(\Phi) = q^{-\mu \binom{r}{2}} x^{-\mu r} P(x^\mu \Phi)$$

(caution: it is not true that $Q(T) = q^{-\mu \binom{r}{2}} x^{-\mu r} P(x^\mu T)$; the variable T commutes with x^μ while Φ does not!). The polynomial Q has smallest slope 0, so by the Lemma

$$q^{-\mu \binom{r}{2}} x^{-\mu r} P(x^\mu \Phi) = \tau(b_0)^{-1} (\Phi - c) Q_1(\Phi)$$

where $Q_1(\Phi) = \sum_{i=0}^{r-1} b_i \Phi^i$. Consider the automorphism of the non-commutative ring $k \langle \Phi \rangle$ carrying Φ to $x^{-\mu} \Phi$ and leaving k fixed. (Note that it is *not* obtained by substituting Φ in a similar automorphism of $k[T]$.) Applying it to the above identity we get

$$\begin{aligned} P(\Phi) &= q^{\mu \binom{r}{2}} x^{\mu r} \tau(b_0)^{-1} (x^{-\mu} \Phi - c) Q_1(x^{-\mu} \Phi) \\ &= \tau(b_0)^{-1} q^{\mu \binom{r}{2} - \mu(r-1)} (\Phi - cq^{\mu(r-1)} x^\mu) x^{\mu(r-1)} Q_1(x^{-\mu} \Phi) \\ &= \tau(b_0)^{-1} (\Phi - cq^{\mu(r-1)} x^\mu) q^{\mu \binom{r-1}{2}} x^{\mu(r-1)} Q_1(x^{-\mu} \Phi) \\ &= \tau(b_0)^{-1} (\Phi - cq^{\mu(r-1)} x^\mu) P_1(\Phi) \end{aligned}$$

where the leading coefficient of P_1 is b_0 . The claim follows, with $u = b_0$ and c replaced by $cq^{\mu(r-1)}$. Note that c is anyhow only determined by M up to a power of q , since $I_{\lambda,c} \simeq I_{\lambda,qc}$. \square

Consider the vector $e_1 = uP_1(\Phi)v \neq 0$. Then

$$\Phi e_1 = cx^\mu e_1$$

so $\tilde{k}e_1 \simeq I_{\mu,c}$.

It is easy to see that the slopes of P are the slopes of P_1 and μ (with multiplicities).

2.1.4. The structure theorem for a q -difference module over \tilde{k} .

Proposition 12. *Let M be an arbitrary q -difference module over \tilde{k} . Then M has an ascending filtration with one-dimensional graded pieces of the form I_{μ_i, c_i} with rational slopes $\mu_1 \leq \mu_2 \leq \dots \leq \mu_r$.*

Proof. It is enough to prove that M contains a rank 1 submodule M_1 , because then we continue by induction on M/M_1 . For that we may assume that M is cyclic, and the claim follows from what was done above. \square

Since the Jordan-Hölder factors of M are intrinsic to M we deduce that if M is cyclic the slopes are independent of the cyclic vector used in the proof.

Corollary 13 (Birkhoff's cyclic vector lemma). *Every q -difference module over \tilde{k} has a cyclic vector.*

Proof. We prove the corollary by induction on the rank, the rank 1 case being obvious. Let $M_1 \subset M$ be a submodule of rank 1 and $v \in M$ a vector projecting to a cyclic vector of M/M_1 . Let e be a basis element of M_1 . Let $P(\Phi)$ be a polynomial in Φ with coefficients in \tilde{k} annihilating v . For an appropriate λ , $P(\Phi)(x^\lambda e) \neq 0$. Replacing e by $x^\lambda e$ we may assume that $P(\Phi)e \neq 0$. But then $u = v + e$ is a cyclic vector for M , as the module generated by it contains $P(\Phi)u = P(\Phi)e$, hence M_1 , and modulo M_1 it contains the image of v , hence projects onto M/M_1 . \square

Theorem 14 (Structure theorem for formal q -difference modules). *Let M be a q -difference module over \tilde{k} . Let $\lambda_1 < \lambda_2 < \dots < \lambda_m$ be the distinct slopes of M in increasing order. Then there are \mathbb{C} -vector spaces $N_i \subset M$ with endomorphisms $\phi_i \in GL_{\mathbb{C}}(N_i)$ so that*

$$M = \bigoplus_{i=1}^m \tilde{k} \otimes_{\mathbb{C}} N_i$$

and $\Phi(1 \otimes v_i) = x^{\lambda_i} \otimes \phi_i(v_i)$ for $v_i \in N_i$. If M is defined over k then the same is true if we extend scalars to k_s where s is the least common denominator of the λ_i .

Proof. We may assume that M is defined over k and that all the slopes are integral. If this is not the case, simply replace the variable x by $x^{1/s}$. In view of the last corollary we may assume that M is generated by a cyclic vector v and we let $P(T)$ be the unique monic polynomial of degree $r = rk(M)$ such that $P(\Phi)v = 0$.

We shall prove the theorem in two stages. First, we show that there exists a basis of M with respect to which Φ is represented by a matrix $A = (a_{ij}) \in GL_r(k)$ where $a_{ii} = c_i x^{\mu_i}$ ($c_i \in \mathbb{C}^\times$, $\mu_1 \leq \mu_2 \leq \dots \leq \mu_r$ are the λ_i with multiplicities), and $a_{ij} = 0$ unless $j = i$ or $j = i + 1$. In particular, A is upper triangular.

Indeed, using Lemma 10 and its Corollary repeatedly we may write

$$P(\Phi) = u_0(\Phi - c_1 x^{\mu_1})u_1(\Phi - c_2 x^{\mu_2})u_2 \cdots u_{r-1}(\Phi - c_r x^{\mu_r})u_r$$

with $u_i \in \mathbb{C}[[x]]^\times$. Let

$$e_i = \{u_i(\Phi - c_{i+1} x^{\mu_{i+1}})u_{i+1} \cdots u_{r-1}(\Phi - c_r x^{\mu_r})u_r\} v.$$

Since $v, \Phi v, \dots, \Phi^{r-1} v$ is a k -basis of M , so is e_1, e_2, \dots, e_r . For $1 \leq i \leq r$

$$(\Phi - c_i x^{\mu_i})e_i = u_{i-1}^{-1} e_{i-1}$$

(where $e_0 = 0$) so the matrix of Φ in the basis $\{e_i\}$ satisfies $a_{ii} = c_i x^{\mu_i}$ and $a_{i-1,i} = u_{i-1}^{-1}$, while all the other $a_{ij} = 0$.

We may assume, without loss of generality, that if $\mu_i = \mu_j$ then $c_i/c_j \notin q^{\mathbb{Z}}$, unless $c_i = c_j$. Indeed, if this were the case, and say $j > i$, replace e_j by some $x^n e_j$, replacing c_j by c_i . We may no longer be able to assume that the $a_{i-1,i}$ are units, but we shall not be using this.

It is now enough to prove the following. Let $\mu_1 \leq \dots \leq \mu_r$ be integers. Let $c_i \in \mathbb{C}^\times$ be such that whenever $\mu_i = \mu_j$, $c_i/c_j \notin q^{\mathbb{Z}}$, unless $c_i = c_j$. Assume that e_1, \dots, e_r is a basis of M , $1 < n \leq r$ and for all $j < n$

$$\Phi(e_j) = x^{\mu_j} \left(c_j e_j + \sum_{i=1}^{j-1} a_{ij} e_i \right)$$

where $a_{ij} \in \mathbb{C}$ and $a_{ij} = 0$ unless $(\mu_i, c_i) = (\mu_j, c_j)$. Assume that

$$\Phi(e_n) = x^{\mu_n} \left(c_n e_n + \sum_{i=1}^{n-1} g_i e_i \right)$$

($g_i \in k$). Then there exists an

$$\tilde{e}_n = e_n + \sum_{i=1}^{n-1} h_i e_i$$

such that

$$\Phi(\tilde{e}_n) = x^{\mu_n} \left(c_n \tilde{e}_n + \sum_{i=1}^{n-1} a_{in} e_i \right)$$

with $a_{in} \in \mathbb{C}$ and $a_{in} = 0$ unless $(\mu_i, c_i) = (\mu_n, c_n)$. Using this inductively we modify the basis with which we started until we get a basis w.r.t. which Φ has the form described in the theorem.

Twisting M by $I_{-\mu_n, 1}$ we may assume that $\mu_n = 0$.

We consider the $h_i \in k$ and the a_{in} as variables and solve for them inductively, starting with $i = n - 1$ and going down. Collecting terms (including the terms arising from the h_j for $j > i$) we get that we have to solve

$$c_n h_i - c_i \tau(h_i) x^{\mu_i} = f_i - a_{in}$$

for some $f_i \in k$. Recall that $\mu_i \leq 0 = \mu_n$. Now if $\mu_i < 0$ this has a solution h_i , with $a_{in} = 0$. If $\mu_i = 0$ but $c_i \neq c_n$ then $c_i q^m \neq c_n$ for all m by assumption and again there is a solution with $a_{in} = 0$. Finally, if $\mu_i = 0$ and $c_i = c_n$ we can cancel out all the terms of f_i except for the constant one, which we kill with a_{in} . This concludes the proof of the theorem. \square

Corollary 15. *A q -difference module over k (or \tilde{k}) descends to \mathbb{C} if and only if all its slopes are 0.*

Borrowing terminology from differential equations, such a module is also called *regular-singular*. We shall not be using this terminology.

Remark 16. The \mathbb{C} -subspaces N_i are not unique. In fact, N_i can be replaced by $x^{\mu_i} N_i$ ($\mu_i \in \mathbb{Q}$) and ϕ_i by $q^{\mu_i} \phi_i$. Two \mathbb{C} -subspaces related in this way will be called *resonants* of each other. It can be checked that this is the only source of non-uniqueness in Theorem 14.

2.1.5. *The structure theorem for a (p, q) -difference module over k .* We now introduce a second operator $\sigma f(x) = f(px)$ for $p \in \mathbb{C}^\times$ such that p and q are multiplicatively independent. We let $\Gamma = \langle \sigma, \tau \rangle \subset \text{Aut}(k)$ and call a Γ -difference module over k a (formal) (p, q) -difference module. Such a module is clearly a q -difference module, and we shall show that the introduction of the second operator Φ_σ , commuting with Φ_τ , imposes serious restrictions on its structure, and forces it to descend to \mathbb{C} .

Theorem 17. *Let M be a (p, q) -difference module over k , for multiplicatively independent p and q . Then M descends to \mathbb{C} .*

Proof. Consider the extension of scalars $M_{\tilde{k}}$ and a decomposition

$$M_{\tilde{k}} = \bigoplus_{i=1}^m \tilde{k} \otimes_{\mathbb{C}} N_i,$$

as given by Theorem 14. Let λ be a slope of $M_{\tilde{k}}$. Then there exists a vector v with $\Phi_{\tau}(v) = cx^{\lambda}v$, for some $c \in \mathbb{C}^{\times}$. Applying Φ_{σ} we have

$$\Phi_{\tau}(\Phi_{\sigma}v) = \Phi_{\sigma}(\Phi_{\tau}v) = \Phi_{\sigma}(cx^{\lambda}v) = cp^{\lambda}x^{\lambda}\Phi_{\sigma}(v).$$

That is, $\Phi_{\sigma}(v)$ is an eigenvector with eigenvalue $cp^{\lambda}x^{\lambda}$. Iterating we find that $I_{\lambda, cp^{\lambda}}$ appears as a Jordan-Hölder constituent of M for all $n \in \mathbb{N}$. Since there are finitely many Jordan-Hölder factors, we have that for some $n > m \geq 0$, $I_{\lambda, cp^{\lambda}} \simeq I_{\lambda, cp^{n\lambda}}$. Thus for some $\alpha \in \mathbb{Q}$ we must have $p^{(n-m)\lambda}q^{\alpha} = 1$. It follows that $\lambda = \alpha = 0$, since p and q are multiplicatively independent, i.e. the only possible q -slope is 0.

Theorem 14 implies that there is a basis e_1, \dots, e_r of M over \tilde{k} such that $[\Phi_{\tau}]_e = A_{\tau}^{-1}$, with $A_{\tau} \in GL_r(\mathbb{C})$. Furthermore, the basis can be chosen so that if c and c' are two distinct eigenvalues of A_{τ} , $c/c' \notin q^{\mathbb{Z}}$. Let A_{σ}^{-1} be the matrix of Φ_{σ} in the same basis. We claim that it is also a constant matrix. In fact, if we write $A_{\sigma}(x) = \sum B_n x^n$ with $B_n \in M_r(\mathbb{C})$, the consistency equation

$$A_{\tau}A_{\sigma}(x) = A_{\sigma}(qx)A_{\tau}$$

implies that

$$A_{\tau}B_nA_{\tau}^{-1} = q^n B_n.$$

The eigenvalues of A_{τ} in its adjoint action are all of the form c/c' for eigenvalues c, c' of A_{τ} . By our assumption $c/c' = q^n$ only if $n = 0$. Hence $B_n = 0$ unless $n = 0$, and A_{σ} is a constant matrix. We conclude that

$$M_0 = \bigoplus_{i=1}^r \mathbb{C}e_i$$

is an underlying \mathbb{C} -structure of M . □

As in theorem 14, the \mathbb{C} -structure M_0 is not unique, because of the existence of resonants.

2.2. Formal (p, q) -Mahler modules.

2.2.1. Rank-1 formal q -Mahler modules. We turn our attention to the case of formal Mahler modules. These are Γ -difference modules over the field of Puiseux series \tilde{k} , where $\Gamma = \langle \tau \rangle$ or $\Gamma = \langle \sigma, \tau \rangle$, and $\sigma f(x) = f(x^p)$, $\tau f(x) = f(x^q)$ are Mahler operators for $p, q \in \mathbb{N}$ multiplicatively independent. We shall call such Γ -difference modules (formal) q -Mahler modules, when $\Gamma = \langle \tau \rangle$, and (formal) (p, q) -Mahler modules when $\Gamma = \langle \sigma, \tau \rangle$. As before, we start by studying the structure of q -Mahler modules, and then examine the restriction imposed by the introduction of the second operator Φ_{σ} .

Rank-1 q -Mahler modules are classified by $\tilde{k}^{\times}/(\tilde{k}^{\times})^{\tau-1}$. Once again, we pick $f \in \tilde{k}^{\times}$ and write $f = cx^{\lambda}f_1$ where $c \in \mathbb{C}^{\times}$ and $f_1 \in U_1(k_s)$. This time both f_1 and x^{λ} are in $(\tilde{k}^{\times})^{\tau-1}$, as $(x^{\mu})^{\tau-1} = x^{\mu(q-1)}$. We get the following easy Proposition.

Proposition 18. *Every rank-1 q -Mahler module over \tilde{k} descends to \mathbb{C} . Let I_c , for $c \in \mathbb{C}^{\times}$, be the rank-1 module $\tilde{k}e$, with $\Phi_{\tau}(e) = ce$. Then every rank-1 module is isomorphic to I_c for a unique c .*

2.2.2. *Factorization in $\tilde{k}\langle\Phi\rangle$.* We consider the twisted polynomial ring $\tilde{k}\langle\Phi\rangle$ consisting of polynomials

$$\sum_{i=0}^n a_i \Phi^{n-i},$$

($a_i \in \tilde{k}$) where $\Phi a = \tau(a)\Phi$.

Lemma 19. *Let $\sum_{i=0}^n a_i \Phi^{n-i} \in \tilde{k}\langle\Phi\rangle$, and assume $a_0 = 1$, $a_n \neq 0$. Then there exist $c \in \mathbb{C}^\times$, $\mu \in \mathbb{Q}$, $b_0, \dots, b_{n-1} \in \tilde{k}$ such that $b_0 \in U_1(\tilde{k})$, $b_{n-1} \neq 0$ and*

$$\sum_{i=0}^n a_i \Phi^{n-i} = \tau(b_0)^{-1} (\Phi - c x^\mu) \sum_{i=0}^{n-1} b_i \Phi^{n-1-i}.$$

[Compare Chapter IV, §4 Lemma 2 in Demazure's *Lectures on p -divisible groups* LNM 302 (1972) Springer-Verlag. That lemma is key to the Manin-Dieudonné classification of F -isocrystals over an algebraically closed field of characteristic p , or - what amounts to the same - the classification of p -divisible groups over such a field up to isogeny.]

Proof. To simplify the notation we write, in the proof of the lemma only, $a^{(i)} = \tau^i(a)$. Write also $u = b_0^{(1)}$. We have to find $\mu, u, b_1, \dots, b_{n-1}$ and c as in the lemma satisfying the equations

$$(2.2) \quad u a_i = b_i^{(1)} - c x^\mu b_{i-1} \quad (0 \leq i \leq n)$$

($b_{-1} = b_n = 0$). Solving successively for b_i we get the equation

$$(2.3) \quad (u a_0) c^n + (u^{(1)} a_1^{(1)} x^{-\mu q}) c^{n-1} + \dots + (u^{(n)} a_n^{(n)} x^{-\mu(q+\dots+q^n)}) = 0,$$

which we have to solve for $u \in U_1(\tilde{k})$ and $c \in \mathbb{C}^\times$. Let

$$\mu = \min_{1 \leq i \leq n} \left(\frac{1 - 1/q}{1 - 1/q^i} \right) \nu(a_i)$$

where ν is the valuation on \tilde{k} , normalized by $\nu(x) = 1$. Note that

$$\nu(a_i^{(i)} x^{-\mu(q+\dots+q^i)}) = q^i \left(\nu(a_i) - \mu \left(\frac{1 - 1/q^i}{1 - 1/q} \right) \right) \geq 0$$

and there exists an index $i \geq 1$ for which this is 0. This means that the expression $a_i^{(i)} x^{-\mu(q+\dots+q^i)}$, appearing together with $u^{(i)}$ as the coefficient of c^{n-i} , is integral, i.e. has no pole, and at least one such expression, besides the leading one, is a unit. Replacing x by x^s for a suitable s , we may assume that all the exponents of x appearing in (2.3) are integral. We solve (2.3) modulo higher and higher powers of x , setting

$$u = 1 + d_1 x + d_2 x^2 + \dots$$

and choosing the d_m successively. By what we have seen, there exists a $c \neq 0$ in \mathbb{C} solving (2.3) modulo x (i.e. substituting $x = 0$). Noting that

$$u^{(i)} = 1 + d_1 x^{q^i} + d_2 x^{2q^i} + \dots$$

it is then an easy matter to solve successively for the d_m . □

Corollary 20. *(Compare with Theorem 15 in [Roq].) Every monic polynomial from $\tilde{k}\langle\Phi\rangle$ factors as*

$$u_0(\Phi - c_1x^{\mu_1})u_1(\Phi - c_2x^{\mu_2})u_2 \cdots u_{n-1}(\Phi - c_nx^{\mu_n})u_n$$

where the $\mu_j \in \mathbb{Q}$, $\mu_j \leq q\mu_{j+1}$, $c_j \in \mathbb{C}^\times$ and $u_j \in U_1(\tilde{k})$.

Proof. Apply the lemma inductively. The relation $\mu_j \leq q\mu_{j+1}$ follows from the inequality

$$q\nu(b_i) \geq \mu\left(1 + \frac{1}{q} + \cdots + \frac{1}{q^{i-1}}\right)$$

which is proved by induction on i , based on (2.2). \square

2.2.3. *The structure theorem for a q -Mahler module over \tilde{k} .* Let M be a q -Mahler module over \tilde{k} . Similarly to Proposition 12 we get the following structure theorem for M .

Theorem 21. *(Compare with Theorem 9 in [Roq].) Let M be a q -Mahler module over \tilde{k} . Then M has an ascending filtration with one-dimensional graded pieces of the form I_c .*

Proof. It is enough to prove that any q -Mahler module over \tilde{k} has a rank 1 submodule. Let u be any non-zero vector in M , and n the minimal number such that $u, \Phi u, \dots, \Phi^n u$ are linearly dependent over \tilde{k} . Let $\sum_{i=0}^n a_i \Phi^{n-i} u = 0$ be a linear dependence with $a_0 = 1$, and decompose the polynomial as in the lemma. Let $v = \sum_{i=0}^{n-1} b_i \Phi^{n-1-i} u$. Note that $v \neq 0$ by our assumption on n . Then $\Phi v = cx^\mu v$. If $\mu \neq 0$ replace v by $x^{-\mu/(q-1)}v$. \square

Contrary to Theorem 14 we do not have at our disposal a more refined structure theorem describing the off-diagonal entries in the resulting upper triangular matrix associated with Φ . One can not expect to have all the entries in \mathbb{C} , because a general q -Mahler module need not descend to \mathbb{C} . However, the theorem we have just proved suffices to obtain the Mahler analogue of Theorem 17.

2.2.4. *The structure theorem for a (p, q) -Mahler module over \tilde{k} .* We now consider a pair of operators σ and τ as above

$$\sigma f(x) = f(x^p), \quad \tau f(x) = f(x^q)$$

for multiplicatively independent $p, q \in \mathbb{N}$. Let $\Gamma = \langle \sigma, \tau \rangle \subset \text{Aut}(\tilde{k})$. A Γ -difference module over \tilde{k} will be called a (formal) (p, q) -Mahler module.

Theorem 22. *Every (p, q) -Mahler module over \tilde{k} admits a unique \mathbb{C} -structure.*

Proof. In terms of matrices, we have to show that any two matrices $A = A_\tau$ and $B = A_\sigma$ in $GL_r(\tilde{k})$ satisfying the consistency condition

$$\sigma(A)B = \tau(B)A$$

are gauge-equivalent to a pair of commuting constant matrices, unique up to conjugation.

The uniqueness is easy. Suppose (A, B) is a commuting pair of constant matrices and $C \in GL_r(\tilde{k})$ is such that

$$(A', B') = (\tau(C)^{-1}AC, \sigma(C)^{-1}BC)$$

are also constant. Replacing x by some x^s we may assume that the entries of C are all in k . Then

$$C = A^{-1}\tau(C)A' = A^{-2}\tau^2(C)A'^2 = \dots$$

so $C \in GL_r(\mathbb{C})$, because its Laurent expansion is supported in degrees divisible by q^n for every n .

We next remark that if (A, B) is a consistent pair in $GL_r(\tilde{k})$ with $A \in GL_r(\mathbb{C})$ then $B \in GL_r(\mathbb{C})$ as well. Indeed, if A is constant the consistency equation takes the form

$$AB = \tau(B)A.$$

Under a change of variable we may assume that the entries of B are all in k . As above, this yields

$$B = A^{-1}\tau(B)A = A^{-2}\tau^2(B)A^2 = \dots$$

so $B \in GL_r(\mathbb{C})$.

Let M be a (p, q) -Mahler module over \tilde{k} . Theorem 21 guarantees that for some $c \in \mathbb{C}^\times$ the space

$$W = \{v \in M \mid \Phi_\tau v = cv\}$$

is non-zero. It is easily seen that vectors in W which are linearly independent over \mathbb{C} are also linearly independent over \tilde{k} . Indeed, if $\sum_{i=1}^m a_i v_i = 0$ is a shortest linear dependence over \tilde{k} between some \mathbb{C} -independent vectors in W , with $a_1 = 1$, apply Φ_τ to get (after dividing by c) $\sum_{i=1}^m \tau(a_i) v_i = 0$. This shows that all $\tau(a_i) = a_i$, hence $a_i \in \mathbb{C}$, or else we get by subtraction a shorter linear dependence. But this contradicts the linear independence of the v_i over \mathbb{C} . It follows that W is finite dimensional over \mathbb{C} . It is evidently preserved by Φ_σ . Thus we may find an eigenvector $e \in W$ for Φ_σ , namely $\Phi_\sigma(e) = de$, for $d \in \mathbb{C}^\times$. This means that $\tilde{k}e = M_1$ is a rank-1 (p, q) -Mahler submodule of M . Continuing in this way with M/M_1 etc. we arrive at a filtration of M by (p, q) -Mahler submodules, whose graded pieces are of rank 1 and admit a \mathbb{C} -structure.

In terms of the matrices A, B with which we started, this means that we may assume that they are lower triangular, with diagonal entries in \mathbb{C}^\times . It remains to prove that they are gauge-equivalent to a lower triangular pair (A', B') with A' constant. As mentioned above, the fact that B' is also constant will follow suit. Write

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}$$

with $A_{11} \in \mathbb{C}^\times$, $A_{21} \in M_{r-1,1}(\tilde{k})$, $A_{22} \in GL_{r-1, r-1}(\tilde{k})$, and similarly for B . The consistency equation for A and B implies the same equation for A_{22} and B_{22} . Hence by induction we may assume that A_{22} and B_{22} are constant lower triangular. It remains to descend the constants in A_{21} .

The consistency equation now takes the form

$$(2.4) \quad A_{21}(x^p)B_{11} + A_{22}B_{21}(x) = B_{21}(x^q)A_{11} + B_{22}A_{21}(x).$$

After a change of variable we may assume that all the exponents appearing in the equations are integers. We will show that if $A_{21}(x)$ or $B_{21}(x)$ have a pole at 0, replacing the pair (A, B) by an equivalent pair, without affecting the diagonal blocks, we can reduce the order of the pole, until we get rid of the polar parts

altogether. To simplify the argument we shall assume that $(p, q) = 1$. We shall explain how to get rid of this assumption at the end of the proof.

Let Mx^{-m} be the lowest term in $A_{21}(x)$ and Nx^{-n} the lowest term in $B_{21}(x)$ where $M, N \in M_{r-1,1}(\mathbb{C})$. Assume that there is a pole, i.e. $n, m \geq 1$ (otherwise there is nothing to prove). Then looking at the lowest order terms in (2.4) gives $qn = pm$ and $MB_{11} = NA_{11}$. By our assumption that $(p, q) = 1$, $m/q = n/p$ is an integer. Let

$$C(x) = \begin{pmatrix} I & 0 \\ -MA_{11}^{-1}x^{-m/q} & I \end{pmatrix}.$$

Then the pair $(\tilde{A}, \tilde{B}) = (\tau(C)AC^{-1}, \sigma(C)BC^{-1})$ has the same shape of (A, B) with

$$\begin{aligned} \tilde{A}_{21}(x) &= -Mx^{-m} + A_{21}(x) + A_{22}MA_{11}^{-1}x^{-m/q}, \\ \tilde{B}_{21}(x) &= -Nx^{-n} + B_{21}(x) + B_{22}NB_{11}^{-1}x^{-n/p}. \end{aligned}$$

The order of the poles of \tilde{A} and \tilde{B} is smaller than their order in A and B . Continuing inductively we can eliminate the polar parts altogether.

We may therefore assume that the pair (A, B) has no poles. To conclude we need to solve the equation

$$\tilde{A}_{21}(x) = A_{21}(x) + C_{21}(x^q)A_{11} - A_{22}C_{21}(x),$$

for $C_{21}(x)$ so that $\tilde{A}_{21}(x)$ is constant. Taking the left hand side to be $A_{21}(0)$, we can find $C_{21} \in M_{r-1,1}(x\mathbb{C}[[x]])$ solving succesively for the coefficients of x^n , $n \geq 1$. This concludes the proof under the assumption that $(p, q) = 1$.

If $(p, q) = \ell > 1$ the Laurent expansions of \tilde{A}_{21} and \tilde{B}_{21} which were constructed in the first step might have a term with fractional degree, with denominator dividing ℓ (the denominator in $m/q = n/p$). Ignore this issue and continue inductively as before, each time removing the terms of lowest degrees. As long as we have not reached the terms of degree $-m/q$ in \tilde{A}_{21} , the lowest terms in it will have integral degree $-m < -m' < -m/q$, and we will be able to remove these lowest terms by a gauge transformation as above, introducing a term with fractional degree with denominator at worst ℓ , in degree $-m/q < -m'/q$. Symmetrically, we may remove all the polar part of \tilde{B}_{21} up to degree $-n/p$, introducing at worst ℓ in the denominators of the exponents of x . Once we reach the first fractional degree, we substitute ξ^ℓ for x , getting new power series with *integral degrees* in ξ (i.e. matrix entries in $\mathbb{C}((\xi))$ satisfying (2.4)). If $p = \ell p'$ and $q = \ell q'$ then the lowest degree in the new $A_{21}(\xi)$ will be $-m/q'$ and the lowest degree in the new B_{21} will be $-n/p'$. As at least one of p', q' is > 1 , we can continue by induction until we remove all the polar part as before. \square

Remark 23. A careful analysis of the proof of the theorem shows that the role of the second Mahler operator Φ_σ in it was minor. It was only used to guarantee that the process of reducing the matrix C to a constant matrix by means of gauge equivalence transformations terminates after finitely many steps. If we replace the field of Puiseux series by the field of Hahn series we can get rid of the polar parts of the entries in C in *countably* many steps that yield a *convergent* Hahn series (with matrix coefficients). Once the polar parts have been eliminated, the rest of the proof is the same. Thus the above proof can be modified to prove the main theorem (Theorem 2) of [Roq], that any q -Mahler module over the field of Hahn series descends to \mathbb{C} . Moreover, if the original module was defined over $\mathbb{C}((x))$ then the

matrix C needed to descend its structure to \mathbb{C} (i.e. to make $C(x^q)^{-1}A(x)C(x) = A_0$ constant) would have entries in Hahn series whose supports are well-ordered subsets of $\mathbb{Z}[1/q]$. In essence, this is the approach taken by Julien Roques.

This remark should be contrasted with Theorem 17. In the case $2\mathbf{Q}$ the second (p -difference) operator was used in a more substantial way, to guarantee that the slopes of the first (q -difference) operator were all 0, and vice versa.

3. THE STRUCTURE OF RATIONAL Γ -DIFFERENCE MODULES

In this part we follow the method of [Sch-Si1, Sch-Si2].

3.1. (p, q) -Mahler modules. In this section we prove Theorem 7 in the case $2\mathbf{M}$. Recall that

$$\tilde{K} = \bigcup_{s=1}^{\infty} \mathbb{C}(x^{1/s}), \quad \tilde{k} = \bigcup_{s=1}^{\infty} \mathbb{C}((x^{1/s})),$$

and the two Mahler operators are

$$\sigma(x^{1/s}) = x^{p/s}, \quad \tau(x^{1/s}) = x^{q/s}$$

where p and q are multiplicatively independent natural numbers.

Let $\Gamma = \langle \sigma, \tau \rangle \subset \text{Aut}(\tilde{K})$ (or $\text{Aut}(\tilde{k})$). Then Γ is free abelian of rank 2. Let M be a rank r Γ -difference module over \tilde{K} (called also a (p, q) -Mahler module). Fix a basis of M and let $A = A_\sigma$ and $B = A_\tau$ be the matrices attached to Φ_σ and Φ_τ in this basis as in §1.2. Changing variables, and writing x for $x^{1/s}$ if necessary, we may assume that $A, B \in GL_r(K)$, where $K = \mathbb{C}(x)$.

Let $i = 0, 1$ or ∞ . Let $t_0 = x$, $t_1 = x - 1$ and $t_\infty = 1/x$ be local parameters at the corresponding point. At the point 1 we shall also use $z = \log(x) = \log(1 + t_1)$ as a formal parameter. Let $k_i = \mathbb{C}((t_i))$ be the completion of $K = \mathbb{C}(x)$ at the point i . By base-change we may regard $M_{\tilde{k}_i}$ for $i = 0, \infty$ as formal (p, q) -Mahler modules over \tilde{k}_i , and M_{k_1} as a formal (p, q) -difference module over k_1 . In the latter case we use the variable z , in terms of which $\sigma(z) = pz$ and $\tau(z) = qz$.

- **Step I.** After writing x for $x^{1/s}$ if necessary, there exist matrices $C_i \in GL_r(k_i)$ ($i = 0, 1, \infty$) such that

$$\begin{cases} \sigma(C_i)^{-1}AC_i = A_i \\ \tau(C_i)^{-1}BC_i = B_i \end{cases}$$

are constant matrices.

By Theorems 17 and 21 there exist such matrices over $\tilde{k}_0, \tilde{k}_\infty$ and k_1 . After a change of variable, substituting x for $x^{1/s}$, we may assume that C_0 and C_∞ have entries in k_0 and k_∞ . Such a move leaves the field k_1 unchanged.

- **Step II.** We may assume that $C_i \in GL_r(\mathcal{O}_i)$ where \mathcal{O}_i is the ring of integers of k_i , and

$$C_i \equiv I \pmod{t_i}$$

A global gauge transformation replaces (A, B) by $(\sigma(C)^{-1}AC, \tau(C)^{-1}BC)$ and C_i by $C^{-1}C_i$ for some $C \in GL_r(K)$. The constant matrices (A_i, B_i) are unchanged. By weak approximation in the field K , we may find a $C \in GL_r(K)$ such that $C^{-1}C_i$ are, simultaneously, as close as we wish to $I \in GL_r(k_i)$, and in particular are in the open set $GL_r(\mathcal{O}_i)$, and congruent to I modulo the maximal ideal.

Observe that once $C_i \in GL_r(\mathcal{O}_i)$, then also $A \in GL_r(\mathcal{O}_i)$. Since A is meromorphic, it is holomorphic at the point i . The same applies to the matrix B . Furthermore, the assumption $C_i \equiv I \pmod{t_i}$ implies $A \equiv A_i \pmod{t_i}$, $B \equiv B_i \pmod{t_i}$.

- **Step III.** Each C_i is holomorphic at some neighborhood of the point $i = 0, 1, \infty$.

To prove this we use estimates on the coefficients in the formal Taylor expansion. For example, at $i = 0$, write

$$C_0(x) = M_0 + M_1x + M_2x^2 + \cdots, \quad A(x) = A_0 + N_1x + N_2x^2 + \cdots$$

($M_0 = I$). From $A(x)C_0(x) = C_0(x^p)A_0$ we get the recursion formula

$$A_0M_n + \sum_{i=1}^n N_iM_{n-i} = M_{n/p}A_0$$

where $M_{n/p} = 0$ if p does not divide n . Let $\|\cdot\|$ be any norm on the space of $r \times r$ complex matrices (they are all equivalent). The analyticity of $A(x)$ at 0 implies that there exists a $c_1 > 0$ such that $\|N_i\| < c_1^i$. It follows easily from this and from the recursion formula that $\|M_i\| < c_2^i$ for some $c_2 > 0$, hence that $C_0(x)$ converges absolutely in $|x| < c_2^{-1}$. The point $i = \infty$ is treated similarly.

At $i = 1$, using expansions in the local parameter $z = \log(x)$, we have

$$C_1(x) = C_1(e^z) = \tilde{C}_1(z) = M_0 + M_1z + M_2z^2 + \cdots, \quad A(e^z) = A_1 + N_1z + N_2z^2 + \cdots$$

($M_0 = I$). From $A(e^z)\tilde{C}_1(z) = \tilde{C}_1(pz)A_1$ we now get the recursion formula

$$M_n p^n - A_1 M_n A_1^{-1} = \sum_{i=1}^n N_i M_{n-i} A_1^{-1}.$$

Since for $n \gg 0$

$$0.9p^n \|M_n\| \leq \|M_n p^n - A_1 M_n A_1^{-1}\| \leq 1.1p^n \|M_n\|,$$

we may conclude the proof of step III as before.

- **Step IV.** The matrix $C_0(x)$ admits meromorphic continuation to $0 \leq |x| < 1$, and the matrix $C_\infty(x)$ admits meromorphic continuation to $1 < |x| \leq \infty$.

The functional equation $C_0(x) = A(x)^{-1}C_0(x^p)A_0$ shows that if $C_0(x)$ has meromorphic continuation to the disk $D(0, r)$ for some $r < 1$, then it has such a meromorphic continuation to $D(0, r^{1/p})$. Since for r small enough $C_0(x)$ is in fact holomorphic in $D(0, r)$, the claim follows. The same argument holds at ∞ .

- **Step V.** The matrix $C_0(x)$ admits meromorphic continuation to $0 \leq |x| < \infty$. Similarly $C_\infty(x)$ admits meromorphic continuation to $0 < |x| \leq \infty$.

Crossing the natural boundary at $|x| = 1$ is subtle. This is where the expansion around $i = 1$ comes to our rescue. Recall that $C_1(x)$ is a-priori defined and analytic only in $|x-1| < \varepsilon$ for some $0 < \varepsilon$. Trying to use *one* of the two functional equations

$$\begin{cases} A_1 = C_1(x^p)^{-1}A(x)C_1(x) \\ B_1 = C_1(x^q)^{-1}B(x)C_1(x) \end{cases}$$

to meromorphically continue it to $0 < |x| < \infty$ as we did with C_0 or C_∞ leads to issues of monodromy. The key idea, due to [Sch-Si1, Sch-Si2], is to use *both*

functional equations to overcome the monodromy. The arguments below constitute a slight variation on the original arguments.

Write $x = e^z$ and define, for $i = 0, 1$,

$$\tilde{C}_i(z) = C_i(e^z).$$

By the previous step, $\tilde{C}_0(z)$ is meromorphic in $\{Re(z) < 0\}$ and is $2\pi i$ -periodic there, while $\tilde{C}_1(z)$ is a-priori defined and analytic only in a neighborhood of $z = 0$. The functional equation

$$A_1 = \tilde{C}_1(pz)^{-1}A(e^z)\tilde{C}_1(z)$$

gives a meromorphic continuation of $\tilde{C}_1(z)$ to all $z \in \mathbb{C}$.

Let

$$\tilde{C}_{01}(z) = \tilde{C}_0(z)^{-1}\tilde{C}_1(z) \quad (Re(z) < 0).$$

Then

$$\begin{cases} \tilde{C}_{01}(pz) = A_0\tilde{C}_{01}(z)A_1^{-1} \\ \tilde{C}_{01}(qz) = B_0\tilde{C}_{01}(z)B_1^{-1}. \end{cases}$$

Lemma 24. *There exist $\pi/2 < \alpha < \beta < 3\pi/2$ such that $\tilde{C}_{01}(z)$ is analytic in the sector $\{\alpha < Arg(z) < \beta\}$.*

Proof. Since the poles of $\tilde{C}_{01}(z)$ have no accumulation point in $\{Re(z) < 0\}$, there are no poles in

$$S = \{z | p^{-1} \leq |z| \leq p, \quad \alpha < Arg(z) < \beta\},$$

for suitable $\pi/2 < \alpha < \beta < 3\pi/2$. The relation $\tilde{C}_{01}(pz) = A_0\tilde{C}_{01}(z)A_1^{-1}$ now yields the lemma. \square

Lemma 25. *$\tilde{C}_0(z)$ has a meromorphic continuation to all $z \in \mathbb{C}$ and is $2\pi i$ -periodic.*

Proof. Assume that we prove

- $\tilde{C}_{01}(z)$ has an analytic continuation to $\mathbb{C} \setminus [0, \infty)$.

Then $\tilde{C}_0(z) = \tilde{C}_1(z)\tilde{C}_{01}(z)^{-1}$ also admits a meromorphic continuation to $\mathbb{C} \setminus [0, \infty)$. In $\{Re(z) < 0\}$, $\tilde{C}_0(z)$ was $2\pi i$ -periodic. It is therefore $2\pi i$ -periodic in the upper half plane, so extends by periodicity to the whole complex plane, and the lemma is verified.

To prove that $\tilde{C}_{01}(z)$ has an analytic continuation to $\mathbb{C} \setminus [0, \infty)$ consider

$$D(w) = \tilde{C}_{01}(e^w),$$

a-priori analytic in the strip $\{\alpha < Im(w) < \beta\}$. It satisfies there

$$\begin{cases} D(w + \log(p)) = A_0D(w)A_1^{-1} \\ D(w + \log(q)) = B_0D(w)B_1^{-1}. \end{cases}$$

Recall that A_0 and B_0 commute, and so do A_1 and B_1 . Let L_0 and L_1 be matrices commuting with B_0 and B_1 respectively, such that

$$A_0 = e^{L_0}, \quad A_1 = e^{L_1}.$$

Then

$$E(w) = e^{-L_0w/\log p}D(w)e^{L_1w/\log p}$$

satisfies

$$\begin{cases} E(w + \log(p)) = E(w) \\ E(w + \log(q)) = F_0 E(w) F_1^{-1} \end{cases},$$

with

$$F_i = B_i e^{-L_i \log q / \log p}.$$

The first equation implies that in the strip $\{\alpha < \operatorname{Im}(w) < \beta\}$ we have a Fourier expansion

$$E(w) = \sum_{n \in \mathbb{Z}} E_n e^{2\pi i n w / \log p}$$

with constant matrices E_n . The second equation then implies

$$E_n e^{2\pi i n \log q / \log p} = F_0 E_n F_1^{-1}.$$

As the linear transformation $M \mapsto F_0 M F_1^{-1}$ on the space of $r \times r$ matrices can have only finitely many eigenvalues, and as the numbers $e^{2\pi i n \log q / \log p}$ are all distinct (thanks to the multiplicative independence of p and q), we deduce that $E_n = 0$ for all but finitely many n . This shows that $E(w)$, and with it $D(w)$, are entire.

Going back to the definition of $D(w)$ we conclude that $\tilde{C}_{01}(z) = D(\log(z))$, a priori only analytic in the sector $\alpha < \operatorname{Arg}(z) < \beta$, has an analytic continuation to $\mathbb{C} \setminus [0, \infty)$. Here $\log(z)$ denotes the principal branch of \log on the complement of $[0, \infty)$. This concludes the proof of the lemma. \square

The Lemma clearly implies that $C_0(x)$ may be meromorphically continued all the way to the north pole. The matrix $C_\infty(x)$ is treated in the same way. The proof of Theorem 7 is concluded with the following (last) step.

• **Step VI.** The matrix $C_0(x) \in GL_r(K)$.

Consider $C_{0\infty}(x) = C_0(x)^{-1} C_\infty(x)$. This matrix is meromorphic in $0 < |x| < \infty$ and satisfies

$$C_{0\infty}(x^p) = A_0 C_{0\infty}(x) A_\infty^{-1}.$$

In any annulus $V(r_1, r_2) = \{r_1 < |x| < r_2\}$ where $C_{0\infty}(x)$ is analytic, it has a power series expansion

$$C_{0\infty}(x) = \sum_{n \in \mathbb{Z}} M_n x^n.$$

The functional equation relates the expansions on $V(r_1, r_2)$ and $V(r_1^p, r_2^p)$, where $C_{0\infty}(x)$ is also analytic, and shows that $M_n = 0$ unless $p|n$. Iterating, we see that $M_n = 0$ unless $p^k|n$ for $k = 1, 2, \dots$. It follows that the only non-zero coefficient is M_0 and $C_{0\infty}$ is constant. Since $C_\infty(x)$ is meromorphic (in fact analytic) at ∞ , $C_0(x)$ is meromorphic everywhere on $\mathbb{P}^1(\mathbb{C})$, hence is a matrix of rational functions.

3.2. (p, q) -difference modules. In this section we prove Theorem 7 in the case 2Q. Recall that $K = \mathbb{C}(x)$ and the two difference operators are

$$\sigma(x) = px, \quad \tau(x) = qx,$$

where p and q are multiplicatively independent non-zero complex numbers. We make the following assumption:

• (Hyp) At least one of p or q is of absolute value $\neq 1$.

Without loss of generality (replacing σ by σ^{-1} or by $\tau^{\pm 1}$, and afterwards replacing τ by τ^{-1} , if necessary), we may assume that $|p| > 1$ and $|q| \geq 1$. At the end of the proof we shall explain how to eliminate (Hyp).

Let $\Gamma = \langle \sigma, \tau \rangle \subset \text{Aut}(K)$. Then Γ is free abelian of rank 2. Let M be a rank r Γ -difference module over K (called also a (p, q) -difference module). Fix a basis of M and let $A = A_\sigma$ and $B = A_\tau$ be the matrices attached to Φ_σ and Φ_τ in this basis as in §1.2.

For $i = 0$ or ∞ let $t_0 = x$ and $t_\infty = 1/x$ be local parameters at the point i . Let $k_i = \mathbb{C}((t_i))$ be the completion of $K = \mathbb{C}(x)$ at the point i . By base-change we may regard M_{k_i} for $i = 0, \infty$ as formal (p, q) -difference modules over k_i . The proofs of the first three steps below are exactly the same as in the case 2M, so we omit them.

- **Step I.** There exist matrices $C_i \in GL_r(k_i)$ ($i = 0, \infty$) such that

$$\begin{cases} \sigma(C_i)^{-1}AC_i = A_i \\ \tau(C_i)^{-1}BC_i = B_i \end{cases}$$

are constant matrices.

- **Step II.** We may assume that $C_i \in GL_r(\mathcal{O}_i)$ where \mathcal{O}_i is the ring of integers of k_i , and

$$C_i \equiv I \pmod{t_i}.$$

- **Step III.** C_i is holomorphic at some neighborhood of the point $i = 0, \infty$.
- **Step IV.** The matrix $C_0(x)$ admits meromorphic continuation to $0 \leq |x| < \infty$, and the matrix $C_\infty(x)$ admits meromorphic continuation to $0 < |x| \leq \infty$.

As before, we use the functional equation $C_0(px) = A(x)C_0(x)A_0^{-1}$ to meromorphically continue $C_0(x)$ from $D(0, r)$ to $D(0, pr)$. Here the assumption $|p| > 1$ is used. A similar argument works for $C_\infty(x)$.

- **Step V.** The matrix $C_0(x) \in GL_r(K)$.

Consider the functional equations

$$\begin{cases} C_0(px) = A(x)C_0(x)A_0^{-1} \\ C_0(qx) = B(x)C_0(x)B_0^{-1} \end{cases}.$$

Let R be large enough so that $A(x)$ and $B(x)$ and their inverses have no poles in $U = \{R < |x| < \infty\}$. Let S be the set of poles of $C_0(x)$ in U . The functional equations imply that if x and px , or x and qx , are both in U , then they are either both in S or both not in S . As p and q are multiplicatively independent, we see that if S is not empty then for a suitable R' the compact subset $Z = \{R' \leq |x| \leq p^2R'\} \subset U$ contains infinitely many distinct points of the form $p^{-a}q^b x_0$ for some $x_0 \in S$ and $a, b \geq 0$. Indeed, if $|q| = 1$ we may take the points $q^b x_0$ where R' is chosen so that $Z \cap S$ is non-empty, and $x_0 \in Z \cap S$. If $|q| > 1$ we take $q^b x_0$ ($b \geq 0$) and then find for each b an a such that $p^{-a}q^b x_0 \in Z$. This implies however that Z contains infinitely many points in S . It follows that S is empty, and $C_0(x)$ is analytic in U .

Consider, as in the case 2M, the function $C_{0\infty}(x) = C_0(x)^{-1}C_\infty(x)$. By choosing R large enough we see that $C_{0\infty}(x)$ is analytic in U , so admits there a power series expansion

$$C_{0\infty}(x) = \sum_{n \in \mathbb{Z}} M_n x^n.$$

Furthermore, it satisfies in U the functional equation

$$C_{0\infty}(px) = A_0 C_{0\infty}(x) A_\infty^{-1},$$

implying $p^n M_n = A_0 M_n A_\infty^{-1}$. As the linear transformation $M \mapsto A_0 M A_\infty^{-1}$ can have only finitely many eigenvalues, $M_n = 0$ for all but finitely many values of n . It follows that $C_{0\infty}(x)$, and with it $C_0(x)$, is meromorphic at ∞ . Thus the entries of $C_0(x)$ are everywhere meromorphic on $\mathbb{P}^1(\mathbb{C})$, so belong to $K = \mathbb{C}(x)$. This concludes the proof of the last step, and with it of the main theorem, under the assumption (Hyp).

• **Step VI.** Elimination of the assumption (Hyp).

As we have seen in Remark (iv) following Theorem 7, while the proof of Step IV above used the dynamics of $z \mapsto pz$ (namely the fact that by iterating this map an arbitrarily small open neighborhood of 0 eventually covered the whole of \mathbb{C}), the statement of the Main Theorem is purely algebraic. Thus (Hyp) can be weakened to assume that *under some abstract automorphism* ι of \mathbb{C} one of $\iota(p)$ or $\iota(q)$ does not lie on the unit circle. There are still algebraic numbers for which this can not be achieved. For example, if E is a CM field and $p = P/\bar{P}$, $q = Q/\bar{Q}$ for some $P, Q \in E$ (it is an easy exercise that we can make such p and q multiplicatively independent).

However, let ℓ be an auxiliary rational prime, let \mathbb{C}_ℓ be the completion of an algebraic closure of \mathbb{Q}_ℓ , and consider an abstract algebraic isomorphism

$$\iota : \mathbb{C} \simeq \mathbb{C}_\ell.$$

Such a ι exists because both fields have the same transcendence cardinality and are algebraically closed. Now, the entire proof given above works, *mutatis mutandis*, over \mathbb{C}_ℓ instead of \mathbb{C} , provided (Hyp) is replaced by (Hyp) $_\ell$: *At least one of $\iota(p)$ or $\iota(q)$ is of absolute value $\neq 1$* . One should understand “analytic” or “meromorphic” in the rigid analytic sense. Note that the only step where Calculus was used was Step III, and this step becomes even easier over \mathbb{C}_ℓ thanks to the ultrametric inequality.

It follows that the only case not covered by the above proof is when p , and similarly q , maps to the unit circle under any field isomorphism $\iota : \mathbb{C} \simeq \mathbb{C}_\ell$ for any prime ℓ , including ∞ . It is well-known that this happens if and only if p and q are both roots of unity, a case ruled out by the assumption on multiplicative independence.

4. p -MAHLER q -DIFFERENCE MODULES

In this part we illustrate the same approach used in cases 2M and 2Q in a third example, where the group Γ is generated by one q -difference operator and one Mahler operator, and turns out to be generalized dihedral. We therefore call this *Case 1M1Q*.

4.1. Formal p -Mahler q -difference modules.

4.1.1. *The group Γ .* Let \tilde{K} be as before, let $p \geq 2$ be a natural number and $q \in \mathbb{C}^\times$ a complex number which is not a root of unity. No assumption of independence is made on p and q . Fix a compatible sequence of roots $q^{1/s}$ as before.

Let $\Gamma = \langle \sigma, \tau \rangle \subset \text{Aut}(\tilde{K})$ where

$$\sigma(x^{1/s}) = x^{p/s}, \quad \tau(x^{1/s}) = q^{1/s} x^{1/s}.$$

The easily verified relation

$$\sigma^{-1} \circ \tau \circ \sigma = \tau^p$$

yields

$$\Gamma \simeq \mathbb{Z} \ltimes \mathbb{Z}\left[\frac{1}{p}\right]$$

where $(-n, 0)(0, a)(n, 0) = (0, p^n a)$. Here $\sigma \mapsto (1, 0)$ and $\tau \mapsto (0, 1)$. Thus Γ is *generalized dihedral* rather than abelian.

Lemma 26. *Every element of Γ is of the form $\sigma^a \tau^b \sigma^c$ for $a, b, c \in \mathbb{Z}$.*

Proof. Every element of Γ is of the form $\sigma^i \tau^{j/p^n}$ for some $i, j, n \in \mathbb{Z}$. But

$$\sigma^i \tau^{j/p^n} = \sigma^{i+n} \tau^j \sigma^{-n}.$$

□

4.1.2. *p-Mahler q-difference modules.* We shall call a Γ -difference module over \tilde{K} (or \tilde{k}) a *p-Mahler q-difference module*. Let $f \in \tilde{k}$, and assume that the $\tilde{K}\langle\Gamma\rangle$ -submodule of \tilde{k} generated by f is finite dimensional over \tilde{K} . Then this module is a *p-Mahler q-difference module* over \tilde{K} , and arguments similar to those of Proposition 8 may be applied.

We label this new case by 1M1Q. Unlike cases 2M and 2Q, for $M = \tilde{K}\langle\Gamma\rangle f$ to be finite dimensional over \tilde{K} , the (necessary) condition that f satisfies both a σ -Mahler equation and a τ -difference equation is not sufficient. This is because Γ is not abelian anymore.

The best we can say with regard to *equations* is that since every element of Γ is of the form $\sigma^a \tau^b \sigma^c$ for $a, b, c \in \mathbb{Z}$, a *finite* number of equations will suffice to guarantee $\dim_{\tilde{K}} M < \infty$. One will need, for example, a σ -Mahler equation for f , say of degree n , then for each $0 \leq c \leq n-1$ a τ -difference equation for $\sigma^c f$, and if m , say, is the maximum of the degrees of these equations, for each $0 \leq c \leq n-1$ and $0 \leq b \leq m-1$ a σ -Mahler equation for the power series $\tau^b \sigma^c f$. This collection of equations will guarantee that the elements $\sigma^a \tau^b \sigma^c f$, for a, b, c in a bounded range, will span M over \tilde{K} .

The discussion above makes it clear that for our generalized dihedral Γ , the natural condition for a result in the style of Theorems 1 and 2 is the finite dimensionality of $M = \tilde{K}\langle\Gamma\rangle f$. Its formulation in terms of equations can be cumbersome.

4.1.3. *Formal p-Mahler q-difference modules.*

Theorem 27. *Let M be a p-Mahler q-difference module over \tilde{k} . Then M has a unique \mathbb{C} -structure M_0 preserved by both Φ_σ and Φ_τ , such that Φ_τ acts potentially unipotently on M_0 .*

Proof. Suppose $\lambda \in \mathbb{Q}$ is a slope of M , considered as a q -difference module. Then there exists a $c \in \mathbb{C}^\times$, uniquely determined up to multiplication by q^α , $\alpha \in \mathbb{Q}$, and a $0 \neq v \in M$, such that $\Phi_\tau v = cx^\lambda v$. Since $\Phi_\tau^p v = c^p q^{\binom{p}{2}\lambda} x^{p\lambda} v$, the equation $\Phi_\tau \circ \Phi_\sigma = \Phi_\sigma \circ \Phi_\tau^p$ yields

$$\Phi_\tau(\Phi_\sigma v) = c^p q^{\binom{p}{2}\lambda} x^{p^2\lambda} \cdot \Phi_\sigma v.$$

It follows that $p^2\lambda$ is also a slope of M as a q -difference module. We can repeat this argument, and since the number of slopes is finite, $\lambda = 0$. This means that M descends to \mathbb{C} , i.e. $M = \tilde{k} \otimes_{\mathbb{C}} M_0$, as a q -difference module. Furthermore, if c is an

eigenvalue of Φ_τ on M , the above computation shows that so is c^p . Since there are only finitely many eigenvalues modulo $q^{\mathbb{Q}}$ it follows that for some $m > n \geq 1$ and $\alpha \in \mathbb{Q}$ we must have

$$c^{p^m} = c^{p^n} q^\alpha.$$

This means that $c = \zeta q^\mu$ for some rational number μ , and a root of unity ζ . Let $M_0(c)$ be the direct summand of M_0 with generalized Φ_τ -eigenvalue c . Replacing it by its “resonant” $x^{-\mu} M_0(c)$, we may assume that $c = \zeta$. Going over all the eigenvalues of Φ_τ on M_0 in this way, we may assume that they have all been replaced by roots of unity, so some power Φ_τ^m acts unipotently on M_0 . This pins down M_0 , namely

$$M_0 = \{v \in M \mid \exists n (\Phi_\tau^m - 1)^n v = 0\}.$$

Substituting ζ for c in the computation above we see that Φ_σ preserves $M_0[\Phi_\tau^m - 1]$, hence by dévissage preserves also M_0 . This concludes the proof of the theorem. \square

4.2. Rational p -Mahler q -difference modules. The analogue of Theorem 7 in case 1M1Q is the following.

Theorem 28. *Let M be a p -Mahler q -difference module over \tilde{K} . Then M has a unique \mathbb{C} -structure M_0 preserved by both Φ_σ and Φ_τ , such that Φ_τ acts potentially unipotently on M_0 .*

Proof. As in cases 2Q and 2M, choose a basis of M over \tilde{K} and let $A = A_\sigma$ and $B = A_\tau$ represent Φ_σ and Φ_τ in this basis. Our goal is to show that the pair (A, B) is gauge-equivalent to a pair of constant matrices (A_0, B_0) , and moreover that all the eigenvalues of B_0 are roots of unity.

Without loss of generality we may assume that $|q| > 1$. The reduction to this case is done precisely as in case 2Q; see step VI in §3.2, *elimination of the assumption (Hyp)*.

We consider the points $i = 0, \infty$ and proceed as in case 2Q. Invoking theorem 27 and repeating the arguments in steps I-IV there we get:

Steps I-IV: After a change of variables, writing x for $x^{1/s}$ for a suitable s , there exists an invertible matrix $C_0(x)$, meromorphic in $0 \leq |x| < \infty$ and holomorphic at 0, and constant matrices A_0, B_0 , such that the following equations hold

$$\begin{cases} C_0(x^p)^{-1} A(x) C_0(x) = A_0 \\ C_0(qx)^{-1} B(x) C_0(x) = B_0 \end{cases}.$$

Furthermore, all the eigenvalues of B_0 are roots of unity.

Likewise, there exists an invertible matrix $C_\infty(x)$, meromorphic in $0 < |x| \leq \infty$ and holomorphic at ∞ , and constant matrices A_∞, B_∞ , such that

$$\begin{cases} C_\infty(x^p)^{-1} A(x) C_\infty(x) = A_\infty \\ C_\infty(qx)^{-1} B(x) C_\infty(x) = B_\infty \end{cases}.$$

Step V: The matrix $C_0(x) \in GL_r(K)$.

Consider

$$C_{0\infty}(x) = C_\infty(x)^{-1} C_0(x),$$

which is meromorphic in $0 < |x| < \infty$. It satisfies there the functional equation

$$C_{0\infty}(x^p) A_0 = A_\infty C_{0\infty}(x).$$

Arguing as in Step VI in case 2M, on the power-series expansions of $C_{0\infty}(x)$ in annuli of analyticity, we deduce that $C_{0\infty}$ is constant. It follows that $C_0(x)$ is meromorphic also at ∞ , hence is rational.

This concludes the proof of the theorem. \square

As in Proposition 8 we can withdraw from the last theorem the following consequence.

Theorem 29. *Let $f \in \mathbb{C}((x))$ and assume that $f \in M \subset \tilde{k} = \bigcup_{s \in \mathbb{N}} \mathbb{C}((x^{1/s}))$, where M is a finite dimensional \tilde{K} -vector space closed under σ and τ . Then $f \in \mathbb{C}(x)$.*

Note that the assumption on the finite dimensionality of M replaces the (insufficient) assumption that f satisfies a p -Mahler equation and a q -difference equation simultaneously. As remarked before, it is possible to encode this assumption in a finite number of equations, but their number will depend, in general, on the power series f , and they will be of mixed type, iterations of both σ and τ figuring in the same equation.

5. FINITE CHARACTERISTIC

In this section we briefly explain how to modify the proof of Theorem 2 and Theorem 7 (in the case 2Q), when \mathbb{C} is replaced by an arbitrary algebraically closed field. We thus prove the following.

Theorem 30. *Theorems 2 and 7 (case 2Q) remain valid as stated, when \mathbb{C} is replaced by an arbitrary algebraically closed field C .*

Proof. If C has characteristic 0 one can apply the Lefschetz principle and assume it is \mathbb{C} . Let therefore $\text{char.}(C) = \ell > 0$. The proof of Proposition 8, deducing Theorem 2 from Theorem 7, did not use any property of \mathbb{C} , besides it being a field. We therefore only have to explain how to modify the proof of Theorem 7.

Theorem 17, giving the structure of a formal (p, q) -difference module, also did not use any property of the field of constants, and works equally well if C has finite characteristic. This provides the starting point for the proof, and steps I-II of §3.2 hold true with C replacing \mathbb{C} . We now use the following lemma.

Lemma 31. *Let C be an algebraically closed field of characteristic ℓ and $p \in C^\times$ not a root of unity. Then there exists an algebraically closed complete valued field $(\widehat{C}, |\cdot|)$ containing C , such that $|p| > 1$.*

Proof. As p is transcendental over \mathbb{F}_ℓ we can complete it to a transcendental basis $\{z_\alpha\}$ of C over \mathbb{F}_ℓ , with $z_0 = p$. Let F be the field generated over \mathbb{F}_ℓ by the z_α for $\alpha \neq 0$, and consider $F(z_0)$ with a valuation which is trivial on F and satisfies $|z_0| > 1$. Let \widehat{C} be an algebraically closed complete extension of $F(z_0)$ to which $|\cdot|$ extends. Since C is algebraic over $F(z_0)$, it embeds in \widehat{C} . \square

We continue as in §3.2, reserving the terms “holomorphic” and “meromorphic” to mean “rigid holomorphic (resp. meromorphic) over \widehat{C} ”. Steps III-V, concluding the proof, are carried out now in the same way as over \mathbb{C} , taking advantage of the fact that $|p| > 1$. Compare with the use of \mathbb{C}_ℓ to eliminate assumption (hyp) in characteristic 0, in loc.cit., Step VI. \square

Remark 32. The extension of cases 2M and 1M1Q to finite characteristic demands special attention, for the following reason. The substitution $z = \log(x)$, which allowed us to delegate the formal study of a rational Mahler module at the fixed point $x = 1$ to the realm of q -difference modules, is no longer valid in finite characteristic. In fact, the formal multiplicative group is not isomorphic to the formal additive group, and therefore the results of §2 have to be recast in a new setup. Notwithstanding this remark, we believe that the main theorems in case 2M remain valid in finite characteristic ℓ , at least if $(\ell, pq) = 1$.

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