FOLIATIONS ON UNITARY SHIMURA VARIETIES IN POSITIVE CHARACTERISTIC

EHUD DE SHALIT AND EYAL Z. GOREN

Abstract. When p is inert in the quadratic imaginary field E and $m < n$, unitary Shimura varieties of signature (n, m) and a hyperspecial level subgroup at p, carry a natural *foliation of height* 1 and rank m^2 in the tangent bundle of their special fiber S. We study this foliation and show that it acquires singularities at deep Ekedahl-Oort strata, but that these singularities are resolved if we pass to a natural smooth moduli problem $S^\sharp$, a successive blow-up of S. Over the (μ-)ordinary locus we relate the foliation to Moonen's generalized Serre-Tate coordinates. We study the quotient of $S^\sharp$ by the foliation, and identify it as the Zariski closure of the ordinary-étale locus in the special fiber $S_0(p)$ of a certain Shimura variety with parahoric level structure at p. As a result, we get that this "horizontal component" of $S_0(p)$, as well as its multiplicative counterpart, are non-singular (formerly they were only known to be normal and Cohen-Macaulay). We study two kinds of integral manifolds of the foliation: unitary Shimura subvarieties of signature (m, m), and a certain Ekedahl-Oort stratum that we denote S_{fol}. We conjecture that these are the only integral submanifolds.

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1. Introduction

Inseparable morphisms in characteristic p have long become an important tool for obtaining deep results in algebraic geometry. The striking difference between algebraic differential geometry in characteristic 0 and in characteristic p is the source of many important applications. We cite the proof by Rudakov and Shafarevich [Ru-Sh] of the non-existence of global vector fields on a K3 surface, and the characteristic p proof by Deligne and Illusie [De-Ill] of the degeneration of the Hodge spectral sequence, as two outstanding examples.

The present paper originated from observations made in a special case in [dS-G2]. Its goal is to study the geometry of unitary Shimura varieties modulo p, making use of the relation between inseparable morphisms of height 1 and height 1 foliations in the tangent bundle. This relation can be traced back, at the birational level, to Jacobson's "Galois theory" for inseparable field extensions by means of derivations [Ja]. It was further developed by Rudakov and Shafarevich in [Ru-Sh], and by Ekedahl [Ek] and Miyaoka [Mi]. The latter extended the notion of a foliation from first-order foliations to any order, to deal with the problem of non-uniqueness of solutions of linear differential equations in characteristic p.

To explain our main results, let E be a quadratic imaginary field and p an odd prime which is inert in E. Let S_K be a Shimura variety associated with a unitary group $\boldsymbol{G}_{/\mathbb{Q}}$, split by E, of signature (n,m) $(0<m<n)$, and with an adèlic level subgroup $K\subset \boldsymbol{G}(\mathbb{A})$. Then S_K is a smooth quasi-projective variety of dimension nm over E (the reflex field). We assume that $K=K_\infty K_p K^p$ where K_p is hyperspecial maximal compact at p. Under this assumption, Kottwitz [Ko] has defined a smooth integral model $\mathcal{S}$ for S_K over $\mathcal{O}_{E,(p)}$, whose special fiber over the residue field κ we denote simply by S. This integral model $\mathcal{S}$ is a moduli space for certain $n+m$ dimensional abelian schemes with PEL structure (the endomorphisms coming from $\mathcal{O}_E$) and we let $\mathcal{A}$ denote the universal abelian scheme over it.

The special fiber S admits a stratification by the isomorphism type of the p-torsion of the abelian varieties making up the family $\mathcal{A}$. This is the Ekedahl-Oort (EO) stratification [Oo], [Mo2], [We], see also [V-W]. It has a unique open dense stratum S^{ord}, which coincides with the unique open dense stratum in the Newton polygon (NP) stratification. Under our assumption that p is inert and $m<n$ the abelian varieties parametrized by S^{ord} are not ordinary, but only μ-ordinary. This means, roughly speaking, that they are "as ordinary as the PEL data permits them to be". This observation is where our journey begins.

When p is split, or when p is inert but $n=m$, S^{ord} classifies ordinary abelian varieties, in the usual sense. As Serre and Tate have shown more than 50 years ago, a formal neighborhood of a point $x\in S^{\mathrm{ord}}$ then carries a canonical structure of a formal torus, hence S^{ord} is locally isotropic, i.e. "looks the same in all directions". When p is inert and $m<n$ this breaks down. In fact, Moonen has introduced in [Mo1] "generalized Serre-Tate coordinates" on S^{ord}, and showed that under our assumptions, a formal neighborhood of a point $x\in S^{\mathrm{ord}}$ has a canonical structure as the part fixed under an involution in a "3-cascade". Vasiu [Va] has also done related work, that we will not need to use in this paper. As a result of Moonen's work, a canonical m^2-dimensional subspace $\mathcal{T}S_x^+$ is singled out in the tangent space $\mathcal{T}S_x$, and these subspaces make up a sub-bundle $\mathcal{T}S^+$ over S^{ord}, which is in fact a p-Lie sub-algebra, i.e. a height 1 foliation.

One can define $\mathcal{T}S^+$ without any appeal to the results of Moonen as follows. The Hodge bundle $\omega_{\mathcal{A}/S} = R^0\pi_*\Omega^1_{\mathcal{A}/S}$ breaks up as a direct sum $\mathcal{P} \oplus \mathcal{Q}$ according to types, where $\mathcal{P}$, the part on which the endomorphisms act via the natural map $\mathcal{O}_E \hookrightarrow \mathcal{O}_{E,(p)} \twoheadrightarrow \kappa$, is of rank n, and $\mathcal{Q}$, the part on which they act via the Galois conjugate map, is of rank m. The Kodaira-Spencer map supplies us with an isomorphism

$$KS : \mathcal{P} \otimes \mathcal{Q} \simeq \Omega^1_S.$$

Once we situate ourselves in the special fiber, we can make use of the Verschiebung isogeny Ver : $\mathcal{A}^{(p)} \to \mathcal{A}$, and we denote by V the map it induces on de-Rham cohomology. We denote by $\mathcal{P}[V]$ the subsheaf which is the kernel of $V|_{\mathcal{P}}$. Over S^{ord} it constitutes a sub-bundle $\mathcal{P}_0$ of rank $n-m$. We let

$$\mathcal{T}S^+ = KS(\mathcal{P}_0 \otimes \mathcal{Q})^{\perp}$$

be the annihilator of $KS(\mathcal{P}_0 \otimes \mathcal{Q})$ in $\mathcal{T}S$ under the natural pairing between the tangent and cotangent bundles. The proof that this sub-bundle is in fact a foliation (closed under Lie bracket and raising to power p) becomes a pleasant exercise involving the Gauss-Manin connection and the notion of p-curvature (see Proposition 3). The foliation $\mathcal{T}S^+$ is the one appearing in the title of our paper. In Theorem 13 we prove that it coincides with what might be obtained from [Mo1].

Besides its simplicity, our definition of the foliation has two other advantages. First, according to the dictionary between foliations of height 1 and inseparable morphisms of height 1, reviewed in §2.2, $\mathcal{T}S^+$ corresponds to a certain quotient variety of S^{ord}. In §3.2 we identify this variety as a Zariski open subset (*the ordinary-étale locus*) in the special fiber of an integral model of a certain Shimura variety of parahoric level at p. We call this Shimura variety $S_{K_0(p)}$. In characteristic 0 it is a finite étale covering of S_K. Its integral model over $\mathcal{O}_{E,(p)}$, denoted $\mathcal{S}_0(p)$, was defined by Rapoport and Zink in Chapter 6 of [Ra-Zi] and studied further by several authors. For instance, Görtz [Gö] proved that it is flat over $\mathcal{O}_{E,(p)}$, and that if we denote its special fiber by $S_0(p)$, the local rings of the irreducible components of $S_0(p)$ are Cohen-Macaulay and normal. See also the work of Pappas and Zhu [P-Z]. We make strong use of these results later on. The special fiber $S_0(p)$ classifies abelian schemes in characteristic p, with PEL structure as in S, equipped with a finite flat, isotropic, $\mathcal{O}_E$-stable "Raynaud" subgroup scheme H of rank p^{2m}. The ordinary-étale locus of $S_0(p)$, denoted $S_0(p)^{\mathrm{ord}}_{et}$, is the open subset lying over S^{ord} classifying such objects in which H is étale. The alleged quotient map $S^{\mathrm{ord}} \to S_0(p)^{\mathrm{ord}}_{et}$ is such that when we compose it with the natural projection $S_0(p)^{\mathrm{ord}}_{et} \to S^{\mathrm{ord}}$, in any order, we get the map Fr_p^2.

More important, perhaps, is that we are able to extend $\mathcal{T}S^+$ into the deeper EO strata, something absent from Moonen's theory of generalized Serre-Tate coordinates. This is tied up with the study of the closure $S_0(p)_{et}$ of the ordinary-étale locus $S_0(p)^{\mathrm{ord}}_{et}$, one of the horizontal relatively irreducible[1] components of $S_0(p)$. It is also tied up with a certain moduli-scheme $S^\sharp$, special to characteristic p, which is a "successive blow-up" of S at deep enough EO strata. In §4.1 we define $S^\sharp$ as a moduli problem and prove that it is representable by a smooth scheme over κ.

[1]A *relatively irreducible* component Y is a union of irreducible components of $S_0(p)$ for which the projection $\pi : S_0(p) \to S$ induces a bijection between the irreducible components of Y and those of $\pi(Y)$. We call a relatively irreducible component *horizontal* if an open subset of it maps finite-flat to S. The special fiber has non-horizontal components too.

We also determine the dimensions of the fibers of the morphism $f : S^\sharp \to S$, and the open set $S_\sharp \subset S$ over which f is an isomorphism. The set $S_\sharp$ "interpolates" between S^{ord} and a unique minimal EO stratum, of dimension m^2, contained in it, which we call S_{fol}. All this information is described in terms of the combinatorics of (n,m)-shuffles in the symmetric group $\mathfrak{S}_{n+m}$.

The height 1 foliation $\mathcal{T}S^+$ extends canonically to a height 1 foliation $\mathcal{T}S^{\sharp+}$ on $S^\sharp$. Over $S_\sharp$ it can be considered to lie in S, but outside $S_\sharp$ it would acquire singularities, and it is necessary to introduce the successive blow up $S^\sharp$ to extend it everywhere.

Having constructed $S_0(p)_{et}$ and $S^\sharp$, we describe purely inseparable morphisms

$$S_0(p)_{et} \overset{\pi^\sharp_{et}}{\to} S^\sharp \overset{\rho}{\to} S_0(p)_{et}$$

whose composition is Fr_p^2. The map $\pi^\sharp_{et}$ extends the natural projection π_{et} from $S_0(p)^{\mathrm{ord}}_{et}$ to S^{ord}, and ρ extends the quotient map $S^{\mathrm{ord}} \to S_0(p)^{\mathrm{ord}}_{et}$ obtained from the foliation $\mathcal{T}S^+$. Using the theorem of Görtz mentioned above we deduce that since $S_0(p)_{et}$ is Cohen-Macaulay, $\pi^\sharp_{et}$ is finite and flat. Using the normality of $S_0(p)_{et}$ we conclude that ρ is also finite and flat, and that $S_0(p)_{et}$ is in fact *non-singular.*

At the other extreme we have the multiplicative horizontal component $S_0(p)_m$, the closure of the ordinary-multiplicative locus $S_0(p)^{\mathrm{ord}}_m$. It too maps to $S^\sharp$ and this map is in fact an isomorphism, proving that $S_0(p)_m$ is non-singular. Note that the projection from $S_0(p)_m$ to S is *not* everywhere finite, and does not admit a section.

We do not know if similar results hold for the other, "mixed" horizontal components of $S_0(p)$, of which there are many in general. We also stress that although $S_0(p)_m$ and $S_0(p)_{et}$ intersect, the maps that we have constructed from them to $S^\sharp$ do not agree on the intersection. This is manifested already in signature $(2,1)$, see [dS-G2].

The upshot of all this for the foliation $\mathcal{T}S^+$ is that it extends uniquely to a rank m^2 foliation on $S^\sharp$, the quotient of $S^\sharp$ by which is $S_0(p)_{et}$.

So far we said nothing about the behavior of the foliation at the cusps. As mentioned above, S, hence also $S^\sharp$, admit smooth compactifications at the cusps. It seems clear that the Hodge bundle, with its decomposition $\mathcal{P} \oplus Q$, as well as $\mathcal{P}_0 = \mathcal{P}[V]$, extend as locally free sheaves to suitable toroidal compactifications of the special fiber. The Kodaira-Spencer isomorphism, on the other hand, acquires poles along subvarieties of the boundary, whose nature reflects in a subtle way the compactification. Thus while $\mathcal{T}S^+$ can be extended simply by taking Zariski closures, it may become singular over the boundary, and we do not attempt to study it there. See [dS-G1] for an analysis of the case $(n,m) = (2,1)$.

In the last section we turn our attention to integral subvarieties of $\mathcal{T}S^{\sharp+}$ in $S_\sharp$. Embedded Shimura varieties associated to $U(m,m)$, or to an inner form of $U(m,m)$, are easily seen to be such integral subvarieties. So is the EO stratum S_{fol} (Theorem 25). The proof of this last fact uses the *canonical filtration of* $\mathcal{A}[p]$ over the EO stratum S_{fol}, and requires some effort, although the idea behind it is simple.

We end the paper with a discussion of a conjecture "of André-Oort type" that Shimura varieties of signature (m,m), and the EO stratum S_{fol}, are the only global integral subvarieties of $\mathcal{T}S^{\sharp+}$. Despite the fact that we do not know if the foliation lifts to a height h foliation for $h > 1$ in any natural way, hence locally formally the integral subvarieties are not unique, we believe that its global nature makes the conjecture plausible.

The results obtained in this paper generalize results that have been obtained for Picard modular surfaces, associated with a unitary group of signature $(2, 1)$, in [dS-G2]. On the other hand it seems that with some extra effort they should generalize to all Shimura varieties of PEL type.

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2. Background

2.1. **Unitary Shimura varieties.**

2.1.1. *The Shimura variety and its integral model at a good prime.* Let E be a quadratic imaginary field, $0 \leq m \leq n$ and $\Lambda = \mathcal{O}_E^{n+m}$, equipped with the hermitian pairing

$$(u, v) = {}^t\overline{u} \begin{pmatrix} & & 1_m \\ & 1_{n-m} & \\ 1_m & & \end{pmatrix} v.$$

Here 1_k is the identity matrix of size k. Let δ be a square root of the discriminant of E, so that $\overline{\delta} = -\delta$, and denote by $\mathrm{Im}_\delta(z) = (z - \overline{z})/\delta$. Then

$$\langle u, v\rangle = \mathrm{Im}_\delta(u, v)$$

is $\mathbb{Q}$-bilinear, skew-symmetric, satisfies $\langle au, v\rangle = \langle u, \overline{a}v\rangle$, and Λ is self-dual, i.e. $\langle ., .\rangle$ induces $\Lambda \simeq \mathrm{Hom}(\Lambda, \mathbb{Z})$. Let $\boldsymbol{G}$ be the general unitary group of $(\Lambda, (., .))$, viewed as a group scheme over $\mathbb{Z}$. For every commutative ring R

$$\boldsymbol{G}(R) = \left\{ g \in GL_{n+m}(R) | \, \exists \mu(g) \in R^\times, \ (gu, gv) = \mu(g)(u, v) \right\}.$$

Fix an *odd* prime p which is *unramified* in E, and an integer $N \geq 3$ relatively prime to p. Let $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ be the adèle ring of $\mathbb{Q}$, where $\mathbb{A}_f = \mathbb{Q} \cdot \widehat{\mathbb{Z}}$ are the finite adèles. Let $K_f \subset \boldsymbol{G}(\widehat{\mathbb{Z}})$ be an open subgroup of the form $K_f = K^p K_p$, where $K^p \subset \boldsymbol{G}(\mathbb{A}^p)$ is the principal congruence subgroup of level N, and

$$K_p = \boldsymbol{G}(\mathbb{Z}_p) \subset \boldsymbol{G}(\mathbb{Q}_p)$$

the hyperspecial maximal compact subgroup at p. Let $K_\infty \subset \boldsymbol{G}(\mathbb{R})$ be the stabilizer of the negative definite subspace spanned by $\{-e_i + e_{n+i};\, 1 \leq i \leq m\}$ in $\Lambda_\mathbb{R} = \mathbb{C}^{n+m}$, where $\{e_i\}$ stands for the standard basis. This K_∞ is a maximal compact-modulo-center subgroup, isomorphic to $G(U(m) \times U(n))$. By $G(U(m) \times U(n))$ we mean the pairs of matrices $(g_1, g_2) \in GU(m) \times GU(n)$ having the same similitude factor. Let $K = K_\infty K_f \subset \boldsymbol{G}(\mathbb{A})$ and $X = \boldsymbol{G}(\mathbb{R})/K_\infty$.

To the Shimura datum $(\boldsymbol{G}, X)$ there is associated a Shimura variety Sh_K. It is a quasi-projective smooth variety of dimension nm over E. If $m = n$ our Shimura variety is even defined over $\mathbb{Q}$, but we still denote by Sh_K its base-change to E. The complex points of Sh_K are identified, as a complex manifold, with

$$Sh_K(\mathbb{C}) = \boldsymbol{G}(\mathbb{Q})\backslash \boldsymbol{G}(\mathbb{A})/K.$$

Fix an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ and let $v|p$ be the prime of E induced by it. Following Kottwitz [Ko], but using a somewhat more restrictive set-up suitable for principally polarized abelian varieties, we define a scheme $\mathcal{S}$ over the localization $\mathcal{O}_{E,v}$ of $\mathcal{O}_E$

at v. This $\mathcal{S}$ is a fine moduli space whose R-points, for every $\mathcal{O}_{E,v}$-algebra R, classify isomorphism types of tuples $\underline{A} = (A, \iota, \phi, \eta)$ where

- A is an abelian scheme of dimension $n+m$ over R.
- $\iota : \mathcal{O}_E \hookrightarrow \mathrm{End}(A)$ has signature (n, m) on the Lie algebra of A.
- $\phi : A \xrightarrow{\sim} A^t$ is a principal polarization whose Rosati involution induces $\iota(a) \mapsto \iota(\overline{a})$ on the image of ι.
- η is an $\mathcal{O}_E$-linear full level-N structure on A compatible with $(\Lambda, \langle ., . \rangle)$ and ϕ ([Lan], 1.3.6).

We shall summarize the above requirements by saying that $\underline{A}$ is a *structure of type* $\mathscr{D}$ over R. See also [Lan] for the comparison of the various languages used to define the moduli problem.

The generic fiber S_K of $\mathcal{S}$ is, in general, a union of *several* Shimura varieties of the type Sh_K. This is due to the failure of the Hasse principle, which can happen when $m+n$ is odd. We also remark that the assumption $N \geq 3$ could be avoided if we were willing to use the language of stacks. As this is not essential to the present paper, we keep the scope slightly limited for the sake of clarity.

As shown by Kottwitz, $\mathcal{S}$ is *smooth* of relative dimension nm over $\mathcal{O}_{E,v}$. It even admits smooth (toroidal) compactifications at the cusps, *cf.* [Lan].

2.1.2. *The universal abelian variety and the Kodaira-Spencer isomorphism.* The moduli space $\mathcal{S}$ carries a universal abelian scheme $\mathcal{A}$ of dimension $n+m$, equipped with ι, ϕ, η as above. Let $\pi : \mathcal{A} \to \mathcal{S}$ be the structure morphism. We denote by $\mathcal{A}^t$ the dual abelian scheme.

We let Σ denote the identity embedding of $\mathcal{O}_E$ in $\mathcal{O}_{E,v}$ and $\overline{\Sigma}$ its complex conjugate. Since p is unramified in E, the locally free sheaves $H^1_{dR}(\mathcal{A}/\mathcal{S})$, $\omega_{\mathcal{A}/\mathcal{S}} = R^0\pi_*\Omega^1_{\mathcal{A}/\mathcal{S}}$ and $\omega^\vee_{\mathcal{A}^t/\mathcal{S}} = R^1\pi_*\mathcal{O}_\mathcal{A}$ decompose as direct sums of their Σ and $\overline{\Sigma}$-parts under the action of $\iota(\mathcal{O}_E)$. We write

$$\mathcal{P} = \omega_{\mathcal{A}/\mathcal{S}}(\Sigma), \;\; \mathcal{Q} = \omega_{\mathcal{A}/\mathcal{S}}(\overline{\Sigma}).$$

These are locally free sheaves of ranks n and m respectively on $\mathcal{S}$.

The Kodaira-Spencer map is the *sheaf homomorphism,*

$$KS : \mathcal{P} = \omega_{\mathcal{A}/\mathcal{S}}(\Sigma) \to \Omega^1_\mathcal{S} \otimes \omega^\vee_{\mathcal{A}^t/\mathcal{S}}(\Sigma),$$

obtained (on the Σ-parts) by embedding $\omega_{\mathcal{A}/\mathcal{S}}$ in $H^1_{dR}(\mathcal{A}/\mathcal{S})$, applying the Gauss Manin connection

$$\nabla : H^1_{dR}(\mathcal{A}/\mathcal{S}) \to \Omega^1_\mathcal{S} \otimes H^1_{dR}(\mathcal{A}/\mathcal{S}),$$

and finally projecting $H^1_{dR}(\mathcal{A}/\mathcal{S})$ to $\omega^\vee_{\mathcal{A}^t/\mathcal{S}}$. Since the polarization ϕ induces an identification

$$\phi^{*\vee} : \omega^\vee_{\mathcal{A}^t/\mathcal{S}}(\Sigma) \xleftarrow{\sim} \omega^\vee_{\mathcal{A}/\mathcal{S}}(\overline{\Sigma}) = \mathcal{Q}^\vee$$

the Kodaira-Spencer map yields a homomorphism, which we denote by the same symbol

$$KS : \mathcal{P} \otimes \mathcal{Q} \to \Omega^1_\mathcal{S}.$$

This map turns out to be an *isomorphism.*

2.1.3. *The NP and EO stratifications of the special fiber of $\mathcal{S}$.* We briefly review some facts about these two stratifications, as the EO stratification is going to play a central role later in the paper.

Let S be the special fiber of $\mathcal{S}$. It is a smooth variety over $\kappa = \kappa_v$, the residue field of v. Let k be an algebraically closed field containing κ, and $x \in S(k)$. Let NP_x be the Newton polygon of the p-divisible group $\mathcal{A}_x[p^\infty]$. It is lower convex, starts at $(0,0)$, ends at $(2(n+m), n+m)$, and has integral break-points. Then NP_x classifies the *k-isogeny class* of $\mathcal{A}_x[p^\infty]$. The set of Newton polygons is partially ordered, where $P' \geq P$ if P' lies on or above P. For every Newton polygon P there is a locally closed stratum S_P in S, defined over κ, whose geometric points are precisely those with $NP_x = P$. The closure of a non-empty S_P is the union of the $S_{P'}$ for all P' satisfying $P' \geq P$ ([V-W], §11). This gives the Newton polygon (NP) stratification of S.

The Ekedahl-Oort (EO) stratification of S is another stratification, by the *isomorphism type* of $\mathcal{A}_x[p]$. In addition to the references already cited in the introduction, see also [Woo] for a thorough discussion of the case at hand. The EO strata S_w are locally closed subsets labeled by certain elements w in the Weyl group W of $\boldsymbol{G}$. More precisely, the w are distinguished representatives for the cosets $W_J \backslash W$, where the subgroup W_J is determined by the signature condition. The S_w are equidimensional, smooth and quasi-affine. The dimension of S_w is $l(w)$, the length of w relative to the Bruhat order on W. The closure of S_w is the union of $S_{w'}$ for $w' \preceq w$ under a certain rather complicated order (related to, but different from the usual Bruhat order; see [V-W], Theorems 2 and 3). We call it the *EO order* on the Weyl group elements indexing the strata. See below for a full description when p is inert in E and $m < n$.

Wedhorn and Moonen have proved the following [We], [Mo1].

Fact. There is a unique largest NP stratum, and a unique largest EO stratum. These two strata coincide, and form an open dense subset $S^{\mathrm{ord}} \subset S$, called[2] the *ordinary* locus of S. The isomorphism type of the whole p-divisible group $\mathcal{A}_x[p^\infty]$ (with its endomorphisms and polarization) is constant as x varies along S^{ord}, and can be given explicitly in terms of the data $\mathscr{D}$.

If p is split in E or $n = m$ then $\mathcal{A}_x[p^\infty] \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{n+m} \times \mu_{p^\infty}^{n+m}$ for all $x \in S^{\mathrm{ord}}(k)$, so $\mathcal{A}_x$ is ordinary. If p is inert in E and $m < n$ this is not the case, and for $x \in S^{\mathrm{ord}}$

$$\mathcal{A}_x[p^\infty] \simeq (\mathcal{O}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p)^m \times \mathscr{G}_\Sigma^{n-m} \times (\mathcal{O}_E \otimes \mu_{p^\infty})^m \tag{2.1}$$

where $\mathscr{G}_\Sigma$ is the unique 1-dimensional, height 2, slope $1/2$, self-dual p-divisible group over k. The subscript Σ means that the embedding of $\mathcal{O}_E$ in $\mathrm{End}_k(\mathscr{G})$ via ι induces on $\mathrm{Lie}(\mathscr{G})$ the type Σ, rather than $\overline{\Sigma}$. In this case it is customary to call $\mathcal{A}_x$, for x in the ordinary locus, *μ-ordinary.*

$$\boxed{\text{From now on we assume that } p \text{ is inert in } E \text{ and } m < n.}$$

Under this assumption $v = (p)$, so we write $\mathcal{O}_{E,(p)}$ instead of $\mathcal{O}_{E,v}$, and $\kappa = \mathbb{F}_{p^2}$. The lowest EO strata S_{id} (labeled by $w = id$) is 0-dimensional and when p is inert it classifies *superspecial* abelian varieties, i.e. those for which $\mathcal{A}_x[p^\infty] \simeq \mathscr{G}_\Sigma^n \times \mathscr{G}_{\overline{\Sigma}}^m$. We call this stratum the *core points.*

[2]In [dS-G2] S^{ord} was denoted S_μ and called the μ-ordinary locus.

If $m = 1$ the EO stratification has been worked out completely by Bültel and Wedhorn [B-W]. The strata are linearly ordered, their dimensions dropping by 1 each time. Thus there are $n + 1$ EO strata altogether. As long as the dimension of the stratum is strictly larger than $\lfloor n/2 \rfloor$, the EO strata are also NP strata. In fact, the isomorphism type of the whole p-divisible group (with its endomorphisms and polarization) is constant along these strata, as it was on S^{ord}. Half the way through, in dimension $\lfloor n/2 \rfloor$, one reaches the supersingular NP stratum, which is stratified further by EO strata. We remark also that from dimension $\lfloor n/2 \rfloor$ down, the isomorphism type of $\mathcal{A}_x[p^\infty]$ is no longer constant along the EO strata, only that of $\mathcal{A}_x[p]$.

In general, we may identify the Weyl group of $\boldsymbol{G}$ with $\mathfrak{S}_{n+m}$, the group of permutations of $\{1, \dots, n+m\}$. Let $W_J = \mathfrak{S}_n \times \mathfrak{S}_m$. The elements w indexing the EO strata belong then to the set $\Pi(n,m)$ of (n,m)-*shuffles* in $\mathfrak{S}_{n+m}$. A permutation w is called an (n,m)-shuffle if

$$w^{-1}(1) < \cdots < w^{-1}(n), \quad w^{-1}(n+1) < \cdots < w^{-1}(n+m).$$

The set $\Pi(n,m)$ is clearly a set of representatives for $W_J \backslash W$.

The dimension of S_w is given by the formula

$$l(w) = \sum_{i=1}^{n} (w^{-1}(i) - i). \tag{2.2}$$

We shall also need to know a formula for

$$a_\Sigma(w) = \dim \mathcal{P}_x[V]$$

($x \in S_w(k)$). Here V is the map induced on cohomology by Verschiebung, see below. This number is the Σ-part of Oort's a-number of $\mathcal{A}_x$. It turns out that it is given by

$$a_\Sigma(w) = |\{i | \, 1 \le i \le n, \ \ 1 \le w^{-1}(i) \le n\}|. \tag{2.3}$$

For the formulae (2.2), (2.3) see [Woo], §3.4 and §3.5. For example, if $m = 1$, $a_\Sigma(w) = n-1$ except if $w = 1$ (corresponding to the core points), where it becomes n. In general, for w the longest (n,m)-shuffle (of length nm), where $S_w = S^{\mathrm{ord}}$, $a_\Sigma(w) = n - m$. For $w = id$, corresponding to the core points, $a_\Sigma(id) = n$.

Finally, we make explicit the EO order relation $w' \preceq w$ on the set $\Pi(n,m)$, following [Woo], Example 3.1.3. Recall that $w' \preceq w$ if and only if $S_{w'} \subset \overline{S_w}$. Let

$$w_{0,J} = \begin{pmatrix} 1 & \dots & n & n+1 & \dots & n+m \\ n & \dots & 1 & n+m & \dots & n+1 \end{pmatrix}.$$

Let $\le$ be the usual Bruhat order on $\mathfrak{S}_{n+m}$ with respect to the standard set of reflections $s_i = (i, i+1)$ $(1 \le i < n+m)$. Note that $w_{0,J}$ is the element of maximal length in W_J. Then $w' \preceq w$ if and only if there exists a $y \in W_J$ such that

$$y w' w_{0,J} y^{-1} w_{0,J} \le w.$$

Taking $y = 1$ we see that if $w' \le w$ then also $w' \preceq w$. This is the only property of the EO order relation that will be used in the paper.

2.1.4. *Frobenius, Verschiebung and the Hasse invariant.* For any scheme X in characteristic p we denote by Φ_X the absolute Frobenius morphism of degree p of X. Let

$$\mathcal{A}^{(p)} = S \times_{\Phi_S, S} \mathcal{A}$$

be the base-change of the universal abelian variety. Let

$$\mathrm{Fr} = Fr_{\mathcal{A}/S} : \mathcal{A} \to \mathcal{A}^{(p)}$$

be the relative Frobenius morphism. It is an isogeny of abelian schemes over S, of degree p^{n+m}. The isogeny dual to $Fr_{\mathcal{A}^t/S}$ is called the Verschiebung of $\mathcal{A}$ and is denoted

$$\mathrm{Ver} = Ver_{\mathcal{A}/S} : \mathcal{A}^{(p)} \to \mathcal{A}.$$

It too is of degree p^{n+m} and $Ver_{\mathcal{A}/S} \circ Fr_{\mathcal{A}/S}$ is multiplication by p on $\mathcal{A}$.

The maps induced by $Fr_{\mathcal{A}/S}$ and $Ver_{\mathcal{A}/S}$ on cohomology will be denoted F and V. It is well-known that

$$F : H^1_{dR}(\mathcal{A}^{(p)}/S) \to H^1_{dR}(\mathcal{A}/S)$$

is a homomorphism of vector bundles of constant rank $n + m$. Its image is a sub-bundle (i.e. a locally free sub-sheaf, the quotient of $H^1_{dR}(\mathcal{A}/S)$ by which is also locally free) and coincides with $H^1_{dR}(\mathcal{A}/S)[V]$. Similarly, the image of

$$V : H^1_{dR}(\mathcal{A}/S) \to H^1_{dR}(\mathcal{A}^{(p)}/S)$$

is a sub-bundle, equal to $H^1_{dR}(\mathcal{A}^{(p)}/S)[F]$.

The same can not be said about the restriction of V to $\omega_{\mathcal{A}/S}$. While $\omega_{\mathcal{A}/S}[V]$ is clearly a saturated sub-sheaf of $\omega_{\mathcal{A}/S}$, its rank may increase when we move from one EO stratum to a smaller one, contained in its closure. Hence, $\omega_{\mathcal{A}/S}[V]$ is in general not a sub-bundle. It is, however, a sub-bundle of rank p^{n-m}, if we restrict it to the ordinary locus.

Since F and V commute with the endomorphisms, they induce maps between the Σ and the $\overline{\Sigma}$-parts. Note however that

$$H^1_{dR}(\mathcal{A}^{(p)}/S)(\Sigma) = H^1_{dR}(\mathcal{A}/S)^{(p)}(\Sigma) = H^1_{dR}(\mathcal{A}/S)(\overline{\Sigma})^{(p)}.$$

In particular we get maps

$$V_{\mathcal{P}} : \mathcal{P} \to \mathcal{Q}^{(p)}, \quad V_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{P}^{(p)}.$$

The sheaf homomorphism

$$H_{\mathcal{A}/S} = V_{\mathcal{P}}^{(p)} \circ V_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}^{(p^2)}$$

is called the *Hasse map.* Let $\mathcal{L} = \det(\mathcal{Q})$, a line bundle on S. Note that for every line bundle $\mathcal{L}$ there is a canonical isomorphism $\mathcal{L}^{(p)} \simeq \mathcal{L}^p$, sending $1 \otimes s$ to $s \otimes \cdots \otimes s$ (here s is a section of $\mathcal{L}$). Thus $h_{\mathcal{A}/S} = \det(H_{\mathcal{A}/S})$ is a homomorphism from $\mathcal{L}$ to $\mathcal{L}^{p^2}$, which is the same as a global section

$$h_{\mathcal{A}/S} \in H^0(S, \mathcal{L}^{p^2-1}),$$

i.e. a modular form "of weight $\mathcal{L}^{p^2-1}$" called the *Hasse invariant.* It plays an important role in the study of p-adic modular forms. Its relation with the stratifications of S is the following.

Fact. [Woo] Let $S^{\rm no}$ be the complement of $S^{\rm ord}$ in S, endowed with its reduced subscheme structure. Then $S^{\rm no}$ is a Cartier divisor and

$$S^{\rm no} = \mathrm{div}(h_{\mathcal{A}/S}).$$

2.1.5. *Pairings in de Rham cohomology of abelian varieties.* We review some general facts on de Rham cohomology of abelian varieties. If A/k is an abelian variety over a field k (or more generally, an abelian scheme over a ring) we let A^t denote the dual abelian variety. There is then a canonical perfect bilinear pairing

$$\{.,.\} = \{.,.\}_A : H^1_{dR}(A/k) \times H^1_{dR}(A^t/k) \to k.$$

When we use the canonical identification of A with $(A^t)^t$ we have

$$\{u,v\}_A = -\{v,u\}_{A^t}.$$

Indeed, one may identify $H^1_{dR}(A^t/k)$ with $H^{2g-1}_{dR}(A/k)$ ($g = \dim A$), and then the pairing is given by cup product, followed by the trace. If $\alpha : A \to B$ is an isogeny, we let $\alpha^t : B^t \to A^t$ be the dual isogeny, and then

$$\{\alpha^* u, v\}_A = \{u, (\alpha^t)^* v\}_B.$$

If $\phi : A \overset{\sim}{\to} A^t$ is a principal polarization then $\phi = \phi^t$ (using the identification of $(A^t)^t$ with A). The *polarization pairing*

$$\{u,v\}_\phi = \{u, (\phi^{-1})^* v\}_A : H^1_{dR}(A) \times H^1_{dR}(A) \to k$$

is *skew-symmetric*, as follows from the preceding two properties.

If $\alpha \in \mathrm{End}(A)$ then $Ros_\phi(\alpha) \in \mathrm{End}(A)$ is defined by

$$Ros_\phi(\alpha) = \phi^{-1} \circ \alpha^t \circ \phi.$$

The previous properties imply then

$$\{\alpha^* u, v\}_\phi = \{u, Ros_\phi(\alpha)^* v\}_\phi.$$

If we apply the above for our A's, figuring in a tuple $\underline{A} \in S(k)$, we get that $\{,\}$ pairs $H^1_{dR}(A/k)(\Sigma)$ non trivially with $H^1_{dR}(A^t/k)(\Sigma)$, while $\{,\}_\phi$ pairs $H^1_{dR}(A/k)(\Sigma)$ non-trivially with $H^1_{dR}(A/k)(\overline{\Sigma})$.

Finally we recall that $\omega_{A/k}$ and $\omega_{A^t/k}$ are mutual annihilators of each other under $\{,\}$. This induces a perfect pairing between $\omega_{A/k}$ with $H^1(A^t, \mathcal{O})$, hence the identification of $H^1(A^t, \mathcal{O})$ with the Lie algebra of A.

2.2. **Foliations and inseparable morphisms of height 1.**

2.2.1. *Foliations of height 1.* In this section we review some general facts from algebraic geometry in characteristic p, due to Rudakov and Shafarevich [Ru-Sh], Ekedahl [Ek] and Miyaoka [Mi]. At the birational level they should be traced back, as mentioned in the introduction, to Jacobson's theorem which establishes a "Galois theory" for finite purely inseparable field extensions using derivations [Ja].

Let k be an algebraically closed field of characteristic p, and X a non-singular n-dimensional variety over k. Let $\mathcal{T}X$ be the tangent sheaf of X, a locally free sheaf of rank n. Recall that $\mathcal{T}X$ becomes a p-Lie algebra over k if for any two vector fields ξ, η defined in some open set U and regarded as operators on $\mathcal{O}_X(U)$, we let

$$[\xi,\eta] = \xi \circ \eta - \eta \circ \xi, \quad \xi^{(p)} = \xi \circ \xi \circ \cdots \circ \xi$$

(composition p times).

Definition. A *foliation of height 1* on X is a sub-bundle $\mathcal{E} \subset \mathcal{T}X$ (i.e. locally a direct summand), which is a p-Lie subalgebra, i.e. involutive (closed under the Lie bracket) and closed under $\xi \mapsto \xi^{(p)}$.

Foliations of higher height (as in [Ek]) will not show up in this paper, until we discuss integral subvarieties at the end. We shall therefore refer to height 1 foliations simply as "foliations". If $\mathcal{E}$ is a line sub-bundle then $\mathcal{E}$ is automatically involutive, as any two sections of $\mathcal{E}$ are proportional. But even in rank 1 the condition of being p-closed is non-void, as the following example shows. Let $X = \mathbb{A}_k^2$ and let $\mathcal{E} = \mathcal{O}_X \cdot \xi$ where

$$\xi = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

It is easily checked that $\xi^{(p)} = x\frac{\partial}{\partial x}$, but this is not a section of $\mathcal{E}$.

If $Y \subset X$ is a non-singular subvariety then $\mathcal{T}Y \subset \mathcal{T}X|_Y$ may be considered "a foliation of height 1 along Y". We call Y an *integral subvariety* for the foliation $\mathcal{E}$ if $\mathcal{E}|_Y = \mathcal{T}Y$. Integral subvarieties always exist in a formal neighborhood of a point ([Ek], Proposition 3.2), but they need not be unique, even if they are global. The foliation generated by the vector field $\partial/\partial y$ in $\mathbb{A}_k^2$ admits all the curves $x = a + by^p$ as integral curves, and infinitely many such curves pass through any given point. As another example, if X is a simple abelian surface, then any non-zero tangent vector at the origin generates a unique translation-invariant foliation on X. If Y were an integral curve for this foliation then Y would be an elliptic curve, contradicting the fact that X was assumed to be simple. Thus the given foliation has no integral curves at all.

2.2.2. *The relation between foliations and inseparable morphisms of height 1.* Let X be a non-singular k-variety. A finite k-morphism $X \xrightarrow{f} Y$ from X to a k-variety Y is called *of height 1* if there is a k-morphism $Y \xrightarrow{g} X^{(p)}$ such that the composition

$$X \xrightarrow{f} Y \xrightarrow{g} X^{(p)}$$

is $Fr_{X/k}$. Here $X^{(p)} = \mathrm{Spec}(k) \times_{\Phi_k, \mathrm{Spec}(k)} X$. If f is also flat, then Y is non-singular, since the property of being regular descends under flat morphisms between locally Noetherian schemes. In this case f is faithfully flat, since it is surjective on k-points, hence surjective. Therefore g is also finite and faithfully flat. Since $f^{(p)} \circ g \circ f = f^{(p)} \circ Fr_{X/k} = Fr_{Y/k} \circ f$ we get also $f^{(p)} \circ g = Fr_{Y/k}$. The link between foliations and height 1 morphisms is given by the following proposition.

Proposition 1. [Ek], *Proposition 2.4. Let X be a non-singular k-variety. There is a natural 1-1 correspondence between finite flat height 1 morphisms $f : X \to Y$ and height 1 foliations $\mathcal{E} \subset \mathcal{T}X$. One has*

$$\deg(f) = p^{rk(\mathcal{E})}.$$

Given f, $\Omega_{X/Y}$ is a locally free sheaf, hence the short exact sequence

$$f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$$

splits and we let $\mathcal{E} = \Omega_{X/Y}^\vee \subset \Omega_X^\vee = \mathcal{T}X$, a height 1 foliation. Conversely, given $\mathcal{E}$, we let $\mathcal{O}_Y = \mathcal{O}_X^{\mathcal{E}=0}$, the sheaf of functions annihilated by the derivations in $\mathcal{E}$.

The fact that $\mathcal{E}$ is a sub-bundle, and not only a saturated subsheaf, is essential. Consider, for example, the subsheaf $\mathcal{E}$ of $\mathcal{T}\mathbb{A}^2$ generated by $\xi = x \cdot \partial/\partial x + y \cdot \partial/\partial y$, which is saturated, but fails to be a sub-bundle at the origin. The quotient of $\mathbb{A}^2$ by $\mathcal{E}$ is the scheme Y for which $\mathcal{O}_Y$ is the sheaf of functions h on $\mathbb{A}^2$ satisfying $\xi(h) = 0$. This Y is singular at the origin, and the quotient map is not flat. Note that $h = x^i y^{p-i}$ is such a function for $0 \le i \le p$.

Let $f : X \to Y$ be as in the proposition, $x \in X$ and $y = f(x)$. Then one has the following local description of the completed local rings.

Proposition 2. [Ek], *Proposition 3.2. There is a system of formal parameters $t_1, \dots, t_n$ at x such that $t_1^p, \dots, t_r^p, t_{r+1}, \dots t_n$ is a system of formal parameters at the point y. In a formal neighborhood of x the foliation $\mathcal{E}$ is generated by $\partial/\partial t_i$ for $1 \le i \le r$.*

3. The foliation over the ordinary locus

3.1. **Definition and first properties.** Let notation be as in §2.1. Over S^{ord} the fibers of the subsheaf

$$\mathcal{P}_0 = \mathcal{P}[V] = \ker(V_{\mathcal{P}} : \mathcal{P} \to \mathcal{Q}^{(p)})$$

have constant rank $n - m$, hence, as S^{ord} is reduced, $\mathcal{P}_0$ is a sub-bundle of $\mathcal{P}$. The sub-bundle $KS(\mathcal{P}_0 \otimes \mathcal{Q})$ of Ω^1_S has accordingly rank $(n-m)m$. We define a rank-m^2 sub-bundle $\mathcal{T}S^+ \subset \mathcal{T}S^{\mathrm{ord}}$ by

$$\mathcal{T}S^+ = KS(\mathcal{P}_0 \otimes \mathcal{Q})^\perp.$$

Proposition 3. *$\mathcal{T}S^+$ is a foliation of height 1.*

The following two results will be used in the proof of the Proposition. In the next lemma we use the identification

$$\phi^* : \omega_{\mathcal{A}^t/S}(\Sigma) \simeq \omega_{\mathcal{A}/S}(\overline{\Sigma}) = \mathcal{Q}$$

induced by the polarization. Note that ϕ^* is type-reversing because the Rosati involution induced by ϕ is $\iota(a) \mapsto \iota(\overline{a})$. By $\nabla_\xi \alpha$ we denote, as usual, the contraction of $\nabla\alpha$ with the tangent vector ξ.

Lemma 4. *Let $\alpha \in \mathcal{P} = \omega_{\mathcal{A}/S}(\Sigma)$ and $\beta \in \mathcal{Q} \simeq \omega_{\mathcal{A}^t/S}(\Sigma)$. Denote by ∇ the Gauss-Manin connection and by*

$$\{,\} : H^1_{dR}(\mathcal{A}/S) \times H^1_{dR}(\mathcal{A}^t/S) \to \mathcal{O}_S$$

the canonical pairing in de Rham cohomology. Then for $\xi \in \mathcal{T}S$ we have

$$\langle KS(\alpha \otimes \beta), \xi \rangle = \{\nabla_\xi \alpha, \beta\}.$$

Proof. The lemma follows immediately from the definitions. Note that the identification $R^1\pi_*\mathcal{O}_{\mathcal{A}} = \omega^\vee_{\mathcal{A}^t/S}$, used in the definition of KS, results from the perfect pairing $\{,\}$ and from the fact that under this pairing $\omega_{\mathcal{A}/S}$ and $\omega_{\mathcal{A}^t/S}$ are exact annihilators of each other. □

Corollary 5. *$\xi \in \mathcal{T}S^+$ if and only if $\nabla_\xi(\mathcal{P}_0) \subset \mathcal{P}_0$.*

Proof. If $\xi \in \mathcal{T}S^+$ then by the lemma $\nabla_\xi(\mathcal{P}_0)$ is orthogonal under the pairing $\{,\}$ to $\mathcal{Q} \simeq \omega_{\mathcal{A}^t/S}(\Sigma)$. It is also orthogonal to $\omega_{\mathcal{A}^t/S}(\overline{\Sigma})$ for reasons of type. It is therefore orthogonal to the whole of $\omega_{\mathcal{A}^t/S}$, so $\nabla_\xi(\mathcal{P}_0) \subset \omega_{\mathcal{A}/S}$. But the Gauss-Manin connection commutes with isogenies and endomorphisms, so ∇_ξ preserves the subspaces $H^1_{dR}(\mathcal{A}/S)(\Sigma)$ and $H^1_{dR}(\mathcal{A}/S)[V]$. It follows that

$$\nabla_\xi(\mathcal{P}_0) \subset \omega_{\mathcal{A}/S}(\Sigma)[V] = \mathcal{P}_0.$$

The converse is obvious. □

We can now prove the proposition.

Proof. The involutivity of $\mathcal{T}S^+$ follows from the corollary, since the Gauss-Manin connection is integrable, i.e.

$$\nabla_{[\xi,\eta]} = \nabla_\xi \circ \nabla_\eta - \nabla_\eta \circ \nabla_\xi.$$

The fact that $\mathcal{T}S^+$ is closed under $\xi \mapsto \xi^{(p)}$ is more subtle as the p-curvature

$$\psi(\xi) := \nabla_{\xi^{(p)}} - \nabla_\xi^{(p)}$$

does not vanish identically, but is only a nilpotent endomorphism of $H^1_{dR}(\mathcal{A}/S)$ ([Ka], Theorem 5.10). However, on the sub-module with connection $H^1_{dR}(\mathcal{A}/S)[V]$ the p-curvature vanishes. This is because the kernel of V is the image of F, so (the easy direction of) Cartier's theorem ([Ka], Theorem 5.1) implies that $\psi(\xi) = 0$ there. Since $\mathcal{P}_0 \subset H^1_{dR}(\mathcal{A}/S)[V]$ we can conclude the proof as before, using the corollary. □

3.2. **The Shimura variety of parahoric level structure.** By Proposition 1, the height 1 foliation $\mathcal{T}S^+$ on S^{ord} corresponds to a finite flat purely inseparable quotient of S^{ord}. Our purpose in this section is to identify this quotient as the ordinary-étale component of (the special fiber of) a certain Shimura variety of parahoric level structure. This will allow us in §4 to extend the foliation to the non-ordinary locus.

3.2.1. *The Shimura variety $S_{K_0(p)}$.* In addition to the lattice $\Lambda = \mathcal{O}_E^{n+m}$ considered in §2.1, consider also the $\mathcal{O}_E$-lattices

$$\Lambda \supset \Lambda' \supset \Lambda'' \supset p\Lambda$$

where

$$\Lambda' = \langle pe_1, \dots pe_m, e_{m+1}, \dots, e_{n+m} \rangle, \quad \Lambda'' = \langle pe_1, \dots pe_n, e_{n+1}, \dots, e_{n+m} \rangle.$$

Note that the dual of Λ' is $p^{-1}\Lambda''$. Let $\mathscr{L}$ be the lattice chain in K_p^{n+m}

$$\cdots \supset \Lambda_p \supset \Lambda'_p \supset \Lambda''_p \supset p\Lambda_p \supset \cdots$$

obtained by tensoring with $\mathbb{Z}_p$ and extending by periodicity, and let $K_0(p)_p$ be its stabilizer in $K_p = \boldsymbol{G}(\mathbb{Z}_p)$. This is a parahoric subgroup, and if we let $K_0(p)$ be the adèlic level subgroup corresponding to it (and to full level N as usual), we get the Shimura variety $S_{K_0(p)}$, which is again defined over E, and is an étale cover of S_K.

3.2.2. *The moduli problem $\mathcal{S}_0(p)$.* Let $\mathcal{S}_0(p)$ be the integral model of $S_{K_0(p)}$ over $\mathcal{O}_{E,(p)}$ which was constructed by Rapoport and Zink in [Ra-Zi], §6.9. We want to give a more concrete description of the moduli problem parametrized by $\mathcal{S}_0(p)$. Let R be an $\mathcal{O}_{E,(p)}$-algebra and $\underline{A} \in \mathcal{S}(R)$ as in §2.1. A finite flat $\mathcal{O}_E$-subgroup scheme $H \subset A[p]$ is called *Raynaud* if for every characteristic p geometric point $x : R \to k$ of $\mathrm{Spec}(R)$, the Dieudonné module $M(H_x)$ is balanced, in the sense that

$$\dim_k M(H_x)(\Sigma) = \dim_k M(H_x)(\overline{\Sigma}).$$

By $M(H_x)$ we denote the covariant Dieudonné module of H_x. It coincides with the contravariant Dieudonné module of the Cartier dual H_x^D of H_x. See [dS-G2] §1.2.1 for a discussion of the Raynaud condition, and its relation to the original condition imposed by Raynaud when $m = 1$.

Proposition 6. *The scheme $\mathcal{S}_0(p)$ is a moduli space for pairs $(\underline{A}, H)$ where $\underline{A} \in \mathcal{S}(R)$ and $H \subset A[p]$ is a finite flat group scheme of rank p^{2m}, which is isotropic for the Weil pairing on $A[p]$ induced by ϕ, $\mathcal{O}_E$-stable and Raynaud.*

Proof. See [dS-G2], §1.3. The proof given there for $(n, m) = (2, 1)$ can be adapted to the general case *mutatis mutandis.* □

We denote[3] by $\pi : \mathcal{S}_0(p) \to \mathcal{S}$ the morphism which on the moduli problem is "forget H". The scheme $\mathcal{S}_0(p)$ is not smooth over $\mathcal{O}_{E,(p)}$ but Görtz [Gö], and later Pappas and Zhu [P-Z], proved the following.

Proposition 7. *(*[P-Z]*, Theorem 0.2) The scheme $\mathcal{S}_0(p)$ is proper and flat over $\mathcal{O}_{E,(p)}$, the irreducible components of its special fiber are reduced, and their local rings are Cohen-Macaulay and normal.*

3.2.3. *The ordinary-multiplicative and ordinary-étale loci.* Let $S_0(p)$ be the special fiber of $\mathcal{S}_0(p)$. Let $S_0(p)^{\mathrm{ord}} = \pi^{-1}(S^{\mathrm{ord}})$. If $x \in S^{\mathrm{ord}}(k)$ is a geometric point, then $\mathcal{A}_x[p^\infty]$ is given by (2.1), and any isotropic Raynaud $\mathcal{O}_E$-subgroup scheme of $\mathcal{A}_x[p]$ of rank p^{2m} is of the form

$$H \simeq (\mathcal{O}_E \otimes \mu_p)^r \times \mathscr{G}_\Sigma[p]^s \times (\mathcal{O}_E \otimes \mathbb{Z}/p\mathbb{Z})^t$$

for an "ordinary type" (r, s, t), $r + s + t = m$. The type (r, s, t) is locally constant on $S_0(p)^{\mathrm{ord}}$ in the Zariski topology. Indeed, p^{2r+2s} is the rank of the connected part H^{conn}, which can only go up under specialization, by duality the same is true of p^{2s+2t}, but $r + s + t$ is constant. Thus under specialization the only possibility is for r and t to go down, and for s to go up. But the same must be true of $A[p]/H$. If we specialize to a μ-ordinary point, the type of $A[p]/H$ is $(m-r, n-m-s, m-t)$, hence $m - r$ and $m - t$ must also go down, forcing r, s and t to remain constant.

It follows that the discrete invariants (r, s, t) decompose $S_0(p)^{\mathrm{ord}}$ into disjoint open sets. We denote by $S_0(p)^{\mathrm{ord}}_m$ the locus where H is of multiplicative type $(s = t = 0)$ and by $S_0(p)^{\mathrm{ord}}_{et}$ the locus where H is étale $(r = s = 0)$. The other loci are denoted by $S_0(p)^{\mathrm{ord}}_{r,s,t}$, and the projection from them to S^{ord} will be denoted $\pi_{r,s,t}$.

Proposition 8. *The loci $S_0(p)^{\mathrm{ord}}_m$ and $S_0(p)^{\mathrm{ord}}_{et}$ are non-singular and relatively irreducible over S^{ord}.*

Proof. This must be well-known. We shall see below that $S_0(p)^{\mathrm{ord}}_m$ is isomorphic to S^{ord}. The relative irreducibility of $S_0(p)^{\mathrm{ord}}_{et}$ is a consequence of the surjectivity of the map $\rho : S^{\mathrm{ord}} \to S_0(p)^{\mathrm{ord}}_{et}$ constructed in §3.2.4. Regularity can be proven by direct computation of the tangent space using deformation theory, as outlined in [dS-G2] when $(n, m) = (2, 1)$ (following Bellaïche). Alternatively, one can argue as follows. The input leading to the computation of the tangent space (or even the local model) at a closed point $x \in S_0(p)^{\mathrm{ord}}_{et}$ is everywhere the same. This is because the p-divisible group is μ-ordinary and H is the kernel of p in its ètale part. Thus either all points of $S_0(p)^{\mathrm{ord}}_{et}$ are non-singular or all are singular. The latter case is excluded since it is reduced, by Proposition 7. □

Denote by π_m and π_{et} the restriction of the map π to S^{ord}_m and S^{ord}_{et} (i.e. the maps $\pi_{m,0,0}$ and $\pi_{0,0,m}$). The map π_m is an *isomorphism.* In fact, it has the section

[3] No confusion should arise from the fact that we denote by π also the structure map $\mathcal{A} \to \mathcal{S}$, or the ratio of the circumference of the circle to its diameter.

associating to every $\underline{A} \in S^{\mathrm{ord}}(R)$ the R-point $(\underline{A}, A[p]^{mult}) \in S_0(p)_m^{\mathrm{ord}}(R)$ where $A[p]^{mult}$ is the maximal finite flat subgroup scheme of $A[p]$ which is of multiplicative type (connected with étale dual). This subgroup is automatically isotropic and Raynaud. Denote this section by

$$\sigma_m : S^{\mathrm{ord}} \to S_0(p)_m^{\mathrm{ord}}.$$

3.2.4. *Morphisms between ordinary-étale components.* We shall define morphisms θ, θ', ρ and ρ' that fit into the commutative diagram below. The scheme $(S_0(p)_{et}^{\mathrm{ord}})^{(p)}$ appearing in the diagram classifies the same objects as $S_0(p)_{et}^{\mathrm{ord}}$, except that the signature of the $\mathcal{O}_E$ action is (m, n) instead of (n, m).

$$\begin{tikzcd}
& (S_0(p)_{et}^{\mathrm{ord}})^{(p)} \arrow[d, "\rho'"] \arrow[dr, "Fr_p"] & \\
S_0(p)_{et}^{\mathrm{ord}} \arrow[ur, "Fr_p"] \arrow[r, "\pi_{et}"] \arrow[dr, "\theta'"'] & S^{\mathrm{ord}} \arrow[r, "\rho"] \arrow[d, shift right, "\sigma_m"'] & S_0(p)_{et}^{\mathrm{ord}} \\
& S_0(p)_m^{\mathrm{ord}} \arrow[u, shift right, "\pi_m"'] \arrow[ur, "\theta"'] &
\end{tikzcd}$$

Before we embark on the definition, we want to make a notational remark.

Remark about Frobenii: We denote by Fr_p the Frobenius of the scheme S (or $S_0(p)$) relative to κ. Thus, Fr_p is a morphism of κ-schemes

$$Fr_p : S \to S^{(p)},$$

where $S^{(p)} = \Phi_\kappa^* S$ is the base change of S with respect to Φ_κ. If $\xi \in S(R)$ for some κ-algebra R and ξ corresponds to the tuple $\underline{A} = \xi^* \underline{\mathcal{A}}$, then $Fr_p(\xi) \in S^{(p)}(R)$ corresponds to $\underline{A}^{(p)} = \Phi_R^* \underline{A}$, the base change of A (with the associated PEL structure) *with respect to* Φ_R. Note that $A^{(p)}$ has type (m, n). We write in short

$$Fr_p(\underline{A}) = \underline{A}^{(p)}.$$

This should not be confused with the isogeny $\mathrm{Fr} : A \to A^{(p)}$, which is a morphism of abelian schemes over $\mathrm{Spec}(R)$. A more appropriate convention would have been to denote Fr_p by $Fr_{S/\kappa}$ and the isogeny Fr by $Fr_{A/R}$ or $Fr_{\mathcal{A}/S}$, but this would result in a pretty heavy notation.

To define the map ρ we consider the map

$$\theta : S_0(p)_m^{\mathrm{ord}} \to S_0(p)_{et}^{\mathrm{ord}}$$

defined on the moduli problems as

$$\theta(\underline{A}, H) = (\underline{A}^{(p^2)}, \mathrm{Fr}(\mathrm{Ver}^{-1}(H))).$$

Then we let

$$\rho = \theta \circ \sigma_m.$$

Some words of explanation are in order. Here $\mathrm{Ver}^{-1}(H)$ is the kernel of the isogeny $A^{(p)} \overset{\mathrm{Ver}}{\to} A \to A/H$, hence is finite flat of rank p^{3m+n}. For $H = A[p]^{mult}$, where Ver is an isomorphism from $H^{(p)}$ to H, $\mathrm{Ver}^{-1}(H)$ coincides, as a group functor, with $H^{(p)} + A^{(p)}[\mathrm{Ver}]$, so is seen to be contained in $A^{(p)}[p]$. Also in this case, the kernel of $\mathrm{Fr} : A^{(p)}[p] \to A^{(p^2)}[p]$ is contained in $\mathrm{Ver}^{-1}(H)$, hence the image $\mathrm{Fr}(\mathrm{Ver}^{-1}(H))$

is a finite flat subgroup scheme of rank p^{2m}. It is easily seen that this subgroup scheme is $\mathcal{O}_E$-stable, Raynaud, isotropic and étale, since these properties can be checked on the geometric fibers. We also remark that for $H = A[p]^{mult}$

$$\mathrm{Fr}(\mathrm{Ver}^{-1}(H)) = \mathrm{Fr}(A^{(p)}[\mathrm{Ver}]) = \mathrm{Fr}^2(A[p]).$$

The reason we chose to define θ the way we did is that this is the definition that will generalize later on, in §4.2, when H is no longer multiplicative, to a map between the Zariski closures $S_0(p)_m$ and $S_0(p)_{et}$ in $S_0(p)$.

There is a similar map

$$\theta' : S_0(p)_{et}^{\mathrm{ord}} \to S_0(p)_m^{\mathrm{ord}}$$

defined on the moduli problems as

$$\theta'(\underline{A}, H) = (\underline{A}, \mathrm{Ver}(\mathrm{Fr}^{-1}(H^{(p^2)}))).$$

The proof that it is well-defined is similar to the one for θ. Note that

$$\pi_{et} \circ \theta = Fr_p^2 \circ \pi_m, \quad \pi_m \circ \theta' = \pi_{et}, \quad \theta \circ \theta' = Fr_p^2, \quad \theta' \circ \theta = Fr_p^2.$$

The definition of ρ' is a little more subtle[4]. Let $(\underline{A}_1, H_1) \in (S_0(p)_{et}^{\mathrm{ord}})^{(p)}(R)$. Thanks to the polarization, the subgroup scheme $H_1^{\perp}$ which is the annihilator of H_1 relative to the Weil pairing on $A_1[p]$, is well-defined, and is finite flat of rank p^{2n} over R. We claim that the closed subgroup scheme $H_1^{\perp}[\mathrm{Fr}]$ is finite flat of rank p^{n-m}. Indeed, it is enough to check it for the universal $(\underline{A}_1, H_1)$ over $(S_0(p)_{et}^{\mathrm{ord}})^{(p)}$. But now the base is reduced (by [Gö]) so it is enough to check that all the geometric fibers of $H_1^{\perp}[\mathrm{Fr}]$ are of the same rank, and that this rank is p^{n-m}. This is straightforward, given that over an algebraically closed field we have the description (2.1). For any geometric point $x : \mathrm{Spec}(k) \to \mathrm{Spec}(R)$, $H_1^{\perp}[\mathrm{Fr}]$ becomes the α_p-subgroup which is the kernel of Frobenius in the local-local part of $x^*(A_1[p])$. Seen in another light, giving H_1 not only splits the connected-étale exact sequence over $\mathrm{Spec}(R)$, but allows us to split off the local-local part from the multiplicative part in $A_1[\mathrm{Fr}]$. In particular, $H_1^{\perp}[\mathrm{Fr}]$ does not intersect H_1, so

$$K_1 := H_1^{\perp}[\mathrm{Fr}] + H_1 \simeq H_1^{\perp}[\mathrm{Fr}] \times H_1$$

is finite flat of rank p^{n+m}. This K_1 is a maximal isotropic subgroup of $A_1[p]$, whose tangent space is $n-m$ dimensional, of type $\overline{\Sigma}$ (sic!). Descending the polarization and the endomorphisms to $A = A_1/K_1$ we get a principally polarized abelian scheme over R, of type (n, m). We let

$$\underline{A} = \rho'(\underline{A}_1, H_1) := \langle p \rangle_N^{-1} \underline{A}_1/K_1 \in S^{\mathrm{ord}}(R). \tag{3.1}$$

The underlying principally polarized abelian scheme with endomorphisms by $\mathcal{O}_E$ is A. The level-N structure differs from the one descended from $\underline{A}_1$ by the diamond operator $\langle p \rangle_N^{-1}$. Recall that the *diamond operator* $\langle a \rangle_N$, for $a \in (\mathcal{O}_E/N\mathcal{O}_E)^{\times}$ takes an $\mathcal{O}_E$-level-N structure $\eta : (\mathcal{O}_E/N\mathcal{O}_E)^{n+m} \simeq A[N]$ to $\eta \circ [a]$, where $[a]$ is multiplication by a.

---

[4]We remind the reader that for general finite flat subgroup schemes Γ_1 and Γ_2 of a finite flat group scheme Γ, over an arbitrary locally Noetherian base, the subgroup scheme $\Gamma_1 \cap \Gamma_2$ need not be flat, and the subgroup functor $\Gamma_1 + \Gamma_2$ need not be represented by a group scheme at all. Similarly, for a homomorphism f between two finite flat group schemes, $\ker(f)$ is a subgroup scheme which need not be flat, and the group-functor image of f need not be represented by a group scheme at all.

Having defined the maps in the diagram, we now check its commutativity. We only have to check the commutativity of the top two triangles, the bottom two being obvious. Let $\underline{A}_1$ and $\underline{A}$ be related by (3.1). Consider the morphism $\mathrm{Fr} : \underline{A} \to \underline{A}^{(p)}$ obtained by dividing A by $A[\mathrm{Fr}]$ and descending the polarization, the endomorphisms, and the level structure. Since $A[\mathrm{Fr}] = A_1[p]/K_1$ we get the string of isomorphisms

$$\underline{A}^{(p)} \simeq \underline{A}/A[\mathrm{Fr}] \simeq \left(\langle p\rangle_N^{-1}\, \underline{A}_1/K_1\right) / \left(A_1[p]/K_1\right) \simeq \langle p\rangle_N^{-1}\, \underline{A}_1/A_1[p] \simeq \underline{A}_1. \tag{3.2}$$

(The last isomorphism is multiplication by p, and it is the reason for introducing $\langle p\rangle_N^{-1}$ in the definition of ρ'.) We conclude that to accommodate an étale subgroup scheme like H_1, $\underline{A}_1$ *must be* of the form $\underline{A}^{(p)}$ and $K_1 = A^{(p)}[\mathrm{Ver}]$. This *may not be said* of H_1 itself, in general. However, if this is the case and $(\underline{A}_1, H_1) = (\underline{A}^{(p)}, H^{(p)}) = Fr_p((\underline{A}, H))$ then the above discussion shows that $\rho'(\underline{A}_1, H_1) = \underline{A}$, proving the commutativity of the first triangle in the diagram:

$$\rho' \circ Fr_p = \pi_{et}.$$

For the second triangle consider

$$\begin{aligned}\rho \circ \rho'(\underline{A}_1, H_1) = \rho(\underline{A}) = (\underline{A}^{(p^2)}, \mathrm{Fr}(\mathrm{Ver}^{-1}(A[p]^{mult}))) = (\underline{A}^{(p^2)}, \mathrm{Fr}^2(A[p]))\\ = (\underline{A}_1^{(p)}, H_1^{(p)}) = Fr_p((\underline{A}_1, H_1)).\end{aligned}$$

To justify the transition from the first to the second line, note that $H_1^{(p)} = \mathrm{Fr}(H_1) \subset \mathrm{Fr}^2(A[p])$ as $H_1 \subset \mathrm{Fr}(A[p])$, but $H_1^{(p)}$ and $\mathrm{Fr}^2(A[p])$ are both finite flat of rank p^{2m}, so they coincide.

Lemma 9. *All the morphisms in the diagram are finite and flat.*

Proof. Since Fr_p is a finite morphism between schemes of finite type over a field, all the maps are clearly finite. The schemes are all regular of dimension nm. Finite maps between locally noetherian regular schemes of the same dimension are flat (it is in fact enough to assume that the source is Cohen-Macaulay). Note that by a theorem of Kunz [Ku] the relative Frobenius morphism from X to $X^{(p)}$ is flat *if and only if* X is regular. □

Lemma 10. *The degrees of ρ and ρ' are given by*

$$\deg(\rho) = p^{m^2}, \quad \deg(\rho') = p^{(n-m)m}.$$

Furthermore, $\deg(\theta) = \deg(\rho)$ *and* $\deg(\theta') = \deg(\pi_{et}) = p^{(2n-m)m}$.

Proof. Since $\rho \circ \rho' = Fr_p$ is of degree p^{nm}, it is enough to prove the formula for $\deg(\rho')$. Since $\rho' \circ Fr_p = \pi_{et}$ it is enough to prove that $\deg(\pi_{et}) = p^{2nm-m^2}$. We use a method of degeneration from characteristic 0, based on the flatness of $\pi : S_0(p)^{\mathrm{ord}} \to S^{\mathrm{ord}}$. This map is flat because it is finite, S^{ord} is regular and $S_0(p)^{\mathrm{ord}}$ is Cohen-Macaulay. (In fact, the arguments of Proposition 8 prove that $S_0(p)^{\mathrm{ord}}$ is non-singular.)

Denote by $\mathcal{S}^{\mathrm{ord}}$ the complement in $\mathcal{S}$ of the non-ordinary locus in the special fiber, and similarly $\mathcal{S}_0(p)^{\mathrm{ord}}$. Since $\mathcal{S}_0(p)^{\mathrm{ord}}$ and $\mathcal{S}^{\mathrm{ord}}$ are flat over $\mathcal{O}_{E,(p)}$ and both the generic and special fibers of $\pi : \mathcal{S}_0(p)^{\mathrm{ord}} \to \mathcal{S}^{\mathrm{ord}}$ are flat, then by the criterion for flatness fiber-by-fiber π is flat also on the ordinary parts of the arithmetic schemes.

Fix a $W(k)$-valued point

$$\xi : \mathrm{Spec}(W(k)) \to \mathcal{S}^{\mathrm{ord}},$$

and denote its specialization by $\xi_0 : \mathrm{Spec}(k) \to S^{\mathrm{ord}}$. Consider the pull-back

$$\xi^*\mathcal{S}_0(p) \to \mathrm{Spec}(W(k)).$$

As the base change of the finite flat morphism $\mathcal{S}_0(p)^{\mathrm{ord}} \to \mathcal{S}^{\mathrm{ord}}$, this map is also finite flat. We denote by $\eta : \xi^*\mathcal{S}_0(p) \to \mathcal{S}_0(p)$ the base change of ξ and by η_0 that of ξ_0.

$$\begin{tikzcd}
\xi_0^*\mathcal{S}_0(p) \arrow[r] \arrow[d] \arrow[rr, bend left, "\eta_0"] & \xi^*\mathcal{S}_0(p) \arrow[r, "\eta"] \arrow[d] & \mathcal{S}_0(p) \arrow[d] \\
\mathrm{Spec}(k) \arrow[r] \arrow[rr, bend right, "\xi_0"'] & \mathrm{Spec}(W(k)) \arrow[r, "\xi"] & \mathcal{S}
\end{tikzcd}$$

Let $\xi^*\mathcal{S}_0(p)_{et}$ be the connected component of $\xi^*\mathcal{S}_0(p)$ containing, in the special fiber, $\eta_0^*(S_0(p)_{et}^{\mathrm{ord}})$. It is finite flat over $W(k)$, and the degree of its special fiber is $\deg(\pi_{et})$. We compute this degree in the *generic* fiber. Let $A = \xi^*\mathcal{A}$ be the pull-back of the universal abelian scheme to $W(k)$, and let A_0 be its special fiber. Consider $A(\overline{E}_p)[p]$, the p-torsion in the group of points of A in a fixed algebraic closure $\overline{E}_p$ of the local field E_p. The geometric points in the generic fiber of $\xi^*\mathcal{S}_0(p)_{et}$ are in 1-1 correspondence with the isotropic, $\mathcal{O}_E$-stable subgroups $H \subset A(\overline{E}_p)[p]$ of rank p^{2m} specializing to $A_0[p]^{et} \subset A_0[p]$.

Denote by $A(\overline{E}_p)^0$ the kernel of the reduction map $A(\overline{E}_p) \to A_0(k)$. We have to count isotropic, $\mathcal{O}_E$-stable subgroups $H \subset A(\overline{E}_p)[p]$ of rank p^{2m} satisfying

$$H + A(\overline{E}_p)^0[p] = A(\overline{E}_p)[p].$$

We are now reduced to linear algebra. The $\mathcal{O}_E$-module $A(\overline{E}_p)[p]$ with the hermitian pairing derived from the polarization is isomorphic to κ^{n+m} with the κ-hermitian form

$$(u, v) = {}^t u^{(p)} \begin{pmatrix} & & 1_m \\ & 1_{n-m} & \\ 1_m & & \end{pmatrix} v$$

and we may choose the isomorphism so that $A(\overline{E}_p)^0[p]$ is the subspace with the last m entries 0. We thus have to count equivalence classes of κ-linear maps $\lambda : \kappa^m \hookrightarrow \kappa^{n+m}$ satisfying (1) the image of λ is isotropic, and (2) the projection of the image of λ on the last m coordinates is an isomorphism. Two such maps λ_1 and λ_2 are equivalent if $\lambda_2 = \lambda_1 \circ \alpha$ for $\alpha \in GL_m(\kappa)$. This is the same as counting matrices $\Gamma \in M_{n\times m}(\kappa)$ satisfying

$$({}^t\Gamma^{(p)}, 1_m) \begin{pmatrix} & & 1_m \\ & 1_{n-m} & \\ 1_m & & \end{pmatrix} \begin{pmatrix} \Gamma \\ 1_m \end{pmatrix} = 0,$$

or equivalently, counting pairs $(\Gamma_1, \Gamma_2) \in M_{m\times m}(\kappa) \times M_{(n-m)\times m}(\kappa)$ satisfying

$$\Gamma_1 + {}^t\Gamma_1^{(p)} + {}^t\Gamma_2^{(p)}\Gamma_2 = 0.$$

This number is easily seen to be p^{2nm-m^2}. Indeed, there are $p^{2(n-m)m}$ choices for Γ_2. For each choice of Γ_2 there are $p^{2\frac{m(m-1)}{2}}$ choices for the entries of Γ_1 above the diagonal, which are arbitrary and determine the entries below the diagonal uniquely, and p^m choices for the entries on the diagonal. This concludes the proof of the lemma. □

3.2.5. *The relation between the foliation and* ρ. In §3.1 we have constructed the foliation $\mathcal{T}S^+$ in the tangent bundle of S^{ord}, while in §3.2.4 we have constructed a flat height 1 morphism $\rho : S^{\mathrm{ord}} \to S_0(p)^{\mathrm{ord}}_{et}$. We shall now prove that the two correspond to each other under the dictionary between height 1 morphisms and height 1 foliations discussed in Proposition 1.

Theorem 11. *The quotient of* S^{ord} *by the height 1 foliation* $\mathcal{T}S^+$ *is the height 1 morphism* $\rho : S^{\mathrm{ord}} \to S_0(p)^{\mathrm{ord}}_{et}$.

Proof. We have to show that the image of $\rho^*(\Omega_{S_0(p)^{\mathrm{ord}}_{et}})$ in $\Omega_{S^{\mathrm{ord}}}$ is $KS(\mathcal{P}_0 \otimes \mathcal{Q})$. Since ρ is a finite flat height 1 morphism of degree p^{m^2} we know, from the general theory explained in §2.2, that the image of $\rho^*(\Omega_{S_0(p)^{\mathrm{ord}}_{et}})$ in Ω_S is a sub-bundle of $\Omega_{S^{\mathrm{ord}}}$ of rank $(n-m)m$. Since the same is true of $KS(\mathcal{P}_0 \otimes \mathcal{Q})$, it is enough to prove the inclusion

$$KS(\mathcal{P}_0 \otimes \mathcal{Q}) \subset \mathrm{Im}(\rho^*(\Omega_{S_0(p)^{\mathrm{ord}}_{et}}) \to \Omega_{S^{\mathrm{ord}}}).$$

As the right hand side is equal to $\ker(\rho'^* : \Omega_{S^{\mathrm{ord}}} \to \Omega_{(S_0(p)^{\mathrm{ord}}_{et})^{(p)}})$ (they are both sub-bundles of rank $(n-m)m$ and the image of ρ^* is contained in the kernel of ρ'^* because $\rho \circ \rho' = Fr_p$), it will be enough to prove that $KS(\mathcal{P}_0 \otimes \mathcal{Q})$ is contained in the latter. More precisely, we have to show that

$$\rho'^* KS(\mathcal{P}_0 \otimes \mathcal{Q}) \subset \ker(\rho'^* : \rho'^* \Omega_{S^{\mathrm{ord}}} \to \Omega_{(S_0(p)^{\mathrm{ord}}_{et})^{(p)}}).$$

For this purpose consider the universal pair $(\mathcal{A}_1, \mathcal{H}_1)$ over $(S_0(p)^{\mathrm{ord}}_{et})^{(p)}$. In (3.2) we have constructed $\mathcal{A}$ such that, at the level of points, $\rho'((\mathcal{A}_1, \mathcal{H}_1)) = \mathcal{A}$ (where to simplify typesetting, we omit the underline symbol). Note that $\mathcal{A}$ is a scheme over S^{ord} and, letting $\mathcal{B} = \rho'^* \mathcal{A}$, the relations obtained in (3.2) imply a canonical isomorphism $\mathcal{A}_1 = \mathcal{B}^{(p)}$, as abelian schemes over $(S_0(p)^{\mathrm{ord}}_{et})^{(p)}$. The construction also provides a canonical isogeny $\mathcal{A}_1 \to \mathcal{B}$, which is nothing else than Ver.

The kernel of Ver contains the finite flat group scheme $\mathcal{H}_1$. Thus, letting $\mathcal{C} = \mathcal{A}_1/\mathcal{H}_1$, we get a decomposition of Ver as the composition of two isogenies between abelian schemes over $(S_0(p)^{\mathrm{ord}}_{et})^{(p)}$:

$$\begin{tikzcd} \mathcal{A}_1 \arrow[r, "\psi"] \arrow[rr, bend right, "\mathrm{Ver}"'] & \mathcal{C} \arrow[r, "\varphi"] & \mathcal{B} \end{tikzcd}.$$

Here ψ is the isogeny with kernel $\mathcal{H}_1$, and φ is the isogeny with kernel $\mathcal{A}_1[\mathrm{Ver}]/\mathcal{H}_1$. Note that although $\mathrm{Ver} : \mathcal{A}_1 \to \mathcal{B}$ is a pull-back by ρ' of a similar isogeny over S^{ord}, only over $(S_0(p)^{\mathrm{ord}}_{et})^{(p)}$ does it factor through $\mathcal{C}$, because $\mathcal{H}_1$ is *not* the pull-back of a group scheme on S^{ord}.

Consider now the commutative diagram

$$\begin{array}{ccc} \rho'^* \mathcal{P} = \omega_{\mathcal{B}}(\Sigma) & \stackrel{KS_{\mathcal{B}}}{\longrightarrow} & \Omega_{(S_0(p)^{\mathrm{ord}}_{et})^{(p)}} \otimes R^1\pi_* \mathcal{O}_{\mathcal{B}}(\Sigma) \\ \downarrow \varphi^* & & \downarrow 1 \otimes \varphi^* \\ \omega_{\mathcal{C}}(\Sigma) & \stackrel{KS_{\mathcal{C}}}{\longrightarrow} & \Omega_{(S_0(p)^{\mathrm{ord}}_{et})^{(p)}} \otimes R^1\pi_* \mathcal{O}_{\mathcal{C}}(\Sigma) \end{array} \tag{3.3}$$

resulting from the functoriality of the Gauss-Manin connection with respect to the isogeny φ. Here $KS_{\mathcal{B}}$ is the Kodaira-Spencer map for the family $\mathcal{B} \to (S_0(p)_{et}^{\mathrm{ord}})^{(p)}$ and likewise for $\mathcal{C}$.

The kernel of the left vertical arrow φ^* is precisely $\rho'^*(\mathcal{P}_0)$. This is because ψ is étale, so ψ^* is an isomorphism on cotangent spaces, hence

$$\ker(\varphi^*|_{\rho'^*\mathcal{P}}) = \ker(\mathrm{Ver}^*|_{\rho'^*\mathcal{P}}) = \rho'^*(\mathcal{P}_0).$$

On the right side of (3.3), we claim that $1 \otimes \varphi^*$ is injective. To verify it, consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \omega_{\mathcal{B}}(\Sigma) & \to & H^1_{dR}(\mathcal{B}/S)(\Sigma) & \to & R^1\pi_*\mathcal{O}_{\mathcal{B}}(\Sigma) & \to & 0 \\ & & \downarrow \varphi^* & & \downarrow \varphi^* & & \downarrow \varphi^* & & \\ 0 & \to & \omega_{\mathcal{C}}(\Sigma) & \to & H^1_{dR}(\mathcal{C}/S)(\Sigma) & \to & R^1\pi_*\mathcal{O}_{\mathcal{C}}(\Sigma) & \to & 0 \end{array} \tag{3.4}$$

The right vertical arrow may be identified with the Σ-component of the map $\varphi^t_* : \mathrm{Lie}(\mathcal{B}^t) \to \mathrm{Lie}(\mathcal{C}^t)$. The signature of $\mathcal{B}^t$ is (m,n), and at every geometric point x

$$\mathcal{B}^t_x[p] \simeq (\mathcal{O}_E \otimes \mu_p)^m \oplus (\mathscr{G}_{\overline{\Sigma}}[p])^{n-m} \oplus (\mathcal{O}_E \otimes \mathbb{Z}/p\mathbb{Z})^m.$$

As $\ker(\varphi) = \mathcal{A}_1[\mathrm{Ver}]/\mathcal{H}_1$ is local-local, so is its dual $\ker(\varphi^t)$; in fact, its geometric fibers are all isomorphic to $\alpha_p^{n-m} \subset (\mathscr{G}_{\overline{\Sigma}}[p])^{n-m}$. The Lie algebra of $\ker(\varphi^t)$, i.e. $\ker(\varphi^t_*)$, *is therefore of type* $\overline{\Sigma}$. We conclude that the right vertical arrow of (3.4), and with it the right vertical arrow of (3.3), are injective, as claimed.

As $\mathcal{B} = \rho'^*\mathcal{A}$, the morphism $KS_{\mathcal{B}}$ is the composition of the map

$$\rho'^*(KS_{\mathcal{A}}) : \rho'^*(\mathcal{P}) \to \rho'^*(\Omega_{S^{\mathrm{ord}}}) \otimes \rho'^*(\mathcal{Q}^\vee)$$

(we identify $\mathcal{Q} = \omega_{\mathcal{A}}(\overline{\Sigma})$ with $\omega_{\mathcal{A}^t}(\Sigma)$ via the polarization as usual, hence also $R^1\pi_*\mathcal{O}_{\mathcal{A}}(\Sigma) = \mathrm{Lie}(\mathcal{A}^t)(\Sigma)$ with $\mathcal{Q}^\vee$) and the map induced by

$$\rho'^* : \rho'^*(\Omega_{S^{\mathrm{ord}}}) \to \Omega_{(S_0(p)_{et}^{\mathrm{ord}})^{(p)}}.$$

From the commutativity of (3.3) we conclude that $KS_{\mathcal{B}}(\rho'^*(\mathcal{P}_0)) = 0$, hence the desired inclusion

$$\rho'^*(KS(\mathcal{P}_0 \otimes \mathcal{Q})) \subset \ker(\rho'^* : \rho'^*\Omega_{S^{\mathrm{ord}}} \to \Omega_{(S_0(p)_{et}^{\mathrm{ord}})^{(p)}}).$$

$\square$

3.3. **Moonen's generalized Serre-Tate coordinates.** Although not necessary for the rest of the paper, we digress to explain the relation between $\mathcal{T}S^+$ and the generalized Serre-Tate coordinates introduced by Moonen. For the following proposition see [Mo1], the remark at the end of Example 3.3.2, and 3.3.3(d) (case AU, $r = 3$).

Proposition 12. *Let $x \in S^{\mathrm{ord}}(k)$. Let $\widehat{\mathscr{G}}$ be the formal group over k associated with the p-divisible group $\mathscr{G}$ and let $\widehat{\mathbb{G}}_m$ be the formal multiplicative group over k. Then the formal neighborhood $Spf(\widehat{\mathcal{O}}_{S,x})$ of x has a natural structure of a $\widehat{\mathbb{G}}_m^{m^2}$-torsor over $\widehat{\mathscr{G}}^{(n-m)m}$. This torsor is obtained as the set of symmetric elements under the involution induced by the polarization on a certain bi-extension of $\widehat{\mathscr{G}}^{(n-m)m} \times \widehat{\mathscr{G}}^{(n-m)m}$, hence it contains a canonical formal torus $\widehat{T}_x$ sitting over the origin of $\widehat{\mathscr{G}}^{(n-m)m}$.*

Theorem 13. *Let $x \in S^{\mathrm{ord}}(k)$. Then $\mathcal{T}S^+|_x$ is the tangent space to $\widehat{T}_x \subset Spf(\widehat{\mathcal{O}}_{S,x})$.*

Proof. Let $i : \widehat{T}_x \hookrightarrow Spf(\widehat{\mathcal{O}}_{S,x})$ be the embedding of formal schemes given by Proposition 12. It sends the origin e of $\widehat{T}_x$ to x. Let i_* be the induced map on tangent spaces

$$i_* : \mathcal{T}\widehat{T}_x|_e \hookrightarrow \mathcal{T}S|_x.$$

We have to show that $i_*(\mathcal{T}\widehat{T}_x|_e)$ annihilates $KS(\mathcal{P}_0 \otimes \mathcal{Q})|_x$. This is equivalent to saying that when we consider the pull back $i^*\mathcal{A}$ of the universal abelian scheme to $\widehat{T}_x$, *its* Kodaira-Spencer map kills $\mathcal{P}_0 \otimes \mathcal{Q}|_e$. For this recall the definition of $KS = KS(\Sigma)$.

Let $\mathfrak{S} = \widehat{T}_x$ and write for simplicity $\mathcal{A}$ for $i^*\mathcal{A}$. We then have the following commutative diagram

$$\begin{array}{ccc} \mathcal{P} = \omega_{\mathcal{A}/\mathfrak{S}}(\Sigma) & \hookrightarrow & H^1_{dR}(\mathcal{A}/\mathfrak{S})(\Sigma) \\ \downarrow KS & & \downarrow \nabla \\ \mathcal{Q}^\vee \otimes \Omega^1_{\mathfrak{S}} \simeq \omega^\vee_{\mathcal{A}^t/\mathfrak{S}}(\Sigma) \otimes \Omega^1_{\mathfrak{S}} & \longleftarrow & H^1_{dR}(\mathcal{A}/\mathfrak{S})(\Sigma) \otimes \Omega^1_{\mathfrak{S}} \end{array} \tag{3.5}$$

in which we identified $H^1(\mathcal{A}, \mathcal{O})$ with $H^0(\mathcal{A}^t, \Omega^1_{\mathcal{A}^t/\mathfrak{S}})^\vee$ and used the polarization to identify the latter with $\omega^\vee_{\mathcal{A}/\mathfrak{S}}$, reversing types. Here ∇ is the Gauss-Manin connection, and the tensor product is over $\widehat{\mathcal{O}}_{\mathfrak{S}}$. Although ∇ is a derivation, KS is a homomorphism of vector bundles over $\widehat{\mathcal{O}}_{\mathfrak{S}}$. We shall show that $KS(\mathcal{P}_0) = 0$, where $\mathcal{P}_0 = \ker(V : \omega_{\mathcal{A}/\mathfrak{S}} \to \omega^{(p)}_{\mathcal{A}/\mathfrak{S}}) \cap \mathcal{P}$.

At this point recall the filtration

$$0 \subset Fil^2 = \mathcal{A}[p^\infty]^{mult} \subset Fil^1 = \mathcal{A}[p^\infty]^{conn} \subset Fil^0 = \mathcal{A}[p^\infty]$$

of the p-divisible group of $\mathcal{A}$ over $\mathfrak{S}$. The graded pieces are of height $2m$, $2(n-m)$ and $2m$ respectively, and $\mathcal{O}_E$-stable. They are given by

$$gr^2 = (\mathcal{O}_E \otimes \mu_{p^\infty})^m, \ \ gr^1 = \mathscr{G}_\Sigma^{n-m}, \ \ gr^0 = (\mathcal{O}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p)^m.$$

For any p-divisible group G over $\mathfrak{S}$ denote by $\mathbb{D}(G)$ the Dieudonné crystal associated to G, and let $D(G) = \mathbb{D}(G)_{\mathfrak{S}}$, *cf.* [Gro]. The $\widehat{\mathcal{O}}_{\mathfrak{S}}$-module $D(G)$ is endowed with an integrable connection ∇ and the pair $(D(G), \nabla)$ determines $\mathbb{D}(G)$.

In our case, we can identify $D(\mathcal{A}[p^\infty])$ with $H^1_{dR}(\mathcal{A}/\mathfrak{S})$, and the connection with the Gauss-Manin connection. The above filtration on $\mathcal{A}[p^\infty]$ induces therefore a filtration $Fil^\bullet$ on $H^1_{dR}(\mathcal{A}/\mathfrak{S})$ which is preserved by ∇. Since the functor $\mathbb{D}$ is contravariant, we write the filtration as

$$0 \subset Fil^1 H^1_{dR}(\mathcal{A}/\mathfrak{S}) \subset Fil^2 H^1_{dR}(\mathcal{A}/\mathfrak{S}) \subset Fil^3 = H^1_{dR}(\mathcal{A}/\mathfrak{S})$$

where

$$Fil^i H^1_{dR}(\mathcal{A}/\mathfrak{S}) = D(\mathcal{A}[p^\infty]/Fil^i \mathcal{A}[p^\infty]).$$

For example, $Fil^1 H^1_{dR}(\mathcal{A}/\mathfrak{S})$ is sometimes referred to as the "unit root subspace". As $Fil^2\mathcal{A}[p^\infty]$ is of multiplicative type, $\ker(V : H^1_{dR}(\mathcal{A}/\mathfrak{S}) \to H^1_{dR}(\mathcal{A}/\mathfrak{S})^{(p)})$ is contained in $Fil^2 H^1_{dR}(\mathcal{A}/\mathfrak{S})$. In particular,

$$\mathcal{P}_0 \subset Fil^2 H^1_{dR}(\mathcal{A}/\mathfrak{S}).$$

Let $G = \mathcal{A}[p^\infty]/\mathcal{A}[p^\infty]^{mult}$, so that $Fil^2 H^1_{dR}(\mathcal{A}/\mathfrak{S}) = D(G)$. It follows that in computing KS on $\mathcal{P}_0$ we may use the following diagram instead of (3.5):

$$\begin{array}{ccc} \mathcal{P}_0 & \hookrightarrow & D(G)(\Sigma) \\ \downarrow KS & & \downarrow \nabla \\ \mathcal{Q}^\vee \otimes \Omega^1_{\mathfrak{S}} & \longleftarrow & D(G)(\Sigma) \otimes \Omega^1_{\mathfrak{S}} \end{array} \tag{3.6}$$

Finally, we have to use the description of the formal neighborhood of x as given in [Mo1]. Since we are considering the pull-back of $\mathcal{A}$ to $\mathfrak{S}$ only, and not the full deformation over $Spf(\widehat{\mathcal{O}}_{S,x})$, it follows from the construction of the 3-cascade (biextension) in *loc.cit.* §2.3.6 that the p-divisible groups $Fil^1\mathcal{A}[p^\infty]$, and dually $G = \mathcal{A}[p^\infty]/Fil^2$, are *constant* over $\mathfrak{S}$. Thus over $\mathfrak{S}$

$$G \simeq \mathscr{G}^{n-m} \times (\mathcal{O}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p)^m,$$

and ∇ maps $D(\mathscr{G}^{n-m})$ to $D(\mathscr{G}^{n-m}) \otimes \Omega^1_{\mathfrak{S}}$. Since

$$\mathcal{P}_0 = \omega_{\mathscr{G}^{n-m}} = D(\mathscr{G}^{n-m})(\Sigma)$$

as subspaces of $H^1_{dR}(\mathcal{A}/\mathfrak{S})$,

$$\nabla(\mathcal{P}_0) \subset \mathcal{P}_0 \otimes \Omega^1_{\mathfrak{S}}.$$

The bottom arrow in (3.6) comes from the homomorphism

$$D(G)(\Sigma) \hookrightarrow H^1_{dR}(\mathcal{A}/\mathfrak{S})(\Sigma) \overset{pr}{\to} H^1(\mathcal{A},\mathcal{O})(\Sigma) \overset{\phi}{\simeq} H^1(\mathcal{A}^t,\mathcal{O})(\overline{\Sigma}) = \mathcal{Q}^\vee.$$

But the projection pr kills $\mathcal{P}_0 \subset \omega_{\mathcal{A}/\mathfrak{S}}$. This concludes the proof. □

Remark. Proposition 12 yields a natural integral "formal submanifold" to the height 1 foliation $\mathcal{T}S^+$ in a formal neighborhood of any ordinary point. As mentioned in §2.2, integral submanifolds to height 1 foliations are ubiquitous. On the other hand we were not able to lift $\mathcal{T}S^+$ to an h-foliation in the sense of [Ek] for $h > 1$, and we do not believe that they lift to characteristic 0 as in [Mi]. The meaning of these "natural" formal submanifolds from the point of view of foliations remains mysterious.

4. Extending the foliation beyond the ordinary locus

In this section we discuss the extension of the foliation $\mathcal{T}S^+$ from S^{ord} to a certain "successive blow-up" $S^\sharp$ of S. We define a finite flat morphism from $S^\sharp$ to the Zariski closure $S_0(p)_{et}$ of $S_0(p)^{\mathrm{ord}}_{et}$, extending the morphism from S^{ord} to $S_0(p)^{\mathrm{ord}}_{et}$, and show that this map is the quotient by the extended foliation. In this section we shall use the results on the Ekedahl-Oort stratification summarized in §2.1.3.

4.1. The moduli scheme $S^\sharp$.

4.1.1. *Definition and general properties.* Recall (§2.1.4) that the map $V_{\mathcal{P}}$ induced by Verschiebung maps $\mathcal{P}$ to $\mathcal{Q}^{(p)}$, hence its kernel at any point of S is at least $(n-m)$-dimensional. Over S^{ord}, but not only there, $V_{\mathcal{P}}$ maps $\mathcal{P}$ onto $\mathcal{Q}^{(p)}$, so the kernel $\mathcal{P}[V]$ is precisely of dimension $n-m$.

Define a moduli problem $S^\sharp$ on κ-algebras R by setting

$$S^\sharp(R) = \{(\underline{A},\mathcal{P}_0)|\ \underline{A} \in S(R),\ \mathcal{P}_0 \subset \mathcal{P}[V] \text{ a subbundle of rank } n-m\}.$$

There is a forgetful map $f : S^\sharp \to S$, which is bijective over S^{ord}. Let $Gr(n-m,\mathcal{P})$ be the relative Grassmanian over S classifying sub-bundles $\mathcal{N}$ of rank $n-m$ in $\mathcal{P}$. It is a smooth scheme over S, of relative dimension $(n-m)m$. As the condition $V(\mathcal{N}) = 0$ is closed, the moduli problem $S^\sharp$ is representable by a closed subscheme of $Gr(n-m,\mathcal{P})$. The fiber $S^\sharp_x = f^{-1}(x)$ is the Grassmanian of $(n-m)$-dimensional subspaces in $\mathcal{P}_x[V]$, and if $x \in S_w$ its dimension is, in the notation of §2.1.3,

$$\dim S^\sharp_x = (n-m)(a_\Sigma(w) - n + m).$$

Denote by $S_\sharp$ the open subset of S where f is an isomorphism, i.e. where $a_\Sigma(w) = n - m$. It is a union of EO strata, containing S^{ord}.

For any (n,m)-shuffle w denote the pre-image of the EO stratum S_w by

$$S_w^\sharp = f^{-1}(S_w).$$

Proposition 14. *The open set $S_\sharp$ contains $\binom{n}{m}$ EO strata. It contains a unique minimal stratum in the EO order, denoted S_{fol}, which is of dimension m^2.*

Proof. Using the labeling of the EO strata by the set $\Pi(n,m)$ of (n,m)-shuffles in $\mathfrak{S}_{n+m}$, and formula (2.3), we see that $a_\Sigma(w) = n - m$ if and only if

$$w^{-1}(n - m + j) = n + j$$

for all $1 \le j \le m$. Thus, the set of w satisfying $a_\Sigma(w) = n - m$ is in bijection with $\Pi(n-m,m)$, the set of $(n-m,m)$-shuffles in $\mathfrak{S}_n$. More precisely, we have to arrange the numbers

$$\{1,\ldots,n-m; n+1,\ldots n+m\}$$

in the interval $[1,n]$, preserving the order within each block. There are $\binom{n}{m}$ such shuffles.

Let $\iota : \Pi(n-m,m) \hookrightarrow \Pi(n,m)$ be the inclusion described above. The element

$$w_{\mathrm{fol}} = \begin{pmatrix} 1 & \ldots & n-m & n-m+1 & \ldots & n & n+1 & \ldots & n+m \\ 1 & \ldots & n-m & n+1 & \ldots & n+m & n-m+1 & \ldots & n \end{pmatrix}$$

belongs to $\iota(\Pi(n-m,m))$ and is the unique minimal element there in the usual Bruhat order. From the remark at the end of §2.1.3 we deduce that it is also the unique minimal element among $\iota(\Pi(n-m,m))$ in the EO order $\preceq$. This w_{fol} must represent a stratum $S_{\mathrm{fol}} = S_{w_{\mathrm{fol}}}$ of minimal dimension among the EO strata in $S_\sharp$. Since $\dim S_w = l(w)$ we conclude that its dimension is $l(w_{\mathrm{fol}}) = m^2$. □

In Figure 4.1, taken from [Woo], we illustrate the EO stratification when $(n,m) = (4,2)$. There are 15 EO strata altogether, labeled by $(4,2)$-shuffles w. We write the $(4,2)$-shuffle w as $w(1)\ldots w(6)$. The strata are arranged from top to bottom in rows, according to their dimension (equal to the length of w). The top row contains only S^{ord}, whose dimension is 8, and the bottom row contains only the core stratum in dimension 0. The EO order relation is represented by downward lines. The 6 strata in $S_\sharp$ are those in which w ends with (...34). The lowest one, S_{fol}, has dimension 4. Note that $S_\sharp$ contains two 6-dimensional strata.

By construction, $S^\sharp$ carries a tautological sub-bundle

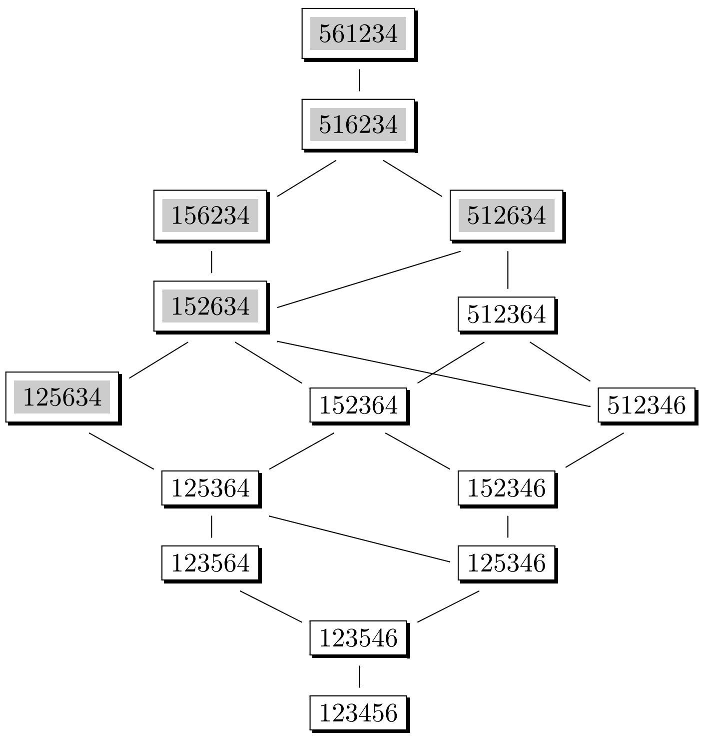
$$\mathcal{P}_0 \subset f^*\mathcal{P}$$

of rank $n-m$, which extends the sub-bundle $\mathcal{P}_0$ defined on S^{ord}. As long as we are above $S_\sharp$ it can be viewed as a bundle on S.

4.1.2. *Example: the case of U(n,1).* This case is particularly simple. There are $n+1$ EO strata, and all of them, except for the core points, lie in $S_\sharp$. The fibers of $S^\sharp$ at the core points are projective spaces of dimension $n-1$. In fact, at such a point $x \in S$ we have $\mathcal{P}_x[V] = \mathcal{P}_x$, because $\mathcal{A}_x$ is superspecial, so there is a *canonical* identification

$$S_x^\sharp = Gr(n-1, \mathcal{P}_x) = Gr(n-1, \mathcal{P}_x \otimes \mathcal{Q}_x)$$

FIGURE 4.1. The EO strata for $(n, m) = (4, 2)$



because $\mathcal{Q}$ is a line bundle. But under the Kodaira-Spencer map $\mathcal{P}_x \otimes \mathcal{Q}_x$ is identified with the cotangent space of S at x, so by duality we have a canonical identification of $S^\sharp_x$ with $Gr(1, \mathcal{T}S_x)$. In fact, $S^\sharp$ is the blow-up of S at the core points.

4.1.3. *Smoothness and irreducibility.*

Theorem 15. *The scheme $S^\sharp$ is non-singular and f induces a bijection on irreducible components.*

Proof. We work over an algebraically closed field k containing κ. Let $y \in S^\sharp(k)$ and $x = f(y)$. Let $k[\epsilon]$ be the ring of dual numbers. Denote by $S^\sharp(k[\epsilon])_y$ the tangent space at y to the moduli problem $S^\sharp$. This is the set of elements in $S^\sharp(k[\epsilon])$ mapping to y modulo ϵ, equipped with the natural structure of a k-vector space. We shall show that

$$\dim S^\sharp(k[\epsilon])_y = nm. \tag{4.1}$$

Let us first see how this implies the theorem. Let Y be the Zariski closure of an irreducible component of S^{ord} in $S^\sharp$. It is nm dimensional, hence (4.1), applied to $y \in Y$, shows that Y is non-singular, and any other irreducible component of $S^\sharp$ is disjoint from Y. Since the fibers of f are connected, there do not exist any other irreducible components in $f^{-1}(f(Y))$. It remains to prove (4.1).

Standard techniques in deformation theory show that we have to compute the tangent space to a certain incidence variety between Grassmanians (see [Har], Example 16.2). We introduce the following notation:

$$W = \omega_{\mathcal{A}_x/k} \subset H = H^1_{dR}(\mathcal{A}_x), \quad H_0 = H^1_{dR}(\mathcal{A}_x)[V]$$

and

$$P_0 = \mathcal{P}_{0,y} \subset H_0 \cap W.$$

These are k-vector spaces with κ action. The polarization pairing $\{,\}_\phi : H \times H \to k$ induces a perfect pairing

$$\{,\}_\phi : W \times H/W \to k,$$

satisfying $\{\iota(a)u, v\}_\phi = \{u, \iota(\overline{a})v\}_\phi$. We claim that $S^\sharp(k[\epsilon])_y$ is identified with

$$\{(\varphi,\psi)|\ \varphi \in \mathrm{Hom}_\kappa(W, H/W)^{sym},\ \psi \in \mathrm{Hom}_\kappa(P_0, H_0/P_0),\ \varphi|_{P_0} = \psi \mod W\}.$$

Indeed, by Grothendieck's crystalline deformation theory [Gro], $S(k[\epsilon])_x$ is identified with $\mathrm{Hom}_\kappa(W, H/W)^{sym}$. The superscript sym refers to homomorphisms symmetric with respect to $\{,\}_\phi$, i.e. satisfying $\{w, \varphi(w')\}_\phi = \{w', \varphi(w)\}_\phi$ for all $w, w' \in W$.

The space $\mathrm{Hom}_\kappa(P_0, H_0/P_0)$ classifies infinitesimal deformations of P_0 preserving the type Σ and the property of being killed by V. This is because under the canonical identification

$$H^1_{cris}(\mathcal{A}_x)_{\mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(k[\epsilon])} = H^1_{dR}(\mathcal{A}_x) \otimes_k k[\epsilon]$$

the map induced on the left hand side by functoriality from $\mathrm{Ver} : \mathcal{A}_x^{(p)} \to \mathcal{A}_x$ is $V_{cris} = V_{dR} \otimes 1$.

Finally the condition $\varphi|_{P_0} = \psi \mod W$ means that the infinitesimal deformation of P_0 stays in the Hodge filtration.

Our problem is now reduced to linear algebra. Note first that

$$\varphi \in \mathrm{Hom}_\kappa(W, H/W)^{sym} = \mathrm{Hom}(P, H(\Sigma)/P),$$

where we have written $P = W(\Sigma)$, the symmetry condition with respect to the pairing $\{,\}_\phi$ then determining uniquely the component in $\mathrm{Hom}(Q, H(\overline{\Sigma})/Q)$, where $Q = W(\overline{\Sigma})$. Likewise,

$$\psi \in \mathrm{Hom}_\kappa(P_0, H_0/P_0) = \mathrm{Hom}(P_0, H_0(\Sigma)/P_0).$$

The dimension of $H_0(\Sigma)$ is n. Indeed, $H_0 = H[V]$ is the image of the map $F : H^{(p)} \to H$, whose kernel is $W^{(p)}$. As H itself is balanced (of type $(m+n, m+n)$), and $H^{(p)}[F] = W^{(p)}$ is of type (m,n),

$$H_0 = H[V] \simeq H^{(p)}/W^{(p)}$$

is of type (n,m). Thus, ψ varies in a space of dimension $(n-m)m$.

Given ψ, $\varphi|_{P_0}$ is determined, and by this we take care of the constraint $\varphi|_{P_0} = \psi$ mod W. It remains to extend φ from P_0 to P. As the codimension of P_0 in P is m and the dimension of $H(\Sigma)/P$ is $(n+m) - n = m$, this adds m^2 dimensions to the tangent space. Altogether

$$\dim S^\sharp(k[\epsilon])_y = (n-m)m + m^2 = nm$$

as desired. $\square$

4.2. **The maps from $S_0(p)_m$ and $S_0(p)_{et}$ to $S^\sharp$.** We denote by $S_0(p)_m$ and $S_0(p)_{et}$ the Zariski closures in $S_0(p)$ of $S_0(p)_m^{\mathrm{ord}}$ and $S_0(p)_{et}^{\mathrm{ord}}$. Our purpose is to define *finite flat* morphisms

$$\pi_m^\sharp : S_0(p)_m \to S^\sharp,\ \ \pi_{et}^\sharp : S_0(p)_{et} \to S^\sharp$$

which extend the restrictions $\pi_m : S_0(p)_m^{\mathrm{ord}} \to S^{\mathrm{ord}}$ and $\pi_{et} : S_0(p)_{et}^{\mathrm{ord}} \to S^{\mathrm{ord}}$ of π to the ordinary-multiplicative and ordinary-étale loci. In fact, $\pi_m^\sharp$ will be an isomorphism, and our main interest will be in $\pi_{et}^\sharp$.

We stress that although the compositions of these maps with the projection from $S^\sharp$ to S both agree with π, as maps to $S^\sharp$ they do not agree on the intersection of $S_0(p)_m$ and $S_0(p)_{et}$, except for the part lying over $S_\sharp$.

4.2.1. *The multiplicative component.*

Lemma 16. *Let R be a κ-algebra and $(\underline{A}, H) \in S_0(p)_m(R)$. Then: (i)* $\mathrm{Fr}(H) = 0$, *(ii) $\omega_{H/R}(\Sigma)$ is locally free of rank m, and (iii) the subsheaf*

$$\mathcal{P}_0 := \ker(\omega_{A/R}(\Sigma) \to \omega_{H/R}(\Sigma)) \tag{4.2}$$

agrees with $\mathcal{P}_0 = \mathcal{P}[V]$ over S^{ord}, is locally free of rank $n-m$ and killed by $V =$ $\mathrm{Ver}^*_{A/R}$.

Proof. (i) This is a closed condition and it holds on $S_0(p)_m^{\mathrm{ord}}$, so it holds by continuity on its Zariski closure $S_0(p)$.

(ii) By reduction to the universal case we may assume, since $S_0(p)_m$ is reduced by Proposition 7, that R is reduced. It is therefore enough to prove that all the geometric fibers of $\omega_{H/R}(\Sigma)$ are of the same dimension m. We may therefore assume that $R = k$ is an algebraically closed field.

Let $M = M(H)$ be the covariant Dieudonné module of H/k. Recall that $\mathrm{Lie}(H) = M[V]$, where V is the map $M(H) \to M(H^{(p)})$ induced by Fr. By (i) $\mathrm{Lie}(H) = M$. But the Dieudonné module is $2m$-dimensional and balanced, so $M(\Sigma)$ is m-dimensional. Hence, $\mathrm{Lie}(H)(\Sigma)$ and its dual $\omega_{H/k}(\Sigma)$ are m-dimensional.

(iii) Since the map $\omega_{A/R} \to \omega_{H/R}$ is surjective, the assertion on the rank follows from (ii). The condition that V kills $\mathcal{P}_0$ holds over $S_0(p)_m^{\mathrm{ord}}$ (where $\mathcal{P}_0 = \mathcal{P}[V]$), so being a closed condition, continues to hold over $S_0(p)_m$. □

Define the map $\pi_m^\sharp : S_0(p)_m \to S^\sharp$ by

$$\pi_m^\sharp(\underline{A}, H) = (\underline{A}, \mathcal{P}_0),$$

where $\mathcal{P}_0$ is given by (4.2). By the lemma, it is well defined, and it clearly extends the isomorphism $\pi_m : S_0(p)_m^{\mathrm{ord}} \simeq S^{\mathrm{ord}}$.

Proposition 17. *The map $\pi_m^\sharp$ is an isomorphism $S_0(p)_m \simeq S^\sharp$.*

Proof. We first check that the map is 1-1 on k-points where k is an algebraically closed field. Let $(\underline{A}, H) \in S_0(p)_m(k)$ and $\pi_m^\sharp(\underline{A}, H) = (\underline{A}, \mathcal{P}_0)$. The proof of Lemma 16 shows that $\mathrm{Lie}(H)(\Sigma)$ is uniquely determined by $\mathcal{P}_0$ as the annihilator of $\mathcal{P}_0$ in $\mathrm{Lie}(A)(\Sigma)$. On the other hand $\mathrm{Lie}(H)(\overline{\Sigma}) = \mathrm{Lie}(A)(\overline{\Sigma})$ since both are m-dimensional. We conclude that $\mathrm{Lie}(H)$ is uniquely determined as a subspace of $\mathrm{Lie}(A) = M(A[p])[V]$. But the proof of Lemma 16 also shows that $\mathrm{Lie}(H) = M(H)$, hence $M(H) \subset M(A[p])$ is uniquely determined, so $H \subset A[p]$ is uniquely determined as a subgroup scheme.

Since $\pi_m^\sharp$ is clearly proper and quasi-finite, it is finite. It is also birational. But $S^\sharp$ is smooth, so by Zariski's Main Theorem $\pi_m^\sharp$ is an isomorphism. □

We let $\sigma_m^\sharp : S^\sharp \to S_0(p)$ be the section inverse to $\pi_m^\sharp$.

4.2.2. *The étale component.* We are now ready to extend the diagram which was constructed in §3.2.4 from the ordinary locus to its Zariski closure.

$$\begin{tikzcd}
& S_0(p)_{et}^{(p)} \arrow[d, "\rho'"] \arrow[dr, "Fr_p"] & \\
S_0(p)_{et} \arrow[ur, "Fr_p"] \arrow[r, "\pi^\sharp_{et}"] \arrow[dr, "\theta'"'] & S^\sharp \arrow[r, "\rho"] \arrow[d, shift right, "\sigma^\sharp_m"'] & S_0(p)_{et} \\
& S_0(p)_m \arrow[u, shift right, "\pi^\sharp_m"'] \arrow[ur, "\theta"'] &
\end{tikzcd}$$

Theorem 18. *(i) The maps ρ, ρ', θ and θ' can be extended, and a map $\pi^\sharp_{et}$ can be defined, so that the diagram above is commutative. (ii) All the morphisms in the diagram are finite and surjective. The maps $\pi^\sharp_{et}$, θ' and ρ' are also flat.*

This theorem can be strengthened, as we shall see in Theorem 21 below: The maps ρ and θ are also flat, and $S_0(p)_{et}$ is non-singular. However, this will result from considerations involving the extension of the foliation $\mathcal{T}S^+$ to $S^\sharp$ and not merely from the constructions outlined here.

In the proof we shall use repeatedly the fact that if H_1 and H_2 are two finite flat subgroup schemes of a finite flat group scheme G over a separated base, then the locus in the base where $H_1 \subset H_2$ is closed.

Proof. We first extend our map θ (*cf.* §3.2.4) from the ordinary locus to a morphism

$$\theta : S_0(p)_m \to S_0(p)_{et}, \quad \theta(\underline{A}, H) = (\underline{A}^{(p^2)}, \mathrm{Fr}(\mathrm{Ver}^{-1}(H))).$$

We must show that this is well-defined. We have proved above that for $(\underline{A}, H) \in S_0(p)_m(R)$, $\mathrm{Fr}(H) = 0$, which means $p\mathrm{Ver}^{-1}(H) = 0$, or $\mathrm{Ver}^{-1}(H) \subset A^{(p)}[p]$. Thus $\mathrm{Ver}^{-1}(H)$ is a finite flat subgroup scheme of rank p^{n+3m} of $A^{(p)}[p]$.

We claim that $A^{(p)}[\mathrm{Fr}] \subset \mathrm{Ver}^{-1}(H)$. This holds, as we have seen before, over $S_0(p)_m^{\mathrm{ord}}$, so by the remark preceding the proof, it persists over the Zariski closure $S_0(p)_m$.

We may now conclude that $J = \mathrm{Fr}(\mathrm{Ver}^{-1}(H))$ is a finite flat subgroup scheme of rank p^{2m} of $A^{(p^2)}[p]$. That it is isotropic follows from the fact that $\mathrm{Ver}^{-1}(H) \subset A^{(p)}[p]$, because for $u, v \in \mathrm{Ver}^{-1}(H)$

$$\langle \mathrm{Fr}(u), \mathrm{Fr}(v) \rangle_{A^{(p^2)}} = \langle u, \mathrm{Ver} \circ \mathrm{Fr}(v) \rangle_{A^{(p)}} = \langle u, pv \rangle_{A^{(p)}} = 0.$$

Clearly J is $\mathcal{O}_E$-stable. To check that it is Raynaud we may assume, as usual, that $R = k$ is an algebraically closed field. The exact sequences of covariant Dieudonné modules

$$0 \to M(A^{(p)}[\mathrm{Ver}]) \to M(\mathrm{Ver}^{-1}(H)) \to M(H) \to 0$$

and

$$0 \to M(A^{(p)}[\mathrm{Fr}]) \to M(\mathrm{Ver}^{-1}(H)) \to M(J) \to 0$$

show, since $M(H)$ is balanced and $M(A^{(p)}[\mathrm{Ver}])$ and $M(A^{(p)}[\mathrm{Fr}])$ have the same signature, that J is Raynaud. To see this last point, from the exact sequence

$$0 \to M(A[\mathrm{Fr}]) \to M(A[p]) \to M(A^{(p)}[\mathrm{Ver}]) \to 0$$

and the fact that $M(A[p])$ is balanced, we get that the types of $M(A[\mathrm{Fr}])$ and $M(A^{(p)}[\mathrm{Ver}])$ are opposite, hence the types of $M(A^{(p)}[\mathrm{Fr}])$ and $M(A^{(p)}[\mathrm{Ver}])$ are the same.

We conclude that θ is well-defined and maps $S_0(p)_m$ into $S_0(p)$. Since it maps $S_0(p)_m^{\mathrm{ord}}$ into $S_0(p)_{et}^{\mathrm{ord}}$, it actually maps $S_0(p)_m$ into $S_0(p)_{et}$.

As before, we define

$$\rho : S^\sharp \to S_0(p)_{et}, \quad \rho = \theta \circ \sigma_m^\sharp.$$

We shall next define a similar extension of θ' (*cf.* §3.2.4) to a map

$$\theta' : S_0(p)_{et} \to S_0(p)_m, \;\; \theta'(\underline{A}, H) = (\underline{A}, \mathrm{Ver}(\mathrm{Fr}^{-1}(H^{(p^2)}))),$$

and let

$$\pi_{et}^\sharp = \pi_m^\sharp \circ \theta' : S_0(p)_{et} \to S^\sharp.$$

Let $(\underline{A}, H) \in S_0(p)_{et}(R)$. Consider $H' = \mathrm{Ver}(\mathrm{Fr}^{-1}(H^{(p^2)})) \subset A$. We claim that H' is a finite flat subgroup scheme of $A[p]$ of rank p^{2m}. To see it, note first that $\mathrm{Fr}^{-1}(H^{(p^2)})$ is finite flat of rank p^{n+3m}, being the kernel of the isogeny

$$\psi : A^{(p)} \xrightarrow{\mathrm{Fr}} A^{(p^2)} \to A^{(p^2)}/H^{(p^2)}.$$

Second, note that $\mathrm{Fr}^{-1}(H^{(p^2)})$ is contained in $A^{(p)}[p]$. Indeed, this holds over $S_0(p)_{et}^{\mathrm{ord}}$, so it holds by continuity over the whole of $S_0(p)_{et}$. Third, we claim that

$$A^{(p)}[\mathrm{Ver}] \subset \mathrm{Fr}^{-1}(H^{(p^2)}).$$

This too follows by continuity, since it clearly holds over $S_0(p)_{et}^{\mathrm{ord}}$. We conclude that H' is finite flat of rank p^{2m}. Moreover $\mathrm{Fr}(H') = 0$, since $\mathrm{Fr} \circ \mathrm{Ver} = p \cdot id_{A^{(p)}}$ and $\mathrm{Fr}^{-1}(H^{(p^2)}) \subset A^{(p)}[p]$. One checks now, as before, that H' is isotropic, $\mathcal{O}_E$-stable and Raynaud. Setting $\theta'(\underline{A}, H) = (\underline{A}, H')$ defines a map from $S_0(p)_{et}$ to $S_0(p)$. As it maps $S_0(p)_{et}^{\mathrm{ord}}$ to $S_0(p)_m^{\mathrm{ord}}$ its image is in $S_0(p)_m$ and θ' extends the morphism between the ordinary parts constructed in §3.2.4.

This concludes the definition of the maps in the lower triangles. It is easily checked that

$$\theta \circ \theta' = \rho \circ \pi_{et}^\sharp = Fr_p^2.$$

As $S^\sharp$ is non-singular, $S_0(p)_{et}$ is Cohen-Macaulay and $\pi_{et}^\sharp$ is finite and onto, we deduce from [Eis] 18.17 that $\pi_{et}^\sharp$ is flat. Hence $\theta' = \sigma_m^\sharp \circ \pi_{et}^\sharp$ is also flat.

It remains to define ρ'. This has been done over the ordinary locus, via the modular interpretation, in §3.2.4. Extending the definition of ρ' via the modular interpretation to $S_0(p)_{et}^{(p)}$ is possible, but painful. Instead, we conclude the proof of the theorem with the help of the following general lemma. It follows from it that ρ' extends to the whole of $S_0(p)_{et}^{(p)}$. The commutativity of the diagram follows by continuity from the fact that it is commutative over the ordinary locus. The fact that ρ' is flat follows again from [Eis] 18.17 since $S_0(p)_{et}^{(p)}$ is Cohen-Macaulay, $S^\sharp$ is non-singular and ρ' is finite and onto. Surjectivity of the maps follows from the fact that they are finite and dominant. □

Lemma 19. *Let X and Y be irreducible varieties over a perfect field κ of characteristic p, with Y normal. Suppose that we are given a finite morphism ρ and a rational map ρ'*

$$Y \overset{\rho'}{\dashrightarrow} X \xrightarrow{\rho} Y^{(p)}$$

such that $\rho \circ \rho' = Fr_p$. Then ρ' extends to a morphism on the whole of Y.

Proof. We may assume that X and Y are affine. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. Let K and L be the fields of fractions of A and B, respectively. Let $B^{(p)} \subset L^{(p)}$ be the image of B and L under Frobenius. Then $B^{(p)} \subset A$ and $L^{(p)} \subset K \subset L$. Since A is integral over $B^{(p)}$ and B is integrally closed in L, we get $A \subset B$, which proves the lemma, and concludes the proof of Theorem 18. $\square$

4.3. **The extended foliation.** Via the map $f : S^\sharp \to S$ we can pull back the universal abelian scheme $\mathcal{A}$ and its de-Rham cohomology, and get a locally free sheaf $f^*H^1_{dR}(\mathcal{A}/S)$ endowed with its own Gauss-Manin connection

$$\nabla : f^*H^1_{dR}(\mathcal{A}/S) \to f^*H^1_{dR}(\mathcal{A}/S) \otimes \Omega^1_{S^\sharp}.$$

Over $S^\sharp$ we find the tautological vector bundle

$$\mathcal{P}_0 \subset f^*\mathcal{P} \subset f^*\omega_{\mathcal{A}/S} \subset f^*H^1_{dR}(\mathcal{A}/S).$$

Let $y \in S^\sharp$. In the proof of Theorem 15 we identified each tangent vector $\eta \in \mathcal{T}S^\sharp_y$ with a pair $\eta = (\varphi, \psi)$ such that, in the notation introduced there,

$$\varphi \in \mathrm{Hom}_\kappa(W, H/W)^{sym}, \ \psi \in \mathrm{Hom}_\kappa(P_0, H_0/P_0), \ \varphi|_{P_0} = \psi \mod W.$$

We define the subsheaf $\mathcal{T}S^{\sharp+} \subset \mathcal{T}S^\sharp$ by the condition

$$\eta \in \mathcal{T}S^{\sharp+}_y \Leftrightarrow \psi = 0.$$

Proposition 20. *The subsheaf $\mathcal{T}S^{\sharp+}$ is a height 1 foliation of rank m^2 which extends $\mathcal{T}S^+$. It is transversal to the fibers of $f : S^\sharp \to S$.*

Proof. In the proof of Theorem 15 we found that $\dim \mathcal{T}S^{\sharp+}_y = m^2$. As the base $S^\sharp$ is reduced and the dimensions of its fibers are constant, $\mathcal{T}S^{\sharp+}$ is a sub-bundle of rank m^2. The tangent space to the fiber of f through y is the set of pairs (φ, ψ) with $\varphi = 0$. Thus $\mathcal{T}S^{\sharp+}$ is transversal to it.

If $y \in S^{\mathrm{ord}}$, Corollary 5 shows that $\psi = 0$ is equivalent to $\eta \in \mathcal{T}S^+$. Finally, the fact that $\mathcal{T}S^{\sharp+}$ is a p-Lie subalgebra follows by continuity from the fact that $\mathcal{T}S^+$ is closed under Lie bracket and raising to power p, since S^{ord} is dense in $S^\sharp$. $\square$

We can now state the main theorem of this section.

Theorem 21. *The variety $S_0(p)_{et}$ is non-singular, the morphism ρ is finite and flat, and identifies $S_0(p)_{et}$ with the quotient of $S^\sharp$ by the foliation $\mathcal{T}S^{\sharp+}$.*

Proof. By Proposition 1 we know that $\mathcal{T}S^{\sharp+}$ corresponds to a finite flat quotient map

$$S^\sharp \xrightarrow{\widetilde{\rho}} S_0(p)^\sim_{et}$$

onto a non-singular variety $S_0(p)^\sim_{et}$, which, thanks to Theorem 11, coincides with $S_0(p)_{et}$ over S^{ord}. Denoting, for simplicity, $X = S^\sharp$, $Y = S_0(p)_{et}$ and $Y^\sim = S_0(p)^\sim_{et}$ we get (by definition) that $\mathcal{O}_{Y^\sim}$ is the subsheaf of $\mathcal{O}_X$ killed by $\mathcal{T}X^+$. If s is a section of $\mathcal{O}_Y$ over a Zariski open U and $\xi \in \mathcal{T}X^+(U)$ then over $U \cap Y^{\mathrm{ord}}$ s is killed by ξ, hence by continuity $\xi s = 0$ on all of U. This shows $\mathcal{O}_Y \subset \mathcal{O}_{Y^\sim} \subset \mathcal{O}_X$ so the morphism ρ factors as $\sigma \circ \widetilde{\rho}$ for a unique finite birational morphism $\sigma : S_0(p)^\sim_{et} \to S_0(p)_{et}$. However, according to Proposition 7, $S_0(p)_{et}$ is normal. Zariski's Main Theorem implies now that σ is an isomorphism, completing the proof. $\square$

5. Integral subvarieties

Recall that an integral subvariety of the foliation $\mathcal{T}S^{\sharp+}$ is a non-singular subvariety $Y \subset S^{\sharp}$ for which $\mathcal{T}S^{\sharp+}|_Y = \mathcal{T}Y$. In this section we find two types of integral subvarieties: Shimura varieties of signature (m,m) embedded in S, and the EO stratum S_{fol}. We end the paper with the natural question whether these are the only global integral subvarieties.

5.1. **Shimura subvarieties of signature** (m,m)**.** There are many ways to embed Shimura varieties associated with unitary groups of signature (m,m) in our unitary Shimura variety S_K. These smaller Shimura varieties can be associated with a quasi-split unitary group, or with an inner form of it. The embeddings extend to the integral models, hence to their special fibers, and can be described in terms of the respective moduli problems. For $(n,m)=(2,1)$ and the resulting embeddings of modular curves or Shimura curves in Picard modular surfaces, see [dS-G1] §4.2.2 or [dS-G2] §1.4.

Theorem 22. *Let S' be the special fiber of a unitary Shimura variety of signature (m,m) embedded in S. Then $S' \cap S^{\mathrm{ord}}$ is an integral subvariety of $\mathcal{T}S^+$.*

Proof. The proof of Theorem 2.3(ii) in [dS-G2] can be easily generalized, once the embedding of the appropriate moduli problems is written down explicitly. A different approach is to use Theorem 13. The set $S' \cap S^{\mathrm{ord}}$ is open and dense in S' and the $2m$-dimensional abelian varieties which it parametrizes are ordinary (in the usual sense). The classical Serre-Tate theorem attaches a structure of a formal torus to the formal neighborhood $\widehat{S'_x} = Spf(\widehat{\mathcal{O}}_{S',x})$ in S' of a point $x \in S' \cap S^{\mathrm{ord}}$. The compatibility of Moonen's generalized Serre-Tate coordinates under embeddings of Shimura varieties shows that under the embedding ι of S' in S the formal neighborhood $\widehat{S'_x}$ gets mapped to $\widehat{T}_{\iota(x)} \subset Spf(\widehat{\mathcal{O}}_{S,\iota(x)})$. The theorem follows now from Theorem 13. □

5.2. **EO strata.** The proof that the EO stratum S_{fol} is an integral subvariety of the foliation $\mathcal{T}S^+$ is more difficult. We follow the strategy outlined in [dS-G1], §3.4, in particular Lemma 3.10 there, but the generalization from signature $(2,1)$ to the general case requires some work.

Recall that we denoted by $S_\sharp$ the open subset of S where $f : S^\sharp \to S$ is an isomorphism, and that S_{fol} is the unique minimal EO stratum in $S_\sharp$, so we are justified in writing $\mathcal{T}S^+$ instead of $\mathcal{T}S^{\sharp+}$ when we refer to $\mathcal{T}S^{\sharp+}|_{S_{\mathrm{fol}}}$. Recall also that $\dim(S_{\mathrm{fol}}) = m^2 = \mathrm{rk}(\mathcal{T}S^+)$, a hint that we are on the right track.

5.2.1. *The Dieudonné module at a point of* S_{fol}. The following Proposition describes the structure of the *contravariant* Dieudonné module $D_0 = D(\mathcal{A}_x[p])$ at a point $x \in S_{\mathrm{fol}}(k)$ (k, as usual, algebraically closed and containing κ). It can be deduced from [Mo2], §4.9, see also [Woo], §3.5. Recall that there exists a canonical identification

$$D_0 = H^1_{dR}(\mathcal{A}_x/k),$$

and that the skew-symmetric pairing $\{,\}_\phi$ on D_0, induced by the polarization, becomes under this identification the pairing $\{x,y\}_\phi = \{x, (\phi^{-1})^* y\}$ where $\{,\}$ is the canonical pairing on $H^1_{dR}(\mathcal{A}_x/k) \times H^1_{dR}(\mathcal{A}_x^t/k)$.

Proposition 23. *Let $x \in S_{\mathrm{fol}}(k)$ and $D_0 = D(\mathcal{A}_x[p])$. There exists a basis*

$$\{e_1, \ldots, e_{n+m}, f_1, \ldots, f_{n+m}\}$$

of D_0 with the following properties.

(i) κ acts on the e_i via Σ and on the f_j via $\overline{\Sigma}$.

(ii) $\{e_i, f_{n+m+1-i}\}_\phi = 1$, and the other $\{e_i, f_j\}_\phi$, as well as $\{e_i, e_j\}_\phi$ and $\{f_i, f_j\}_\phi$, are all 0.

(iii) The maps $F : D_0^{(p)} \to D_0$ and $V : D_0 \to D_0^{(p)}$ induced by Fr *and* Ver *are given by the following tables. We abbreviate the list $e_a, \ldots, e_b$ as $e_{[a,b]}$ and similarly with $f_{[a,b]}$.*

$i \in$	$[1, n-m]$	$[n-m+1, n]$	$[n+1, n+m]$
$F(e_i^{(p)})$	0	$-f_{i-n+m}$	0
$V(e_i)$	0	0	$f_{i-n+m}^{(p)}$

$j \in$	$[1, m]$	$[m+1, 2m]$	$[2m+1, n+m]$
$F(f_j^{(p)})$	$-e_j$	0	$-e_{j-m}$

$j \in$	$[1, m]$	$[m+1, n]$	$[n+1, n+m]$
$V(f_j)$	0	$e_{j-m}^{(p)}$	$e_j^{(p)}$

In particular, $\omega_{\mathcal{A}_x/k} = (D_0^{(p)}[F])^{(p^{-1})} = Span_k\{e_{[1,n-m]}, e_{[n+1,n+m]}, f_{[m+1,2m]}\}$ and $\ker(V_{\mathcal{P}})_x = \mathcal{P}_{0,x} = Span_k\{e_{[1,n-m]}\}$.

Corollary 24. *Notation as above, if $x \in S_{\mathrm{fol}}$, then $V(\mathcal{Q})_x = Span_k\{e_1^{(p)}, \ldots, e_m^{(p)}\}$ if $2m \leq n$, and $V(\mathcal{Q})_x = Span_k\{e_1^{(p)}, \ldots, e_{n-m}^{(p)}, e_{n+1}^{(p)}, \ldots, e_{2m}^{(p)}\}$ if $n < 2m$.*

In passing, we note that the Hasse matrix $H_{\mathcal{A}/S} = V_{\mathcal{P}}^{(p)} \circ V_{\mathcal{Q}} = 0$ over S_{fol} if $2m \leq n$ but not if $n < 2m$. The Hasse invariant $h_{\mathcal{A}/S} = \det(H_{\mathcal{A}/S})$ always vanishes, of course.

5.2.2. *Proof of the main theorem.* In the proof of the following theorem we shall separate the three cases (i) $n = 2m$ (ii) $2m < n$ (iii) $n < 2m$. Although the idea of the proof is the same, the three cases become progressively more complicated. Thus, for the sake of exposition, we felt it was better to treat them separately, at the price of some repetition.

As a matter of notation, if $f : T \to S$ is a morphism of schemes, and $\mathcal{F}$ is a coherent sheaf on S, we denote by $\mathcal{F}(T)$ the global sections of $f^*\mathcal{F}$ on T. We shall employ this notation in particular when T is an infinitesimal neighborhood of a closed point of S, or a closed subscheme of such an infinitesimal neighborhood.

Theorem 25. *The EO stratum S_{fol} is an integral subvariety of the foliation $\mathcal{T}S^+$, i.e. $\mathcal{T}S^+|_{S_{\mathrm{fol}}} = \mathcal{T}S_{\mathrm{fol}}$.*

Proof. **1**. Let $x \in S_{\mathrm{fol}}$ and $R = \mathcal{O}_{S,x}/\mathfrak{m}_{S,x}^2$. Let $D = H_{dR}^1(\mathcal{A}/R)$ be the infinitesimal deformation of D_0. Although $\mathrm{Spec}(R)$ is not smooth over k, the R-module D inherits, by base-change, the Gauss-Manin connection

$$\nabla : D \to D \otimes_R (R \otimes_{\mathcal{O}_S} \Omega^1_{\mathcal{A}/S}).$$

As it admits a basis of horizontal sections over $\mathrm{Spec}(R)$, we may write $D = D_0 \otimes_k R$, the horizontal sections being $D_0 \otimes_k k = D_0$. Since the Gauss-Manin connection is compatible with isogenies, F and V take horizontal sections to horizontal sections. Thus Proposition 23(iii) holds also for $F : D^{(p)} \to D$ and $V : D \to D^{(p)}$. The pairing $\{,\}_\phi$ is horizontal for the Gauss-Manin connection, i.e.

$$d\{x, y\}_\phi = \{\nabla x, y\}_\phi + \{x, \nabla y\}_\phi,$$

so the formulae from part (ii) of the Proposition also persist in D. What *does change*, and, according to Grothendieck, completely determines the infinitesimal deformation, is the Hodge filtration $\omega_{\mathcal{A}/R}$. One sees that the most general deformation of $\omega_{\mathcal{A}_x/k}$ is given by

$$\omega_{\mathcal{A}/R}(\Sigma) = \mathrm{Span}_R\{e_i + \sum_{j=n-m+1}^{n} u_{ij}e_j,\ \ e_\ell + \sum_{j=n-m+1}^{n} v_{\ell j}e_j\ :\ 1 \le i \le n-m,\ n+1 \le \ell \le n+m\}, \tag{5.1}$$

where the $nm = (n-m)m + m^2$ variables u_{ij} and $v_{\ell j}$ are local parameters at x, and their residues modulo $\mathfrak{m}_{S,x}^2$ form a basis for $\mathcal{T}S_x^\vee$. The deformation $\omega_{\mathcal{A}/R}(\overline{\Sigma})$ is then completely determined by $\omega_{\mathcal{A}/R}(\Sigma)$ and the condition that $\omega_{\mathcal{A}/R}$ is isotropic for $\{,\}_\phi$. A small computation reveals that it is given by

$$\omega_{\mathcal{A}/R}(\overline{\Sigma}) = \mathrm{Span}_R\{f_{n+m+1-j} - \sum_{i=1}^{n-m} u_{ij}f_{n+m+1-i} - \sum_{\ell=n+1}^{n+m} v_{\ell j}f_{n+m+1-\ell}\ :\ n-m+1 \le j \le n\}. \tag{5.2}$$

Compare the proof of Theorem 15. The data encoded in the matrices u and v is just the data denoted there by

$$\varphi \in \mathrm{Hom}_\kappa(W, H/W)^{sym} = \mathrm{Hom}(P, H(\Sigma)/P).$$

2. Consider the abelian scheme $\mathcal{A}^{(p)}$ over $\mathrm{Spec}(R)$, and note that it is constant:

$$\mathcal{A}^{(p)} = \mathrm{Spec}(R) \times_{\Phi_R, \mathrm{Spec}(R)} \mathcal{A} = \mathrm{Spec}(R) \times_{\mathrm{Spec}(k)} \mathcal{A}_x^{(p)},$$

since the absolute Frobenius ϕ_R of the ring R factors as

$$R \twoheadrightarrow k \xrightarrow{\phi_k} k \hookrightarrow R. \tag{5.3}$$

Inside $D^{(p)} = H^1_{dR}(\mathcal{A}^{(p)}/R) = R \otimes_{\phi,R} D$ we therefore get

$$\omega_{\mathcal{A}^{(p)}/R}(\overline{\Sigma}) = \omega_{\mathcal{A}/R}(\Sigma)^{(p)} = Span_R\{e_i^{(p)}, e_\ell^{(p)};\ 1 \le i \le n-m,\ n+1 \le \ell \le n+m\}.$$

In particular,

$$\mathcal{P}_0^{(p)}(\mathrm{Spec}(R)) = R \otimes_k \mathcal{P}_{0,x}^{(p)} = Span_R\{e_i^{(p)};\ 1 \le i \le n-m\}.$$

3. Let us compute the image of a typical generator of $\mathcal{Q}(\mathrm{Spec}(R)) = \omega_{\mathcal{A}/R}(\overline{\Sigma})$ under V.

$$V(f_{n+m+1-j} - \sum_{i=1}^{n-m} u_{ij}f_{n+m+1-i} - \sum_{\ell=n+1}^{n+m} v_{\ell j}f_{n+m+1-\ell}) \equiv \tag{5.4}$$

$$-\sum_{i=1}^{\min(m,n-m)} u_{ij}e^{(p)}_{n+m+1-i} - \sum_{i=\min(m,n-m)+1}^{n-m} u_{ij}e^{(p)}_{n+1-i} \mod R\otimes V(\mathcal{Q})_x.$$

By Corollary 24 this image is contained in $R\otimes V(\mathcal{Q})_x$ if and only if all $u_{ij}=0$.

4. If $n=2m$ we can finish the proof as follows. Corollary 24 implies that over S_{fol} we then have $V(\mathcal{Q})=\mathcal{P}_0^{(p)}$, because the same holds at every k-valued point of S_{fol} and the base is reduced. Let R_{fol} be the quotient of R defined by

$$\mathrm{Spec}(R_{\mathrm{fol}})=\mathrm{Spec}(R)\cap S_{\mathrm{fol}}.$$

We get

$$V(\mathcal{Q})(\mathrm{Spec}(R_{\mathrm{fol}}))=\mathcal{P}_0^{(p)}(\mathrm{Spec}(R_{\mathrm{fol}}))=R_{\mathrm{fol}}\otimes_k\mathcal{P}^{(p)}_{0,x}=R_{\mathrm{fol}}\otimes_k V(\mathcal{Q})_x.$$

By Point 3 this means that over $\mathrm{Spec}(R_{\mathrm{fol}})$ we must have all $u_{ij}=0$. A dimension count shows that $\{u_{ij}=0\}$ is actually the set of equations defining S_{fol} infinitesimally, i.e. $R_{\mathrm{fol}}=R/(u_{ij})$. Thus $\mathcal{T}S_{\mathrm{fol},x}$ is spanned by $\{\partial/\partial v_{\ell j};\ n+1\le\ell\le n+m,\ n-m+1\le j\le n\}$.

On the other hand, from the explicit description of $\mathcal{P}(\mathrm{Spec}(R))=\omega_{\mathcal{A}/R}(\Sigma)$, and from the characterization of $\mathcal{T}S^+$ given in Corollary 5, we find that $\mathcal{T}S^+_x$ is also spanned by $\partial/\partial v_{\ell j}$ $(n+1\le\ell\le n+m,\ n-m+1\le j\le n)$. Indeed, $\mathcal{P}_0(\mathrm{Spec}(R))$ is spanned over R by the sections

$$e_i+\sum_{j=n-m+1}^{n}u_{ij}e_j,$$

which are killed by $\nabla_{\partial/\partial v_{\ell j}}$, but are sent to sections which are outside $\mathcal{P}_0(\mathrm{Spec}(R))$ by $\nabla_{\partial/\partial u_{ij}}$. Thus, given $\xi\in\mathcal{T}S_x$, ∇_ξ preserves $\mathcal{P}_0(\mathrm{Spec}(R))$ if and only if ξ is a linear combination of the $\partial/\partial v_{\ell j}$. This concludes the proof of the theorem when $n=2m$.

5. To finish the proof under the more general assumption $2m\le n$ we must show that $\{u_{ij}=0\}$ is *always* the system of infinitesimal equations for S_{fol}. For that it is enough to prove the following claim, that generalizes what we have found for $n=2m$. Let

$$\mathcal{D}=H^1_{dR}(\mathcal{A}/S),$$

endowed with endomorphisms by $\mathcal{O}_E$ and the bilinear form $\{,\}_\phi$. This $\mathcal{D}$ is the unitary Dieudonné space of $\mathcal{A}[p]$ over S, in the sense of [We] (5.5). It is a locally free $\mathcal{O}_S$-module of rank $2(n+m)$, $\omega_{\mathcal{A}/S}$ is a maximal isotropic sub-bundle, and $D=\mathcal{D}(\mathrm{Spec}(R))$ is the base-change of $\mathcal{D}$ under $\mathcal{O}_S\to\mathcal{O}_{S,x}/\mathfrak{m}^2_{S,x}=R$.

Claim. *Let $2m\le n$. Over S_{fol} there is a sub-bundle $\mathcal{M}\subset\mathcal{D}$ such that at each geometric point $x\in S_{\mathrm{fol}}(k)$, $V(\mathcal{Q})_x=\mathcal{M}_x^{(p)}\subset\mathcal{D}_x^{(p)}$.* (In fact, $\mathcal{M}$ will be a sub-bundle of $\omega_{\mathcal{A}/S_{\mathrm{fol}}}$.)

Assuming the claim has been proved, we proceed as in the case $n=2m$, when we identified $\mathcal{M}$ with $\mathcal{P}_0$. As the base is reduced, $V(\mathcal{Q})=\mathcal{M}^{(p)}$. Since the absolute Frobenius ϕ_R of the ring R factors as in (5.3) and similarly for its quotient ring R_{fol}, we get

$$\mathcal{M}^{(p)}(\mathrm{Spec}(R_{\mathrm{fol}}))=R_{\mathrm{fol}}\otimes_k\mathcal{M}_x^{(p)},$$

as sub-modules of $\mathcal{D}^{(p)}(\mathrm{Spec}(R_{\mathrm{fol}}))=R_{\mathrm{fol}}\otimes_k D_0^{(p)}$.

This means that $V(\mathcal{Q})(\mathrm{Spec}(R_{\mathrm{fol}}))$ lies in the subspace $R_{\mathrm{fol}} \otimes V(\mathcal{Q})_x$. But we have seen that a typical generator of $\mathcal{Q}(\mathrm{Spec}(R_{\mathrm{fol}}))$ maps to a vector outside $R_{\mathrm{fol}} \otimes_k V(\mathcal{Q})_x$, unless all $u_{ij} = 0$. We conclude that $R_{\mathrm{fol}} = R/(u_{ij})$ as before.

6. *Proof of Claim.* The key for proving the claim is the observation that if $2m \leq n$

$$V(\mathcal{Q})_x = \{F(D_0[V](\overline{\Sigma})^{(p)})\}^{(p)}.$$

Indeed,

$$D_0[V](\overline{\Sigma}) = Span_k\{f_1, \ldots, f_m\},$$

$$F(Span_k\{f_1^{(p)}, \ldots, f_m^{(p)}\}) = Span_k\{e_1, \ldots, e_m\},$$

so we may use Corollary 24. Now $F(D_0[V]^{(p)})$ is part of the *canonical filtration* of D_0 in the sense of [Mo2] 2.5 (the part commonly denoted "$FV^{-1}(0)$"). It is therefore the (contravarient) Dieudonné module of $\mathcal{A}_x[p]/\mathcal{N}_x$ for a certain subgroup scheme $\mathcal{N}_x$ of $\mathcal{A}_x[p]$ which belongs to the *canonical filtration* of the latter, in the sense of [Oo] (2.2).

The point is that over any EO stratum, in particular over S_{fol}, the canonical filtration of $\mathcal{A}[p]$ *exists* as a filtration by finite flat subgroup schemes, and yields the canonical filtration at each geometric point by specialization. See Proposition (3.2) in [Oo]. Thus the $\mathcal{N}_x$ are the specializations of a finite flat group scheme $\mathcal{N}$ over S_{fol}. Letting $\mathcal{M}$ be the Σ-part of the Dieudonné module of $\mathcal{A}[p]/\mathcal{N}$ proves the claim. Alternatively, we can define $\mathcal{M}$ directly as

$$\mathcal{M} = F(\mathcal{D}[V](\overline{\Sigma})^{(p)})$$

and use the constancy of fiber ranks over the reduced base S_{fol} to show that this is a sub-bundle of $\mathcal{D}$.

7. The key idea when $2m \leq n$ was the observation that over S_{fol} the sub-bundle $V(\mathcal{Q}) \subset \mathcal{D}^{(p)}$ was of the form $\mathcal{M}^{(p)}$ for a sub-bundle $\mathcal{M} \subset \mathcal{D}$. This $\mathcal{M}$ was obtained as the Σ-part of a certain piece in the canonical filtration of $\mathcal{D}$, namely $\mathcal{M} = FV^{-1}(0)(\Sigma)$. No such piece of the canonical filtration works if $n < 2m$. We are able however to replace the equality $V(\mathcal{Q}) = \mathcal{M}^{(p)}$ by an *inclusion* $V(\mathcal{Q}) \subset \mathcal{M}^{(p)}$ for a carefully chosen $\mathcal{M}$, and modify the arguments accordingly.

Let the natural number $r \geq 1$ satisfy

$$\frac{r}{r+1} < \frac{m}{n} \leq \frac{r+1}{r+2}.$$

Let

$$\mathcal{M} = V^{-2r}F^{2r+1}V^{-1}(0)(\Sigma).$$

More precisely, we consider $V^{-2r}\{(F^{2r+1}(\mathcal{D}[V]^{(p^{2r+1})}))^{(p^{2r})}\} \subset \mathcal{D}$. That this is a well-defined sub-bundle of $\mathcal{D}$, over any EO stratum, and in particular over S_{fol}, follows as before from Proposition (3.2) in [Oo]. Hence the same applies to its Σ-part, which is $\mathcal{M}$.

Claim. Let r and $\mathcal{M}$ be as above. Then, using the notation of Proposition 23:
(i) For any $x \in S_{\mathrm{fol}}$

$$\mathcal{M}_x = Span_k\{e_1, \ldots, e_{2m}\}.$$

(ii) Over S_{fol} we have $V(\mathcal{Q}) \subset \mathcal{M}^{(p)}$.

Part (i) will be proved in Lemma 26 below. Part (ii) follows from Corollary 24. By the corollary, the inclusion $V(\mathcal{Q})_x \subset \mathcal{M}_x^{(p)}$ holds between the *fibers* of the two sub-bundles at any geometric point $x \in S_{\mathrm{fol}}(k)$, and the base is reduced.

We can now apply a small variation on the case $2m \leq n$. From the Claim we obtain

$$V(\mathcal{Q})(\mathrm{Spec}(R_{\mathrm{fol}})) \subset \mathcal{M}^{(p)}(\mathrm{Spec}(R_{\mathrm{fol}})) = R_{\mathrm{fol}} \otimes_k \mathcal{M}_x^{(p)} = Span_{R_{\mathrm{fol}}}\{e_1^{(p)}, \ldots, e_{2m}^{(p)}\}.$$

However, when $n < 2m$ (5.4) implies

$$V(f_{n+m+1-j} - \sum_{i=1}^{n-m} u_{ij} f_{n+m+1-i} - \sum_{\ell=n+1}^{n+m} v_{\ell j} f_{n+m+1-\ell}) \equiv$$

$$- \sum_{i=1}^{n-m} u_{ij} e_{n+m+1-i}^{(p)} \mod R \otimes_k \mathcal{M}_x^{(p)}.$$

Since $e_{n+m+1-i}^{(p)}$ $(1 \leq i \leq n-m)$ *remain linearly independent* modulo $\mathcal{M}_x^{(p)}$ we conclude that in R_{fol} we must have $\overline{u}_{ij} = 0$. As before, this implies that $R_{\mathrm{fol}} = R/(u_{ij})$, and concludes the proof of the theorem. $\square$

5.2.3. *A Dieudonné module computation.* To complete the proof of Theorem 25 when $n < 2m$ we need to prove the following.

Lemma 26. *Let notation be as in Proposition 23, and let $r \geq 1$ satisfy $r/(r+1) < m/n \leq (r+1)/(r+2)$. Then*

$$V^{-2r} F^{2r+1} V^{-1}(0) = Span_k\{e_{[1,2m]}, f_{[1,rn-(r-1)m]}\}.$$

Proof. Let $D_0(a,b) = Span_k\{e_{[1,a]}, f_{[1,b]}\}$. We first observe that

$$FD_0(a,b) = D_0(a^-, b^-), \quad V^{-1}D_0(a,b) = D_0(a^+, b^+)$$

where

$$a^- = \begin{cases} b & 0 \leq b \leq m \\ m & m < b \leq 2m \\ b-m & 2m < b \leq n+m \end{cases}$$

$$b^- = \begin{cases} 0 & 0 \leq a \leq n-m \\ a-n+m & n-m < a \leq n \\ m & n < a \leq n+m \end{cases}$$

$$a^+ = \begin{cases} n & 0 \leq b \leq m \\ b+n-m & m < b \leq 2m \\ n+m & 2m < b \leq n+m \end{cases}$$

$$b^+ = \begin{cases} a+m & 0 \leq a \leq n-m \\ n & n-m < a \leq n \\ a & n < a \leq n+m \end{cases}.$$

To be precise, we should have written $FD_0(a,b)^{(p)} = D_0(a^-, b^-)$ etc., but from now on we omit the relevant Frobenius twists to simplify the notation. We now compute, using these formulae, and leaving out straightforward verifications:

1. $V^{-1}(0) = D_0(n,m)$.

2. Let $0 \le i \le r$. One proves inductively that

$$F^{2i}V^{-1}(0) = D_0(im-(i-1)n,\,(i+1)m-in)$$

$$F^{2i+1}V^{-1}(0) = D_0((i+1)m-in,\,(i+1)m-in).$$

3. Let $1 \le j \le r$. Using induction on j one shows

$$V^{-2j+1}F^{2r+1}V^{-1}(0) = D_0(jn-(j-1)m,\,(r+3-j)m-(r+1-j)n)$$

$$V^{-2j}F^{2r+1}V^{-1}(0) = D_0((r+2-j)m-(r-j)n,\,jn-(j-1)m).$$

The assumption that $r/(r+1) < m/n \le (r+1)/(r+2)$ is used repeatedly in these computations. Putting $j=r$ proves the Lemma. □

5.3. **A conjecture of André-Oort type.** Given a foliation in a real manifold, the celebrated theorem of Frobenius says that integral subvarieties exist, and are unique, in sufficiently small neighborhoods of any given point. Working in the algebraic category, in characteristic p, one has to impose, in addition to the integrability condition, also being closed under the p-power operation. Integral subvarieties then exist in *formal* neighborhoods, but are far from being unique. For that purpose Ekedahl introduced in [Ek] the notion of height h foliations for any $h \ge 1$, a notion that we do not discuss here, as our height 1 foliation does not seem to extend to higher height foliations. Nor does the foliation lift to characteristic 0 in any natural way; thus, the approach taken by Miyaoka in [Mi] to deal with the same problem does not apply here.

Despite this lack of formal uniqueness, the global nature of our foliation imposes a severe restriction on integral subvarieties. Thus, we dare to make the following conjecture.

Conjecture. *The only integral subvarieties of the foliation $\mathcal{T}S^+$ in $S_\sharp$ are Shimura varieties of signature (m,m) or S_{fol}.*

References

[B-W] O. Bültel, T. Wedhorn: Congruence relations for Shimura varieties associated to some unitary groups, J. Instit. Math. Jussieu **5** (2006), pp. 229-261.

[De-Ill] P. Deligne, L. Illusie: Relèvements modulo p^2 et décomposition du complexe de de Rham, Invent. Math. **89**, 247-270 (1987).

[dS-G1] E. de Shalit, E.Z. Goren: A theta operator on Picard modular forms modulo an inert prime, Res. Math. Sci. **3:**28 (2016), arXiv:1412.5494.

[dS-G2] E. de Shalit, E.Z. Goren: On the bad reduction of certain $U(2,1)$ Shimura varieties, *preprint*, arXiv:1703.05720.

[Eis] D. Eisenbud: *Commutative Algebra with a View Toward Algebraic Geometry*, GTM 150, Springer-Verlag, New York, 1995.

[Ek] T. Ekedahl: Foliations and inseparable morphisms, in: *Algebraic Geometry, Bowdin 1985,* Proc. Symp. Pure Math. **46** (2), 139-149 (1987).

[Gö] U. Görtz: On the flatness of models of certain Shimura varieties of PEL type, Math. Ann. **321** (2001), pp. 689-727.

[Gro] A. Grothendieck: Groupes de Barsotti-Tate et cristaux de Dieudonné, Univ. Montréal, Montréal, 1974.

[Har] J. Harris: *Algebraic Geometry: A first course,* Springer, New York, 1992.

[Ja] N. Jacobson: *Lie Algebras,* Interscience, New York, 1962.

[Ka] N. Katz: Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin, Publ. Math IHES, **39** (1970), 175–232.

[Ko] R. E. Kottwitz: Points on some Shimura varieties over finite fields, J. AMS **5** (1992), 373–444.

[Ku] E. Kunz: Characterizations of regular local rings of characteristic p. American Journal of Mathematics, **91**(3) (1969), 772–784.

[Lan] K.-W. Lan: Arithmetic compactifications of PEL-type Shimura varieties, London Mathematical Society Monographs **36**, Princeton, 2013.

[Mi] Y. Miyaoka: Deformation of a morphism along a foliation, in: *Algebraic Geometry, Bowdin 1985,* Proc. Symp. Pure Math. **46** (1), 245-268 (1987).

[Mo1] B. Moonen: Serre-Tate theory for moduli spaces of PEL type, Ann. Scient. Éc. Norm. Sup., 4e série, t. **37** (2004), pp. 223-269.

[Mo2] B. Moonen: Group schemes with additional structures and Weyl group elements, in: *Moduli of abelian varieties,* C. Faber, G. van der Geer, F. Oort, eds., Progress in mathematics 195, Birkhauser, 2001, 255-298.

[Oo] F. Oort: A stratification of a moduli space of abelian varieties, in: *Moduli of abelian varieties,* C Faber, G. van der Geer, F. Oort, eds., Progress in mathematics 195, Birkhauser, 2001, 345-416.

[P-Z] G. Pappas, X. Zhu: Local models of Shimura varieties and a conjecture of Kottwitz, Invent. Math. **194** (2013), 147-254

[Ra-Zi] M. Rapoport, T. Zink: Period Spaces for p-divisible Groups, Annals of Mathematics Studies **141**, Princeton University Press, Princeton (1996).

[Ru-Sh] A. N. Rudakov, I. R. Shafarevich: Inseparable morphisms of algebraic surfaces, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), pp. 1269–1307.

[Va] A. Vasiu: Manin problems for Shimura varieties of Hodge type, J. Ramanujan Math. Soc. **26** (2011), 31–84.

[V-W] E. Viehmann, T. Wedhorn: Ekedahl-Oort and Newton strata for Shimura varieties of PEL type, Math. Ann. **356** (2013), 1493–1550.

[We] T. Wedhorn: The dimension of Oort strata of Shimura varieties of PEL-type, in: *Moduli of abelian varieties,* C Faber, G. van der Geer, F. Oort, eds., Progress in mathematics 195, Birkhauser, 2001, 441-471.

[Woo] A. Woodin: The Ekedahl-Oort stratification of unitary Shimura varieties, Ph.D. thesis, McGill University, 2016.

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