F-isocrystals

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CHAPTER 1

F-isocrystals

1. Semilinear algebra

1.1. What is semilinear algebra? Let K be a field. Linear algebra over K is the study of linear transformations in finite dimensional vector spaces over K. More generally, if Γ is a semigroup or a group, one studies representations of Γ in finite dimensional vector spaces over K, the case of a single (invertible) linear transformation being the case of $\Gamma = \mathbb{N}$ ($\Gamma = \mathbb{Z}$). Equivalently, one studies $K[T]$- or $K[\Gamma]$-modules.

Suppose that K itself is equipped with an automorphism σ (or more generally, with a group action for some group Γ). Semilinear algebra studies finite dimensional vector spaces V over K with an additive transformation φ satisfying $\varphi(av) = \sigma(a)\varphi(v)$ for all $v \in V$ and $a \in K$. (More generally, V is equipped with a semilinear action of Γ.) Equivalently, this is the study of $K\langle T\rangle$ or $K\langle\Gamma\rangle$-modules, where $K\langle T\rangle$ or $K\langle\Gamma\rangle$ is the twisted polynomial ring.

Lemma 1.1. *Semilinear endomorphism of an n-dimensional vector space over K are classified by* twisted conjugacy classes *in $M_n(K)$: equivalence classes under the relation $A \sim_\sigma B$ if there exists a $P \in GL_n(K)$ such that $B = P^{-1}A\sigma(P)$.*

Proof. With respect to a basis $\{e_i\}$ we have $\varphi e_j = \sum_k a_{kj}e_k$. If $e'_i = \sum_j p_{ji}e_j$ is another basis, and $e_k = \sum_l p'_{lk}e'_l$ where $P' = P^{-1}$, then

$$\varphi e'_i = \sum_l \sum_k \sum_j \sigma(p_{ji})a_{kj}p'_{lk}e'_l \tag{1}$$

so the matrix representing φ in the new basis is $P^{-1}A\sigma(P)$. □

Let Γ be a group acting on K. A 1-cocycle of Γ in $GL_n(K)$ is a collection A_γ, for $\gamma \in \Gamma$, satisfying

$$A_\sigma\sigma(A_\tau) = A_{\sigma\tau}. \tag{2}$$

Another 1-cocycle B is cohomologous to A if there is an invertible P such that

$$B_\sigma = P^{-1}A_\sigma\sigma(P). \tag{3}$$

Being cohomologous is an equivalence relation and the equivalence classes form the cohomology *set* $H^1(\Gamma, GL_n(K))$. Precisely as in the proof of the last lemma, we have the following.

Lemma 1.2. *Semilinear actions of Γ on an n-dimensional vector space over K are in one-to-one correspondence with the cohomology set $H^1(\Gamma, GL_n(K))$.*

Variant 1. Just as it is possible to talk about Γ-representations with additional structure (symplectic, orthogonal...) it is possible to talk about semilinear Γ-actions with *structure group* G which is not the full GL_n, and they are then classified by $H^1(\Gamma, G(K))$.

Variant 2. If Γ is a topological group, K a topological field, and the action of Γ on K is continuous, the semilinear *continuous* actions of Γ on K-vector spaces of dimension n are classified by $H^1_{cont}(\Gamma, GL_n(K))$.

1.2. Example: Hilbert's theorem 90. Let K/F be a finite Galois extension with Galois group Γ. Hilbert's theorem 90 (see below) is the assertion that every semilinear action of Γ over K is *trivial* in the sense that it comes from a trivial Γ-module over F after a semilinear extension of scalars.

More generally, K/F may be infinite, and one is looking for *continuous semilinear actions* of $\Gamma = Gal(K/F)$. Here continuity is with respect to the Krull topology on Γ and the discrete topology on K. Hilbert's theorem 90 remains valid.

1.3. Example: Lang's theorem. Let K be a perfect field in characteristic p, and $\sigma(x) = x^p$ the Frobenius automorphism. One would like to classify all (invertible, say) semilinear endomorphisms φ of V. If K is *algebraically closed* Lang's theorem says that all such pairs (V, φ) are trivial. They are obtained by semilinear base extension from the identity on a vector space over $\mathbb{F}_p$.

1.4. Example: F-isocrystals. See below. Here one starts with F a p-adic field, K the completion of its maximal *unramified* extension (or any intermediate unramified extension) and looks for Frobenius-semilinear bijective endomorphisms of V. Note that this means a semilinear action of $\mathbb{Z} = W(K/F)$ and *not* of $Gal(K/F)$.

1.5. Example: Tate-Sen theory. Once again K/F is a Galois extension of a local field F, but this time it is assumed to be *totally ramified* with $\Gamma = Gal(K/F) \simeq \mathbb{Z}_p$. One is looking for *continuous* semilinear actions of Γ where K is endowed with the p-adic topology, and Γ with the Krull topology. Since these actions do not necessarily factor through finite quotients of Γ, unlike the first example, there are non-trivial actions. Tate-Sen theory completely classifies them in terms of Sen's operator, and proves that this is the same as classifying continuous semilinear structures over $\mathbb{C}_p$.

1.6. Example: Fontaine's (Φ, Γ)-modules. This is, in a precise sense, the amalgamation of the previous two examples.

2. F-crystals and F-isocrystals over perfect fields

2.1. Definitions. Let k be a *perfect* field of characteristic p, and $\sigma(x) = x^p$ its Frobenius automorphism. Let $W = W(k)$ and K its field of fractions. Denote by σ also the Frobenius of W and K.

DEFINITION 2.1. *(i) An F-*crystal *(M, φ)* defined over *k is a free W-module M of finite rank, together with an injective σ-linear endomorphism $\varphi : M \to M$.*

*(ii) An F-*isocrystal *(E, φ)* defined over *k is a finite dimensional vector space E over K, together with an injective σ-linear endomorphism $\varphi : E \to E$.*

The rank of M (or the dimension of E) is also called its height. *An F-isocrystal E is called* effective *if it comes from an F-crystal (i.e. if it contains a W-submodule of maximal rank M stable under φ). Two such underlying F-crystals need not be isomorphic.*

REMARK 2.1. *If M is an F-crystal, $\varphi(M)$ is of full rank in M, and if E is an F-isocrystal, φ is bijective.*

DEFINITION 2.2. *A* homomorphism $f : M \to M'$ *is a* W-linear *map commuting with φ. The group of homomorphisms is denoted $Hom(M, M')$. The same definition applies to F-isocrystals. An* isogeny *between F-crystals is a homomorphism that induces an isomorphism on the F-isocrystals.*

The map φ is not an endomorphism of M, but it extends W-linearly to a homomorphism Φ from $M^{(p)} = W \otimes_{\sigma,W} M$ to M. Multiplication by a is an endomorphism only if $a \in \mathbb{Q}_p$.

REMARK 2.2. *For any integer m one can define a σ^m-crystal (or isocrystal) requiring φ to be σ^m-linear. If (E, φ) is an F-isocrystal, then (E, φ^m) is a σ^m-isocrystal.*

Let $W\langle F\rangle$ be the twisted polynomial ring over W and similarly $K\langle F\rangle$, where $Fa = \sigma(a)F$ for $a \in K$. An F-crystal is the same as a left $W\langle F\rangle$ module (F acts like φ), free of finite rank over W, on which F is injective. The same holds for F-isocrystals.

LEMMA 2.1. *(i) The ring $K\langle F\rangle$ is a* non-commutative Euclidean domain*: for f and h, $h \neq 0$, $\deg(f) \geq \deg(h)$, there are unique q and r, $\deg(r) < \deg(h)$ such that $f = qh + r$ (same for division on the right, but q and r may be different).*

(ii) Every left ideal in $K\langle F\rangle$ is principal: $I = K\langle F\rangle h$. If h is monic, it is unique.

The proof of the lemma is the same as in the commutative case.

EXAMPLE 2.1. *A* cyclic *F-isocrystal of rank n is of the form*

$$E = K\langle F\rangle / K\langle F\rangle h \tag{4}$$

where $h = F^n + a_1 F^{n-1} + \cdots + a_n$ with $a_i \in K$ and $a_n \neq 0$ (the last assumption is necessary and sufficient for φ to be injective).

Note that h is not unique. If $g \in K\langle F\rangle$ is such that $1 = ug + vh$ for some u and v, then $E = K\langle F\rangle \bar{g}$ and the annihilator of $\bar{g}$ is

$$\{f \in K\langle F\rangle\, ;\, fg \in K\langle F\rangle h\} = K\langle F\rangle h' \tag{5}$$

so $E \simeq K\langle F\rangle / K\langle F\rangle h'$. In particular if $g \in K$, $u = g^{-1}$, we get

$$h' = hu = \sigma^n(u)F^n + \sigma^{n-1}(u)a_1F^{n-1} + \cdots + ua_n. \tag{6}$$

2.2. The category of F-isocrystals. The category of F-isocrystals is an abelian $\mathbb{Q}_p$-linear category. It has a tensor product (with $\varphi = \varphi \otimes \varphi$) and internal Hom : $\underline{Hom}(E, E')$ is the usual space of K-homomorphisms between E and E' with $\varphi(u) = \varphi \circ u \circ \varphi^{-1}$. Do not confuse it with the categorical $Hom(E, E')$ which is $\underline{Hom}(E, E')^{\varphi=1}$, a vector space over $\mathbb{Q}_p$ (not over K) which may be 0. The F-isocrystal (K, σ) is the neutral element in the category with respect to tensor product, and $E^\vee = \underline{Hom}(E, (K, \sigma))$ is the *dual* of E. The n^{th} *twist* of E, denoted $E(n)$ is the space E with φ replaced by $p^{-n}\varphi$. For every E, $E(n)$ is effective for $n << 0$. The exterior power $\bigwedge^i E$ is also defined as usual. These definitions apply to F-crystals as well, although the category of F-crystals is not abelian (cokernels do not exist!).

If k' is a perfect extension of k, there is an obvious "extension of scalars" functor between crystals (isocrystals) over k to the same objects over k'.

3. The F-isocrystals E_k^λ

3.1. The F-isocrystals E_k^λ and their underlying F-crystals. Let $\lambda = s/r$ be a rational number, $r > 0$, $(s,r) = 1$. The F-isocrystal E^λ is defined over $\mathbb{F}_p$. It is

$$E^\lambda = \mathbb{Q}_p \langle F \rangle / (F^r - p^s). \tag{7}$$

It has a basis $1, F, \ldots, F^{r-1}$.

If $s \geq 0$, then E^λ is effective. It contains the F-crystal $M^\lambda = \mathbb{Z}_p \langle F \rangle / \mathbb{Z}_p \langle F \rangle \cdot (F^r - p^s)$. In general, $E^\lambda(1) = E^{\lambda - 1}$. Indeed,

$$E^\lambda(1) = \mathbb{Q}_p \langle p^{-1} F \rangle / ((p^{-1}F)^r - p^{s-r}) \tag{8}$$

with φ acting like multiplication by $p^{-1}F$, so changing notation we recover $E^{\lambda-1}$. We conclude that after an appropriate twist we can always bring λ to the interval $[0,1)$. If $\lambda \in (0,1)$, then E^λ contains also another F-crystal which is

$$\bar{M}^\lambda = \mathbb{Z}_p[F,V]/(FV - p, F^{r-s} - V^s) \tag{9}$$

(map V to $p^{1-s}F^{r-1}$ in $\mathbb{Q}_p \langle F \rangle$). Here $0 < s < r$ and a $\mathbb{Z}_p$ basis for $\bar{M}^\lambda$ consists of $V^{s-1}, \ldots, V, 1, F, \ldots, F^{r-s}$. Note that $\bar{M}^\lambda / \varphi(\bar{M}^\lambda)$ is $(\mathbb{Z}/p\mathbb{Z})^s$ (the residue classes of $V^{s-1}, \ldots, V, 1$ give a basis) while $M^\lambda / \varphi(M^\lambda)$ is $\mathbb{Z}/p^s\mathbb{Z}$. [This example shows two different Hodge polygons associated with the same Newton polygon].

Let $s > 0$. Since $(r,s) = 1$ the F-isocrystal E^λ can be identifed with the *field* $\mathbb{Q}_p(p^{s/r}) = \mathbb{Q}_p(p^{1/r})$ with φ acting like multiplication by $p^{s/r}$.

For any perfect field k, we define E_k^λ by extension of scalars from $\mathbb{F}_p$ to k. It may be identified with $K\langle F \rangle / K \langle F \rangle (F^r - p^s)$, or with the field $K(p^{1/r})$, where now φ acts by multiplication by $p^{s/r}$ twisted by the σ-action on K.

3.2. Computing $D^\lambda = End(E_k^\lambda)$.

Lemma 3.1. *Let E be an arbitrary F-isocrystal over k. Let $\lambda = s/r$ as before. Then the map $\alpha \mapsto \alpha(1)$ is a bijection from*

$$Hom(E_k^\lambda, E) \tag{10}$$

to $\{x \in E;\ \varphi^r x = p^s x\}$.

The lemma is clear from the definition of E_k^λ. Note that $Hom(E_k^\lambda, E)$ is only a $\mathbb{Q}_p$-vector space, although the set $\{x \in E;\ \varphi^r x = p^s x\}$ is a vector space over K_r if $\mathbb{F}_{p^r} \subset k$.

Let $\alpha \in End(E_k^\lambda)$. If $\alpha(1) = \sum_{i=0}^{r-1} a_i F^i$ we write

$$\alpha = \sum_{i=0}^{r-1} a_i \xi^i. \tag{11}$$

Clearly $\alpha(F^j) = \alpha \circ \varphi^j(1) = \varphi^j(\alpha(1)) = \sum_{i=0}^{r} \sigma^j(a_i) F^{i+j}$. In particular, $\alpha(F^r) = \alpha(p^s) = p^s \alpha(1)$ imposes the restriction that $a_i \in K \cap K_r$, where K_r is the field of fractions of $W(\mathbb{F}_{p^r})$. By the lemma, this is the only restriction.

If α and β are endomorphism, and $\beta(1) = \sum_{i=0}^{r-1} b_i F^i$, then

$$\alpha \circ \beta(1) = \alpha \left(\sum_j b_j F^j \right) = \sum_j b_j \sum_i \sigma^j(a_i) F^{i+j}. \tag{12}$$

Let the ring $D^\lambda = End(E_k^\lambda)$ act on E_k^λ *on the right* (this is natural since it commutes with the left action of K). Then $\alpha \circ \beta = \beta\alpha$. We obtain the following lemma.

LEMMA 3.2. *If $\mathbb{F}_{p^r} \subset k$, then*

$$D^\lambda = \left\{ \sum_{i=0}^{r-1} a_i \xi^i;\ a_i \in K_r,\ \xi a = \sigma(a)\xi,\ \xi^r = p^s \right\} \tag{13}$$

is the unique division ring of invariant λ *over $\mathbb{Q}_p$. It depends only on λ mod 1. On the other hand if $k = \mathbb{F}_p$, $End(E^\lambda) = \mathbb{Q}_p(p^{1/r})$.*

3.3. Homomorphisms between E_k^λ and E_k^μ ($\lambda \neq \mu$).

PROPOSITION 3.3. *If $\lambda \neq \mu$, then $Hom(E_k^\lambda, E_k^\mu) = 0$.*

PROOF. Let $\mu = s'/r'$ as usual, and take $x \in E_k^\mu$. If $\varphi^r x = p^s x$ then clearly $\varphi^{rr'} x = p^{sr'} x$. Writing x with respect to the standard basis of E_k^μ, $x = \sum x_i e_i'$ we get

$$p^{s'r} \sum \sigma^{rr'}(x_i) e_i' = p^{sr'} \sum x_i e_i'. \tag{14}$$

Since $s'r \neq sr'$ this is impossible (compare valuations of coefficients). □

If $m = \gcd(r, r')$ then the denominator of $\lambda + \lambda'$ is rr'/m and

$$E_k^\lambda \otimes E_k^{\lambda'} \simeq (E_k^{\lambda+\lambda'})^m. \tag{15}$$

This agrees with $D^\lambda \otimes D^{\lambda'} \simeq M_m(D^{\lambda+\lambda'})$.

3.4. Extensions between E_k^λ.

LEMMA 3.4. *If $\lambda \neq \lambda'$, then any extension of E_k^λ by $E_k^{\lambda'}$ splits. If k is algebraically closed, the same is true even if $\lambda = \lambda'$.*

PROOF. It is enough to show that $\varphi^r - p^s : E_k^{\lambda'} \to E_k^{\lambda'}$ is surjective, because by the previous lemma to split a short exact sequence

$$0 \to E_k^{\lambda'} \to E \to E_k^\lambda \to 0 \tag{16}$$

it is enough to find $x \in E$ mapping to $1 \in E_k^\lambda$ and satisfying $\varphi^r x = p^s x$. We can take an arbitrary x mapping to 1, and then to correct it by an element of $E_k^{\lambda'}$ so that is satisfies the required equation, we precisely need the surjectivity of $\varphi^r - p^s$ on $E_k^{\lambda'}$.

Twisting by an integer we may assume that both λ and λ' are positive. It is also clearly enough to prove the surjectivity of $\varphi^{rr'} - p^{sr'}$. But in terms of the standard basis $e_i' = F^{i-1} mod(F^{r'} - p^{s'})$ we have

$$\left(\varphi^{rr'} - p^{sr'}\right)\left(\sum x_i e_i'\right) = \sum \left(p^{rs'} \sigma^{rr'}(x_i) - p^{sr'} x_i\right) e_i'. \tag{17}$$

To prove the surjectivity we have to solve equations of the form

$$p^a \sigma^b(x) - x = y \tag{18}$$

for a given $y \in K$, where $b \neq 0$. If $\lambda \neq \lambda'$ then $rs' \neq sr'$ and this corresponds to an equation with $a \neq 0$. Since $1 - p^a \sigma^b$ (if $a > 0$) or $1 - p^{-a}\sigma^{-b}$ (if $a < 0$) are invertible operators on K, such a solution always exists. If $\lambda = \lambda'$ we are reduced to solving an equation of the type $\sigma^b(x) - x = y$, where $b > 0$. This reduces to the solvability of $x^{p^b} - x = y$ in k, which is guaranteed by the fact that k was assumed to be algebraically closed. □

4. F-isocrystals over algebraically closed fields (Manin)

4.1. Preliminary lemmas.

LEMMA 4.1. *If* $b_0 + \cdots + b_n = 0$ *then* $F^n b_0 + F^{n-1} b_1 + \cdots + b_n = (F-1)g$ *for* $g \in K\langle F\rangle$ *of degree* $n-1$.

PROOF. Simply note that $F^i - 1$ is divisible by $F - 1$ on the left. □

LEMMA 4.2. *Assume that* k *is algebraically closed. Suppose that* $b_i \in W$ $(1 \leq i \leq n)$*, and at least one of them is a unit. Then for some unit* $v \in W^\times$,

$$\sigma^n(v) + \sigma^{n-1}(v) b_1 + \cdots + v b_n = 0. \tag{19}$$

PROOF. By succesive approximations it is enough to solve the same equation modulo p, for $v \in k^\times$, where it becomes obvious, since k is algebraically closed. □

LEMMA 4.3. *Assume that* k *is algebraically closed. Suppose* $f \in W\langle F\rangle$ *is monic of degree* n*. Let* $\lambda = s/r$ *be the minimal slope of the Newton polygon of* f :

$$\lambda = \inf\left\{ord_p(a_i)/i\right\}. \tag{20}$$

Then in $W[p^{1/r}]\langle F\rangle$ *(where* F *commutes with* $p^{1/r}$*) we have*

$$f = u \cdot (F - p^{s/r}) \cdot g \tag{21}$$

for some $u \in W^\times$.

PROOF. Let $f(F) = F^n + a_1 F^{n-1} + \cdots + a_n = F^n + F^{n-1} b_1 + \cdots + b_n$. Suppose $s/r = ord_p(a_i)/i$, and consider

$$f_1(F) = p^{-ns/r} f(p^{s/r} F). \tag{22}$$

This is still a monic polynomial in $W[p^{1/r}]\langle F\rangle$ and now the i^{th} coefficient is a unit. Multiplying f on the left by an appropriate $v = u^{-1} \in W[p^{1/r}]^\times$ we may assume that the sum of the b_i in f_1 is 0 and therefore that $f_1 = (F-1)g_1$. (We use the previous two lemmas in $W[p^{1/r}]$ instead of W. They clearly remain valid there.) We get that $f(F) = p^{ns/r} f_1(p^{-s/r} F)$ is divisible on the left by $(p^{-s/r} F - 1)$, or, what is the same, by $(F - p^{s/r})$. □

PROPOSITION 4.4. *Assume that* k *is algebraically closed. Let* E *be an* F*-isocrystal. Then for some* λ *there exists a non-zero homomorphism from* E_k^λ *to* E.

PROOF. We may assume that E is a *simple* left $K\langle F\rangle$ module, where F acts like φ. It is therefore cyclic, and may be identified with $K\langle F\rangle / K\langle F\rangle f$ for a monic f of degree $n = \dim_K E$. Twisting E we may assume that f is in $W\langle F\rangle$. Multiplying f on the left by a unit we may assume that over $W[p^{1/r}]$, $f = (F - p^{s/r}) \cdot g$. The polynomial g represents then a nonzero element x of $\mathbb{Q}_p(p^{1/r}) \otimes E$ for which $\varphi x = p^{s/r} x$, hence $\varphi^r x = p^s x$. Writing $x = \sum_{i=0}^{r-1} p^{i/r} \otimes x_i$ we see that $\varphi^r x_i = p^s x_i$, hence there exists an $x \in E$ with the same property, and this defines a map from E_k^λ to E. □

COROLLARY 4.5. *The* E_k^λ *are simple objects.*

PROOF. It is enough to prove this over an algebraically closed k. Let E be a subobject of E_k^λ. Then by the previous lemma there is a non-zero map from E_k^μ to E, hence to E_k^λ. By another lemma that we proved, $\mu = \lambda$, so this map is a non-zero endomorphism of E_k^λ. But D^λ is a division ring, so our map is an isomorphism, and E is all of E_k^λ. □

4.2. The main theorem.

THEOREM 4.6. *(Manin-Dieudonne). Let k be algebraically closed. Then the category of F-isocrystals is semisimple. Its simple objects are the E_k^λ.*

PROOF. We have shown that the E_k^λ are simple, that any E contains a copy of at least one E_k^λ, and that any extension between two E_k^λ splits. The theorem follows formally from these facts. □

5. Slopes, Newton and Hodge polygons (Katz, Mazur)

5.1. Newton polygon of an F-isocrystal. Let $f = a_0X^n + \cdots + a_n \in K[X]$ be a polynomial. The Newton polygon *Newton*(f) is the boundary of the convex hull of the points $(i, ord_p(a_i))$ ($i \in \mathbb{Z}$, $ord_p(0) = \infty$). Its *slopes* are called the slopes of f (infinity excluded). It is well-known that if *Newton*(f) has a side of slope λ and horizontal projection r then f has precisely r roots α with $ord_p(\alpha) = \lambda$, in an algebraic closure of K.

The Newton polygon *Newton*(E) of an F-isocrystal E over a perfect field k is defined as follows. Let $E_{\bar{k}}$ be the extension of scalars of E to an algebraically closed field. Let

$$E_{\bar{k}} = \bigoplus_i \left(E_{\bar{k}}^{\lambda_i}\right)^{m_i} \tag{23}$$

where $\lambda_1 < \lambda_2 < \cdots$. Write $\lambda_i = s_i/r_i$ in reduced terms with $r_i > 0$. Then *Newton*(E) is the convex polygon starting at $(0,0)$ with sides of slope λ_i and horizontal projection m_ir_i. Note that its end point is $(\sum m_ir_i, \sum m_is_i)$ and that $\sum m_ir_i = n$ is the height of E.

Suppose that E is defined over $\mathbb{F}_p$. Then φ is *linear* and it makes sense to talk about the Newton polygon of the characteristic polynomial of φ. It can be shown (exercise!) that it coincides with the Newton polygon of E.

However, in general it is not true that the slopes of E can be computed as the valuations of the eigenvalues of a matrix representing φ in some basis of E over K. Indeed, such a matrix is determined not up to conjugation, but up to a twisted conjugation. If Φ is the matrix representing φ in some basis, then the matrix representing φ in another basis is $\sigma(P)\Phi P^{-1}$ for an invertible matrix P. Not only the eigenvalues themselves, but even their valuations, are not invariant under twisted conjugation. Gross gave the following 2×2 example. Let $p \equiv 3 mod 4$, $K = \mathbb{Q}_p(i)$, $i^2 = -1$ and consider

$$\Phi = \begin{pmatrix} 1-p & i(p+1) \\ i(p+1) & p-1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad \sigma(P)\Phi P^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 2p \end{pmatrix}. \tag{24}$$

It follows that the slopes of E are 0 and 1, while the slopes of $char.\Phi$ are 1/2 and 1/2.

Let M be an F-crystal of height n, and let $\lambda_1, \ldots, \lambda_n$ be the sequence of slopes in increasing order, where each $\lambda_i = s_i/r_i$ is repeated m_ir_i times (m_i is its multiplicity).

PROPOSITION 5.1. *Let N be an integer divisible by all the r_i (for example $n!$). Then over $R = W(\bar{k})[p^{1/N}]$, $R \otimes_W M$ admits a basis in terms of which $\sigma \otimes \varphi$ is upper triangular, with $p^{\lambda_1}, \ldots, p^{\lambda_n}$ along the diagonal.*

PROOF. Let L be the field of fractions of R. The module $L \otimes_K E$ (where E is the F-isocrystal associated to M) admits a basis $\varepsilon_1, \ldots, \varepsilon_n$ w.r.t. which φ is diagonal with p^{λ_i} along the diagonal. Normalizing ε_1 we may assume that it lies in $R \otimes_W M$ and that $R \otimes_W M/R\varepsilon_1$ is torsion free. We continue by induction. □

Note that if $L \otimes_K E = L\varepsilon_1 \oplus E'$ then $E' \cap R \otimes_W M$ need not be a direct complement to $R\varepsilon_1$ in $R \otimes_W M$: the sum will be direct, but it will only be a full lattice in $R \otimes_W M$, and not necessarily the whole thing. As an example consider the F-crystal over $\mathbb{F}_p$ which is $M = \mathbb{Z}_p e_1 \oplus \mathbb{Z}_p e_2$, where $\varphi e_1 = pe_1$ and $\varphi e_2 = p^2 e_2 + e_1$. Then over $\mathbb{Q}_p$, letting $e_1' = e_1$ and $e_2' = e_2 + (1/p(p-1))e_1$ we have $\varphi e_2' = p^2 e_2'$. However,

$$M \cap \mathbb{Q}_p e_1' + M \cap \mathbb{Q}_p e_2' \tag{25}$$

is of index p in M. This will not change by replacing $(\mathbb{Z}_p, \mathbb{Q}_p)$ by (R, L). Note that E contains *another* F-crystal, namely $\mathbb{Z}_p e_1' + \mathbb{Z}_p e_2'$, which is diagonalizable.

COROLLARY 5.2. *If we repeat the slopes as above, then $\lambda_1 + \cdots + \lambda_i$ is the minimal slope of $\bigwedge^i M$. In particular, the end point of the Newton polygon of E is $(height(E),\ ord_p(\det(\varphi))$.*

5.2. Slope decomposition of an F-isocrystal. Let k be any perfect field, and E an F-isocrystal.

PROPOSITION 5.3. *There exists a unique direct sum decomposition*

$$E = \bigoplus_{\lambda} E_\lambda \tag{26}$$

where E_λ is an F-isocrystal, isoclinical of slope λ. It is characterized as follows: If $\lambda = s/r$, then E_λ is the sum of all the $\mathcal{O}_K$-submodules M in E for which $\varphi^r(M) = p^s M$.

PROOF. The sum of two submodules satisfying $\varphi^r(M) = p^s M$ is of the same nature. There is therefore a maximal subspace E_λ which is the sum of all such modules. Over $\bar{k}$, the Manin decomposition shows that $E_{\bar{k},\lambda}$ is the λ-isoclinic component (the sum of the factors of slope λ). Let $M_{\bar{k}}$ be a lattice in $E_{\bar{k},\lambda}$ which spans it rationally and satisfies $\varphi^r(M_{\bar{k}}) = p^s M_{\bar{k}}$. Since $\tau M_{\bar{k}}$ is also such a lattice, we may (taking the sum of all the Galois conjugates) assume that it is Galois stable, hence comes from a lattice M defined over k. Clearly $\varphi^r(M) = p^s M$ too, so it follows that $E_{\bar{k},\lambda}$ is the extension of scalars of E_λ, defined as in the proposition, and the slope decomposition descends to E. □

5.3. The Hodge polygon of an F-crystal. Let M be an F-crystal over k of height n. Let $p^{a_1}, \ldots, p^{a_n}$ be the elementary divisors of $M/\varphi(M)$, where $0 \leq a_1 \leq a_2 \leq \ldots$. The Hodge polygon of M, *Hodge*(M), is the polygon connecting the points $(i, a_1 + \cdots + a_i)$. Its initial point is $(0,0)$ and its end point is $(n, ord_p \det(\varphi))$. They coincide with the initial and end points of *Newton*(M). Unlike the Newton polygon, it depends not only on the F-isocrystal, but on M. For example, in E^λ we found the F-crystals M^λ, whose Hodge polygon has slopes $(0, \ldots, 0, s)$ ($r-1$ times 0) and another F-crystal $\bar{M}^\lambda$, whose Hodge polygon has slopes $(0, \ldots, 0, 1, \ldots, 1)$

($r-s$ times 0 and s times 1). Note that both lie *under* the Newton polygon, which is the straight line of slope $\lambda = s/r$ and horizontal length r.

Another way to characterize the Hodge polygon is to introduce $ord_p(M)$ as the greatest integer a such that

$$\varphi M \subset p^a M. \tag{27}$$

Then $a_1 + \cdots + a_i$ is $ord_p(\bigwedge^i M)$.

THEOREM 5.4. *(Mazur) The Hodge polygon always lies* under *the Newton polygon.*

PROOF. Both polygons do not change if we pass to $\bar{k}$, so assume that k is algebraically closed. We have to show that if we label the Newton slopes $\lambda_1 \leq \lambda_2 \leq \ldots$, with $\lambda_i = s_i/r_i$ repeated $m_i r_i$ times, then

$$a_1 + \cdots + a_i \leq \lambda_1 + \cdots + \lambda_i. \tag{28}$$

However, $a_1 + \cdots + a_i$ is the first Hodge slope of $\bigwedge^i M$ and $\lambda_1 + \cdots + \lambda_i$ is the first Newton slope of $\bigwedge^i M$. It is therefore enough to prove that $a_1 \leq \lambda_1$ *for every* M. Let $R = W[p^{1/N}]$ for some large N and pick a basis for $R \otimes_W M$ in terms of which $\sigma \otimes \varphi$ is upper triangular, with $p^{\lambda_1}, \ldots, p^{\lambda_n}$ along the diagonal. Then a_1 is the largest integer such that *all* the entries of the matrix representing $\sigma \otimes \varphi$ are divisible by p^{a_1}. Clearly $a_1 \leq \lambda_1$. □

5.4. Newton and Hodge polygons of extensions.

PROPOSITION 5.5. *Let*

$$0 \to M' \to M \to M'' \to 0 \tag{29}$$

be a short exact sequence of F-crystals (meaning that M' is a full sub-F-crystal of M). Then the Newton polygon of $M' \oplus M''$ coincides with the Newton polygon of M, while the Hodge polygon of $M' \oplus M''$ lies above the Hodge polygon of M.

PROOF. We may assume that k is algebraically closed. Newton polygons depend only on the F-isocrystals, but the category of F-isocrystals is semisimple, so every short exact sequence splits, and we are done.

For the Hodge polygons consider an arbitrary filtration $0 = M_0 \subset \cdots \subset M_i \subset M_{i+1} \subset \cdots \subset M$, such that M_i are φ-stable, and M_{i+1}/M_i is torsion free. Let grM be the associated graded F-crystal. Then clearly

$$ord_p(M) \leq ord_p(grM) \tag{30}$$

so the first Hodge slope of grM is greater or equal to the first Hodge slope of M. Let M be as in the proposition. For each j, $\bigwedge^j M$ has a filtration induced from the two-step filtration on M (the Koszul filtration). Moreover, $gr^i(\bigwedge^j M) = \bigwedge^{j-i} M' \otimes \bigwedge^i M''$, so

$$gr(\bigwedge^j M) = \bigwedge^j (M' \oplus M''). \tag{31}$$

It follows that $ord_p \bigwedge^j M \leq ord_p \bigwedge^j (M' \oplus M'')$, which is the assertion about Hodge polygons. □

5.5. Newton polygon as the limit of Hodge polygons. Let M be an F-crystal. One can define $Hodge(\varphi^k)$, the Hodge polygon of φ^k, by considering the elementary divisors of $M/\varphi^k M$.

PROPOSITION 5.6. $\mathrm{Newton}(\varphi) = \lim_{k\to\infty} \frac{1}{k}\mathrm{Hodge}(\varphi^k)$.

PROOF. As usual, we may assume that the field k is algebraically closed. It is enough to prove that the sum of the first i slopes of $\frac{1}{k}Hodge(\varphi^k)$, which is a piecewise linear function on the interval $[0, n]$, tends to the sum of the first i slopes of $Newton(\varphi)$. Replacing M by $\bigwedge^i M$ as usual, it is enough to prove the assertion for the first slope. Adjoining $p^{1/N}$ for some N we may assume that φ is given by upper triangular matrix Φ with p^{λ_i} on the diagonal. The assertion to be proved is then that if $ord_p(\Phi)$ is the highest a such that all the entries of Φ are divisible by p^a, then

$$\lim_{k\to\infty}\frac{1}{k} ord_p(\Phi^{(k)}) = \lambda_1. \tag{32}$$

Here

$$\Phi^{(k)} = \Phi \circ \sigma\Phi \circ \cdots \circ \sigma^{k-1}\Phi. \tag{33}$$

This is left as an exercise on matrices. □

REMARK 5.1. *It is not true that the convergence is monotone, and it is also not true that the limit is attained at a finite level. However, during the process of convergence the end points are fixed and the Hodge polygons always lie under the Newton polygon. Note that the Hodge slopes of φ are integers. This limits the possibilities for the Hodge polygons associated with a given Newton polygon.*

CHAPTER 2

Filtrations and semistability

1. Flags, flag varieties and semistable points

1.1. Flags, flag varieties, and parabolic subgroups. Let V be an n-dimensional vector space over a field k. For any field K containing k we write V_K for $K \otimes_k V$. Let

$$n = \sum_{i=1}^{r} n_i. \tag{34}$$

A *flag of type* $\underline{n} = (n_1, \dots, n_r)$ in V_K is a descending sequence of vector spaces

$$x : V_K = V_1 \supset \dots V_r \supset 0 \tag{35}$$

such that $\dim V_i/V_{i+1} = n_i$. The flags of type $\underline{n}$ in V_K are the K-rational points of a smooth projective variety $\mathcal{F} = \mathcal{F}_{\underline{n}}$ defined over k, called a *flag variety.* The group

$$G = GL(V) \tag{36}$$

acts transitively on $\mathcal{F}$ and the stabilizer of each flag is a *parabolic* $P = P_x$. If we fix a basis $\{e_i\}$ such that $V_i = Span\,\{e_j;\, j > n_1 + \dots + n_{i-1}\}$ then P becomes the standard (lower) parabolic of type $\underline{n}$. We may therefore write, having chosen a "base flag",

$$\mathcal{F} \simeq G/P. \tag{37}$$

All parabolics of a given type are conjugate in G. Put

$$gr_x V = \bigoplus_{i=1}^{r} V_i/V_{i+1}, \tag{38}$$

and if $g \in P_x$ let $gr_x(g)$ be the induced automorphism of $gr_x V$. The *unipotent radical* $N = N_x$ of P is the subgroup of all g for which $gr_x(g) = 1$. It is normal and the quotient P/N is called the Levi quotient. Any semidirect complement M such that $P = MN$ is called a *Levi factor* of P.

A *full flag* is one where $r = n$ and all $n_i = 1$. The corresponding P is then called a *Borel subgroup.* If a flag x' is a *refinement* of a flag x then $P_{x'} \subset P_x$ but $N_{x'} \supset N_x$.

1.2. Splittings and decompositions. A *splitting* of a flag x is a decomposition

$$V = \bigoplus_{i=1}^{r} W_i \tag{39}$$

such that $V_i = W_i \bigoplus V_{i+1}$. Splittings of a flag x correspond 1:1 to Levi decompositions

$$P = MN \tag{40}$$

of $P = P_x$, where $M = \prod GL(W_i)$.

A *decomposition of a flag* x is a decomposition

$$V = \bigoplus_{j \in J} L_j \tag{41}$$

of V into a direct sum of lines such that each V_i is the sum of a some of the L_j. Decompositions of V into lines are in 1:1 correspondence with maximal split tori $T \subset G$. Those that decompose the flag x correspond to the $T \subset P_x$.

1.3. Bruhat's lemma.

LEMMA 1.1. *For every two flags there exists a decomposition of V into lines compatible with both. Alternatively, every two parabolics share a common maximal split torus.*

PROOF. It is enough to prove that if

$$V = V_1 \supset V_2 \supset \cdots \supset V_n \supset 0 \tag{42}$$

and

$$V = W_1 \supset W_2 \supset \cdots \supset W_n \supset 0 \tag{43}$$

are two full flags, then there is a basis compatible with both. We construct by decreasing induction on i a vector $e_i \in V_i - V_{i+1}$ for which there exists a $\pi(i)$ such that $e_i \in W_{\pi(i)} - W_{\pi(i)+1}$ and π is a permutation of $\{1, \ldots, n\}$. Let e_n be a basis of V_n and $\pi(n)$ the slot where it sits in the second filtration. Suppose $e_{i+1}, \ldots, e_n$ have been constructed. Consider any $e \in V_i - V_{i+1}$ and let $j = j(e)$ be such that $e \in W_j - W_{j+1}$ Of all the possible e pick one with maximal $j = j(e)$, denote it by e_i, and $j(e)$ by $\pi(i)$. We have to show that $\pi(i)$ is not $\pi(i')$ for some $i' \geq i+1$. But if it were, we could replace e_i by $e_i - \lambda e_{i'}$ for a suitable scalar λ to push it into W_{j+1}, contradicting the maximality of $j(e)$. □

1.4. G-line bundles on $\mathcal{F}$. Let $\mathcal{F} = \mathcal{F}_{\underline{n}}$. There are r *tautological line bundles* $\mathcal{L}_i$ on $\mathcal{F}$, which at a point $x \in \mathcal{F}$ given by the flag $x : V = V_1 \supset \ldots V_r \supset 0$ are simply

$$\mathcal{L}_i|_x = \det(V_i/V_{i+1}). \tag{44}$$

These line bundles carry a G-action covering the action of G on $\mathcal{F}$ so they belong to $Pic^G\mathcal{F}$, the group of G-line bundles over $\mathcal{F}$. Note that

$$\bigotimes_{i=1}^{r} \mathcal{L}_i = \det(V) \tag{45}$$

is trivial as an abstract line bundle, but not as a G-bundle. The action of G on it is via the determinant.

For every $\underline{p} = (p_1, \ldots, p_r) \in \mathbb{Z}$ let

$$\mathcal{L}(\underline{p}) = \bigotimes_{i=1}^{r} \mathcal{L}_i^{\otimes -p_i}. \tag{46}$$

Note that

$$\mathcal{L}(\underline{p}) = \det(V)^{\otimes -p_1} \otimes \bigotimes_{i=2}^{r} \det(V_i)^{\otimes(p_{i-1}-p_i)}$$

LEMMA 1.2. *(i) Every $\mathcal{L} \in Pic^G\mathcal{F}$ is $\mathcal{L}(p_1,\dots,p_r)$ for a unique $\underline{p}$.*
(ii) $\mathcal{L}(p_1,\dots,p_r)$ is ample if and only if $p_1 < p_2 < \cdots < p_r$.
(iii) The center of G acts on $\mathcal{L}(\underline{p})$ via the character $t \mapsto t^{-\sum p_i n_i}$.

PROOF. (i) We show that every G-line bundle is of this sort. Write $\mathcal{F} = G/P$. A G-line bundle is trivialized under pull-back to G (because the G-action on it allows one to identify canonically the fiber over an arbitary point with the fiber at 1). It follows that G-line bundles are in 1:1 correspondence with *characters* $\chi : P \to \mathbb{G}_m$. Such a character must vanish on the unipotent radical N of P, so amounts to a product of characters on the GL_{n_i} making up the Levi quotient, and this gives the p_i, since a character of GL_m is a power of the determinant. Since different sequences of p_i give different characters, $\mathcal{L}$ determines $\underline{p}$ uniquely.

(ii) A line bundle $\mathcal{L}$ on a projective variety X is ample if for every coherent sheaf $\mathcal{M}$ the sheaf $\mathcal{M} \otimes \mathcal{L}^N$ is generated by global sections for all $N >> 0$, namely the sheaf map

$$\Gamma(\mathcal{M} \otimes \mathcal{L}^N) \otimes \mathcal{O}_X \to \mathcal{M} \otimes \mathcal{L}^N \tag{47}$$

is surjective. This definition makes it evident that the restriction of an ample line bundle to a closed subvariety is ample, and the product of ample line bundles on a product variety is ample. For projective space, the dual $\mathcal{O}(1)$ of the tautological line bundle $\mathcal{O}(-1)$ is ample.

We first treat the case of the Grassmanian $\mathcal{F} = Gr(m,n)$ which corresponds to flags of type $(n-m, m)$. In this case we have to show that if $\mathcal{L}$ is the tautological line bundle whose value at an m-dimensional subspace W is $\det(W)$, then $\mathcal{L}^{\otimes -p}$ is ample if and only if $p > 0$. Fix coordinates and represent a point $x \in Gr(m,n)$ by an $m \times n$ matrix up to the action of $GL(m)$ on the left (the rows spanning the subspace W represented by x). Now map such a matrix to the point in $\mathbb{P}^{\binom{n}{m}}$ via the determinants of all the m-minors. This gives the standard embedding of $Gr(m,n)$ in projective space. Since the pull-back under an embedding of an ample line bundle is ample, it is enough to note that $\mathcal{L}$ is just the pull-back of the tautological line bundle $\mathcal{O}(-1)$ on projective space.

For a general $\mathcal{F}$ embed it as a closed subvariety of

$$\mathcal{G} = \prod_{i=2}^{r} Gr(\sum_{j=i}^{r} n_j, n) \tag{48}$$

and observe that if $p_1 < p_2 < \cdots < p_r$, then

$$\mathcal{L}(\underline{p}) = \det(V)^{\otimes -p_1} \otimes \bigotimes_{i=2}^{r} \det(V_i)^{\otimes(p_{i-1}-p_i)} \tag{49}$$

is the product of a trivial line bundle with the pull back of an ample line bindle on $\mathcal{G}$. On the other hand if $\mathcal{L}(\underline{p})$ is ample, fix $V_i = V_i^0$ for all $i \neq j$ and let only V_j vary. We get a copy of $Gr(n_j, n_{j-1} + n_j)$ inside $\mathcal{F}$ and the restriction of $\mathcal{L}(\underline{p})$ to it, which must be ample, is $\mathcal{L}_{j-1}^{\otimes -p_{j-1}} \otimes \mathcal{L}_j^{\otimes -p_j}$ so we must have $p_{j-1} < p_j$.

(iii) This is clear. □

For example, if $\mathcal{F}$ is the flag variety of type $(1, n-1)$ classifying hyperplanes in V, or what is the same, lines in $V^\vee$ (equipped with the contragredient action of G), then $\mathcal{F} = \mathbb{P}(V^\vee)$. One computes

$$\mathcal{L}(-1, 0) = \mathcal{O}(1) \tag{50}$$

where $\mathcal{O}(1)$ is the *dual* of the tautological line bundle $\mathcal{L}$ on $\mathbb{P}(V^\vee)$ which assigns to the point $[V^\vee \supset L \supset 0]$ the line L.

1.5. One-parameter-subgroups, semistability, and Mumford's invariant $\mu^{\mathcal{L}}(x, \lambda)$. A 1-PS of G is a morphism $\lambda : \mathbb{G}_m \to G$. It induces a splitting of V according to the characters of $\mathbb{G}_m$:

$$V = \bigoplus_{i \in \mathbb{Z}} V(i) \tag{51}$$

where $\lambda(t)v = t^i v$ for all $v \in V(i)$. The center of the centralizer of λ, denoted T_λ, is a split torus containing $Im(\lambda)$. It is the product of the centers of $GL(V(i))$. Call λ *regular* if T_λ is a maximal torus, i.e. all the $V(i)$ are one-dimensional.

Consider the *orbit*

$$\mathbb{O}_\lambda(x) = \lambda(\mathbb{G}_m)x \tag{52}$$

of a closed point $x \in \mathcal{F}$. This is x itself (x is a *fixed point* of λ) if and only if $Im(\lambda) \subset P_x$. This happens if and only if $T_\lambda \subset P_x$. If λ is regular, its fixed points are finite in number. Otherwise, they make up a finite union of subvarieties. We denote the set of fixed points of λ by $Fix(\lambda) \subset \mathcal{F}$.

If x is not a fixed point, $\mathbb{O}_\lambda(x)$ is a quotient of $\mathbb{G}_m$ by a finite group, so can't be closed in $\mathcal{F}$. By the valuative criterion for properness, the map $t \mapsto \lambda(t)x$ extends to $t = 0$ (view $\mathbb{G}_m \subset \mathbb{A}^1$) and we may think of its value at 0, denoted $\lambda(0)x$, as $\lim_{t\to 0} \lambda(t)x$. Clearly $\lambda(0)x \in Fix(\lambda)$.

Let $\mathcal{L} \in Pic^G\mathcal{F}$. Then $\lambda(\mathbb{G}_m)$ acts on the fiber $\mathcal{L}_{\lambda(0)x}$ by a character $t \mapsto t^r$ and we set[1]

$$\mu^{\mathcal{L}}(x, \lambda) = r. \tag{53}$$

The point x is called *semistable* w.r.t. λ and $\mathcal{L}$ if $\mu^{\mathcal{L}}(x, \lambda) \geq 0$ and *stable* if $\mu^{\mathcal{L}}(x, \lambda) > 0$. Geometrically, stability means that if x^* is any nonzero point above x in the line bundle $\mathcal{L}^\vee$ (on which G acts via the contragredient action), then $\lambda(t)x^* \to \infty$ as $t \to 0$ (the orbit of x^* remains closed as $t \to 0$). Semistability simply means that $\lambda(t)x^*$ stays away from 0 as $t \to 0$.

The following properties of $\mu^{\mathcal{L}}(x, \lambda)$ are immediate:

- $\mu^{\mathcal{L}}(x, \lambda) = \mu^{\mathcal{L}}(y, \lambda)$ for every $y \in \mathbb{O}_\lambda(x)$
- $\mu^{\mathcal{L}}(x, \lambda) = \mu^{\mathcal{L}}(\lambda(0)x, \lambda)$.
- For every $g \in G$, $\mu^{\mathcal{L}}(x, \lambda) = \mu^{\mathcal{L}}(gx, {}^g\lambda)$, where ${}^g\lambda(t) = g\lambda(t)g^{-1}$.
- For fixed x and λ, $\mu^\bullet(x, \lambda) : Pic^G\mathcal{F} \to \mathbb{Z}$ is a homomorphism.

Definition 1.1. *Let $J \subset G(k)$ be a subgroup. We say that x is* semistable *w.r.t. J and $\mathcal{L}$ if it is semistable w.r.t. every 1-PS contained in J.*

We say that x is stable *w.r.t. J and $\mathcal{L}$ if for every 1-PS λ contained in J, either $x \in Fix(\lambda)$ and $\mu^{\mathcal{L}}(x, \lambda) \geq 0$, or $\mu^{\mathcal{L}}(x, \lambda) > 0$. (Note that if $x \in Fix(\lambda)$, then because we must consider λ^{-1} too, we must have $\mu^{\mathcal{L}}(x, \lambda) = 0$.)*

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[1] This differs by a sign from [GIT]. We believe that there is a sign mistake in [GIT], ch.2, section1.

Clearly the set of such points is invariant under the action of J (and even under the action of the *normalizer* of J). Stability w.r.t. J and $\mathcal{L}$ means that the orbit of any nonzero $x^* \in \mathcal{L}^\vee$ above x, under any $\mathbb{G}_m \subset J$, is closed (it stays closed as $t \to 0$, and considering the inverse 1-PS, also as $t \to \infty$).

If J contains the scalar matrices, then using the 1-PS $t \mapsto diag(t, \dots, t)$ and its inverse we see that no point can be semistable w.r.t. J and $\mathcal{L} = \mathcal{L}(p_1, \dots, p_r)$ unless $\sum p_i n_i = 0$. Indeed, all points of $\mathcal{F}$ are fixed under these 1-PS subgroups, but on the fibers of $\mathcal{L}$ they act via raising to power $\mp \sum p_i n_i$, and both these numbers should be non-positive. In the future, whenever we discuss semistability, either we shall assume that $J \subset SL(V)$, or that $\sum p_i n_i = 0$ (the center of G acts trivially on $\mathcal{L}$).

Example. Let $\dim V = n$, and $\mathcal{F} = \mathbb{P}(V^\vee)$ the space of hyperplanes in V (lines in $V^\vee$). We take for $\mathcal{L}$ the ample line bundle $\mathcal{L}(-1, 0) = \mathcal{O}(1)$ which to a point $x = [V \supset W \supset 0]$ attaches $\mathcal{L}_x = V/W$. Let λ be a 1-PS into $GL(V)$. Choose a basis of V so that $\lambda(t)e_i = t^{r_i}e_i$. Identify V with column vectors ${}^t(x_1, \dots, x_n)$. Points in $\mathcal{F}$ are then

$$[\xi_1, \dots, \xi_n] \tag{54}$$

homothety classes of non-zero row vectors, and the action of $g \in GL(V)$ on them is the right action of the matrix g^{-1}. The action of $\lambda(t)$ is therefore

$$\lambda(t)[\xi_1, \dots, \xi_n] = [t^{-r_1}\xi_1, \dots, t^{-r_n}\xi_n]. \tag{55}$$

Fix $\xi \in \mathcal{F}$ and let

$$\mu = \max\{r_i;\ \xi_i \neq 0\}. \tag{56}$$

Let I be the index set where $r_i = \mu$. It is clear that $\eta = \lambda(0)\xi$ is the point whose coordinates are ξ_i if $i \in I$ and 0 otherwise. As a non-zero representative of $\mathcal{L}_\eta$ we may take any vector of V not annihilated by the linear functionals in the line η. For example, we may take any e_i, $i \in I$. But then $\lambda(t)$ acts via $\mu = r_i$ so

$$\mu^{\mathcal{L}}(\xi, \lambda) = \mu. \tag{57}$$

Suppose now that λ is into $J = SL(V_k)$. Then $\sum r_i = 0$. If we want our point ξ to be semistable w.r.t. to all such λ then we must have, in any *k-rational coordinates*, $\xi_i \neq 0$. It turns out that the $(SL(V_k), \mathcal{O}(1))$-semistable points of $\mathcal{F}$ are the lines ξ that do not lie in any k-rational hyperplane in $V^\vee$, or dually, the hyperplanes in V that do not contain any k-rational line.

2. Filtrations and semistability

2.1. Filtrations. A (real, descending) *filtration* α on a (finite dimensional) vector space V is a collection of subspaces$V^p = V^p_\alpha$ satisfying:

- $V^p \supset V^q$ if $p < q$, $V^p = 0$ for $0 << p$ and $V^p = V$ for $p << 0$.
- (left continuity) $V^p = V^{p-\varepsilon}$ for ε small enough.

Put $gr^pV = V^p/V^{p+\varepsilon}$ (ε small...) and call p a *break point* if $gr^pV \neq 0$. Call the filtration rational or integral if its break points are in $\mathbb{Q}$ or $\mathbb{Z}$. A filtration is determined by its underlying flag *and* the break points.

The parabolic subgroup stabilizing the flag x underlying a filtration α will also be denoted P_α.

Suppose a decomposition $V = \oplus L_i$ into lines is given, and for every i there is a real number p_i such that

$$V^p = \bigoplus_{p_i \geq p} L_i. \tag{58}$$

Then we say that the decomposition and the filtration are *compatible.* (This is the same as being compatible with the underlying flag.) The set of filtrations compatible with a given decomposition naturally forms a Euclidean space $\mathbb{R}^n$.

2.2. The category of filtered vector spaces. Filtered vector spaces form an *additive, but non-abelian category.* It has tensor products and duals, where

$$(V \otimes W)^p = \sum_q V^q \otimes W^{p-q} \tag{59}$$

and

$$(V^*)^p = (V/V^{-p})^*. \tag{60}$$

If $V = \bigoplus L_i$ and $W = \bigoplus M_j$ are decompositions compatible with the filtrations on V and W then

$$V \otimes W = \bigoplus L_i \otimes M_j \tag{61}$$

is compatible with the filtration on the tensor product, and the dual decomposition of V^* is compatible with the dual filtration.

If V is a filtered vector space, and W is a subspace, there are *induced filtrations* on W and on V/W. However, even if $W \to V$ is an injection, it does not mean that the filtration on W is the one induced from V. It only means that each W^p gets mapped into V^p.

2.3. An inner product on all filtrations. Let α and β be two filtrations on V. Define their "inner product"

$$\langle \alpha, \beta \rangle = \sum_p \sum_q pq \dim gr^p_\alpha gr^q_\beta V. \tag{62}$$

LEMMA 2.1. *Let $V = \bigoplus L_i$ be a decomposition of V into lines compatible with both α and β (such a decomposition exists by Bruhat's lemma) and let p_i (resp. q_i) be the real numbers such that $V^p_\alpha = \sum_{p_i \geq p} L_i$ and $V^q_\beta = \sum_{q_i \geq q} L_i$. Then $\langle \alpha, \beta \rangle = \sum p_i q_i$. In particular the inner product is symmetric, and the Euclidean distance*

$$d(\alpha, \beta) = \sqrt{\sum (p_i - q_i)^2} = \sqrt{\langle \alpha, \alpha \rangle + \langle \beta, \beta \rangle - 2 \langle \alpha, \beta \rangle} \tag{63}$$

is independent of the decomposition used to define it.

PROOF. A basis for a given $gr^p_\alpha gr^q_\beta V$ is given by the images of generators of the lines L_i with $p_i = p$ and $q_i = q$. □

Although the set of filtrations is not a vector space, this inner product has many good properties. For example, it satisfies the Cauchy-Schwartz inequality

$$|\langle \alpha, \beta \rangle| \leq |\alpha||\beta| \tag{64}$$

where $|\alpha|^2 = \sum p^2 \dim gr^p_\alpha V$.

??Does the Euclidean distance between filtrations turn the set of all filtrations into a metric space ??

2.4. The relation between filtrations and non-archimedean norms on V. Put on k the trivial absolute value: $|a| = 1$ if $a \neq 0$, $|0| = 0$. A *norm* on V is a function $|\bullet| : V \to [0, \infty)$ satisfying (i) $|av| = |a||v|$ (ii) $|u + v| \leq \max\{|u|, |v|\}$ and (iii) $|v| = 0$ if and only if $v = 0$.

Norms on V are in one-to-one correspondence with filtrations. Given a norm, we put

$$V^p = \left\{v; |v| \leq e^{-p}\right\}. \tag{65}$$

2.5. Integral filtration associated to a 1-PS. Let λ be a 1-PS. Define

$$V^p_{\alpha(\lambda)} = \bigoplus_{p \leq i} V(i) \tag{66}$$

where $V(i)$ is the i eigenspace w.r.t. λ. This is an integral filtration $\alpha(\lambda)$ called *the filtration associated to* λ.

2.6. Integral filtration associated to x and $\mathcal{L}$. An integral filtration α of type $\underline{n} = (n_1, \ldots, n_r)$ gives us a point x_α in the flag variety $\mathcal{F} = \mathcal{F}_{\underline{n}}$. Its break points $p_1 < \cdots < p_r$ define the ample G-line bundle $\mathcal{L}_\alpha = \mathcal{L}(p_1, \ldots, p_r)$ on the flag variety $\mathcal{F}$. Conversely, given a point $x \in \mathcal{F}$ and an ample G-line bundle $\mathcal{L} \in Pic^G\mathcal{F}$, we get a unique filtration $\alpha(x, \mathcal{L})$. We therefore have a 1:1 correspondence

$$\{\text{integral filtrations } \alpha \text{ of type } \underline{n}\} \longleftrightarrow \left\{\begin{array}{c}\text{pairs } (x, \mathcal{L}) \text{ where } x \in \mathcal{F} \\ \text{and } \mathcal{L} \text{ is an ample } G\text{-line bundle on } \mathcal{F}\end{array}\right\}$$

The parabolic P_α acts on the line $\mathcal{L}_\alpha|_{x_\alpha}$ via a character, which we denote χ_α.

Example. If λ is a 1-PS then $\lambda : \mathbb{G}_m \to P_{\alpha(\lambda)}$ and $\chi_{\alpha(\lambda)} \circ \lambda(t) = t^{-\sum n_i p_i^2} = t^{-|\alpha(\lambda)|^2}$.

2.7. The computation of $\mu^{\mathcal{L}}(x, \lambda)$.

THEOREM 2.2. *(Mumford) Let $\mathcal{F}$ be the flag variety of type $\underline{n} = (n_1, \ldots, n_r)$, and $\mathcal{L} \in Pic^G\mathcal{F}$ an ample G-line bundle. Let $x \in \mathcal{F}$. Let λ be a 1-PS in G. Then Mumford's invariant $\mu^{\mathcal{L}}(x, \lambda)$ can be computed from the filtrations associated to λ and to the pair $(x, \mathcal{L})$ as follows:*

$$\mu^{\mathcal{L}}(x, \lambda) = -\langle \alpha(\lambda), \alpha(x, \mathcal{L})\rangle. \tag{67}$$

PROOF. Put $\alpha = \alpha(\lambda)$ and $\beta = \alpha(x, \mathcal{L})$. We have to prove that

$$\mu^{\mathcal{L}}(x, \lambda) = -\sum_{w,p} wp \dim gr^w_\alpha gr^p_\beta V. \tag{68}$$

If the break points in $\alpha(\lambda)$ are w_i (these are the weights of λ acting on V), $\mathcal{L} = \mathcal{L}(p_1, \ldots, p_r)$ and x is the flag $x : V = V_1 \supset \cdots \supset V_r \supset 0$, then we have to prove

$$\mu^{\mathcal{L}}(x, \lambda) = -\sum_{i,j} w_i p_j \dim gr^{w_i}_\alpha (V_j/V_{j+1}). \tag{69}$$

This formula will be proved for *every* sequence of $(p_1, \ldots, p_r)$, whether $\mathcal{L}$ is ample or not (the ampleness is not essential for the theorem, it is only needed in the dictionary between pairs $(x, \mathcal{L})$ and filtrations). By additivity, it is enough to show that for the tautological line bundle $\mathcal{L}_j$ (whose fiber at x is $\det(V_j/V_{j+1})$)

$$\mu^{\mathcal{L}_j}(x, \lambda) = \sum_i w_i \dim gr^{w_i}_\alpha (V_j/V_{j+1}). \tag{70}$$

Arrange the weights of λ in increasing order:

$$w_1 < w_2 < \cdots < w_s. \tag{71}$$

Choose a basis $e_1, \ldots, e_n$ of V diagonalizing the action of $\lambda : \lambda(t)e_l = t^{w_i} e_l$ for

$$m_1 + \cdots + m_{i-1} < l \leq m_1 + \cdots + m_{i-1} + m_i, \tag{72}$$

(we say that l is in the i^{th} block). Choose vectors

$$v_k = \sum_{l=1}^{n} a_{lk} e_l \tag{73}$$

$(1 \leq k \leq n_j)$ in V_j representing a basis of V_j/V_{j+1} in such a way that v_k for

$$q_1 + \cdots + q_{i-1} < k \leq q_1 + \cdots + q_{i-1} + q_i \tag{74}$$

(k in the i^{th} block) represent a basis of $gr_\alpha^{w_i}(V_j/V_{j+1})$, where

$$q_i = \dim gr_\alpha^{w_i}(V_j/V_{j+1}), \tag{75}$$

and of course $\sum_{i=1}^{s} q_i = n_j$ (some of the q_i may be 0). This simply means that $a_{lk} = 0$ if k is in the i^{th} block of indices and $l \leq m_1 + \cdots + m_{i-1}$. Moreover, the block just below the zeros, namely the matrix (a_{lk}) for k and l both in the i^{th} block, has rank q_i.

Now consider the limit of the lines $\lambda(t)v_k$ as $t \to 0$, for k in the i^{th} block. Since $w_{i'} > w_i$ if $i' > i$ the (l,k) entry for l below the i^{th} block of indices will go to 0 much faster then the entries where both k and l are in the i^{th} block (and the entries for l above the i^{th} block all vanish by our choices). Renormalizing by multiplication by the scalar t^{-w_i} we see that the flag $\lambda(0)x$ is given by $V_1^0 \supset \cdots \supset V_r^0 \supset 0$ where a basis of V_j^0/V_{j+1}^0 is represented by the vectors v_k^0 obtained from the vectors v_k simply by erasing the (l,k) entries where l is below the i^{th} block. It is now clear that $\lambda(t)$ acts on $\mathcal{L}_j|_{\lambda(0)x} = \det(V_j^0/V_{j+1}^0)$ by the character

$$t \mapsto t^{\sum w_i q_i} \tag{76}$$

hence

$$\mu^{\mathcal{L}_j}(x,\lambda) = \sum_i w_i q_i \tag{77}$$

as we had to show. □

3. Slopes and semistability

3.1. Rank, degree and slope. Define the *rank* $rk(\alpha)$ of a filtration α, to be $\dim V$, its *degree*

$$\deg(\alpha) = \sum_p p \dim gr_\alpha^p \tag{78}$$

and its *slope*

$$\mu(\alpha) = \frac{\deg(\alpha)}{rk(\alpha)}. \tag{79}$$

If β is any filtration and $\beta[m]$ is the *shift* of β by m, namely

$$V_{\beta[m]}^p = V_\beta^{p-m} \tag{80}$$

then

$$\langle \alpha, \beta[m] \rangle = \langle \alpha, \beta \rangle + m \deg(\alpha), \tag{81}$$

and

$$\deg(\beta[m]) = \deg(\beta) + m.rk(\beta). \tag{82}$$

From the non-negativity of $\langle \beta[m], \beta[m] \rangle$ we obtain the following inequality

$$0 \leq \langle \beta, \beta \rangle + 2m \deg(\beta) + m^2 rk(\beta) \tag{83}$$

for all m, so

$$|\mu(\beta)| \leq |\beta| \sqrt{rk(\beta)}. \tag{84}$$

It helps to think of the degree as the price one has to pay for picking a basis compatible with the filtration. The more special the subspace is (the higher up it is in the filtration) the more expensive its vectors are. The slope is then the average price per vector.

If W is a subspace of a filtered vector space V then in the induced filtrations

$$\deg(V) = \deg(W) + \deg(V/W) \tag{85}$$

and

$$\mu(V) = \frac{rkW}{rkV} \mu(W) + \frac{rk(V/W)}{rkV} \mu(V/W) \tag{86}$$

is a *weighted average* of the slopes of W and V/W. If $(V, \alpha) \to (W, \beta\ )$ is a morphism of filtered vector spaces which is an isomorphism on the vector spaces, then $\deg(\alpha) \leq \deg(\beta)$. Using splittings compatible with given filtrations it is easy to show that

$$\deg(V \otimes W) = rk(V) \deg(W) + rk(W) \deg(V) \tag{87}$$

hence

$$\mu(V \otimes W) = \mu(V) + \mu(W). \tag{88}$$

Denoting by $\det(V)$ the highest exterior power of V we also see that

$$\deg(\det(V)) = \deg(V). \tag{89}$$

The following relation is also obvious. If α is an integral filtration on V and $\mathcal{L}_\alpha$ the corresponding line bundle, then the *central character* of $\mathcal{L}_\alpha$ (the character by which the center of G acts on $\mathcal{L}_\alpha$) is

$$t \mapsto t^{-\deg(\alpha)}. \tag{90}$$

3.2. K/k-semistability. Let K be a field extension of k. Let V be a vector space defined over k together with a filtration α defined over V_K. The basic definition is the following.

DEFINITION 3.1. *The filtration α is called (K/k-)* semistable *if for every subspace W of V (over k)*

$$\mu(W_K) \leq \mu(V_K) \tag{91}$$

in the induced filtration on V_K.

Alternatively, $\mu(V_K/W_K) \geq \mu(V_K)$.

Examples. (1) If $k = K$ then the only semistable filtrations are the trivial ones: with only one break point.

(2) Let α be a filtration with only two break points $r < s$ with $\dim gr_\alpha^s = 1$. Then α is a semistable if and only if the line $V_\alpha^s \subset V_K$ is not contained in a k-rational hyperplane.

(3) Let α be a filtration as in (2) with $\dim gr_\alpha^r = n-2$, $\dim gr_\alpha^s = 2$ $(n \geq 3)$. Then α is semistable if and only if (i) the plane V_α^s is not contained in any k-rational hyperplane *and* (ii) any line on V_α^s is not contained in a k-rational subspace of dimension $< n/2$.

(4) Let $\dim V = 3$ and consider a full flag filtration on V_K with breaks at $r < s < t$. Then

(i) If $t - s > s - r$ the condition for semistability is that V^t is not contained in any k-rational plane.

(ii) If $t - s < s - r$ the condition is that V^s does not contain any k-rational line.

(iii) If $t-s = s-r$ the condition is that V^s and V^t are not themselves k-rational.

3.3. The relation to the GIT notion of semistability.

PROPOSITION 3.1. *(Totaro) Let $x \in \mathcal{F}(K)$ be a K-point in the flag variety of type $\underline{n}$. Let $\mathcal{L} = \mathcal{L}(\underline{p})$ be an ample line bundle and let $\alpha = \alpha(x, \mathcal{L})$ be the corresponding integral filtration on V_K. Then the following are equivalent:*

(i) α is K/k-semistable and $\mu(V_K, \alpha) = 0$.

(ii) x is semistable w.r.t. $\mathcal{L}$ and $GL(V_k)$.

PROOF. Property (ii) is equivalent to $0 \leq \mu^{\mathcal{L}}(x, \lambda) = -\langle \alpha(\lambda), \alpha(x, \mathcal{L})\rangle$ for all 1-PS λ into $GL(V_k)$, or

$$\langle \alpha(\lambda), \alpha(x, \mathcal{L})\rangle \leq 0 \tag{92}$$

for all such λ. Applying it to $\lambda(t) = t^q$ we see that $q \sum p_i n_i \leq 0$ for all q, hence $\deg(\alpha) = \sum p_i n_i = 0$ and $\mu(\alpha) = 0$. If W is a k-rational subspace of V, decompose $V = W \bigoplus W'$ and let $\lambda(t)$ act by 1 on W' and by t on W. The filtration $\alpha(\lambda)$ then has only two break points, 0 and 1, with graded pieces W' and W, and the condition on the inner product becomes precisely $\deg(W_K, \alpha) \leq 0$, or $\mu(W_K, \alpha) \leq \mu(V_K, \alpha)$.

Conversely, assume (i) holds. Let λ be a 1-PS into $GL(V_k)$, let $V = W_1 \supset \cdots \supset W_s \supset 0$ be the flag of $\beta = \alpha(\lambda)$, and w_i the corresponding weights in increasing order. One has

$$\langle \alpha(\lambda), \alpha(x, \mathcal{L})\rangle = \sum w_i p_j \dim gr_\beta^i gr_\alpha^j V_K = -\int_{-\infty}^{\infty} l.d(\deg(V_{\beta,K}^l, \alpha)) \tag{93}$$

and the function $l \mapsto \deg(V_{\beta,K}^l, \alpha)$ is locally constant, and vanishes at $l >> 0$ (because $V_\beta^l = 0$) or at $l << 0$ (because $\mu(V_K, \alpha) = 0$). Integration by parts gives

$$\langle \alpha(\lambda), \alpha(x, \mathcal{L})\rangle = \int_{-\infty}^{\infty} \deg(V_{\beta,K}^l, \alpha).dl \leq 0 \tag{94}$$

because we assumed that $\mu(V_{\beta,K}^l, \alpha) \leq \mu(V_K, \alpha) = 0$. □

A similar proof gives

PROPOSITION 3.2. *Under the same circumstances (i) α is K/k-semistable* if and only if *(ii) x is semistable w.r.t. $\mathcal{L}$ and $SL(V_k)$.*

3.4. Harder-Narasimhan filtration.

PROPOSITION 3.3. *Let α be a filtration on V_K for an extension K/k. There exists a filtration of V, $V = V_m \supset \cdots \supset V_1 \supset 0$ such that each V_i/V_{i-1} (with the filtration induced by α after base change to K) is semistable, and*

$$\mu_\alpha(V_1) > \mu_\alpha(V_2/V_1) > \cdots > \mu_\alpha(V_m/V_{m-1}). \tag{95}$$

This filtration is unique.

PROOF. Let μ_1 be the maximal slope of a subspace of V (defined over k) and let V_1 be a maximal subspace with that slope. If W is any subspace of V_1 then $\mu(W) \leq \mu_1 = \mu(V_1)$, so V_1 is semistable. Suppose W is a subspace strictly containing V_1. Then the slope of W is a weighted average of the slopes of V_1 and of W/V_1. Since it is strictly less than μ_1, the slope of W/V_1 must be less than μ_1. Now if W is a subspace with slope μ_1 consider $W + V_1 \supset V_1 \supset 0$. Since $(W + V_1)/V_1 \simeq W/W \cap V_1$ and since $\mu(W) = \mu_1$ is a weighted average of $\mu(W/W \cap V_1)$ and $\mu(W \cap V_1) \leq \mu_1$, we obtain

$$\mu((W + V_1)/V_1) = \mu(W/W \cap V_1) \geq \mu_1. \tag{96}$$

But then $\mu(W + V_1) \geq \mu_1$ and $W + V_1 = V_1$ by the maximality of V_1. We conclude that V_1 is unique. We continue by induction on the dimension, replacing V by V/V_1. □

The space V is semistable if and only if its Harder-Narasimhan filtration is trivial. At the other extreme, if $k = K$, so that α is never semistable (unless it is trivial) the Harder Narasimhan filtration is the one given by α (notice the slight abuse of language - by the Harder Narasimhan filtration we only mean the flag of subspaces, not the specification of the "break points" p_i.)

3.5. Flag varieties and period domains. We denote the set of semistable points in $\mathcal{F}(K)$ w.r.t. the data $\underline{p}$ by $\mathcal{F}(K)^{ss}_{\underline{p}}$. These are the flags of type $\underline{n}$ such that when we endow them with the filtration with break points at the p_i they are semistable. The $\mathcal{F}(K)^{ss}_{\underline{p}}$ are called *period domains.* They depend on $\underline{p}$, namely on an ample line bundle, but not in a serious way, as the examples above demonstrate. By varying the p_i we get *finitely many* period domains inside $\mathcal{F}(K)$.

To check if a point $\alpha \in \mathcal{F}(K)^{ss}_{\underline{p}}$ one has to check for *every* k-rational subspace W whether $\mu_\alpha(W) \leq \mu_\alpha(V)$. For a given W this is a set of open conditions on the dimensions of $W \cap V^{p_i}_\alpha$. (Open means that the more generic our point in $\mathcal{F}$ is, the smaller the slope is: the slope *increases* under specialization.) It follows that if k is a finite field there are only finitely many W to check, so there is a Zariski open k-rational subset $\mathcal{F}^{ss}_{\underline{p}}$ such that

$$\mathcal{F}(K)^{ss}_{\underline{p}} = \mathcal{F}^{ss}_{\underline{p}}(K) \tag{97}$$

for any field K containing k. However, for a general k the period domain is not the K-points of an algebraic variety.

4. The Faltings-Totaro theorem

THEOREM 4.1. *Let V and W be two finite dimensional vector spaces over k. Let K be a field extension of k, and let α and β be two K/k-semistable filtrations on V_K and W_K respectively. Then the filtration $\alpha \otimes \beta$ on $V_K \otimes_K W_K$ is K/k-semistable.*

We begin with some preparations.

4.1. Computing degrees of subspaces. Let

$$V = \bigoplus_{i \in I} L_i \tag{98}$$

be a decomposition of a vector space V into a direct sum of lines, and S a subspace of dimension s. For $A \subset I$ let V_A be the sum of the L_i for $i \in A$ and π_A the projection onto V_A along V_{I-A}. Let $\mathcal{A}$ be the family of all subsets A, $\#(A) = s$, such that $\pi_A : S \to V_A$ is an isomorphism.

Let α be a filtration on V compatible with the decomposition. Let p_i be such that

$$V_\alpha^p = \sum_{p \leq p_i} L_i. \tag{99}$$

Lemma 4.2. *We have*

$$\deg_\alpha(S) = \inf_{A \in \mathcal{A}} \deg_\alpha(V_A) = \inf_{A \in \mathcal{A}} \sum_{i \in A} p_i. \tag{100}$$

Proof. For each A the isomorphism $\pi_A : S \to V_A$ is a bijective map of filtered vector spaces, hence $\deg_\alpha(S) \leq \deg_\alpha(V_A)$. We must show that there exists an A (depending on α) for which there is an equality. Let q be a break point in the filtration induced by α on S. Since $gr_\alpha^q S$ is a subspace of $gr_\alpha^q V$, we can choose a subset of the L_i with $p_i = q$ such that the projection of $gr_\alpha^q S$ on the subspace spanned by them is an isomorphism of filtered vector spaces. Taking the unions of these subsets, for all q, we get an A for which π_A is an isomorphism of filtered vector spaces. □

4.2. A lemma in convex geometry.

Lemma 4.3. *(Ramanan-Ramanathan) Let $\{l(x); l \in \mathcal{A}\}$ be a finite collection of linear forms on $\mathbb{R}^n$. Consider the function $f : \mathbb{R}^n \to \mathbb{R}$, given by*

$$f(x) = \inf_{l \in \mathcal{A}} l(x). \tag{101}$$

(i) $f(tx) = tf(x)$ for $t > 0$ and f is concave.

Assume that there exists a point x where $f(x) > 0$. Then:

(ii) $f|S^{n-1}$ attains its maximum at a unique *point $a \in S^{n-1}$. Moreover, there is no other point $b \in S^{n-1}$ where f attains a (weak) local maximum and $f(b) > 0$.*

(iii) If $\mathcal{A}'$ is the collection of $l \in \mathcal{A}$ such that $f(a) = l(a)$ then a is also the unique point where

$$f'(x) = \inf_{l \in \mathcal{A}'} l(x) \tag{102}$$

attains its maximum.

(iv) For every $x \in \mathbb{R}^n$,

$$f(x) \leq f(a) \langle a, x \rangle . \tag{103}$$

Proof. (i) For any x and y, and any $l \in \mathcal{A}$

$$tf(x) + (1-t)f(y) \leq tl(x) + (1-t)l(y) = l(tx + (1-t)y). \tag{104}$$

Since this is true for every $l \in \mathcal{A}$, we get

$$tf(x) + (1-t)f(y) \leq f(tx + (1-t)y). \tag{105}$$

(ii) Let $a \in S^{n-1}$ be a point where f attains its maximum. Suppose f attains a (weak) local maximum at a point $b \in S^{n-1}$ different from a, and $f(b) > 0$. Consider the point $x = ta + (1-t)b$. For every $l \in \mathcal{A}$

$$l(x) = tl(a) + (1-t)l(b) \geq f(b) \tag{106}$$

since $l(a) \geq f(a) \geq f(b)$ and $l(b) \geq f(b)$. For all $0 < t < 1$ we shall then have, since $|x| < 1$ and $f(b) > 0$,

$$l(x/|x|) > f(b) \tag{107}$$

hence $f(x/|x|) > f(b)$, contradicting the assumption that b is a local maximum.

(iii) If $l \in \mathcal{A}'$ and $l' \notin \mathcal{A}'$ then $l'(a) > l(a)$. This means that $l'(x) > l(x)$ for all x close enough to a, so $f'(x) = f(x)$ in the vicinity of a, so a is a local maximum of f' where $f'(a) > 0$. By the previous part of the lemma, a is also the unique global maximum of f'.

(iv) We first show a special case: if $\langle a, x\rangle < 0$ then $f(x) < 0$. Consider $a + tx$. By our assumption

$$|a + tx|^2 = 1 + 2t\langle a, x\rangle + t^2|x|^2 \tag{108}$$

so for all small enough positive t, $|a+tx| < 1$. If $f(x) \geq 0$, then $l(x) \geq 0$ for all l, and for small enough positive t we shall have $l(a+tx) \geq l(a)$, hence $l((a+tx)/|a+tx|) > l(a)$. Taking the infimum over all l we get $f((a+tx)/|a+tx|) > f(a)$, contradiction.

By continuity, if $\langle a, x\rangle \leq 0$, then $f(x) \leq 0$.

We now turn to the desired inequality. Without loss of generality we may assume that all $l \in \mathcal{A}$ attain at a the same value $f(a)$, because if we discard the ones attaining a larger value, and call the resulting function f', we still have a as the unique maximum of f', but $f(x) \leq f'(x)$ in general, so if we prove the inequality for f', it would a fortiori hold for f.

Let

$$g(x) = f(x) - f(a)\langle a, x\rangle = \inf_{l\in\mathcal{A}} \left(l(x) - f(a)\langle a, x\rangle\right). \tag{109}$$

Write $x = y + ta$ with $\langle a, y\rangle = 0$. Then

$$g(x) = \inf_{l\in\mathcal{A}} \left(l(y) + tl(a) - tf(a)\right) = \inf_{l\in\mathcal{A}} l(y) = f(y) \tag{110}$$

since we assumed that $l(a) = f(a)$ for all l. By the special case proved before, $g(x) = f(y) \leq 0$, which is what we had to prove. □

4.3. Proof of the theorem. Step I. Let S be a fixed subspace of $V \bigotimes W$ and assume that there exists a filtration $\alpha = (\alpha_V, \alpha_W)$ of (V, W) such that $\mu_\alpha(V) = \mu_\alpha(W) = 0$ (hence also $\mu_\alpha(V \bigotimes W) = 0$) but $\mu_\alpha(S) > 0$. We shall show that, up to scaling, there exists a unique filtration α of this sort for which $\mu_\alpha(S)/|\alpha|$ is *maximal.* (The maximaizing filtration α is called *Kempf's filtration.*)

We first work in one "appartment". Fix decompositions into a direct sum of lines $V = \bigoplus_{i\in I} L_i$, $W = \bigoplus_{j\in J} M_j$, so that

$$V \bigotimes W = \bigoplus_{(i,j)\in I\times J} L_i \otimes M_j. \tag{111}$$

Let $\mathcal{A}$ be, as before, the collection of subsets A of $I \times J$ of cardinality $s = \dim S$ such that the projection π_A is an isomorphism of S onto $(V \bigotimes W)_A$. Suppose there exists a filtration $\alpha = (\alpha_V, \alpha_W)$ compatible with the decomposition, for which $\mu_\alpha(V) = \mu_\alpha(W) = 0$ (hence also $\mu_\alpha(V \bigotimes W) = 0$) but $\mu_\alpha(S) > 0$. Consider the

Euclidean space of all the filtrations of (V, W) of slope 0 on V and on W separately, which are compatible with the decomposition. A point in it is given by $\alpha = (p_i, q_j)$ where $\sum_{i\in I} p_i = \sum_{j\in J} q_j = 0$. The norm $|\alpha|$ is the usual norm $\left\{\sum p_i^2 + \sum q_j^2\right\}^{1/2}$. Consider the linear form

$$l_A(p_i, q_j) = \sum_{(i,j)\in A} (p_i + q_j). \tag{112}$$

Then by the first lemma (applied in $V \bigotimes W$)

$$\mu_\alpha(S) = \frac{1}{s} \inf_{A\in\mathcal{A}} l_A(\alpha). \tag{113}$$

The second lemma now guarantees that if an α exists for which this is positive, then there is a unique α, up to scaling, where the maximum of $\mu_\alpha(S)/|\alpha|$ ia attained.

Now suppose we move to another appartment, namely change the decomposition. The family $\mathcal{A}$ changes, but the linear functionals are the same, so there are only finitely many of them and there exists an α where the maximum is attained over all the filtrations of (V, W) of (separate) slopes 0. If this maximum is attained at two different filtrations α and β, then by Bruhat's lemma there exists a decomposition compatible with both α and β, but we have already shown that in one appartment, the α where the maximum is attained, is unique up to scaling, so if we normalize them to have norm 1, $\alpha = \beta$.

Step II. Let α be the unique filtration on (V, W) of separate slopes 0 and norm 1, whose existence was shown in step I. Then for any other filtration β of seperate slopes 0 we can choose an appartment containing both α and β, and clearly α maximizes $\mu_\gamma(S)/|\gamma|$ over the γ of separate slopes 0 in that appartment, so by the inequality of the second lemma we have

$$\begin{aligned}\frac{\mu_\beta(S)}{\mu_\alpha(S)} &\leq \langle\alpha, \beta\rangle \\ &= \sum_{k,l} kl \dim gr_\beta^k gr_\alpha^l V + \sum_{k.l} kl \dim gr_\beta^k gr_\alpha^l W \\ &= -\int l.d(\deg_\beta V_\alpha^l) - \int l.d(\deg_\beta W_\alpha^l) \\ &= \int \deg_\beta V_\alpha^l.dl + \int \deg_\beta W_\alpha^l.dl. \end{aligned} \tag{114}$$

In the last step we used integration by parts and the fact that β is of separate slopes 0.

Step III. Assume now that V and W, as well as S, are defined over k. If there is no filtration α of (V, W) of separate slopes 0 for which $\mu_\alpha(S) > 0$, there is nothing to prove (this is for example, the case for a *generic* S, because then no matter how we decompose V and W, the collection $\mathcal{A}$ will contain all the subsets). If there is such a filtration let α be the one maximizing $\mu_\alpha(S)/|\alpha|$.

Let K be an extension of k. By uniquenes of α, it remains the extension maximizing $\mu_\alpha(S)/|\alpha|$ even after we extend scalars to K. (The Galois conjugates of α would also have the same $\mu_\alpha(S)/|\alpha|$, so by uniqueness must be the same.) The V_α^l and W_α^l are therefore all k-rational. Suppose β is K/k-semistable of separate slopes 0. Then $\deg_\beta V_\alpha^l$ and $\deg_\beta W_\alpha^l$ are non-positive and we get $\mu_\beta(S) \leq 0$. Since this holds for every k-rational subspace S of $V \bigotimes W$, $\beta_{V\otimes W}$ is also K/k-semistable.

5. Consequences

5.1. The category of semistable filtered vector spaces. Let K/k be a fixed extension. Consider the additive category of couples (V, α) where V is a k-vector space, and α is a semistable filtration on V_K of total slope (or degree) 0. Morphisms are k-linear homomorphisms which, after base change to K, respect the filtration.

PROPOSITION 5.1. *This category is* abelian.

PROOF. We do not check all the axioms, but show that kernels and cokernels exist. Let (V, α) be an object of the category, and let (W, β) map to (V, α) so that the map on vector spaces is injective. We claim that $\beta = \alpha|W$. In fact, choose a decomposition of $W = \bigoplus L_i$ into lines compatible with both β and $\alpha|W$, and let q_i and p_i be such that

$$W_\alpha^p = \sum_{p_i \geq p} L_i, \;\; W_\beta^q = \sum_{q_i \geq q} L_i. \tag{115}$$

Since the identity map is a morphism of (W, β) to $(W, \alpha|W)$, $q_i \leq p_i$. By semistability, and the fact that W is k-rational, $\sum p_i = \deg_\alpha(W) = \mu_\alpha(W) \dim W \leq \mu_\alpha(V) \dim W = 0$. However, $\sum q_i = \mu_\beta(W) \dim W = 0$. It follows that $p_i = q_i$.

Similarly, if (V, α) maps to (U, γ) and the map of vector spaces is surjective, then γ is the filtration induced from α on U. It follows that if

$$0 \to (W, \beta) \to (V, \alpha) \to (U, \gamma) \to 0 \tag{116}$$

is a sequence which is exact on the level of vector spaces, it is also exact in the category, and this essentially proves that it is abelian. □

The Faltings-Totaro theorem implies that the category is in fact a *tensor category.*

5.2. A variant. One can consider also several filtrations α_i on the same vector space V_K. They are said to be *jointly semistable* if for any k-rational subspace W we have

$$\sum_i \mu_{\alpha_i}(W) \leq \sum \mu_{\alpha_i}(V). \tag{117}$$

We have the following generalization of Faltings-Totaro.

THEOREM 5.2. *Let $\alpha_{i,V}$ and $\alpha_{i,W}$ be filtrations on V and W (over K) which are jointly (K/k-) semistable. Then $\alpha_{i,V\otimes W} = \alpha_{i,V} \otimes \alpha_{i,W}$ are jointly semistable on $V \otimes W$.*

PROOF. Since $\mu_{\beta\otimes\gamma} = \mu_\beta + \mu_\gamma$ we may assume, normalizing all the filtrations with appropriate shifts, that they are of total slope 0. Let S be a k-rational subspace of $V \bigotimes W$. If for any pair of filtrations on V and W of slopes 0, the tensor product filtration has slope ≤ 0 on S (e.g. if S is *generic*), then there is nothing to prove. Otherwise let α be the filtration maximizing $\mu_\alpha(S)/|\alpha|$ as in step I of the proof above. Step II shows that for every i

$$\mu_{\alpha_i}(S) \leq \mu_\alpha(S) \left\{ \int \deg_{\alpha_i} V_\alpha^l . dl + \int \deg_{\alpha_i} W_\alpha^l . dl \right\}. \tag{118}$$

Summing over i we get

$$\sum \mu_{\alpha_i}(S) \leq \mu_\alpha(S) \left\{ \int \sum \deg_{\alpha_i} V_\alpha^l . dl + \int \sum \deg_{\alpha_i} W_\alpha^l . dl \right\} . \tag{119}$$

However, the V_α^l and W_α^l ar k-rational, so the sums under the integral sign are non-positive, by our assumptions of semistability, and of total slope 0. It follows that the left hand side is also ≤ 0. □

5.3. Extra structure. In applications, the vector spaces V will often have extra structure, compatible with tensor products. In such a case, it is reasonable to require the inequality defining the semistability condition only for k-subspaces with the extra structure. In fact, from this point of view "having a k-structure" is an example of such extra structure, and in the presence of a different extra structure, allowing one to distinguish a special collection of subspaces (while the filtration will be by arbitrary subspaces), one may or may not want to keep the extension K/k in the picture.

To prove the analoge of the Faltings-Totaro theorem in the presence of an extra structure, one will have to show that the subspaces in Kempf's filtration possess the extra structure (just as we had to show that they are k-rational here). This will generally follow from the uniqueness of the filtration. We shall see below how all this works for filtered F-isocrystals.

CHAPTER 3

Borel-Weil-Bott theory

1. Generalities on semisimple groups and their roots

1.1. Roots and root lattices. Let G be a semi-simple group over an algebraically closed field k of characteristic 0. Let T be a *maximal torus.* The center Z of G is finite and contained in T (because $C_G(T) = T$).

Let $\Phi = \Phi(G, T)$ the *root system* corresponding to T. Recall that $\Phi \subset X(T)$ is the set of characters of T appearing in $(Ad, \mathfrak{g})$ where $\mathfrak{g} = Lie(G)$ is the Lie algebra of G and $g \mapsto Ad(g)$ is the adjoint representation of G.

If $G \to G'$ is an isogeny with a finite kernel then this kernel lies in Z, hence in T, and T', the image of T in G', is a maximal torus in G'. Since $X(T')$ is a sublattice of finite index in $X(T)$, we shall identify $X(T)_{\mathbb{R}}$ with $X(T')_{\mathbb{R}}$. Under this identification, $\Phi(G, T) = \Phi(G', T')$ since both groups have the same Lie algebra $\mathfrak{g}$ and the adjoint representation of G factors through the adjoint representation of G'. In particular, if $\bar{G} = G/Z$ where Z is the center of G, and $\bar{T} = T/Z$ then $\Phi \subset X(\bar{T})$.

The *root lattice* Λ_r is the sublattice of $X(T)_{\mathbb{R}}$ spanned by Φ.

1.2. Positive and simple roots. Let B be a *Borel subgroup* containing T. Let $\Phi = \Phi^+ \cup \Phi^-$ be the decomposition into *positive* and *negative roots*, so that

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \tag{120}$$

where $\mathfrak{b} = Lie(B)$, $\mathfrak{h} = Lie(T)$ and $\mathfrak{g}_\alpha$ is the one-dimensional space on which G acts via α in the adjoint representation.

Let Δ be the *simple roots* with respect to this decomposition. Then $l = \#(\Delta) = \dim T$ is the *rank* of G.

The simple roots Δ form a basis over $\mathbb{Z}$ for Λ_r. The positive cone is $\Lambda_r^+ = \left\{\sum_{\alpha \in \Delta} c_\alpha \alpha | c_\alpha \in \mathbb{N}\right\}$.

1.3. The Weyl group. The *Weyl group* $W = N_G(T)/T$ acts on $X(T)$ and permutes Φ. If $\sigma \in W$ is represented by $n \in N_G(T)$ then $\sigma(\alpha)(t) = \alpha(n^{-1}tn)$.

Let $(.,.)$ be a W-invariant real inner product on $X(T)_{\mathbb{R}}$. [If G is *simple* the representation of W on $X(T)_{\mathbb{R}}$ is irreducible so $(.,.)$ is unique up to a scalar.] Write

$$\langle x, \alpha \rangle = \frac{2(x, \alpha)}{(\alpha, \alpha)}. \tag{121}$$

Write $\sigma_\alpha(x) = x - \langle x, \alpha \rangle \alpha$ for the *reflection in the root vector* α. The σ_α preserve Φ, and $\{\sigma_\alpha | \alpha \in \Delta\}$ generate W. For a general element $\sigma \in W$ we let $l(\sigma)$, the *length* of σ, be the minimal length of σ as a word in the σ_α, for $\alpha \in \Delta$. We have $l(\sigma) = \#(\Phi^+ \cap \sigma(\Phi^-))$.

The Weyl group has a *unique element of maximal length* w (for the given decomposition of Φ). It is characterized by the fact that it takes Δ to $-\Delta$, and it follows from this fact that $w^2 = 1$. In general it is not a reflection. For example, in the case of SL_{n+1}, $W = \mathfrak{S}_n$, w is the permutation $i \mapsto n+1-i$ and its action on $X(T)_{\mathbb{R}}$ is to reverse the order of the coordinates. Its fixed points form a linear subspace of dimension $n/2$ or $(n+1)/2$ depending on the parity of n, and not $n-1$.

1.4. Weyl chambers. The hyperplane perpendicular to the root α will be denoted Π_α, and $\Pi_\alpha^\pm$ will denote the set of vectors γ in $X(T)_{\mathbb{R}}$ such that (γ, α) is positive or negative. If Δ is a basis of simple roots we denote by $\mathfrak{C}(\Delta) = \bigcap_{\alpha\in\Delta} \Pi_\alpha^+$. These are the vectors that form an acute angle with all the vectors in Δ. The Weyl group acts *simply transitively* on the bases Δ, or what is the same on decompositions $\Phi = \Phi^+ \cup \Phi^-$, or on the *chambers* $\mathfrak{C}(\Delta)$, and $\sigma\mathfrak{C}(\Delta) = \mathfrak{C}(\sigma\Delta)$. The closures of the $\mathfrak{C}(\Delta)$ cover $X(T)_{\mathbb{R}}$. If we fix one Δ we can label the Weyl chambers by the elements of W.

1.5. The Bruhat decomposition. This is the decomposition into disjoint double cosets

$$G = \bigcup_{\sigma\in W} B\tilde{\sigma}B \tag{122}$$

where $\tilde{\sigma}$ is a representative of σ. Let $U(\sigma)$ be the subgroup of U whose tangent space at the origin is the sum of $\mathfrak{g}_\alpha$ for $\alpha \in \Phi^+ \cap \sigma(\Phi^-)$. Note that $\alpha+\beta$ also satisfies this condition whenever α and β do, if it is a root. From $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset [\mathfrak{g}_{\alpha+\beta}]$ we see that we have indeed defined a Lie subalgebra, so $U(\sigma)$ is well-defined. Note that $\dim U(\sigma) = l(\sigma)$. Then $B\tilde{\sigma}B = U(\sigma)\tilde{\sigma}B \simeq U(\sigma) \times B$ as a variety, it is affine and its image in G/B is therefore an affine set isomorphic to $U(\sigma)$. They are called the *Bruhat cells.* Precisely one of them is open and dense, corresponding to the unique $w \in B$ of maximal length. Its dimension is $\dim G/B = \#(\Phi^+)$. It is called the *big cell.*

2. Weights and representations

2.1. Abstract weights. The *weight lattice* $\Lambda_w \subset X(T)_{\mathbb{R}}$ is the lattice of all λ for which $\langle\lambda, \alpha\rangle \in \mathbb{Z}$ for every $\alpha \in \Phi$. Clearly $\Lambda_r \subset \Lambda_w$ and the quotient Λ_w/Λ_r is finite. The elements of Λ_w are called *abstract weights* (or simply *weights*). If $\alpha \in \Phi$, then the α-*string* through $\lambda \in \Lambda_w$ is the set $\{\lambda + k\alpha | k \in \mathbb{Z}\}$.

The *fundamental weights* are the λ_α, $\alpha \in \Delta$, where

$$\langle\lambda_\alpha, \beta\rangle = \delta_{\alpha b}. \tag{123}$$

They form a basis for Λ_w. A weight is *dominant* if it is a linear combination of fundamental weights with non-negative coefficients.

2.2. Weights of rational representations are abstract weights. Let $\rho : G \to GL(V)$ be a representation of G in a finite dimensional vector space over k.

- The *weights* of ρ (the characters of T on V) are in Λ_w, and they are permuted by W.

Proof: Denote by V_λ the weight space of a weight λ. For $\alpha \in \Phi$, letting U_α be the unipotent group isomorphic to $\mathbb{G}_a$ with Lie group $\mathfrak{g}_\alpha$, $\rho(U_\alpha)$ sends the weight space V_λ to $\sum_{k\geq 0} V_{\lambda+k\alpha}$. Similarly $U_{-\alpha}$ sends V_λ to $\sum_{k\leq 0} V_{\lambda+k\alpha}$. Letting Z_α be the subgroup generated by T, U_α and $U_{-\alpha}$ we see that $\sum_k V_{\lambda+k\alpha}$ is stable under

Z_α. In particular, $N_G(T) \cap Z_\alpha/T = \{1, \sigma_\alpha\} \subset W$ permutes the α-string through λ, hence $\sigma_\alpha(\lambda) = \lambda + k\alpha$ for some integer k, and this means that $\lambda \in \Lambda_w$. Finally, if $n \in N_G(T)$ represents $\sigma \in W$, then $nV_\lambda = V_{\sigma(\lambda)}$.

- The weights of the contragredient representation $\rho^\vee$ are the negatives of the weights of ρ.

2.3. Characters and weights. As a corollary,

$$\Lambda_r \subset X(T) \subset \Lambda_w. \tag{124}$$

In fact, we only have to observe that every character of T is a weight of some representation of G. [If this is not clear from general principles, it will follow from the Borel-Weil theory.]

The group G is of *adjoint type* if $X(T) = \Lambda_r$ and is *simply connected* if $X(T) = \Lambda_w$. If we start with any G we can get to the adjoint type $\bar{G}$ upon dividing by the center and to the simply connected one $\tilde{G}$ by taking the universal cover.

2.4. Highest weights of representations. Assume that ρ is an *irreducible* representation.

- There exists a unique *highest weight* λ_ρ in the sense that every other weight λ of ρ is $\lambda_\rho - \mu$ for some $\mu \in \Lambda_r^+$.
- The highest weight is a *dominant* weight, and it determines ρ uniquely. Its eigenspace is one-dimensional. Every non-zero vector in it is called a highest weight vector.

Proof: By Lie-Kolchin there exists a vector v_ρ spanning a line fixed by B. Let λ_ρ be the character of T defined by v_ρ. For ever $\alpha \in \Phi^+$, the group $U_{-\alpha}$ sends V_λ to $\sum_{k \leq 0} V_{\lambda + k\alpha}$, hence the subspace spanned by $U^- v_\rho$ contains only weights $\lambda \leq \lambda_\rho$ (here $\lambda \leq \mu$ if $\mu - \lambda \in \Lambda_r^+$), and the only vector of weight λ_ρ is v_ρ itself, up to a multiple. But this is also the space spanned by $U^- B v_\rho$ and since $U^- B$ is Zariski dense in G, from irreducibility, this is V.

Let λ be any weight, and let $\sigma \in W$ be such that $(\lambda, \sigma^{-1}(\alpha)) \geq 0$ for all $\alpha \in \Delta$. Then $c_\alpha = \langle \sigma(\lambda), \alpha \rangle \geq 0$ for all $\alpha \in \Delta$ so $\sigma(\lambda) = \sum_{\alpha \in \Delta} c_\alpha \lambda_\alpha$ is dominant. Thus *every weight is conjugate under W to a dominant weight*, and if λ is dominant, every other $\sigma(\lambda) \leq \lambda$. Since W permutes the weights of ρ, this shows that λ_ρ is dominant. We have already seen that the only vectors of weight λ_ρ are multiples of v_ρ. The proof of the fact that λ_ρ determines ρ up to isomorphism is standard, see Humphreys, Theorem 31.3(c).

The fact that every dominant weight $\lambda \in X(T)$ indeed corresponds to a representation of G is deeper. We shall get it from the Borel-Weil-Bott theorem.

- The representation ρ also has a unique lowest weight, which is $w(\lambda)$ for w the element of maximal length in W, and the highest weight of $\rho^\vee$ is $-w(\lambda)$.

3. G-line bundles and representations

3.1. G-line bundles on G/B. Let G, T and B be as before, fix $\Phi = \Phi^+ \cup \Phi^-$ and Δ, and let Λ_r and Λ_w be the root and weight lattices in $X(T)_\mathbb{R}$.

If $\lambda \in X(T)$ we denote by L_λ the line bundles over the full flag variety G/B obtained as the quotient of $G \times \mathbb{A}^1$ under the equivalence relation $(g, a) \sim (gb, \lambda(b)^{-1}a)$ for $b \in B$. Note that G acts on it, covering the action on the base, so (equipped with the G-action) it becomes an element of $Pic^G(G/B)$. The group B stabilizes the point $[B] = x_0 \in G/B$ and acts on $L_\lambda|_{x_0}$ via the character λ. This construction sets up an isomorphism between $X(T)$ and $Pic^G(G/B)$ as is easily seen pulling back any G-line-bundle on G/B to G and trivializing it.

3.2. The G-line bundle associated to an irreducible representation. Let (ρ, V) be an irreducible k-rational representation, let $v_0 = v_\rho$ be a highest weight vector (spanning a line fixed by B), and $\lambda = \lambda_\rho \in X(T)$ the corresponding highest weight. The stabilizer of the line spanned by v_0 in $\mathbb{P}(V)$ is a parabolic group denoted by $P \supset B$, and $G/P \hookrightarrow \mathbb{P}(V)$ as a complete subvariety. In fact it is the unique closed orbit for the action of G on $\mathbb{P}(V)$, since by the Borel fixed point theorem any such closed orbit must contain a fixed point for the action of B, but $[v_0]$ is the unique fixed point of B.

Let L_ρ be the pull-back of the tautological line bundle $\mathcal{O}(-1)$ from $\mathbb{P}(V)$ to $G/B \twoheadrightarrow G/P \hookrightarrow \mathbb{P}(V)$. Note that $L_\rho^\vee$ is ample. We claim that $L_\rho \simeq L_\lambda$ as a G-bundle. Indeed

$$G \times_{G/B} L_\rho \simeq G \times \mathbb{A}^1 \tag{125}$$

under $(g, v) \mapsto (g, (v : gv_0))$. Here, if we denote $x_0 = [v_0] \in \mathbb{P}(V)$ (the unique fixed point under B), $L_\rho|_{gx_0} = [gv_0]$ and for every v in this line we let $(v : gv_0) = a$ if $v = agv_0$. Note that $(gb, v) \mapsto (gb, \lambda(b)^{-1}(v : gv_0))$, so to get L_ρ back from the pull back we have to divide $G \times \mathbb{A}^1$ precisely by the equivalence relation defining L_λ.

PROPOSITION 3.1. *Notation as above, $L_\lambda = L_\rho$ and its dual is an ample G-line bundle. Moreover, we have a natural isomorphism of representations*

$$V^\vee \simeq H^0(G/B, L_\rho^\vee). \tag{126}$$

PROOF. We only have to establish the isomorphism. We identify $L_\rho^\vee|_{gx_0}$ with $V^\vee/[gv_0]^\perp$. We send $\alpha \in V^\vee$ to the section s_α defined by $s_\alpha(gx_0) = \alpha mod[gv_0]^\perp$. It is well-defined because if we replace g by gb we get the same thing. For $\gamma \in G$, $\gamma s_\alpha(gx_0) = \gamma(s_\alpha(\gamma^{-1}gx_0)) = \gamma\alpha mod[gv_0]^\perp = s_{\gamma\alpha}(gx_0)$ where γ acts on $V^\vee$ via $\rho^\vee$. Since $V^\vee$ is irreducible this is an embedding of $V^\vee$. The fact that it is onto follows from the irreducibility of $H^0(G/B, L_\rho^\vee)$ (see below). □

3.3. The representation associated to a dominant weight. Let $\lambda \in X(T)$ be a dominant weight.

LEMMA 3.2. *The line bundle $L_\lambda^\vee$ is ample, and $H^0(G/B, L_\lambda^\vee)$ is nonzero.*

THEOREM 3.3. *The representation $H^0(G/B, L_\lambda^\vee)$ is irreducible. It is the* dual *of the representation with highest weight λ.*

PROOF. Identify sections of $L_\lambda^\vee$ with maps $s : G \to \mathbb{A}^1$ which satisfy $s(gb) = \lambda(b)s(g)$ for $b \in B$. The action of $\gamma \in G$ is by left translation: $\gamma s(g) = s(\gamma^{-1}g)$. From the Bruhat decomposition, such an s is determined by its values on the *big cell* $U\tilde{w}B$ where $w \in W$ is the element of maximal length and $\tilde{w} \in N_G(T)$ represents it. This is because the big cell is open and dense in G. Now suppose s_0 is a highest weight vector, namely it spans a line which is invariant under B. Then s_0 is invariant under U and therefore is determined uniquely by $s_0(\tilde{w})$. It follows that s_0 is unique up to a multiple, if it exists. In fact, it is easy to define $s_0(u\tilde{w}b) = \lambda(b)$ on the big cell, but then to extend s_0 to G we need to use the fact that λ is dominant. This is exactly the contents of the lemma.

In any case, suppose $H^0(G/B, L_\lambda^\vee)$ is nonzero, so a nonzero s_0 exists and is unique up to a multiple. If $b = tu \in B$ then $bs_0(\tilde{w}) = s_0(b^{-1}\tilde{w}) = s_0(\tilde{w}\tilde{w}^{-1}t^{-1}\tilde{w}) = \lambda(\tilde{w}^{-1}t^{-1}\tilde{w})s_0(\tilde{w}) = -w(\lambda)(b)s_0(w)$. It follows that $H^0(G/B, L_\lambda^\vee)$ is the irreducible representation associated to $-w(\lambda)$. This shows that it is the dual of the representation with highest weight λ. $\square$

CHAPTER 4

Filtered F-isocrystals

1. Weakly admissible F-isocrystals

1.1. Filtered F-isocrystals. Let k be a perfect field, K_0 the field of fractions of $W(k)$, and K a finite totally ramified extension of K_0. A *filtered F-isocrystal* (V, Φ, λ) over K is an F-isocrystal (V, Φ) over k with a decreasing $\mathbb{Z}$-filtration λ on V_K.

The filtration is not assumed to have any relation to the structure as an isocrystal. In particular, it is *not* a filtration by sub-isocrystals (which anyhow doesn't make sense if K is not K_0).

Recall that V has a (Frobenius-) *slope decomposition*

$$V = \bigoplus_{l \in \mathbb{Q}} V_l \tag{127}$$

where V_l, if $l = s/r$ in reduced terms and $r > 0$, is the sum of all the $W(k)$-lattices M satisfying

$$\Phi^r M = p^s M. \tag{128}$$

The *slope filtration* is the $\mathbb{Q}$-filtration

$$V^x_{\lambda_0} = \bigoplus_{x \leq -l} V_l. \tag{129}$$

If $x = s/r$ as before, $V^x_{\lambda_0}$ is the sum of all the $W(k)$-submodules M satisfying $p^s \Phi^r M \supset M$.

The *total slope* of (V, Φ, λ) is defined

$$\mu(V, \Phi, \lambda) = \mu_{\lambda_0}(V) + \mu_\lambda(V). \tag{130}$$

DEFINITION 1.1. *The filtered F-isocrystal (V, Φ, λ) is called* semistable *if for any sub-isocrystal S (endowed with the induced filtration)*

$$\mu(S, \Phi, \lambda) \leq \mu(V, \Phi, \lambda). \tag{131}$$

It is called weakly admissible *if in addition $\mu(V, \Phi, \lambda) = 0$.*

1.2. Basic results.

- The Harder-Narasimhan filtration of (V, Φ, λ) is by sub-isocrystals.
- The tensor product of two semistable (resp. weakly admissible) filtered F-isocrystals is semistable (resp. weakly admissible)
- The category of weakly admissible filtered F-isocrystals is abelian.