

ELLIPTIC (p, q) -DIFFERENCE MODULES

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ABSTRACT. Let p and q be multiplicatively independent natural numbers, and K the field $\mathbb{C}(x^{1/s} | s = 1, 2, 3, \dots)$. Let p and q act on K as the Mahler operators $x \mapsto x^p$ and $x \mapsto x^q$. In a recent article [Sch-Si] Schäfke and Singer showed that a finite dimensional vector space over K , carrying commuting structures of a p -Mahler module and a q -Mahler module, is obtained via base change from a similar object over \mathbb{C} . As a corollary, they gave a new proof of a conjecture of Loxton and van der Poorten, which had been proved before by Adamczewski and Bell [Ad-Be]. When $K = \mathbb{C}(x)$, and p and q are complex numbers of absolute value greater than 1, acting on K via dilations $x \mapsto px$ and $x \mapsto qx$, a similar theorem has been obtained in [Bez-Bou]. Underlying these two examples is the algebraic group \mathbb{G}_m , resp. \mathbb{G}_a , K is the function field of its universal covering, and p, q act as endomorphisms.

Replacing the multiplicative or additive group by the elliptic curve \mathbb{C}/Λ , and K by the maximal unramified extension of the field of Λ -elliptic functions, we study similar objects, which we call *elliptic (p, q) -difference modules*. Here p and q act on K via isogenies. When p and q are *relatively prime*, we give a structure theorem for elliptic (p, q) -difference modules. The proof is based on a Periodicity Theorem, which we prove in somewhat greater generality. A new feature of the elliptic modules is that their classification turns out to be fibered over Atiyah's classification of vector bundles on elliptic curves [At]. Only the modules whose associated vector bundle is trivial admit a \mathbb{C} -structure as in the case of \mathbb{G}_m or \mathbb{G}_a , but all of them can be described explicitly with the aid of (logarithmic derivatives of) theta functions. We conclude with a proof of an elliptic analogue of the conjecture of Loxton and van der Poorten.

1. INTRODUCTION

1.1. Background.

1.1.1. *Difference equations and difference modules.* A *difference field* (K, σ) is a field K equipped with an automorphism $\sigma \in \text{Aut}(K)$. The fixed field C_K of σ is called its *constant field*. A (linear) *difference equation* over (K, σ) is an equation

$$(1.1) \quad \sigma^n(f) + a_1\sigma^{n-1}(f) + \dots + a_{n-1}\sigma(f) + a_n f = 0$$

where $a_i \in K$. One seeks solutions f in K or in an extension (L, σ) of (K, σ) .

A long-studied example (*of q -difference equations*) occurs when $K = \mathbb{C}(x)$ or when it is replaced by $\hat{K} = \mathbb{C}((x))$, and $\sigma f(x) = f(x/q)$ for $q \in \mathbb{C}^\times$, $|q| > 1$. We call such q -difference equations *rational* (over $\mathbb{C}(x)$) or *formal* (over $\mathbb{C}((x))$). As another example, take $K = \mathbb{C}(x^{1/s} | s = 1, 2, 3, \dots)$ or replace it by the field of Puiseux series, and let σ be the Mahler operator $\sigma(x) = x^q$ where $q > 1$ is a natural number. Such difference equations are called *Mahler equations*, because Mahler studied them

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extensively with relation to transcendence theory (see [Ad] for a survey). In both cases $C_K = \mathbb{C}$.

Behind these two examples lies the algebraic group $\mathbb{G} = \mathbb{G}_{a/\mathbb{C}}$ or $\mathbb{G}_{m/\mathbb{C}}$, respectively. Let $K_0 = \mathbb{C}(x)$ be its function field. The field K is the maximal extension of K_0 which is unramified at the points of \mathbb{G} (in the additive case, the group is simply connected, so $K = K_0$). The automorphism σ is induced by an endomorphism of the group.

The study of difference equations goes back to the beginning of the 20th century. It was considered, with relation to q -hypergeometric functions, by Jackson, Adams, Carmichael and more generally by G.D.Birkhoff [Bi]. By the standard argument used to reduce a linear differential equation of degree n to a vector-valued equation of degree 1, the classification of difference equations reduces to that of *difference modules*, historically introduced much later. We focus from now on on the latter notion.

Definition 1. A *difference module* (M, Φ) over (K, σ) is a finite dimensional K -vector space M , equipped with a σ -linear bijective endomorphism $\Phi : M \rightarrow M$.

The endomorphism Φ satisfies

$$\Phi(av) = \sigma(a)\Phi(v) \quad (v \in M, a \in K).$$

If we fix a basis (e_1, \dots, e_r) of M and let $A^{-1} = (a_{ij})$ be the matrix of Φ in this basis, so that

$$\Phi(e_j) = \sum_{i=1}^r a_{ij}e_i,$$

then we may identify M with K^r , where

$$\Phi(v) = A^{-1}\sigma(v) \quad (v \in K^r).$$

The fixed vectors of Φ , corresponding to the solutions of (1.1), become the solutions to

$$\sigma(v) = Av.$$

A different basis (e'_1, \dots, e'_r) , related to the first by the transition matrix $C = (c_{ij})$

$$e'_j = \sum_{i=1}^r c_{ij}e_i,$$

results in a matrix A' related to A by the *gauge transformation*

$$A' = \sigma(C)^{-1}AC.$$

The classification of difference modules is therefore equivalent to the classification of $A \in GL_r(K)$, up to gauge transformations. If we let $\Gamma = \langle \sigma \rangle \subset Aut(K)$ be the cyclic subgroup generated by σ , this is the same as the determination of the non-abelian cohomology

$$H^1(\Gamma, GL_r(K)).$$

For a comprehensive survey of difference equations and their Galois theory, see [vdP-Si]. If F is a perfect field of characteristic p , $W(F)$ is its ring of Witt vectors, $K = W(F)[1/p]$ and σ is the Frobenius of K (lifting $x \mapsto x^p$), then a difference module over (K, σ) is an *isocrystal*, a notion central to p -adic Hodge theory.

Generalizations are obtained by either of the following two procedures.

- Replace $\langle \sigma \rangle$ by a group $\Gamma \subset \text{Aut}(K)$.
- Replace GL_r by a linear algebraic group G defined over C_K .

The resulting objects might be called “ Γ -difference modules with G -structure”, and are again classified by the non-abelian cohomology

$$H^1(\Gamma, G(K)).$$

1.1.2. *Rational (p, q) -difference modules.* Let $K = \mathbb{C}(x)$, let p and q be complex numbers, $|p| > 1$, $|q| > 1$, and assume that p and q are *multiplicatively independent*, i.e. $p^n q^m = 1$ if and only if $n = m = 0$. We let

$$\sigma f(x) = f(x/p), \quad \tau f(z) = f(x/q).$$

The subgroup $\Gamma = \langle \sigma, \tau \rangle \subset \text{Aut}(K)$ is then free abelian of rank 2. We call a Γ -difference module also a (p, q) -difference module. It is a finite dimensional K -vector space M , equipped with *commuting bijective* endomorphisms Φ_σ, Φ_τ satisfying

$$\Phi_\sigma(av) = \sigma(a)\Phi_\sigma(v), \quad \Phi_\tau(av) = \tau(a)\Phi_\tau(v).$$

Having fixed a basis, M may be replaced by K^r , the endomorphisms Φ_σ and Φ_τ by matrices $A^{-1}, B^{-1} \in GL_r(K)$ as above, and the commutation relation $\Phi_\sigma \circ \Phi_\tau = \Phi_\tau \circ \Phi_\sigma$ by the *consistency condition*

$$(1.2) \quad B(x/p)A(x) = A(x/q)B(x).$$

The consistent pair (A, B) is well-defined up to the gauge transformation

$$(1.3) \quad (C(x/p)^{-1}A(x)C(x), C(x/q)^{-1}B(x)C(x))$$

where $C \in GL_r(K)$.

The multiplicative independence of p and q imposes a remarkable restriction on M .

Theorem 2. ([Bez-Bou],[Sch-Si] *Case 2Q*) *Notation as above, the module M has a basis with respect to which the matrices A and B are in $GL_r(\mathbb{C})$. Equivalently, any two consistent matrices $A, B \in GL_r(K)$ may be reduced by a gauge transformation to a pair (A_0, B_0) of commuting scalar matrices (matrices with entries in \mathbb{C}). Still equivalently, the natural map*

$$H^1(\Gamma, GL_r(\mathbb{C})) \rightarrow H^1(\Gamma, GL_r(K))$$

is surjective.

1.2. Elliptic q - and (p, q) -difference modules and the main result.

1.2.1. *Our set-up.* The goal of the present paper is to study an elliptic analogue¹ of the rational (p, q) -difference modules. Let $\Lambda_0 \subset \mathbb{C}$ be a lattice and K_0 the field of Λ_0 -elliptic functions. We recall that

$$K_0 = \mathbb{C}(\wp(z, \Lambda_0), \wp'(z, \Lambda_0))$$

where $\wp(z, \Lambda_0)$ is the Weierstrass \wp -function of the lattice Λ_0 . Let

$$K = K_0^{nr}$$

be the maximal unramified extension of K_0 . This is the union of the fields K_Λ of Λ -elliptic functions for all sublattices $\Lambda \subset \Lambda_0$. We emphasize that K depends only

¹As explained in [Sa], for example, the study of fuchsian (rational) q -difference equations leads, by a method of Birkhoff, to the consideration of elliptic functions on the elliptic curve $\mathbb{C}^\times / \langle q \rangle$. As far as we can see this is unrelated, and should not be confused, with our set-up.

on the commensurability class of Λ_0 . Replacing Λ_0 by any commensurable lattice, e.g. by a sublattice, leads to the same field K .

Let p, q be multiplicatively independent positive integers. If Λ_0 has complex multiplication we can take any two multiplicatively independent endomorphisms of the elliptic curve $X_0 = \mathbb{C}/\Lambda_0$, but to simplify the presentation we do not treat this case. Then

$$\sigma f(z) = f(z/p), \quad \tau f(z) = f(z/q)$$

are commuting *automorphisms* of the field K , because $K_\Lambda \subset \sigma(K_\Lambda) \subset K_{p\Lambda}$ for every lattice $\Lambda \subset \Lambda_0$ and similarly with τ . The group

$$\Gamma = \langle \sigma, \tau \rangle \subset \text{Aut}(K).$$

is free abelian of rank 2.

An *elliptic (p, q) -module* is defined, exactly as in the rational case, as a finite dimensional K -vector space M , equipped with commuting σ -linear (resp. τ -linear) bijective endomorphisms Φ_σ (resp. Φ_τ). Such a module M is determined, up to isomorphism, by a pair (A, B) of matrices from $GL_r(K)$ satisfying (1.2), up to the gauge transformation (1.3). Thus A and B will be matrices of Λ -elliptic functions for $\Lambda \subset \Lambda_0$ small enough. Explicitly, to (A, B) we associate $M = K^r$ with the endomorphisms

$$\Phi_\sigma v = A^{-1}\sigma(v), \quad \Phi_\tau v = B^{-1}\tau(v).$$

The isomorphism classes of elliptic (p, q) -modules of rank r are classified therefore by

$$H^1(\Gamma, G(K))$$

where, from now on, to simplify the notation, we put $G = GL_r$.

1.2.2. *An example.* In [dS1] we proved the analogue of Theorem 2 when $r = 1$. Namely, we showed that the map

$$H^1(\Gamma, \mathbb{C}^\times) \rightarrow H^1(\Gamma, K^\times)$$

is bijective. Thus every rank-1 module is isomorphic to $M_1(a, b)$ for unique $a, b \in \mathbb{C}^\times$, where the standard module $M_1(a, b)$ is the vector space K with $\Phi_\sigma(v) = a^{-1}\sigma(v)$ and $\Phi_\tau(v) = b^{-1}\tau(v)$.

This is false in higher rank. For $r \geq 2$, the map from $H^1(\Gamma, G(\mathbb{C}))$ to $H^1(\Gamma, G(K))$ is injective, but not surjective. At this point we want to give an example of a rank-2 (p, q) -difference module, which does not arise from a (p, q) -difference module over \mathbb{C} by extension of scalars. This example will turn out to be typical.

Fix a lattice $\Lambda \subset \Lambda_0$ and let

$$\sigma(z, \Lambda) = z \prod_{0 \neq \omega \in \Lambda} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}$$

be the Weierstrass sigma function associated to Λ . Its logarithmic derivative

$$(1.4) \quad \zeta(z, \Lambda) = \frac{\sigma'(z, \Lambda)}{\sigma(z, \Lambda)}$$

is known as the Weierstrass zeta-function. It is holomorphic outside Λ , has a simple pole with residue 1 at every $\omega \in \Lambda$, and satisfies

$$\zeta(z + \omega, \Lambda) = \zeta(z, \Lambda) + \eta(\omega, \Lambda)$$

for some homomorphism $\eta(\cdot, \Lambda) : \Lambda \rightarrow \mathbb{C}$, named after Legendre. Its derivative $\zeta'(z, \Lambda) = -\wp(z, \Lambda)$.

The functions

$$\begin{cases} g_p(z, \Lambda) = p\zeta(qz, \Lambda) - \zeta(pqz, \Lambda) \\ g_q(z, \Lambda) = q\zeta(pz, \Lambda) - \zeta(pqz, \Lambda) \end{cases}$$

are consequently Λ -elliptic. Moreover, g_p is even $q^{-1}\Lambda$ -elliptic, has simple poles only, and its residual divisor $Res_\Lambda(g_p)$ on the curve $X_\Lambda = \mathbb{C}/\Lambda$ satisfies

$$pqRes_\Lambda(g_p) = p^2 \sum_{\xi \in q^{-1}\Lambda/\Lambda} [\xi] - \sum_{\xi \in p^{-1}q^{-1}\Lambda/\Lambda} [\xi].$$

An analogous formula holds for g_q . The relation

$$g_p(z, \Lambda) - qg_p(z/q, \Lambda) = g_q(z, \Lambda) - pg_q(z/p, \Lambda)$$

implies that if we define

$$A(z) = \begin{pmatrix} 1 & g_p(z, \Lambda) \\ 0 & p \end{pmatrix}, \quad B(z) = \begin{pmatrix} 1 & g_q(z, \Lambda) \\ 0 & q \end{pmatrix},$$

the consistency equation (1.2) is satisfied. The *standard special module* M_2^{sp} will have K^2 as an underlying vector space,

$$\Phi_\sigma v = A^{-1}\sigma(v), \quad \Phi_\tau v = B^{-1}\tau(v).$$

Up to isomorphism, this module does not depend on the lattice Λ . As we shall show, it does not arise from a scalar module by extension of scalars from \mathbb{C} to K , and up to a twist by $M_1(a, b)$, is the only such rank-2 (p, q) -difference module.

1.2.3. *Standard modules.* Let (K, σ, τ) be as above. The following construction, important for the formulation of the main theorem below, generalizes the special example above. It will be studied in more detail in section 5.4.

Let $N_r = (n_{ij})$ be the nilpotent $r \times r$ matrix with $n_{ij} = 1$ if $j = i + 1$ and 0 elsewhere. Let

$$U_r(z) = \exp(\zeta(pqz, \Lambda)N_r),$$

and let $T_r^{sp} = \text{diag}[1, p, \dots, p^{r-1}]$, $S_r^{sp} = \text{diag}[1, q, \dots, q^{r-1}]$. Then

$$A_r^{sp}(z) = U_r(z/p)T_r^{sp}U_r(z)^{-1}, \quad B_r^{sp}(z) = U_r(z/q)S_r^{sp}U_r(z)^{-1}$$

lie in $G(K_\Lambda)$ and satisfy the consistency equation. In fact, $A_r^{sp} = (a_{ij})$ is upper-triangular and for $i \leq j$

$$a_{ij} = p^{i-1} \frac{g_p^{j-i}}{(j-i)!}.$$

A similar equation holds for B_r^{sp} . We call the elliptic (p, q) -difference module associated with the pair (A_r^{sp}, B_r^{sp}) the *standard special module* of rank r , and denote it by M_r^{sp} . For $a, b \in \mathbb{C}^\times$ put

$$M_r^{sp}(a, b) = M_r^{sp} \otimes M_1(a, b).$$

1.2.4. *The Main Theorem.* For our main theorem to hold we have to assume, as we shall do from now on, that p and q are relatively prime. We do not know if the weaker assumption of multiplicative independence suffices.

Let M be an elliptic (p, q) -difference module over K . In section 5.3 we explain how to associate with M a vector bundle \mathcal{E} on the elliptic curve $X_\Lambda = \mathbb{C}/\Lambda$ for all $\Lambda \subset \Lambda_0$ sufficiently small. These vector bundles are compatible under pull-back with respect to the maps $X_{\Lambda'} \rightarrow X_\Lambda$ if $\Lambda' \subset \Lambda$, and for all sufficiently small Λ are of the (same) form

$$\mathcal{E} \simeq \bigoplus_{i=1}^k \mathcal{F}_{r_i}$$

for unique $r_1 \leq r_2 \leq \dots \leq r_k$, $\sum r_i = r$. Here \mathcal{F}_r is the unique indecomposable vector bundle of rank r and degree 0 on X_Λ with non-zero global sections, sometimes called Atiyah's bundle of rank r (see section 3). We call (r_1, \dots, r_k) the *type* of M .

Theorem 3 (Structure Theorem). *Let $p \geq 2$ and $q \geq 2$ be relatively prime integers. Let M be an elliptic (p, q) -difference module of rank r , and let (r_1, \dots, r_k) be its type, $r_1 \leq r_2 \leq \dots \leq r_k$, $\sum_{i=1}^k r_i = r$. Let*

$$U(z) = \bigoplus_{i=1}^k U_{r_i}(z)$$

in block-diagonal form. Then, in an appropriate basis, M is represented by a consistent pair (A, B) of matrices from $G(K)$ for which

$$U(z/p)^{-1}A(z)U(z) = T, \quad U(z/q)^{-1}B(z)U(z) = S$$

are commuting scalar matrices (i.e. matrices in $G(\mathbb{C})$).

For a more precise statement, see Theorem 35.

Corollary 4. (i) *The module M admits a \mathbb{C} -structure if and only if its type is $(1, 1, \dots, 1)$.*

(ii) *If the type of M is (r) (equivalently, \mathcal{E} is indecomposable), then $M \simeq M_r^{sp}(a, b)$ for some $a, b \in \mathbb{C}^\times$.*

As an example, we work out a complete classification of the modules of rank $r \leq 3$. In higher rank, such a classification is in principle possible, but becomes unwieldy.

1.3. Contents of the paper.

1.3.1. *The Periodicity Theorem.* The proof of the main theorem rests on a Periodicity Theorem (Theorem 6 below), which is a vast generalization of the criterion proved in [dS1]. The idea of the proof is nevertheless the same, and the reader may want to get acquainted first with the special case treated there. Anticipating future generalizations, in which we replace elliptic curves by higher genus abelian varieties, or the group $G = GL_r$ by a general reductive group, this periodicity theorem is phrased, and proved, in greater generality than needed for the application. We did not see, however, any advantage in restricting its scope, as the proof would have been just the same.

1.3.2. *Vector bundles on elliptic curves.* With the Periodicity Theorem at hand, the proof of Theorem 3 can be described as follows.

Let M be an elliptic (p, q) -difference module, and (A, B) a consistent pair of matrices representing it in some basis. Let $\widehat{K} = \mathbb{C}((z))$ be the completion of K at the origin. Using well-known results, explained in the last chapter of [vdP-Si], and recalled in section 4, the pair (A, B) may be transformed into a scalar commuting pair (A_0, B_0) by a gauge transformation (1.3) with $C \in G(\widehat{K})$. An approximation argument, based on the denseness of K in \widehat{K} , together with standard estimates, show that, after replacing (A, B) by a pair which is gauge equivalent *over* K , C may be taken to be holomorphic in a neighborhood of 0. The equation

$$A_0 = C(z/p)^{-1}A(z)C(z)$$

implies that it is globally meromorphic. Had C been periodic, i.e. a matrix of elliptic functions, we would have been finished. This, unfortunately (or fortunately, depending on one's attitude), is false, as we saw in the rank 2 example above.

The key idea is to interpret the relation between C and A (or B) as suggesting something weaker, but still meaningful. Consider the sheaf $\mathcal{F} = G(\mathcal{M})/G(\mathcal{O})$, where $\mathcal{O} \subset \mathcal{M}$ are the sheaves of holomorphic and meromorphic functions on \mathbb{C} . This is the type of sheaf to which our Periodicity Theorem applies. While C itself is not necessarily periodic, \overline{C} , its image in the global sections of \mathcal{F} , turns out, as a consequence of the Periodicity Theorem, to be Λ -periodic for some lattice $\Lambda \subset \Lambda_0$. (This is slightly inaccurate, because in general we need to modify \overline{C} at 0, but this is a technical point with which we deal in due course.) The Λ -periodic sections of \mathcal{F} may be identified with $G(\mathbb{A}_\Lambda)/G(\mathbb{O}_\Lambda)$, where \mathbb{A}_Λ is the ring of adèles of the field K_Λ and \mathbb{O}_Λ is its maximal compact subring. Let $X_\Lambda = \mathbb{C}/\Lambda$ be the associated elliptic curve. The class of \overline{C} in

$$\text{Bun}_r(X_\Lambda) = G(K_\Lambda) \backslash G(\mathbb{A}_\Lambda)/G(\mathbb{O}_\Lambda)$$

depends only on the gauge-equivalence class of (A, B) , namely on the isomorphism class of M . This double coset space is well-known to classify the isomorphism types of rank- r vector bundles on X_Λ . We have thus attached to M such a vector bundle \mathcal{E}_Λ , and in fact, we did so for every Λ sufficiently small, in a way that is compatible with pull-back. It also follows from the construction that \mathcal{E}_Λ is invariant under pull-back by the isogeny p_Λ or q_Λ of multiplication by p or q . For all sufficiently small Λ , \mathcal{E}_Λ is “the same” vector bundle of rank r and degree 0, which we denote simply by \mathcal{E} .

In passing, we remark that it would be interesting to find a direct, functorial, construction of \mathcal{E} . This would give a richer, “stacky” meaning to the phrase that “the classification of elliptic (p, q) -difference modules is fibered over the classification of vector bundles” (see the abstract). So far, we work naively with matrices and double coset spaces.

Elliptic curves are among the few examples over which a complete classification of vector bundles is known, thanks to work of Atiyah from 1957 [At]. We review the necessary results in section 3, and also perform some explicit computations in matrices, involving the Weierstrass zeta function, that will become instrumental later on. The upshot of Atiyah's classification is that we can attach to M an important invariant, its *type*, which is a partition $r = \sum_{i=1}^k r_i$ of $r = rk(M)$, as explained above.

1.3.3. *The induction step.* Reverting to the language of matrices and canonical forms, we are now able to analyze the matrix C by an inductive process. Two extreme cases are easier to explain. When the type is $(1, 1, \dots, 1)$, \mathcal{E} is trivial and the pair (A, B) turns out to be gauge equivalent over K (not only over \widehat{K}) to a commuting pair of scalar matrices (A_0, B_0) . At the other extreme lies type (r) , where \mathcal{E} is indecomposable. In this case M is a twist of the standard special module of rank r , i.e. of the shape $M_r^{sp}(a, b)$ discussed above. Proving this involves a delicate bootstrapping argument with elliptic functions. The general case, where \mathcal{E} is neither trivial, nor indecomposable, is technically more complicated, and we refer to the text for details.

1.3.4. *An elliptic analogue of the conjecture of Loxton and van der Poorten.* In the last section we explain how to draw from the main theorem a conclusion regarding a formal power series which satisfies, simultaneously, a p -difference equation and a q -difference equation, whose coefficients are (the Laurent expansions at 0 of) Λ -elliptic functions. Our theorem will say that such a function lies in the ring

$$R = K_{\Lambda'}[z, z^{-1}, \zeta(z, \Lambda')]$$

generated over the field of Λ' -elliptic functions by $z^{\pm 1}$ and $\zeta(z, \Lambda')$, for some lattice $\Lambda' \subset \Lambda$. Conversely, every function from this ring satisfies a p -difference equation and a q -difference equation with elliptic functions as coefficients.

While the reason for the inclusion of $z^{\pm 1}$ in the ring R is technical (the need to allow a modification at 0 in the Periodicity Theorem), the appearance of $\zeta(z, \Lambda')$ is fundamental. It is attributed to the fact that, unlike the case of \mathbb{G}_a or \mathbb{G}_m , there are non-trivial vector bundles over X_Λ , namely the \mathcal{F}_r , that are invariant under pull-back by p_Λ and q_Λ .

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2. A PERIODICITY THEOREM

2.1. **Equivariant sheaves of cosets.** Let V be a finite-dimensional vector space over \mathbb{R} . Our goal in this section is to generalize the periodicity criterion of [dS1], Theorem 1, to cover a certain class of sheaves on V (equipped with its classical topology), which will be used in the proof of Theorem 3. From section 3 on we shall specialize to $V = \mathbb{C}$, and the set-up of Example 5 below.

Let \mathcal{G} be a sheaf of groups on V , and \mathcal{H} a sheaf of subgroups of \mathcal{G} . Let $\mathcal{F} = \mathcal{G}/\mathcal{H}$ be the sheaf of right cosets of \mathcal{H} . This is a sheaf of pointed sets (the distinguished point being the trivial coset), equipped with a left action

$$\mathcal{G} \times \mathcal{F} \rightarrow \mathcal{F}.$$

We assume that these sheaves satisfy the following condition:

- (Dis) If $U \subset V$ is an open set and $f \in \mathcal{G}(U)$ then

$$\{x \in U \mid f_x \notin \mathcal{H}_x\}$$

is a *discrete*² subset of U .

²By “discrete” we mean that its intersection with any compact subset of U is finite.

A consequence of this assumption is that sections of \mathcal{F} are *discretely supported*. In other words, if $s \in \mathcal{F}(U)$ then denoting by 0_x the distinguished element of \mathcal{F}_x , the set of $x \in U$ where $s_x \neq 0_x$ is discrete. This in particular holds for global sections.

For $v \in V$ we consider the translation $t_v(x) = x + v$, and assume that there are isomorphisms

$$\iota_v : \mathcal{G} \simeq t_v^* \mathcal{G}$$

satisfying $t_u^*(\iota_v) \circ \iota_u = \iota_{u+v}$. We assume that these isomorphisms restrict to isomorphisms on \mathcal{H} and hence on \mathcal{F} . Note that since $(t_v^* \mathcal{G})_x = \mathcal{G}_{t_v(x)} = \mathcal{G}_{x+v}$, on the stalks these are isomorphisms

$$\iota_{v,x} : \mathcal{G}_x \simeq \mathcal{G}_{x+v}, \quad \mathcal{H}_x \simeq \mathcal{H}_{x+v}$$

satisfying $\iota_{v,x+u} \circ \iota_{u,x} = \iota_{u+v,x}$. From now on we write ι_v for $\iota_{v,x}$. Later on we might even drop ι_v from the notation and *identify* \mathcal{G}_x with \mathcal{G}_{x+v} via translation.

We also consider, for each $0 \neq p \in \mathbb{R}$, the multiplication $m_p(x) = px$, and assume that there are isomorphisms

$$\varphi_p : \mathcal{G} \simeq m_p^* \mathcal{G}$$

satisfying $m_p^*(\varphi_q) \circ \varphi_p = \varphi_{pq}$. We assume that these isomorphisms as well restrict to isomorphisms on \mathcal{H} and hence on \mathcal{F} . On the stalks they are isomorphisms

$$\varphi_{p,x} : \mathcal{G}_x \simeq \mathcal{G}_{px}, \quad \mathcal{H}_x \simeq \mathcal{H}_{px}$$

satisfying the obvious condition with respect to composition. Again we write φ_p for $\varphi_{p,x}$.

Finally, we observe that $m_p \circ t_v = t_{pv} \circ m_p$ gives $t_v^* m_p^* \mathcal{G} \simeq m_p^* t_{pv}^* \mathcal{G}$, and we assume that the relation

$$m_p^*(\iota_{pv}) \circ \varphi_p = t_v^*(\varphi_p) \circ \iota_v$$

holds for any p and v . On the stalks this means that the diagram

$$(2.1) \quad \begin{array}{ccc} \mathcal{G}_x & \xrightarrow{\varphi_p} & \mathcal{G}_{px} \\ \iota_v \downarrow & & \downarrow \iota_{pv} \\ \mathcal{G}_{x+v} & \xrightarrow{\varphi_p} & \mathcal{G}_{px+pv} \end{array}$$

commutes.

We call a system $(\mathcal{G}, \mathcal{H}, \mathcal{F}, \iota_v, \varphi_p)$ as above an *equivariant sheaf of right cosets*.

If $s \in \Gamma(V, \mathcal{F})$ is a global section we denote by $m_p^*(s)$ the section $\varphi_p \circ s \circ m_p^{-1}$, namely

$$m_p^*(s)_x = \varphi_p(s_{x/p}).$$

Similarly, $t_v^*(s)$ is the section $\iota_v \circ s \circ t_v^{-1}$, namely

$$t_v^*(s)_x = \iota_v(s_{x-v}).$$

Example 5. Let $V = \mathbb{C}$, let G be an algebraic group over \mathbb{C} , $\mathcal{G} = G(\mathcal{M})$ where \mathcal{M} is the sheaf of meromorphic functions, and $\mathcal{H} = G(\mathcal{O})$ where \mathcal{O} is the sheaf of holomorphic functions. The condition (Dis) is satisfied. We put

$$\iota_v f = f \circ t_v^{-1}, \quad \varphi_p f = f \circ m_p^{-1}.$$

Then $\Gamma(V, \mathcal{G}) = G(\mathcal{M})$ where \mathcal{M} is the field of meromorphic functions, and if $s(z) \in G(\mathcal{M})$

$$m_p^* s(z) = s(z/p).$$

When $G = \mathbb{G}_m$ the sheaf \mathcal{F} is the sheaf of *divisors* on V . When $G = \mathbb{G}_a$, it is the sheaf of *principal parts*. For the proof of our main theorem we take $G = GL_r$, where the stalks

$$\mathcal{F}_x = G(\mathcal{M}_x)/G(\mathcal{O}_x)$$

are “*affine Grassmanians*”. For an introduction to affine Grassmanians and the stack $Bun_r(X)$ that will appear in section 3, we refer the reader to [Zhu]. However, we shall not be using anything about these concepts besides their naive definitions as coset spaces.

2.2. Global periodic sections. Our interest lies in the set $\Gamma(V, \mathcal{F})$ of global sections of \mathcal{F} . Recall that the supports of these sections intersect any bounded subset of V in a finite set. A section $s \in \Gamma(V, \mathcal{F})$ is said to be Λ -*periodic*, for a lattice $\Lambda \subset V$, if

$$t_\lambda^*(s) = s$$

for any $\lambda \in \Lambda$. The same terminology applies to global sections of \mathcal{G} . Our periodicity theorem is the following. If $s \in \Gamma(V, \mathcal{F})$ then by a *modification of s at 0* we mean a section $s' \in \Gamma(V, \mathcal{F})$ whose restriction to $V \setminus \{0\}$ agrees with the restriction of s to the same set.

Theorem 6. *Let $s \in \Gamma(V, \mathcal{F})$. Let $p, q \geq 2$ be relatively prime natural numbers. Suppose there are $A, B \in \Gamma(V, \mathcal{G})$ such that*

$$m_p^*(s) = As, \quad m_q^*(s) = Bs.$$

If A and B are Λ -periodic, so is a suitable modification s' of s at 0. Furthermore, this modification also satisfies

$$m_p^*(s') = As', \quad m_q^*(s') = Bs'.$$

Easy examples show that we can not forgo the modification at 0 in the statement of the theorem. The section s' is clearly unique, as the difference of any two such modifications is supported at 0, and being also periodic, must vanish identically. To prove the theorem we have to show that $s_{x+\lambda} = \iota_\lambda(s_x)$ for every $x \in V$ and $\lambda \in \Lambda$ such that both x and $x + \lambda$ are not 0. The proof breaks into two cases, depending on whether $x \in \mathbb{Q}\Lambda$ or not.

Before we embark on the proof, let us verify the last claim, which is easy. Indeed, at any point other than the origin, the germs of s' and s agree. At 0 the claim follows from the periodicity of s' and A or B . For example, if $0 \neq \omega \in \Lambda$ and we identify stalks via translation (dropping the identification maps ι_v from the notation)

$$(m_p^*s')_0 = \varphi_p(s'_0) = \varphi_p(s'_\omega) = (m_p^*s')_{p\omega} = A_{p\omega}s'_{p\omega} = A_0s'_0 = (As')_0.$$

2.3. Proof of the periodicity on $\mathbb{Q}\Lambda$. Let $N \geq 1$ be an integer such that A_x and B_x lie in \mathcal{H}_x if $x \in \mathbb{Q}\Lambda \setminus N^{-1}\Lambda$. The existence of such an N follows from the periodicity of A and B and the assumption (Dis). By induction on m we get

$$(2.2) \quad s = A^{-1} \cdot m_p^*(s) = \dots = A^{-1}m_p^*(A)^{-1} \dots (m_p^*)^{m-1}(A)^{-1} \cdot (m_p^*)^m(s).$$

If $x \in \mathbb{Q}\Lambda \setminus N^{-1}\Lambda$ then $x/p^\ell \notin N^{-1}\Lambda$ for any $\ell \geq 0$ so $(m_p^*)^\ell(A)_x = \varphi_p^\ell(A_{x/p^\ell}) \in \mathcal{H}_x$. For m large enough $(m_p^*)^m(s)_x = \varphi_p^m(s_{x/p^m}) = 0_x$ is the distinguished element of \mathcal{F}_x , since the support of s is discrete. It follows that $s_x = 0_x$ as well. In short, the support of $s|_{\mathbb{Q}\Lambda}$ is contained in $N^{-1}\Lambda$.

Changing notation (calling $N^{-1}\Lambda$ from now on Λ) we assume that A_x and B_x lie in \mathcal{H}_x if $x \in \mathbb{Q}\Lambda \setminus \Lambda$, and are $N\Lambda$ -periodic. We have seen that in such a case $s_x = 0_x$ if $x \in \mathbb{Q}\Lambda \setminus \Lambda$, and we need to prove that $s|_\Lambda$ is $N\Lambda$ -periodic, away from 0.

Let $x, y \in \Lambda \setminus \{0\}$ satisfy $x - y \in N\Lambda$. We have to show that $s_x = \iota_{x-y}(s_y)$. We choose a basis of Λ over \mathbb{Z} in which the coordinates of x and y are all non-zero, and identify from now on Λ with \mathbb{Z}^d (where $d = \dim V$). Such a basis, adapted to x and y , is easily seen to exist.

Let $S = \{p_i\}$ be the set of prime divisors of p . Recall the equivalence relation $u \sim_S v$ on \mathbb{Z}^d defined in [dS1]. This equivalence relation depends on N , which we hold fixed. First, if $d = 1$ we say that $u \sim_S v$ if $e_i = \text{ord}_{p_i}(u) = \text{ord}_{p_i}(v)$ for each i , and writing $u'_S = \prod p_i^{-e_i} u$ for the S -deprived part of u , we have, in addition,

$$u'_S \equiv v'_S \pmod{N}.$$

Note that $0 \sim_S v$ implies $v = 0$. If $d \geq 1$ we say that $u \sim_S v$ if for every coordinate $1 \leq \nu \leq d$ we have $u_\nu \sim_S v_\nu$.

Let $T = \{q_j\}$ be the set of primes dividing q . Let $(\mathbb{Z}^d)'$ be the subset of \mathbb{Z}^d consisting of vectors all of whose coordinates are non-zero. In [dS1], Lemma 2.1, it was proved that the equivalence relation on $(\mathbb{Z}^d)'$ generated by \sim_S and \sim_T is $\equiv \pmod{N}$. This uses, of course, the assumption that p and q are relatively prime. Since none of the coordinates of x or y vanishes, to prove the periodicity of $s|_\Lambda$ we may assume, in addition, that $x \sim_S y$ or $x \sim_T y$.

Let us assume therefore, without loss of generality, that $x \sim_S y$, and let $m - 1$ be the highest power of p for which p^{m-1} divides all the coordinates of x . Since $e_{i,\nu} = \text{ord}_{p_i}(x_\nu) = \text{ord}_{p_i}(y_\nu)$ for every $p_i \in S$ and every $1 \leq \nu \leq d$, p^{m-1} is also the highest power of p dividing all the coordinates of y . Since $s_z = 0_z$ if $z \in \mathbb{Q}\Lambda \setminus \Lambda$ we get

$$\begin{aligned} s_x &= A_x^{-1} \cdot \varphi_p(s_{x/p}) = \cdots = A_x^{-1} \varphi_p(A_{x/p})^{-1} \cdots \varphi_p^{m-1}(A_{x/p^{m-1}})^{-1} \cdot \varphi_p^m(s_{x/p^m}) \\ (2.3) \quad &= A_x^{-1} \varphi_p(A_{x/p})^{-1} \cdots \varphi_p^{m-1}(A_{x/p^{m-1}})^{-1} \cdot 0_x. \end{aligned}$$

The same equation, with the same m , holds with x replaced by y . For $0 \leq \ell \leq m - 1$, the condition $x \sim_S y$ implies that $p^{-\ell}x \equiv p^{-\ell}y \pmod{N}$, because for every coordinate $1 \leq \nu \leq d$

$$\prod_i p_i^{-e_{i,\nu}} x_\nu = x'_{\nu,S} \equiv \pmod{N} y'_{\nu,S} = \prod_i p_i^{-e_{i,\nu}} y_\nu$$

and $p^\ell | \prod_i p_i^{e_{i,\nu}}$, so $p^{-\ell}x_\nu \equiv p^{-\ell}y_\nu \pmod{N}$ as well. By the periodicity of A

$$A_{x/p^\ell} = \iota_{(x-y)/p^\ell} A_{y/p^\ell},$$

hence in view of the commutativity of the diagram (2.1)

$$\varphi_p^\ell(A_{x/p^\ell}) = \iota_{x-y}(\varphi_p^\ell(A_{y/p^\ell}))$$

and $s_x = \iota_{x-y}(s_y)$. This concludes the proof of the periodicity on $\mathbb{Q}\Lambda$.

2.4. Periodicity at points of $V \setminus \mathbb{Q}\Lambda$. Notation as in the theorem, let S_p and $S_q \subset V/\Lambda$ be the supports of $A \pmod{\mathcal{H}}$ and $B \pmod{\mathcal{H}}$. If A or B happen to lie in \mathcal{H}_0 we add $0 \in V/\Lambda$ to S_p and S_q even though it does not belong to the support of $A \pmod{\mathcal{H}}$ and $B \pmod{\mathcal{H}}$. By assumption these are finite sets, and

we let \tilde{S}_p and \tilde{S}_q be their pre-images in V . Let \tilde{S} denote the support of the section s . Equation (2.2) implies that for every $m \geq 1$ and every $x \in V$

$$s_x = A_x^{-1} \varphi_p(A_{x/p})^{-1} \cdots \varphi_p^{m-1}(A_{x/p^{m-1}})^{-1} \cdot \varphi_p^m(s_{x/p^m})$$

and similarly

$$s_x = B_x^{-1} \varphi_q(B_{x/q})^{-1} \cdots \varphi_q^{m-1}(B_{x/q^{m-1}})^{-1} \cdot \varphi_q^m(s_{x/q^m}).$$

Since, if $x \neq 0$, ultimately $s_{x/p^m} = 0_{x/p^m}$ and similarly $s_{x/q^m} = 0_{x/q^m}$, while if $x = 0$ it was included in \tilde{S}_p and \tilde{S}_q ,

$$\tilde{S} \subset \bigcup_{n=0}^{\infty} p^n \tilde{S}_p \cap \bigcup_{m=0}^{\infty} q^m \tilde{S}_q.$$

Lemma 7. *The projection of \tilde{S} modulo Λ is finite.*

Proof. See [dS1], Lemma 2.3. It is enough to assume, for this Lemma, that p and q are multiplicatively independent. \square

We write

$$M = \mathbb{Q}\Lambda.$$

Let S be the projection of \tilde{S} modulo Λ . Pick $z \in \tilde{S}_p$, $z \notin M$. We call

$$\{z, pz, p^2z, \dots\} \cap \tilde{S}_p$$

the p -chain through z . Since $z \notin M$ all the $p^n z$ have distinct images modulo Λ , so only finitely many of them belong to \tilde{S}_p . Let $p^{n(z)}z$ be the last one, and call $n(z) \geq 0$ the *exponent* of the p -chain through z . Call a p -chain *primitive* if it is not properly contained in any other p -chain, i.e. if none of the points $p^n z$ for $n < 0$ belongs to \tilde{S}_p . Since \tilde{S}_p is Λ -periodic, $n(z + \lambda) = n(z)$ for $\lambda \in \Lambda$. It follows from the finiteness of S_p that

$$n_p = 1 + \max_{z \in \tilde{S}_p, z \notin M} n(z) < \infty.$$

Lemma 8. *Let $\{z, pz, p^2z, \dots, p^{n(z)}z\} \cap \tilde{S}_p$ be a primitive p -chain through $z \notin M$. Then*

$$A_{p^{n(z)}z}^{-1} \varphi_p(A_{p^{n(z)-1}z})^{-1} \cdots \varphi_p^{n(z)}(A_z)^{-1} \in \mathcal{H}_{p^{n(z)}z}.$$

Proof. First, $s_{z/p} = 0_{z/p}$ since

$$s_{z/p} = A_{z/p}^{-1} \varphi_p(A_{z/p^2})^{-1} \cdots \varphi_p^{m-2}(A_{z/p^{m-1}})^{-1} \cdot \varphi_p^{m-1} s_{z/p^m},$$

all the z/p^ℓ ($\ell \geq 1$) are outside \tilde{S}_p , so $A_{z/p^\ell} \in \mathcal{H}_{z/p^\ell}$, while for m large enough $s_{z/p^m} = 0_{z/p^m}$.

For every $n \geq n(z)$

$$\begin{aligned} s_{p^n z} &= A_{p^n z}^{-1} \varphi_p(A_{p^{n-1}z})^{-1} \cdots \varphi_p^n(A_z)^{-1} \cdot \varphi_p^{n+1}(s_{z/p}) \\ &= A_{p^n z}^{-1} \varphi_p(A_{p^{n-1}z})^{-1} \cdots \varphi_p^n(A_z)^{-1} \cdot 0_{p^n z}. \end{aligned}$$

Since $A_{p^\ell z} \in \mathcal{H}_{p^\ell z}$ for $\ell > n(z)$, were the lemma not valid,

$$A_{p^n z}^{-1} \varphi_p(A_{p^{n-1}z})^{-1} \cdots \varphi_p^n(A_z)^{-1} \notin \mathcal{H}_{p^n z}$$

and so $s_{p^n z} \neq 0_{p^n z}$. But this would mean that for all $n \geq n(z)$, $p^n z \in \tilde{S}$. As $z \notin M$, these points have distinct images in S , contradicting the previous lemma. \square

We conclude that $s_{p^n z} = 0_{p^n z}$ if $n < 0$ or $n \geq n(z)$.

Corollary 9. *For any point $z \notin M = \mathbb{Q}\Lambda$, $n, m \in \mathbb{Z}$, if both $p^n z$ and $p^m z$ belong to \tilde{S} , then $|n - m| < n_p$.*

Proof. (of Theorem 6, concluded). Let $\lambda \in \Lambda$. Assume that $z \notin M$ and $s_z \neq 0_z$. Then by the corollary $s_{z/p^{n_p}} = 0_{z/p^{n_p}}$, so

$$s_z = A_z^{-1} \varphi_p(A_{z/p})^{-1} \cdots \varphi_p^{n_p-1}(A_{z/p^{n_p-1}})^{-1} \cdot 0_z.$$

By the periodicity of A under translation by Λ we now have

$$\iota_{p^{2n_p}\lambda} s_z = A_{(z+p^{2n_p}\lambda)}^{-1} \varphi_p(A_{(z+p^{2n_p}\lambda)/p})^{-1} \cdots \varphi_p^{n_p-1}(A_{(z+p^{2n_p}\lambda)/p^{n_p-1}})^{-1} \cdot 0_{(z+p^{2n_p}\lambda)}.$$

Since $z \in \tilde{S}$, for every $n_p \leq n$ we must have $z/p^n \notin \tilde{S}$ (by the corollary). This implies that $z/p^n \notin \tilde{S}_p$ (by (2.3) and decreasing induction on n), hence $A_{z/p^n} \in \mathcal{H}_{z/p^n}$. If $n_p \leq n < 2n_p$ then by the periodicity of A also $A_{(z+p^{2n_p}\lambda)/p^n} \in \mathcal{H}_{(z+p^{2n_p}\lambda)/p^n}$. We therefore get

$$\begin{aligned} \iota_{p^{2n_p}\lambda} s_z &= \\ & A_{(z+p^{2n_p}\lambda)}^{-1} \varphi_p(A_{(z+p^{2n_p}\lambda)/p})^{-1} \cdots \varphi_p^{2n_p-1}(A_{(z+p^{2n_p}\lambda)/p^{2n_p-1}})^{-1} \cdot 0_{(z+p^{2n_p}\lambda)}. \end{aligned}$$

Now at least one of $A_{(z+p^{2n_p}\lambda)/p^i}$ for $0 \leq i < n_p$ is not in \mathcal{H} , or else all the A_{z/p^i} for i in the same range will be in \mathcal{H} and s_z would be 0_z . By the definition of n_p this implies that $A_{(z+p^{2n_p}\lambda)/p^i}$ is in \mathcal{H} for $i \geq 2n_p$. We thus get that for every $n \geq 2n_p$

$$\iota_{p^{2n_p}\lambda} s_z = A_{(z+p^{2n_p}\lambda)}^{-1} \varphi_p(A_{(z+p^{2n_p}\lambda)/p})^{-1} \cdots \varphi_p^{n-1}(A_{(z+p^{2n_p}\lambda)/p^{n-1}})^{-1} \cdot 0_{(z+p^{2n_p}\lambda)}.$$

But for n large enough this is also

$$\begin{aligned} & A_{(z+p^{2n_p}\lambda)}^{-1} \varphi_p(A_{(z+p^{2n_p}\lambda)/p})^{-1} \cdots \varphi_p^{n-1}(A_{(z+p^{2n_p}\lambda)/p^{n-1}})^{-1} \cdot \varphi_p^n(s_{(z+p^{2n_p}\lambda)/p^n}) \\ &= s_{z+p^{2n_p}\lambda}. \end{aligned}$$

The relation $\iota_{p^{2n_p}\lambda} s_z = s_{z+p^{2n_p}\lambda}$ is therefore proven under the assumption $s_z \neq 0_z$. But it stays valid also if $s_z = 0_z$, because if $s_{z+p^{2n_p}\lambda} \neq 0_{z+p^{2n_p}\lambda}$, replace z by $z + p^{2n_p}\lambda$ and λ by $-\lambda$ and use the previous argument.

We have therefore shown that if $z \notin \mathbb{Q}\Lambda$

$$s_{z+p^{2n_p}\lambda} = \iota_{p^{2n_p}\lambda} s_z.$$

Similarly,

$$s_{z+q^{2n_q}\lambda} = \iota_{q^{2n_q}\lambda} s_z.$$

If p and q are relatively prime the lattice generated by $p^{2n_p}\Lambda$ and $q^{2n_q}\Lambda$ is Λ . We have therefore concluded the proof of the following proposition, and with it of Theorem 6. \square

Proposition 10. *Let $s \in \Gamma(V, \mathcal{F})$ and assume that p and q are multiplicatively independent. Assume that the conditions of Theorem 6 are satisfied. Then there exists a lattice $\Lambda' \subset \Lambda$ (depending on s) such that for every $z \notin M = \mathbb{Q}\Lambda$ and $\lambda \in \Lambda'$*

$$s_{z+\lambda} = \iota_\lambda(s_z).$$

If furthermore $\gcd(p, q) = 1$, we may take $\Lambda' = \Lambda$.

3. VECTOR BUNDLES ON ELLIPTIC CURVES

3.1. Atiyah's classification.

Theorem 11. ([At], Theorem 5, p.432) (i) Let X be an elliptic curve. Every vector bundle on X is a direct sum of indecomposable vector bundles, and the indecomposable components (with their multiplicities) are uniquely determined up to isomorphism.

(ii) Let $\mathcal{E}(r, d)$ be the set of isomorphism classes of indecomposable vector bundles of rank r and degree d . Let $p_X \in \text{End}(X)$ be multiplication by p . Then $p_X^*(\mathcal{E}(r, d)) \subset \mathcal{E}(r, p^2d)$.

(iii) There exists a unique isomorphism class $\mathcal{F}_r \in \mathcal{E}(r, 0)$ characterized by $H^0(X, \mathcal{F}_r) \neq 0$ (a space which is then one-dimensional). For every $p \in \mathbb{Z}$ we have $p_X^*\mathcal{F}_r \simeq \mathcal{F}_r$.

(iv) We have $\mathcal{F}_1 \simeq \mathcal{O}_X$ and for $r \geq 2$ there is a non-split extension

$$0 \rightarrow \mathcal{F}_{r-1} \rightarrow \mathcal{F}_r \rightarrow \mathcal{O}_X \rightarrow 0.$$

(v) For every $\mathcal{E} \in \mathcal{E}(r, 0)$ there exists a unique line bundle $\mathcal{L} \in \mathcal{E}(1, 0) = \text{Pic}^0(X) \simeq X$ such that

$$\mathcal{E} \simeq \mathcal{F}_r \otimes \mathcal{L}.$$

This gives a complete description of $\mathcal{E}(r, d)$ for $r|d$ (twisting by line bundles) and reduces the study of vector bundles of a general degree d to the range $0 \leq d < r$. In loc.cit., Theorem 6, Atiyah related $\mathcal{E}(r, d)$ for $0 \leq d < r$ to $\mathcal{E}(r-d, d)$ via extensions. Using the Euclidean algorithm, and fixing a degree 1 line bundle, he obtained a bijection between $\mathcal{E}(r, d)$ and $\mathcal{E}(r, d, 0)$. We shall not need these results.

Corollary 12. Let \mathcal{E} be a vector bundle on X such that $p_X^*\mathcal{E} \simeq \mathcal{E}$ for some $p > 1$. Then every indecomposable component of \mathcal{E} has degree 0, and, after pulling back to an unramified covering $X' \rightarrow X$, we may assume that every indecomposable component of \mathcal{E} is isomorphic to some \mathcal{F}_r .

Proof. If $\{\mathcal{E}_i\}$ are the indecomposable components, for every i there exists a j such that $p_X^*\mathcal{E}_i \simeq \mathcal{E}_j$. It follows that for every i there are $n, m \geq 1$ such that $(p_X^{n+m})^*\mathcal{E}_i \simeq (p_X^n)^*\mathcal{E}_i$, hence if d were the degree of \mathcal{E}_i , $p^{2(n+m)}d = p^{2n}d$, and $d = 0$. Write $\mathcal{E}_i \simeq \mathcal{F}_r \otimes \mathcal{L}$ for a line bundle $\mathcal{L} \in \text{Pic}^0(X)$. Then since $p_X^*\mathcal{F}_r \simeq \mathcal{F}_r$, while $k_X^*\mathcal{L} \simeq \mathcal{L}^k$ ([Mum], p.75)

$$\mathcal{L}^{p^{n+m}} \simeq (p_X^{n+m})^*\mathcal{L} \simeq (p_X^n)^*\mathcal{L} \simeq \mathcal{L}^{p^n},$$

so \mathcal{L} is torsion of order $p^{n+m} - p^n$. Let $\pi : X' \rightarrow X$ be an unramified covering so that $\pi^*\mathcal{L} \simeq \mathcal{O}_{X'}$. Since $\pi^*\mathcal{F}_{r,X} \simeq \mathcal{F}_{r,X'}$ we draw the desired conclusion. \square

3.2. The stack $Bun_r(X)$. Let X be a complex elliptic curve, K its function field, K_x ($x \in X$ a closed point) the completion of K at x , $O_x \subset K_x$ its valuation ring, \mathbb{A} the ring of adèles, i.e. the restricted product of all (K_x, O_x) for $x \in X$, and $\mathbb{O} = \prod_{x \in X} O_x$ its maximal compact subring. Let $G = GL_r$ and

$$Bun_r(X) = G(K) \backslash G(\mathbb{A})/G(\mathbb{O}).$$

Let η be the generic point of X .

If \mathcal{E} is a vector bundle of rank r over X we choose isomorphisms

$$\forall x \in X : \alpha_x : \widehat{\mathcal{E}}_x \simeq O_x^r, \quad \alpha_\eta : \mathcal{E}_\eta \simeq K^r,$$

and extend them to isomorphisms $\alpha_x : \widehat{\mathcal{E}}_x \otimes_{O_x} K_x \simeq K_x^r$ and $\alpha_\eta : \mathcal{E}_\eta \otimes_K K_x \simeq K_x^r$. For all but finitely many $x \in X$, $\alpha_\eta \circ \alpha_x^{-1} \in G(O_x)$. The double coset

$$\beta(\mathcal{E}) = [(\alpha_\eta \circ \alpha_x^{-1})_{x \in X}] \in Bun_r(X)$$

depends only on the isomorphism class of \mathcal{E} and not on our choices. The following is well-known.

Proposition 13. *The map $\mathcal{E} \mapsto \beta(\mathcal{E})$ is a bijection between isomorphism classes of vector bundles of rank r on X and $Bun_r(X)$.*

Proof. We construct a map in the opposite direction. If $s \in G(\mathbb{A})$ let $\mathcal{E}(s)$ be the following subsheaf of the constant sheaf \underline{K}^r :

$$\mathcal{E}(s)(U) = \{e \in \underline{K}^r(U) \mid \forall x \in U \ e_x \in s_x O_x^r\}.$$

Then $\mathcal{E}(s)$ is a vector bundle, up to isomorphism depends only on the class $[s] \in Bun_r(X)$, $\beta(\mathcal{E}(s)) = [s]$ and $\mathcal{E}(\beta(\mathcal{E})) \simeq \mathcal{E}$. \square

Let $\pi : X' \rightarrow X$ be an unramified covering. It induces maps $K \rightarrow K'$ and $K_{\pi(x')} \rightarrow K'_{x'}$ between the function fields and their completion. The latter map induces a map $K_x \rightarrow \prod_{\pi(x')=x} K'_{x'}$, hence a map $G(\mathbb{A}_K) \rightarrow G(\mathbb{A}_{K'})$, sending $G(\mathbb{O}_K)$ to $G(\mathbb{O}_{K'})$. The resulting map

$$\pi^* : Bun_r(X) \rightarrow Bun_r(X')$$

satisfies

$$\pi^*(\beta_X(\mathcal{E})) = \beta_{X'}(\pi^*(\mathcal{E})).$$

Let $\Lambda' \subset \Lambda$ be two lattices in \mathbb{C} and $\pi : X' = \mathbb{C}/\Lambda' \rightarrow \mathbb{C}/\Lambda = X$ the resulting unramified covering of elliptic curves. If $\xi \in \mathbb{C}$ and $x = \xi \bmod \Lambda$ then K_x is identified with $\mathbb{C}((z-\xi))$, and $K_{x+\omega}$, for $\omega \in \Lambda$, is identified with K_x via translation of the variable z . Since $\ker(\pi) = \Lambda/\Lambda'$, if x' is one point above x and we identify K_x with $K'_{x'}$ via π^* , the diagonal map

$$K_x \rightarrow \prod_{\pi(y)=x} K'_y = \prod_{\omega \in \Lambda/\Lambda'} K'_{x'+\omega}$$

is induced by the identification $K_x \simeq K'_{x'}$ and translation by ω for $\omega \in \Lambda/\Lambda'$. Taking the restricted product of these maps over $x \in \mathbb{C}/\Lambda$ we get the maps $G(\mathbb{A}_K) \rightarrow G(\mathbb{A}_{K'})$ and $Bun_r(X) \rightarrow Bun_r(X')$.

3.3. Vector bundles on elliptic curves associated with periodic sections of \mathcal{F} . We let $\mathcal{O} \subset \mathcal{M}$ be the sheaves of holomorphic and meromorphic functions on \mathbb{C} , $\mathcal{H} = G(\mathcal{O}) \subset \mathcal{G} = G(\mathcal{M})$ as in example 5, and $\mathcal{F} = \mathcal{G}/\mathcal{H}$. For a lattice Λ we denote by $\Gamma_\Lambda(\mathbb{C}, \mathcal{F})$ the Λ -periodic global sections of \mathcal{F} , i.e. the global sections s satisfying $t_\omega^* s = s$ for all $\omega \in \Lambda$. We write K_Λ for the function field of $X_\Lambda = \mathbb{C}/\Lambda$, \mathbb{A}_Λ for its adèles etc. We then have the identification

$$\Gamma_\Lambda(\mathbb{C}, \mathcal{F}) = \prod_{x \in X_\Lambda} G(K_x)/G(O_x) = G(\mathbb{A}_\Lambda)/G(\mathbb{O}_\Lambda).$$

If $s \in \Gamma_\Lambda(\mathbb{C}, \mathcal{F})$ we let $[s] \in Bun_r(X_\Lambda)$ be the associated double coset and $\mathcal{E}(s)$ the associated vector bundle.

The following lemma is easily verified.

Lemma 14. *The vector bundle associated with the class of m_{p-1}^*s is*

$$\mathcal{E}(m_{p-1}^*s) = p_\Lambda^* \mathcal{E}(s)$$

where $p_\Lambda : X_\Lambda \rightarrow X_\Lambda$ is multiplication by p .

Let $C \in \Gamma(\mathbb{C}, \mathcal{G}) = G(\mathcal{M})$ be an invertible $r \times r$ matrix of meromorphic functions. Assume that its image \overline{C} in \mathcal{F} is Λ -periodic, i.e.

$$\overline{C} \in \Gamma_\Lambda(\mathbb{C}, \mathcal{F}).$$

We may then consider the vector bundle $\mathcal{E}(\overline{C})$ on X_Λ associated to the double coset $[\overline{C}] \in \text{Bun}_r(X_\Lambda)$. This double coset is not changed if we multiply C on the left by a matrix from $G(K_\Lambda)$, or from the right by a matrix from $\Gamma(\mathbb{C}, \mathcal{H})$, i.e. an $r \times r$ invertible matrix of holomorphic functions whose inverse is also holomorphic.

Lemma 15. *Let $\zeta(z, \Lambda) \in \mathcal{M}$ be the Weierstrass zeta function defined in (1.4). Let N_r be the $r \times r$ nilpotent matrix (n_{ij}) with $n_{ij} = 1$ if $j = i + 1$ and 0 otherwise. Let $q \geq 1$ be any integer, $z_0 \in \mathbb{C}$ any point, and*

$$U_r(z) = U_r(q, z_0; z) = \exp(\zeta(qz - z_0, \Lambda)N_r) \in \Gamma(\mathbb{C}, \mathcal{G}).$$

Then $\overline{U}_r \in \Gamma_\Lambda(\mathbb{C}, \mathcal{F})$ and $\mathcal{E}(\overline{U}_r) = \mathcal{F}_r$ is Atiyah's vector bundle of rank r and degree 0 on X_Λ (see Theorem 11).

Proof. Since $U_r(z + \omega) = U_r(z) \cdot \exp(\eta(q\omega, \Lambda)N_r)$ for every $\omega \in \Lambda$, the first assertion is clear. We prove the second assertion by induction on r , the case $r = 1$ being trivial. Assume the lemma to hold for $r - 1$ ($r \geq 2$). From the upper-triangular form of U_r and our induction hypothesis we deduce that there is an extension

$$0 \rightarrow \mathcal{F}_{r-1} \rightarrow \mathcal{E}(\overline{U}_r) \rightarrow \mathcal{O}_X \rightarrow 0,$$

where $X = X_\Lambda$. This already shows that $\mathcal{E}(\overline{U}_r)$ is of degree 0. It is known that $\text{Ext}^1(\mathcal{O}_X, \mathcal{F}_{r-1}) \simeq H^1(X, \mathcal{F}_{r-1})$ is 1-dimensional, and the only non-trivial extension is \mathcal{F}_r , so it suffices to show that $\mathcal{E}(\overline{U}_r)$ is a non-trivial extension of \mathcal{O}_X by \mathcal{F}_{r-1} . For that it is enough to show that we get a non-trivial extension of \mathcal{O}_X by \mathcal{F}_{r-1} . For that it is enough to show that we get a non-trivial extension of \mathcal{O}_X after we mod out by $\mathcal{F}_{r-2} \subset \mathcal{F}_{r-1}$. Thus we are reduced to showing that when $r = 2$ we get a non-trivial extension, or that

$$\begin{pmatrix} 1 & \zeta(z) \\ 0 & 1 \end{pmatrix} \notin G(K_\Lambda)G(\mathbb{O}_\Lambda),$$

where we have abbreviated $\zeta(z) = \zeta(qz - z_0, \Lambda)$. If

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \zeta(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G(K_\Lambda)$$

where a, b, c, d are Λ -elliptic and $\alpha, \beta, \gamma, \delta$ are holomorphic, we get that $c = \gamma$ and $d = \delta$, being both elliptic and holomorphic, are constant. Then

$$a = \alpha + \gamma\zeta(z)$$

must have the same residual divisor as that of $\gamma\zeta(z)$. This residual divisor is $q^{-1}\gamma \sum_{q\xi = z_0 \pmod{\Lambda}} [\xi]$. Since a is elliptic, the sum of its residues on X_Λ must vanish, so $\gamma = 0$. By the same argument, applied to the equation

$$b = \beta + \delta\zeta(z),$$

we get $\delta = 0$. This contradiction concludes the proof of the lemma. \square

Note that the class of $\overline{U}_r(q, z_0; z)$ in $Bun_r(X_\Lambda)$ does not depend, as a result, on q or z_0 . This may be also checked directly by matrix arithmetic.

4. FORMAL (p, q) -DIFFERENCE MODULES

4.1. Formal p -difference modules. In this section we recall some known results about formal p -difference modules, see [vdP-Si], Chapter 12, and [Sch-Si], case 2Q. Let

$$\widehat{K} = \mathbb{C}((z))$$

and let \mathcal{K} be the algebraic closure of \widehat{K} . This is the field of formal Puiseux series

$$\mathcal{K} = \bigcup_{s \geq 1} \mathbb{C}((z^{1/s})).$$

We extend the action of Γ to \mathcal{K} by fixing a compatible sequence of s th roots of p and q . To fix ideas, we may take their positive real roots. Thus still

$$\sigma f(z) = f(z/p), \quad \tau f(z) = f(z/q).$$

Theorem 16. (i) *Every p -difference module over \mathcal{K} has a unique direct sum decomposition*

$$M = \bigoplus_{\lambda \in \mathbb{Q}} M_\lambda$$

where $M_\lambda \simeq \mathcal{K}^r$ with

$$(4.1) \quad \Phi_\sigma(v) = z^\lambda A_0^{-1} \sigma(v),$$

and A_0 is a scalar invertible matrix in Jordan canonical form.

(ii) *Let $c_1, \dots, c_k \in \mathbb{C}$ be the eigenvalues of the matrix A_0 appearing in the description of M_λ for some λ . If $v \in M$ and $\Phi_\sigma v = z^\lambda c^{-1} v$ for some $0 \neq c \in \mathbb{C}$, then $v \in M_\lambda$ and there exists an i and a rational number α such that $c = p^\alpha c_i$. Conversely, for any $1 \leq i \leq k$ and $\alpha \in \mathbb{Q}$ there exists a $v \in M_\lambda$ such that $\Phi_\sigma v = z^\lambda p^{-\alpha} c_i^{-1} v$.*

The λ which appear in the decomposition are called the *slopes* of M .

If M is a p -difference module over \widehat{K} , then we can extend scalars to \mathcal{K} and apply the classification theorem over \mathcal{K} . The slopes of M are by definition the slopes of $M_{\mathcal{K}}$. Theorem 16 is supplemented by the following Proposition.

Proposition 17. *Let M be a p -difference module over \widehat{K} . If the only slope of M is 0, then the consequence of Theorem 16 holds already over \widehat{K} . In other words, M has a basis on which the action of Φ_σ is given by a scalar matrix A_0^{-1} where A_0 is in Jordan canonical form.*

Furthermore, A_0 can be taken to be p -restricted, i.e. with eigenvalues c satisfying

$$1 \leq |c| < p.$$

Such an A_0 is then unique up to a permutation of the Jordan blocks.

4.2. Formal (p, q) -difference modules. Let $\Gamma = \langle \sigma, \tau \rangle \subset \text{Aut}(\mathcal{K})$ as before. Let M be a formal (p, q) -difference module over \mathcal{K} . Thus M is simultaneously a p -difference module and a q -difference module, and these structures commute with each other.

Let λ be a p -slope of M . Then there exists a vector $v \in M$ with $\Phi_\sigma v = z^\lambda c^{-1}v$ for some $c \in \mathbb{C}$. Applying Φ_τ we find out that

$$\Phi_\sigma(\Phi_\tau v) = \Phi_\tau(\Phi_\sigma v) = \Phi_\tau(z^\lambda c^{-1}v) = z^\lambda q^{-\lambda} c^{-1}(\Phi_\tau v).$$

Let $I_{\lambda, c}$ denote the rank-1 p -difference module over \mathcal{K} defined by

$$I_{\lambda, c} = \mathcal{K}e, \quad \Phi_\sigma e = z^\lambda c^{-1}e.$$

The above argument shows that if M contains a copy of $I_{\lambda, c}$, it contains also a copy of $I_{\lambda, q^\lambda c}$. Part (ii) of Theorem 16 implies that the only rank-1 submodules of M of slope λ are of the form $I_{\lambda, p^\alpha c_i}$ for a finite list $\{c_1, \dots, c_k\}$. We conclude that for some $i, n \geq 1$ and $\alpha \in \mathbb{Q}$

$$q^{n\lambda} c_i = p^\alpha c_i.$$

If p and q are multiplicatively independent, this forces $\lambda = \alpha = 0$. We conclude that the only possible p -slope of M is 0, and similarly the only q -slope is 0. In the language of difference modules, M is *regular singular*.

Assume now that M is a (p, q) -difference module over \widehat{K} given by a pair of matrices (A, B) satisfying (1.2) and that p and q are multiplicatively independent. Extending scalars to \mathcal{K} it follows from the above discussion that the only slope of M is $(0, 0)$. Proposition 17 implies that already over \widehat{K} the pair (A, B) is gauge-equivalent to a pair (A_0, B_0) where A_0 is a scalar matrix, with eigenvalues in the range $1 \leq |c| < p$. The consistency equation

$$A_0 B_0(z) = B_0(z/p) A_0$$

now forces B_0 to be constant too. To see it write

$$B_0(z) = \sum_{i \in \mathbb{Z}} M_i z^i,$$

with $M_i \in M_r(\mathbb{C})$, so that

$$A_0 M_i A_0^{-1} = p^{-i} M_i.$$

The eigenvalues of A_0 in its action on $M_r(\mathbb{C})$ by conjugation are each a quotient of two eigenvalues of A_0 . By our assumption, p^i is not among them for $i \neq 0$. This proves that $M_i = 0$ for $i \neq 0$ and B_0 is constant as well.

Recall that by definition, the cohomology set $H^1(\Gamma, G(\widehat{K}))$ is the set of equivalence classes of pairs (A, B) of matrices from $G(\widehat{K})$ satisfying the consistency condition (Γ -cocycles), up to gauge equivalence (the relation of being *cohomologous*). Similarly, $H^1(\Gamma, G(\mathbb{C}))$ is the set of equivalence classes of commuting pairs (A, B) of scalar matrices (i.e. homomorphisms $\varphi : \Gamma \rightarrow G(\mathbb{C})$), up to conjugation. Denote by $H^1(\Gamma, G(\mathbb{C}))^{p\text{-restricted}}$ the collection of such homomorphisms φ for which $A_0 = \varphi(\sigma)$ is p -restricted, up to conjugation. We have proved the following.

Theorem 18. *The map $H^1(\Gamma, G(\mathbb{C})) \rightarrow H^1(\Gamma, G(\widehat{K}))$ induces a bijection*

$$H^1(\Gamma, G(\mathbb{C}))^{p\text{-restricted}} \simeq H^1(\Gamma, G(\widehat{K})).$$

Equivalently, any pair (A, B) of matrices from $G(\widehat{K})$ satisfying (1.2) can be reduced by a gauge transformation (1.3) with $C \in G(\widehat{K})$ to a pair (A_0, B_0) of matrices from $G(\mathbb{C})$, where A is p -restricted, and such a pair (A_0, B_0) is unique up to conjugation.

Symmetrically, we may assume that B_0 is q -restricted. In general, however, we can not make A_0 p -restricted and B_0 q -restricted simultaneously.

5. PROOF OF THE MAIN THEOREM

In this section we deduce Theorem 3 from Theorem 6.

5.1. An approximation argument. Let $K = K_0^{nr} = \bigcup K_\Lambda$ be as in the introduction. Let $A, B \in G(K)$ satisfy the consistency condition (1.2). Let $\widehat{K} = \mathbb{C}((z))$ be the completion of K at 0. By Theorem 18 there exists a $C \in G(\widehat{K})$ such that

$$(5.1) \quad A_0 = C(z/p)^{-1}A(z)C(z), \quad B_0 = C(z/q)^{-1}B(z)C(z)$$

are scalar matrices, and A_0 is p -restricted. Let $E \in G(K)$ be such that

$$E^{-1}C \in I + z^R M_r(\mathbb{C}[[z]])$$

where $R \geq 1$ is a fixed large number, yet to be determined. Such an E exists since $G(K)$ is dense in $G(\widehat{K})$. Replacing (A, B) by the gauge-equivalent pair

$$(E(z/p)^{-1}A(z)E(z), E(z/q)^{-1}B(z)E(z))$$

we may assume, without loss of generality, *and without changing A_0 and B_0* , that $C(z) \in I + z^R M_r(\mathbb{C}[[z]])$. In such a case, $A(z)$ and $B(z)$ are also holomorphic at $z = 0$ and congruent to A_0 and B_0 modulo z^R .

The next lemma shows that if C is congruent to I modulo z , then C is uniquely determined.

Lemma 19. *Let $C, C' \in I + z^R M_r(\mathbb{C}[[z]])$ ($R \geq 1$) $A_0, A'_0 \in G(\mathbb{C})$ satisfy*

$$A_0 = C(z/p)^{-1}A(z)C(z), \quad A'_0 = C'(z/p)^{-1}A(z)C'(z).$$

Then $A_0 = A'_0$. If p^i is not an eigenvalue of conjugation by A_0 on $M_r(\mathbb{C})$ for $i \geq R$, then $C = C'$. The last condition holds when $R = 1$ if A_0 is p -restricted.

Proof. Write $C' = CD$. Then $D = I + \sum_{i=R}^{\infty} D_i z^i$ satisfies

$$D(z/p)A'_0 = A_0 D(z).$$

The constant term gives $A_0 = A'_0$ and the higher terms give $A_0^{-1}D_i A_0 = p^i D_i$. If p^i ($i \geq R$) is not an eigenvalue of conjugation by A_0 , all the $D_i = 0$. If A_0 is p -restricted, then p^i cannot be an eigenvalue of conjugation by A_0 for $i \neq 0$. Indeed, any eigenvalue of the map $M \mapsto A_0^{-1} M A_0$ on $M_r(\mathbb{C})$ is the quotient of two eigenvalues of A_0 , and these latter ones are all assumed to lie, in absolute value, in the interval $[1, p)$. \square

5.2. $C(z)$ is everywhere meromorphic.

Proposition 20. *Suppose that $C(z) \in G(\widehat{K})$ satisfies (5.1). Then $C(z)$ is meromorphic on \mathbb{C} .*

Proof. As noted above, we may assume that $C(z) \equiv I \pmod{z^R}$ where $R \geq 1$ is chosen as in Lemma 19. The equation

$$C(z) = A(z)^{-1}C(z/p)A_0$$

and the fact that $A(z)$ is meromorphic, show that it is enough to prove that $C(z) = I + \sum_{i=R}^{\infty} z^i C_i$ converges in $\{|z| < \varepsilon\}$ for *some* $\varepsilon > 0$.

For this fix a norm $\|\cdot\|$ on $M_r(\mathbb{C})$ (they are all equivalent) and let $c_1 > 0$ be such that

$$\|A_0^{-1}MA_0\| \leq c_1\|M\|.$$

Writing $A(z)^{-1}A_0 = I + \sum_{i=R}^{\infty} z^i M_i$, the holomorphicity of $A(z)$ in a neighborhood of 0 implies that there exists a $c_2 > 0$ so that $\|M_i\| \leq c_2^i$. For $m \geq 1$ define

$$A^{(m)}(z) = A_0^{1-m}A(z/p^{m-1})^{-1}A_0^m$$

(so that $A^{(1)}(z) = A(z)^{-1}A_0$) and

$$C^{(m)}(z) = A^{(1)}(z)A^{(2)}(z) \cdots A^{(m)}(z) = A(z)^{-1}A(z/p)^{-1} \cdots A(z/p^{m-1})^{-1}A_0^m.$$

Note that

$$C^{(m+1)}(z) = A(z)^{-1}C^{(m)}(z/p)A_0.$$

Suppose we show that $C^{(m)}(z)$ converges to some $C^{(\infty)}(z)$ which is holomorphic in a neighborhood of 0. Then $C^{(\infty)}(z)$ satisfies (5.1), and is congruent to I modulo z^R , so by Lemma 19 it is equal to C and the proposition will be proved.

Writing

$$A^{(m)}(z) = I + \sum_{i=R}^{\infty} z^i M_i^{(m)}$$

we have

$$\|M_i^{(m+1)}\| = p^{-im}\|A_0^{-m}M_iA_0^m\| \leq p^{-im}c_1^m c_2^i.$$

Choose R large enough so that $c_1/p^R \leq 1/2$. Note that the reduction step that allowed us to take a large R did not affect A_0 , so did not affect c_1 (it did affect $A(z)$, hence c_2). These estimates immediately give the existence of the limit $C^{(\infty)}$ and its convergence in the neighborhood $|z| < c_2^{-1}$ of 0. This completes the proof of the proposition. \square

5.3. Applying the periodicity theorem to get a vector bundle on an elliptic curve. Let $\mathcal{G} = G(\mathcal{M})$, $\mathcal{H} = G(\mathcal{O})$ and $\mathcal{F} = \mathcal{G}/\mathcal{H}$ be as in example 5. As we have seen, we may assume that $C \in G(\mathcal{M})$ (\mathcal{M} is the field of meromorphic functions on \mathbb{C}) is normalized at 0 by

$$C \equiv I \pmod{z}$$

and that A_0 is p -restricted. Lemma 19 shows that C is then uniquely determined by (A, B) . Replacing (A, B) by the gauge equivalent pair

$$(E(z/p)^{-1}A(z)E(z), E(z/q)^{-1}B(z)E(z))$$

for some $E \in G(K)$, results in multiplying C on the left by E^{-1} , leaving (A_0, B_0) unchanged. Thus up to multiplication on the left by a matrix from $G(K)$, C depends only on the module M .

The matrix C is a global section $C \in \Gamma(V, \mathcal{G})$ and we let \bar{C} be its image in $\Gamma(V, \mathcal{F})$. The equation (5.1) yields

$$(5.2) \quad m_p^* \bar{C}(z) = \bar{C}(z/p) = A(z)\bar{C}(z), \quad m_q^* \bar{C}(z) = \bar{C}(z/q) = B(z)\bar{C}(z).$$

Theorem 6 implies that there exists a *modification of \bar{C} at 0*, denoted $s \in \Gamma(V, \mathcal{F})$, which is Λ -periodic for some lattice $\Lambda \subset \Lambda_0$. We may assume that A and B are Λ -periodic as well. Furthermore, this s satisfies

$$m_p^* s = As, \quad m_q^* s = Bs.$$

By the periodicity, s is an element of

$$\Gamma_\Lambda(\mathbb{C}, \mathcal{F}) = G(\mathbb{A}_\Lambda)/G(\mathbb{O}_\Lambda).$$

Replacing C by $E^{-1}C$ for $E \in G(K_\Lambda)$, hence s by $E^{-1}s$, does not change the class $[s]$ of s in

$$Bun_r(X_\Lambda) = G(K_\Lambda) \backslash G(\mathbb{A}_\Lambda)/G(\mathbb{O}_\Lambda).$$

This class is therefore an invariant of the module M . By definition, $(m_{p^{-1}}^* s)_x = \varphi_{p^{-1}}(s_{px})$. Thus if s is Λ -periodic, $m_{p^{-1}}^* s$ is also Λ -periodic.

Let $\mathcal{E} = \mathcal{E}(s)$ be the vector bundle associated with our s , the Λ -periodic modification of \bar{C} . From the equation $m_p^* s = As$ we conclude that $m_{p^{-1}}^* s = (m_{p^{-1}}^* A)^{-1}s$. But $m_{p^{-1}}^* A^{-1}(z) = A^{-1}(pz)$ is in $G(K_\Lambda)$. Thus the classes of s and $m_{p^{-1}}^* s$ in $Bun_r(X_\Lambda)$ are the same, hence, by Lemma 14,

$$p_\Lambda^* \mathcal{E} \simeq \mathcal{E}.$$

Replacing Λ by a sublattice, we may assume, by Corollary 12, that

$$\mathcal{E} \simeq \bigoplus_{i=1}^k \mathcal{F}_{r_i}$$

where $r_1 \leq \dots \leq r_k$ and $\sum_{i=1}^k r_i = r$. Since $[s] \in Bun_r(X_\Lambda)$, hence also the isomorphism type of $\mathcal{E} = \mathcal{E}(s)$, depend only on M , we can make the following definition.

Definition 21. The partition (r_1, \dots, r_k) of r is called the *type* of M .

From Lemma 15 we conclude that the double coset of s in $Bun_r(X)$ and the double coset of \bar{U} , where U is the matrix (in block form)

$$(5.3) \quad U(z) = \bigoplus_{i=1}^k U_{r_i}(pq, z_0; z)$$

are the same. We deduce the following.

Corollary 22. *Let C be the invertible $r \times r$ matrix of everywhere meromorphic functions obtained in §5.2. Then, possibly after replacing C by $E^{-1}C$ ($E \in G(K)$), and the pair (A, B) by a gauge-equivalent pair, we may assume that*

$$C(z) = U(z)D(z)$$

where U is the upper triangular unipotent matrix described in (5.3) and D is an invertible matrix of holomorphic functions, with holomorphic inverse, except possibly at 0.

Proof. Since $[s] = [\bar{U}] \in G(K_\Lambda) \backslash G(\mathbb{A}_\Lambda)/G(\mathbb{O}_\Lambda)$, there exists an $E \in G(K_\Lambda)$ such that $E\bar{U} = s$. Replacing C by $E^{-1}C$, hence s by $E^{-1}s$, we may assume that $\bar{U} = s$ in $G(\mathbb{A}_\Lambda)/G(\mathbb{O}_\Lambda)$. Define D by the equation $C = UD$. Then at any $0 \neq x \in \mathbb{C}$ we have $\bar{U}_x = s_x = \bar{C}_x$ in $\mathcal{F}_x = \mathcal{G}_x/\mathcal{H}_x$. It follows that $D_x \in \mathcal{H}_x = G(O_x)$ for every $0 \neq x$. \square

We emphasize that although the change in C (to $E^{-1}C$) may introduce poles at points of Λ , this change, accompanied by the corresponding gauge equivalence of (A, B) , *does not change* A_0 and B_0 .

Rewrite the first of the two functional equations

$$\begin{cases} A_0 = C(z/p)^{-1}A(z)C(z) \in G(\mathbb{C}) \\ B_0 = C(z/q)^{-1}B(z)C(z) \in G(\mathbb{C}), \end{cases}$$

as

$$(5.4) \quad A(z) = U(z/p)T(z)U(z)^{-1},$$

where $T(z) = D(z/p)A_0D(z)^{-1}$ is everywhere holomorphic (meaning that its inverse is also holomorphic, i.e. its germ lies in $\mathcal{H}_x = G(O_x)$), except possibly at 0. Similarly, with the same $U(z)$, and with $S(z) = D(z/q)B_0D(z)^{-1}$,

$$(5.5) \quad B(z) = U(z/q)S(z)U(z)^{-1}$$

and $S(z)$ is everywhere holomorphic, except possibly at 0.

At last, we get rid of the phrase “except possibly at 0”, forced upon us, so far, since the Periodicity Theorem had the freedom of modification at 0. Recall that the parameter $z_0 \in \mathbb{C}$ introduced in (5.3), see also Lemma 15, is still at our disposal. By an appropriate choice of z_0 we may assume that U is holomorphic at any $\omega \in p^{-1}\Lambda$ and any $\omega \in q^{-1}\Lambda$. So are T and S if $\omega \neq 0$. This means that A and B are holomorphic at any $0 \neq \omega \in \Lambda$. Being Λ -periodic, A and B must be holomorphic at 0 as well, hence $T_0, S_0 \in \mathcal{H}_0$. Since

$$\zeta(pqz - z_0, \Lambda) - \zeta(pqz - z_1, \Lambda) \in K_\Lambda,$$

changing z_0 results in replacing the pair (A, B) in equations (5.4,5.5) by a gauge-equivalent pair, but T and S are unchanged. We therefore conclude that, no matter what z_0 is, $T(z)$ and $S(z)$ are everywhere holomorphic.

We record our intermediate conclusion.

Corollary 23. *Let M be an elliptic (p, q) -difference module, and (r_1, \dots, r_k) its type. Then there exists a lattice $\Lambda \subset \Lambda_0$ such that the module M is represented, in an appropriate basis, by Λ -periodic matrices A and B of the form*

$$A(z) = U(z/p)T(z)U(z)^{-1}, \quad B(z) = U(z/q)S(z)U(z)^{-1}$$

where: (i) $U = \bigoplus_{i=1}^k U_{r_i}(pq, z_0; z)$ (ii) The matrices $T(z)$ and $S(z)$ are everywhere holomorphic with a holomorphic inverse.

5.4. Two extreme cases. If $U = I$ (i.e. the vector bundle \mathcal{E} is trivial) then (5.4) shows that A , being a matrix of elliptic functions which are at the same time everywhere holomorphic, is constant. Similarly B is constant. We draw the following conclusion.

Proposition 24. *Assume that \mathcal{E} is trivial (i.e. the type of M is $(1, 1, \dots, 1)$). Then the elliptic (p, q) -difference module M represented by the pair (A, B) is obtained by base change from a scalar one. Equivalently, the pair (A, B) is gauge-equivalent to a pair (A_0, B_0) of commuting matrices from $G(\mathbb{C})$.*

This proves, in particular, the case $r = 1$ of the main theorem (Theorem 3), proved already in [dS1].

Assume, at the other extreme, that $\mathcal{E} \simeq \mathcal{F}_r$ is indecomposable (i.e. the type of M is (r)), so that $U(z) = U_r(pq, z_0; z)$. Write $U(z)$, $T(z)$ and $A(z)$ in block form where

$$T(z) = \begin{pmatrix} T'(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix}, \quad A(z) = \begin{pmatrix} A'(z) & b(z) \\ c(z) & d(z) \end{pmatrix},$$

T' and A' are of size $(r-1) \times (r-1)$, γ and β are row/column vectors consisting of everywhere holomorphic functions, c and b are similar vectors of elliptic functions, δ is holomorphic and d is elliptic. We get

$$U(z/p) \begin{pmatrix} T'(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix} = \begin{pmatrix} A'(z) & b(z) \\ c(z) & d(z) \end{pmatrix} U(z).$$

Lemma 25. $\gamma = c = 0$ and $\delta = d$ is a constant.

Proof. Recall that $U(z)$ is upper triangular unipotent, and $u_{i,i+1}(z) = \zeta(pqz - z_0, \Lambda)$. We prove by induction on i that $\gamma_i = c_i = 0$. If $i = 1$ then $\gamma_1(z) = c_1(z)$. Being both elliptic and holomorphic, $\gamma_1 = c_1$ is a constant. Next,

$$\gamma_2(z) = c_1 \zeta(pqz - z_0, \Lambda) + c_2(z).$$

The residual divisor of $c_2(z)$ on X_Λ is therefore $-p^{-1}q^{-1}c_1 \sum_{pq\xi = z_0 \pmod{\Lambda}} [\xi]$. As $c_2(z)$ is elliptic, the sum of its residues must vanish, so $c_1 = 0$. Assume that $c_1 = \dots = c_{i-1} = \gamma_1 = \dots = \gamma_{i-1} = 0$ ($2 \leq i \leq r-1$). Then

$$\gamma_i(z) = c_i(z),$$

so by the same argument as before it is constant, and

$$\gamma_{i+1}(z) = c_i \zeta(pqz - z_0, \Lambda) + c_{i+1}(z)$$

(if $i = r-1$ take δ instead of γ_r and d instead of c_r), so as before we conclude that $\gamma_i = c_i = 0$. The same argument shows that $\delta = d$ is constant. \square

Corollary 26. *The matrices A and T are upper triangular, with constants along the diagonal. So are B and S .*

Proof. Use induction on r . \square

The significance of the last corollary is that our elliptic (p, q) -difference module is a successive extension of 1-dimensional ones, and the heart of the classification (at least when \mathcal{E} is indecomposable) is to compute the Ext^\bullet groups between the 1-dimensional objects. As these computations are inevitably based on arguments similar to the ones below, we decided to work directly with canonical forms of matrices, in a somewhat old-fashioned manner.

To continue, and to simplify the notation, it will be convenient to assume from now on that $z_0 = 0$. Write $\zeta(z)$ for $\zeta(z, \Lambda)$, and $U(z)$ for $U_r(pq, 0; z)$.

Let, as in the introduction,

$$g_p(z) = p\zeta(qz) - \zeta(pqz) \in K_\Lambda, \quad g_q(z) = q\zeta(pz) - \zeta(pqz) \in K_\Lambda.$$

In fact, $g_p(z)$ is even $q^{-1}\Lambda$ -elliptic, and $g_q(z)$ is $p^{-1}\Lambda$ -elliptic.

Let

$$A_r^{sp}(z) = (a_{ij})$$

where $a_{ij} = 0$ if $1 \leq j < i \leq r$, and

$$a_{ij} = \frac{p^{i-1}}{(j-i)!} g_p(z)^{j-i}$$

if $1 \leq i \leq j \leq r$. Let $T_r^{sp} = \text{diag}[1, p, \dots, p^{r-1}]$.

Lemma 27. *We have*

$$A_r^{sp}(z) = U(z/p)T_r^{sp}U(z)^{-1}.$$

Proof. Checking the identity amounts to checking, for $1 \leq i \leq k \leq r$, that

$$\sum_{j=i}^k \frac{p^{i-1}}{(j-i)!} g_p(z)^{j-i} \frac{1}{(k-j)!} \zeta(pqz)^{k-j} = \frac{p^{k-1}}{(k-i)!} \zeta(qz)^{k-i}.$$

This follows at once from the binomial theorem. \square

Similarly define B_r^{sp} , reversing the roles of p and q , let $S_r^{sp} = \text{diag}[1, q, \dots, q^{r-1}]$, and the analogous lemma, asserting that

$$B_r^{sp}(z) = U(z/q)S_r^{sp}U(z)^{-1}$$

then holds also. The following lemma is an immediate consequence, since the diagonal matrices T_r^{sp} and S_r^{sp} commute.

Lemma 28. *The consistency equation*

$$A_r^{sp}(z/q)B_r^{sp}(z) = B_r^{sp}(z/p)A_r^{sp}(z)$$

holds.

Definition 29. We denote by M_r^{sp} the elliptic (p, q) -difference module represented by the pair $(A_r^{sp}(z), B_r^{sp}(z))$. We call it the *standard special module of rank r* . Any module isomorphic to it is called special.

Lemma 30. *Assume that $\mathcal{E} \simeq \mathcal{F}_r$, and the notation is as in Corollary 23, with $U(z) = U_r(pq, 0; z)$. Then there exists an upper-triangular unipotent scalar matrix F , commuting with $U(z)$, of the form*

$$F = \exp\left(\sum_{\ell=1}^{r-1} \lambda_\ell N_r^\ell\right),$$

such that

$$FA(z)F^{-1} = aA_r^{sp}(z), \quad FB(z)F^{-1} = bB_r^{sp}(z)$$

for some $a, b \in \mathbb{C}^\times$, and

$$FT(z)F^{-1} = aT_r^{sp}, \quad FS(z)F^{-1} = bS_r^{sp}.$$

In particular, $T(z)$ and $S(z)$ were scalar matrices to begin with, and the pair (A, B) is gauge equivalent to the pair (aA_r^{sp}, bB_r^{sp}) .

This proves the following.

Proposition 31. *Assuming that $\mathcal{E} \simeq \mathcal{F}_r$, the elliptic (p, q) -difference module represented by (A, B) is isomorphic to*

$$M_r^{sp}(a, b) = M_r^{sp} \otimes M_1(a, b).$$

Note that together with the case $\mathcal{E} \simeq \mathcal{O}_X^r$ mentioned before, this completes the classification of (p, q) -difference modules for $r \leq 2$.

Proof. (of the Lemma). We prove our claim by induction on r , the case $r = 1$ being trivial. The matrix F will be of the form

$$F = \exp\left(\sum_{\ell=1}^{r-1} \lambda_{\ell} N_r^{\ell}\right),$$

and will therefore commute with $U(z)$. Since all the matrices are in upper triangular form, the induction hypothesis allows us to assume that the first $r - 1$ columns of $A(z)$ agree with those of $A_r^{sp}(z)$ and similarly for $B(z)$ and $B_r^{sp}(z)$. We may also assume (or, it follows from the formulae) that the first $r - 1$ columns of $T(z)$ and T_r^{sp} , and similarly of $S(z)$ and S_r^{sp} , agree.

Note that in the induction step, if $F' = \exp(\sum_{\ell=1}^{r-2} \lambda_{\ell} N_{r-1}^{\ell})$ is the matrix conjugating the north-west blocks of size $(r - 1) \times (r - 1)$ into their standard form, we must replace all four $r \times r$ matrices by their conjugates by $\exp(\sum_{\ell=1}^{r-2} \lambda_{\ell} N_r^{\ell})$. For example, if $r = 3$ and we have used

$$F' = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

to bring the north-west blocks of size 2×2 into the desired form, we should conjugate A, B, T and S by

$$\begin{pmatrix} 1 & \lambda & \lambda^2/2 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix}$$

before we proceed as below. Since the matrix with which we have conjugated commutes with $U_r(z)$, the equations (5.4, 5.5) remain intact.

Thus we assume (ignoring the trivial twist by $M_1(a, b)$) that

$$A(z) = \begin{pmatrix} 1 & g_p(z) & \frac{1}{2}g_p(z)^2 & \cdots & \frac{1}{(r-2)!}g_p(z)^{r-2} & a_1(z) \\ & p & pg_p(z) & \cdots & \frac{1}{(r-3)!}pg_p(z)^{r-3} & pa_2(z) \\ & & p^2 & \cdots & \vdots & \vdots \\ & & & \ddots & & \\ & & & & p^{r-3} & p^{r-3}g_p(z) & p^{r-3}a_{r-2}(z) \\ & & & & & p^{r-2} & p^{r-2}a_{r-1}(z) \\ & & & & & & p^{r-1}a_r \end{pmatrix}$$

and

$$T(z) = \begin{pmatrix} 1 & & & & t_1(z) \\ & p & & & pt_2(z) \\ & & p^2 & & \vdots \\ & & & \ddots & \\ & & & & p^{r-3} & p^{r-3}t_{r-2}(z) \\ & & & & & p^{r-2} & p^{r-2}t_{r-1}(z) \\ & & & & & & p^{r-1}t_r \end{pmatrix}.$$

Similar equations will hold for $B(z)$ and $S(z)$. We shall prove the following, by *decreasing* induction on i .

- $t_r = a_r = 1$
- $t_i(z) = 0$ and $t_{i-1}(z) = t_{i-1}$ is constant ($2 \leq i \leq r - 1$)
- $a_i(z) = \frac{1}{(r-i)!}g_p(z)^{r-i}$ ($2 \leq i \leq r - 1$).

It will follow that t_1 is constant and $t_2 = \cdots = t_{r-1} = 0$. Similarly s_1 is a constant and $s_2 = \cdots = s_{r-1} = 0$. In particular T and S are scalar, equal to $\text{diag}[1, p, \dots, p^{r-1}]$ and $\text{diag}[1, q, \dots, q^{r-1}]$, except for the north-east corner. Now the consistency equation between A and B implies that T and S commute. Thus

$$s_1 + t_1 q^{r-1} = t_1 + s_1 p^{r-1}.$$

Let $\lambda_{r-1} = -t_1/(p^{r-1}-1) = -s_1/(q^{r-1}-1)$. It is easily verified that by conjugating all our matrices by

$$F_r = \exp(\lambda_{r-1} N_r^{r-1}) = I + \lambda_{r-1} N_r^{r-1},$$

a matrix commuting with $U(z)$, we bring them to the desired form, i.e. $F_r T F_r^{-1} = T_r^{sp}$, $F_r S F_r^{-1} = S_r^{sp}$, and as a result (or by direct computation) $A(z)$ and $B(z)$ get transformed into A_r^{sp} and B_r^{sp} . All that remains is to check the three ‘‘bullets’’. We will do it for A and T , the case of B and S being identical. The method will be the same ‘‘bootstrapping’’ technique used in the proof of Lemma 25. Note that in the $i+1$ step of the induction we only get that t_i is constant, but the i th step (the *next* one, since this is a *decreasing* induction) strengthens it and shows that $t_i = 0$. This explains why we end up with t_1 being only a scalar, which we kill by conjugation with F_r . As a final preparation, we remark that we shall be using repeatedly the same two principles:

- (Hol) an everywhere holomorphic elliptic function is constant,
- (Res) the sum of the residues of an elliptic function over a fundamental domain for the period lattice is 0.

We start working out the consequences of the equation $A(z)U(z) = U(z/p)T(z)$ from the bottom up. Row r gives $a_r = t_r$.

From row $r-1$ we get (after dividing by a suitable power of p)

$$a_{r-1}(z) - t_{r-1}(z) = p t_r \zeta(qz) - \zeta(pqz).$$

By (Res) applied to $a_{r-1}(z)$ ($t_{r-1}(z)$ contributes no residues) we must have $a_r = t_r = 1$, so the RHS of the last equation is the elliptic function $g_p(z)$. Then (Hol) applied to $t_{r-1}(z) = a_{r-1}(z) - g_p(z)$ gives that t_{r-1} is constant.

Row $r-2$ now gives

$$\frac{1}{2} \zeta(pqz)^2 + g_p(z) \zeta(pqz) + a_{r-2}(z) = t_{r-2}(z) + p t_{r-1} \zeta(qz) + \frac{1}{2} p^2 \zeta(qz)^2.$$

Rearranging the terms this gives

$$a_{r-2}(z) - t_{r-2}(z) = \frac{1}{2} g_p(z)^2 + p t_{r-1} \zeta(qz).$$

(Res) gives $t_{r-1} = 0$, hence also $a_{r-1}(z) = g_p(z)$. By (Hol) t_{r-2} is constant and $a_{r-2}(z) - t_{r-2} = \frac{1}{2} g_p(z)^2$.

This was the case $i = r-1$ of the second bullet, the base of the induction. Consider now row $r-k$, $k \geq 3$, corresponding to case $i = r-k+1 \leq r-2$ of the second bullet. By the induction hypothesis (with $i \geq r-k+2$) we know that t_{r-k+1} is constant and $t_{r-k+2} = \cdots = t_{r-1} = 0$.

Cancelling out a power of p we get

$$\sum_{j=0}^{k-1} \frac{1}{j!(k-j)!} \zeta(pqz)^{k-j} g_p(z)^j + a_{r-k}(z) = t_{r-k}(z) + p t_{r-k+1} \zeta(qz) + \frac{1}{k!} (p \zeta(qz))^k.$$

Recalling that $p\zeta(qz) = \zeta(pqz) + g_p(z)$, the binomial theorem gives

$$a_{r-k}(z) - t_{r-k}(z) = pt_{r-k+1}\zeta(qz) + \frac{1}{k!}g_p(z)^k.$$

As before, (Res) gives $t_{r-k+1} = 0$, as well as $a_{r-k+1}(z) = \frac{1}{(k-1)!}g_p(z)^{k-1}$ and then (Hol) yields that t_{r-k} is constant and $a_{r-k}(z) - t_{r-k} = \frac{1}{k!}g_p(z)^k$. The induction step is thereby established, and with it the proof of the Lemma. \square

5.5. Interlude: rank 3 modules. The higher the rank, the more options there are to assemble an elliptic (p, q) -difference module from the special modules M_r^{sp} and the ones obtained from commuting pairs of scalar matrices (A_0, B_0) . We illustrate this by classifying the rank 3 modules.

Proposition 32. *Every rank 3 module belongs to one of the following mutually disjoint classes:*

- (i) *Type (1,1,1): A module represented by a commuting pair of scalar matrices (A_0, B_0) .*
- (ii) *Type (2,1): $M_2^{sp}(a, b) \oplus M_1(a', b')$.*
- (iii) *Type (2,1): a non-split extension of $M_1(a, b)$ by $M_2^{sp}(a, b)$. For every a, b there is a family of pairwise non-isomorphic modules of this type indexed by $\mathbb{P}^1(\mathbb{C})$.*
- (iv) *Type (2,1): a non-split extension of $M_2^{sp}(a, b)$ by $M_1(pa, qb)$. For every a, b there is a family of pairwise non-isomorphic modules of this type indexed by $\mathbb{P}^1(\mathbb{C})$.*
- (v) *Type (3): $M_3^{sp}(a, b)$.*

Proof. We have classified the modules of type (1,1,1) or (3). It remains to classify modules of type (2,1). We shall do it by finding canonical forms for the matrices T, S (thereby for A, B) in the equations

$$A(z) = U(z/p)T(z)U(z)^{-1}, \quad B(z) = U(z/q)S(z)U(z)^{-1},$$

where

$$U(z) = \begin{pmatrix} 1 & \zeta(pqz) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We are allowed to conjugate A, B, T and S by scalar invertible matrices that commute with $U(z)$. Up to the center, they are of the form

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & * & * \end{pmatrix}.$$

We call these matrices *legitimate*.

Writing A, B, T and S in blocks, and applying Lemma 30 for $r = 1$ and 2 we may assume that

$$T = \begin{pmatrix} a & 0 & t_{13}(z) \\ 0 & pa & t_{23}(z) \\ t_{31}(z) & t_{32}(z) & a' \end{pmatrix}, \quad S = \begin{pmatrix} b & 0 & s_{13}(z) \\ 0 & qb & s_{23}(z) \\ s_{31}(z) & s_{32}(z) & b' \end{pmatrix}$$

where $a, b, a', b' \in \mathbb{C}^\times$ and the t_{ij} and s_{ij} are holomorphic functions. For A and B we get

$$A = \begin{pmatrix} a & ag_p(z) & a_{13}(z) \\ 0 & pa & a_{23}(z) \\ a_{31}(z) & a_{32}(z) & a' \end{pmatrix}, \quad B = \begin{pmatrix} b & bg_q(z) & b_{13}(z) \\ 0 & qb & b_{23}(z) \\ b_{31}(z) & b_{32}(z) & b' \end{pmatrix}$$

where the a_{ij} and b_{ij} are elliptic functions. Using

$$A(z)U(z) = U(z/p)T(z)$$

the bottom row gives

$$(a_{31}(z), a_{31}(z)\zeta(pqz) + a_{32}(z), a') = (t_{31}(z), t_{32}(z), a').$$

By (Hol) $a_{31} = t_{31}$ is constant, and then by (Res) $a_{31} = t_{31} = 0$ and $a_{32} = t_{32}$ is constant. From the last column we get

$${}^t(a_{13}(z), a_{23}(z), a') = {}^t(t_{13}(z) + \zeta(qz)t_{23}(z), t_{23}(z), a').$$

This implies, in the same way, that $a_{23} = t_{23} = 0$ and $a_{13} = t_{13}$ is constant. Similarly for S and B . We conclude that

$$T = \begin{pmatrix} a & 0 & t \\ 0 & pa & 0 \\ 0 & t' & a' \end{pmatrix}, \quad S = \begin{pmatrix} b & 0 & s \\ 0 & qb & 0 \\ 0 & s' & b' \end{pmatrix}$$

are *scalar* matrices. The consistency equation for A and B forces T and S to *commute*. This yields

$$(5.6) \quad st' = ts', \quad (b' - b)t = (a' - a)s, \quad (b' - qb)t' = (a' - pa)s'.$$

Assume that $a' \neq a$. Conjugating T by

$$\begin{pmatrix} 1 & \lambda & \\ & 1 & \\ & & 1 \end{pmatrix}$$

replaces t by $t + \lambda(a' - a)$, so an appropriate choice of λ kills it and we may assume $t = 0$. Equation (5.6) forces then $s = 0$ also. Symmetrically, if $b' \neq b$ we may assume $s = 0$ and deduce that $t = 0$. Thus if $(a', b') \neq (a, b)$ we may assume that $s = t = 0$. In the process of killing t and s we might have introduced a non-zero entry at T_{12} and S_{12} , but these may now be killed by conjugation by a matrix of the form

$$\begin{pmatrix} 1 & \mu & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Assume that $(a', b') \neq (pa, qb)$. Similar arguments show that conjugating by a suitable legitimate matrix we may assume that $s' = t' = 0$. Furthermore, if $s = t = 0$, this is unchanged by the conjugation.

We conclude that if $(a', b') \neq (a, b), (pa, qb)$ we may take T and S diagonal. In this case M is of class (ii), i.e. a direct sum of $M_2^{sp}(a, b)$ and $M_1(a', b')$.

If $(a', b') = (a, b)$ we may assume $s' = t' = 0$. In this case

$$T = \begin{pmatrix} a & 0 & t \\ 0 & pa & 0 \\ 0 & 0 & a \end{pmatrix}, \quad S = \begin{pmatrix} b & 0 & s \\ 0 & qb & 0 \\ 0 & 0 & b \end{pmatrix}.$$

If $s = t = 0$ we land again in class (ii). Otherwise, conjugation by $\text{diag}[1, 1, u]$ shows that the only further invariant of M is $(s : t) \in \mathbb{P}^1(\mathbb{C})$. In this case

$$A = \begin{pmatrix} a & ag_p(z) & t \\ 0 & pa & 0 \\ 0 & 0 & a \end{pmatrix}, \quad B = \begin{pmatrix} b & bg_q(z) & s \\ 0 & qb & 0 \\ 0 & 0 & b \end{pmatrix}$$

and we are in class (iii). The module M can be described as the push-out

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & M_1(a, b) & \rightarrow & M_2^{sp}(a, b) & \rightarrow & M_1(pa, qb) \rightarrow 0 \\
 & & \downarrow & \square & \downarrow & & \\
 0 & \rightarrow & M' & \rightarrow & M & & \\
 & & \downarrow & & \downarrow & & \\
 & & M_1(a, b) & = & M_1(a, b) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

where \square is co-cartesian. Here M' is a rank 2 scalar extension with invariant $(s : t)$. By this we mean that there exists a basis of M' with respect to which the matrices of Φ_σ and Φ_τ are the scalar matrices

$$\begin{pmatrix} a & t \\ 0 & a \end{pmatrix}^{-1}, \quad \begin{pmatrix} b & s \\ 0 & b \end{pmatrix}^{-1},$$

respectively. Note that $(s : t)$ is independent of the chosen basis.

Finally, if $(a', b') = (pa, qb)$ we may assume that $s = t = 0$, so

$$T = \begin{pmatrix} a & 0 & 0 \\ 0 & pa & 0 \\ 0 & t' & pa \end{pmatrix}, \quad S = \begin{pmatrix} b & 0 & 0 \\ 0 & qb & 0 \\ 0 & s' & qb \end{pmatrix}.$$

Once again, if $(s', t') \neq (0, 0)$ then

$$A = \begin{pmatrix} a & ag_p(z) & 0 \\ 0 & pa & 0 \\ 0 & t' & pa \end{pmatrix}, \quad B = \begin{pmatrix} b & bg_q(z) & 0 \\ 0 & qb & 0 \\ 0 & s' & qb \end{pmatrix}.$$

Now we are in class (iv), M is the pull-back

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & M_1(pa, qb) & = & M_1(pa, qb) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & M & \rightarrow & M' & \rightarrow & 0 \\
 & & \downarrow & \square & \downarrow & & \\
 0 & \rightarrow & M_1(a, b) & \rightarrow & M_2^{sp}(a, b) & \rightarrow & M_1(pa, qb) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

where \square is cartesian, M' is as before, and has invariant $(s' : t')$. \square

5.6. Conclusion of the proof.

5.6.1. *Legitimate matrices.* We turn to the general case, and assume that

$$U(z) = \oplus_{i=1}^k U_{r_i}(pq, 0; z) = \oplus_{i=1}^k U_{r_i}(z)$$

in block-diagonal form.

Lemma 33. *The scalar matrices commuting with $U(z)$ are the matrices which, in block form (the (i, j) block being of size $r_i \times r_j$), are of the shape*

$$E = (E_{ij})_{1 \leq i, j \leq k}$$

where

$$U_{r_i} E_{ij} = E_{ij} U_{r_j}.$$

Furthermore,

$$E_{ij} = \begin{pmatrix} 0 & E_{ij}^* \\ 0 & 0 \end{pmatrix},$$

where E_{ij}^* is an $s \times s$ invertible, upper triangular matrix ($0 \leq s \leq \min\{r_i, r_j\}$) of the form

$$E_{ij}^* = \alpha \exp\left(\sum_{\ell=1}^{s-1} \lambda_\ell N_s^\ell\right),$$

for some $\lambda_\ell \in \mathbb{C}$ and $\alpha \in \mathbb{C}^\times$. Conversely, any such a matrix E commutes with $U(z)$.

Proof. We omit the proof. \square

We call such matrices E , commuting with $U(z)$, *legitimate*.

5.6.2. *Block arithmetic.* We shall investigate the consequences of the equation

$$A(z)U(z) = U(z/p)T(z),$$

written in a block form (block (i, j) being of size $r_i \times r_j$). Fix (i, j) and write, to simplify the notation, $n = r_i$ and $m = r_j$. We have

$$(5.7) \quad A_{ij}(z)U_m(z) = U_n(z/p)T_{ij}(z).$$

Lemma 34. *The $n \times m$ matrix T_{ij} is scalar. We have*

$$T_{ij} = \begin{pmatrix} 0 & T_{ij}^* \\ 0 & 0 \end{pmatrix},$$

where T_{ij}^* is an $s \times s$ invertible, upper-triangular matrix ($0 \leq s \leq \min\{m, n\}$) of the form

$$T_{ij}^* = \exp\left(-\sum_{\ell=1}^{s-1} \lambda_\ell N_s^\ell\right) \cdot \alpha_{ij} T_s^{sp} \cdot \exp\left(\sum_{\ell=1}^{s-1} \lambda_\ell N_s^\ell\right)$$

for some $\lambda_\ell \in \mathbb{C}$ and $\alpha_{ij} \in \mathbb{C}^\times$.

Similarly, with the same s , λ_ℓ and α_{ij}

$$A_{ij} = \begin{pmatrix} 0 & A_{ij}^* \\ 0 & 0 \end{pmatrix},$$

where

$$A_{ij}^* = \exp\left(-\sum_{\ell=1}^{s-1} \lambda_\ell N_s^\ell\right) \cdot \alpha_{ij} A_s^{sp} \cdot \exp\left(\sum_{\ell=1}^{s-1} \lambda_\ell N_s^\ell\right).$$

Proof. We prove the assertions on T_{ij} by induction on $n + m$, and assume that $n \geq m$, the other case being treated similarly. If $n = m$ all the matrices in (5.7) are square of size $n \times n$. By Lemma 25 and its corollary we get that A_{ij} and T_{ij} are upper-triangular, with constants along the diagonal. Note that the proof of that lemma did not use the fact that T and A were invertible, an assumption that is no longer valid for the *blocks* of our original A and T .

Arguing as in Lemma 30, using (Hol) and (Res), we find that the diagonal of T_{ij} (equal to the diagonal of A_{ij}) is of the form $\alpha(1, p, \dots, p^{n-1})$. If $\alpha \neq 0$ then T_{ij} and A_{ij} are invertible and Lemma 30 gives us the desired form of $T_{ij} = T_{ij}^*$ (in this

case). If $\alpha = 0$ the bottom rows of A_{ij} and T_{ij} vanish, so writing A'_{ij} and T'_{ij} for the matrices of size $(n-1) \times n$ obtained by deleting the last rows of A_{ij} and T_{ij} ,

$$A'_{ij}(z)U_n(z) = U_{n-1}(z/p)T'_{ij}(z),$$

and we finish the proof by induction.

If $n > m$ it is enough to show that the bottom row of T_{ij} vanishes, because then we may use induction in the same way as we have just done when n was equal to m and α was 0. Write

$$T_{ij}(z) = \begin{pmatrix} * \\ T_{ij}^\dagger(z) \end{pmatrix}$$

where T_{ij}^\dagger is of size $m \times m$. Apply the same notation to A_{ij} . We have

$$A_{ij}^\dagger(z)U_m(z) = U_m(z/p)T_{ij}^\dagger(z).$$

By the induction hypothesis,

$$T_{ij}^\dagger = \begin{pmatrix} t & * & * & \cdots & * \\ 0 & pt & * & & * \\ \vdots & 0 & p^2t & & \vdots \\ & & 0 & \ddots & * \\ 0 & \cdots & 0 & & p^{m-1}t \end{pmatrix}.$$

From the first entry in row $n-m$ of (5.7) we get

$$a_{n-m,1}(z) \cdot 1 = 1 \cdot t_{n-m,1}(z) + \zeta(qz) \cdot t.$$

By (Res), we must have $t = 0$, hence the bottom row of T_{ij} vanishes.

Finally, the analogous statements for A_{ij} follow, by block multiplication, from the fact that $\exp(-\sum_{\ell=1}^{s-1} \lambda_\ell N_s^\ell)$ commutes with $U_s(z)$. \square

Similar formulae hold of course for S and B .

5.6.3. Main Structure Theorem.

Theorem 35. *Let $p \geq 2$ and $q \geq 2$ be relatively prime integers. Let M be an elliptic (p, q) -difference module of rank r , and let (r_1, \dots, r_k) be its type, $r_1 \leq r_2 \leq \cdots \leq r_k$, $\sum_{i=1}^k r_i = r$. Let*

$$U(z) = \bigoplus_{i=1}^k U_{r_i}(z) = \bigoplus_{i=1}^k U_{r_i}(pq, 0; z)$$

in block-diagonal form. Then, in an appropriate basis, M is represented by a consistent pair (A, B) of matrices from $G(K)$ for which

$$U(z/p)^{-1}A(z)U(z) = T, \quad U(z/q)^{-1}B(z)U(z) = S$$

are commuting scalar matrices.

Writing $T = (T_{ij})$ and $S = (S_{ij})$ in block form, the (i, j) block of size $r_i \times r_j$, T_{ij} and S_{ij} are then of the form prescribed in Lemma 34.

Conversely, for any collection of scalar matrices T_{ij} and S_{ij} of the above form, such that $T = (T_{ij})$ and $S = (S_{ij})$ commute and are invertible,

$$A(z) = U(z/p)TU(z)^{-1}, \quad B(z) = U(z/q)SU(z)^{-1}$$

is a consistent pair of matrices from $G(K)$.

The matrices T and S are uniquely determined by the module M up to conjugation by an invertible matrix E as in Lemma 33.

The pair (A, B) is gauge-equivalent to a scalar pair if and only if the type of M is $(1, 1, \dots, 1)$. In this case $U = I$ and the above $(A, B) = (T, S)$ are already scalar.

Proof. The results obtained so far yield the first (direct) part of the theorem. For the converse, note that if T and S are invertible and commute, then A and B are invertible and satisfy the consistency equation. Lemma 34 shows that their entries are elliptic functions, i.e. they belong to $G(K)$.

A change of basis of M results in a gauge transformation replacing T and S by $C(z/p)^{-1}TC(z)$ and $C(z/q)^{-1}SC(z)$. If these are constant, say T' and S' , then

$$C(z/p) = TC(z)T'^{-1}.$$

Expanding C at the origin as a Laurent expansion we see that C must be a Laurent polynomial, since $M \mapsto TMT'^{-1}$ can have only finitely many eigenvalues on $M_r(\mathbb{C})$. Since the entries of C are elliptic, we deduce that C is scalar. But C must commute with U too so it must be a legitimate matrix.

Finally, if the type is $(1, 1, \dots, 1)$ then $U = I$ and $(A, B) = (T, S)$. On the other hand, a module M admitting a \mathbb{C} -structure gives rise to the trivial vector bundle \mathcal{E} , so its type must be $(1, 1, \dots, 1)$. \square

5.7. Simple elliptic (p, q) -difference modules. If the type of M is (r) we have seen that M is a successive extension of 1-dimensional modules. The same is true if the type is $(1, 1, \dots, 1)$ because any two commuting scalar matrices can be brought into triangular forms with respect to the same basis.

Problem 36. Is it true that any simple elliptic (p, q) -difference module is 1-dimensional?

In Proposition 32 we have analyzed also modules of type $(2, 1)$, and it follows that the answer to our question is positive in rank ≤ 3 .

In general, the question is the following. Fix a type (r_1, r_2, \dots, r_k) . Given commuting invertible matrices T and S in block form as in Lemma 34, does there exist an invertible legitimate matrix E as in Lemma 33 such that ETE^{-1} and ESE^{-1} are simultaneously upper-triangular?

We shall not pursue this question here, although it need not be too difficult.

6. AN ELLIPTIC ANALOGUE OF THE CONJECTURE OF LOXTON AND VAN DER POORTEN

6.1. The conjecture of Loxton and van der Poorten and its additive analogue. Let $K = \mathbb{C}(x^{1/s} \mid s \in \mathbb{N})$. Let p and q be multiplicatively independent natural numbers. Define $\sigma, \tau \in \text{Aut}(K)$ by

$$\sigma f(x) = f(x^p), \quad \tau f(x) = f(x^q).$$

Extend the definition to the field of Puiseux series $\mathcal{K} = \bigcup_{s \geq 1} \mathbb{C}((x^{1/s}))$ (this field is not complete; it is the algebraic closure of $\mathbb{C}((x))$). The following theorem was conjectured by Loxton and van der Poorten [vdPo] and proved by Adamczewski and Bell [Ad-Be]. The proof was based on Cobham's theorem in the theory of automata [Co], and was quite intricate. Schäfke and Singer supplied a more conceptual proof in [Sch-Si], which in turn yields an elegant proof of Cobham's theorem.

Theorem. [Ad-Be, Sch-Si] *Assume that $f \in \mathcal{K}$ satisfies the two (p, q) -Mahler equations*

$$\begin{cases} a_0\sigma^n(f) + \cdots + a_{n-1}\sigma(f) + a_n f = 0 \\ b_0\tau^m(f) + \cdots + b_{m-1}\tau(f) + b_m f = 0 \end{cases}$$

with coefficients $a_i, b_j \in K$. Then $f \in K$.

It is easy to use Galois descent to derive from the above a similar statement when the pair (K, \mathcal{K}) is replaced by $(\mathbb{C}(x), \mathbb{C}((x)))$. The advantage of working with K and \mathcal{K} as in our formulation is that σ and τ are automorphisms, and not merely endomorphisms, of these fields.

Theorem 2, mentioned in the introduction, has a similar consequence for a Laurent power series satisfying a pair of q -difference equations. Let the notation be as in Theorem 2. In particular $K = \mathbb{C}(x)$ now, and we let $\widehat{K} = \mathbb{C}((x))$.

Theorem. [Bez-Bou, Sch-Si] *Assume that $f \in \widehat{K}$ satisfies the two (p, q) -difference equations*

$$\begin{cases} a_0\sigma^n(f) + \cdots + a_{n-1}\sigma(f) + a_n f = 0 \\ b_0\tau^m(f) + \cdots + b_{m-1}\tau(f) + b_m f = 0 \end{cases}$$

with coefficients $a_i, b_j \in K$. Then $f \in K$.

6.2. An elliptic analogue. We shall now derive from Theorem 3 an elliptic analogue of the above two theorems.

Let $K = \bigcup K_\Lambda$ be as before, where K_Λ is the field of Λ -elliptic functions, and Λ runs over all the sublattices of a fixed lattice $\Lambda_0 \subset \mathbb{C}$. Let R be the ring generated over K by the functions z, z^{-1} and $\zeta(z, \Lambda)$ for all Λ as above. Thus

$$R = \bigcup R_\Lambda, \quad R_\Lambda = K_\Lambda[z, z^{-1}, \zeta(z, \Lambda)].$$

Note that $n\zeta(z, \Lambda) - \zeta(nz, \Lambda) \in K_\Lambda$, so instead of $\zeta(z, \Lambda)$ we could have taken $\zeta(nz, \Lambda)$ for any $n \geq 1$. Note also that if $\Lambda' \subset \Lambda$ then

$$\wp(z, \Lambda) - \sum_{\omega \in \Lambda/\Lambda'} \wp(z + \omega, \Lambda')$$

is Λ -periodic and everywhere holomorphic, hence it is constant. Integrating we get that

$$\zeta(z, \Lambda) - \sum_{\omega \in \Lambda/\Lambda'} \zeta(z + \omega, \Lambda') = az + b$$

for some $a, b \in \mathbb{C}$. Since $\zeta(z + \omega, \Lambda') - \zeta(z, \Lambda') \in K_{\Lambda'}$ we get that

$$\zeta(z, \Lambda) - [\Lambda : \Lambda']\zeta(z, \Lambda') \in K[z, z^{-1}].$$

We conclude that it is enough to adjoin $\zeta(z, \Lambda)$ for one lattice, i.e.

$$R = K[z, z^{-1}, \zeta(z, \Lambda_0)].$$

Let p and q be relatively prime integers greater than 1, and, as before, define $\sigma, \tau \in \text{Aut}(K)$ by

$$\sigma f(z) = f(z/p), \quad \tau f(z) = f(z/q).$$

Let $\widehat{K} = \mathbb{C}((z))$ and extend σ and τ to \widehat{K} . We regard R as a subring of \widehat{K} , associating to any $f \in R$ its Laurent expansion at 0.

Theorem 37. *Let $f \in \widehat{K}$ satisfy*

$$\begin{cases} a_0\sigma^n(f) + \cdots + a_{n-1}\sigma(f) + a_n f = 0 \\ b_0\tau^m(f) + \cdots + b_{m-1}\tau(f) + b_m f = 0 \end{cases}$$

where a_i and $b_j \in K$. Then $f \in R$.

Remark 38. (i) As mentioned before, we do not know if the theorem remains true under the weaker hypothesis that p and q are only multiplicatively independent.

(ii) The theorem is equivalent to the same theorem with $\sigma^{-1}f(z) = f(pz)$ and $\tau^{-1}f(z) = f(qz)$ replacing σ and τ . It may also be phrased as saying that if all the $a_i, b_j \in K_\Lambda$ then there exists a $\Lambda' \subset \Lambda$ such that $f \in R_{\Lambda'}$.

(iii) One may ask for the relation between Λ and Λ' . To be precise, suppose (using the equivalent formulation with σ^{-1} and τ^{-1}) that

$$\begin{cases} a_0(z)f(p^n z) + \cdots + a_{n-1}(z)f(pz) + a_n(z)f(z) = 0 \\ b_0(z)f(q^m z) + \cdots + b_{m-1}(z)f(qz) + b_m(z)f(z) = 0, \end{cases}$$

where a_i and b_j are Λ -periodic. What is the largest lattice Λ' such that $f \in R_{\Lambda'}$? A more careful examination of our proof may shed light on this question.

(iv) The collection of all $f \in \widehat{K}$ satisfying an elliptic p -difference equation and a similar q -difference equation simultaneously, is easily seen to be a *subring* R_0 of \widehat{K} , containing K . It contains $z^{\pm 1}$, hence all Laurent polynomials. If $f = \zeta(z, \Lambda)$ then $p\sigma(f) - f \in K$ and similarly $q\tau(f) - f \in K$. It follows that $\zeta(z, \Lambda) \in R_0$ as well. Thus $R_0 = R$ and our theorem is optimal.

Lemma 39. *Let $h_i \in \widehat{K}$ ($1 \leq i \leq r$) and assume that there is a matrix $T^{-1} = (t_{ij}) \in G(\mathbb{C})$ such that*

$$h_j(z/p) = \sum_{i=1}^r t_{ij} h_i(z).$$

Then every $h_i \in \mathbb{C}[z^{-1}, z]$ is a Laurent polynomial.

Proof. Let $C = (c_{ki}) \in G(\mathbb{C})$ be such that $C^{-1}TC = \widetilde{T}$ is upper triangular. Write $\widetilde{T}^{-1} = (\widetilde{t}_{ij})$. Replacing the column vector $h = {}^t(h_1, \dots, h_r)$ by the vector \widetilde{h} with

$$\widetilde{h}_j = \sum_{k=1}^r c_{kj} h_k$$

we get that

$$\widetilde{h}_j(z/p) = \sum_{i=1}^r \widetilde{t}_{ij} \widetilde{h}_i(z)$$

and if the \widetilde{h}_i are Laurent polynomials, so are the h_i . We may therefore assume, without loss of generality, that T is upper triangular, so $t_{ij} = 0$ for $i > j$. The equation $h_1(z/p) = t_{11}h_1(z)$ is satisfied only if $h_1 = z^n$ for some $n \in \mathbb{Z}$ and $t_{11} = p^{-n}$. We conclude that every h_i is in $\mathbb{C}[z^{-1}, z]$ by induction on i , noting that if $g \in \mathbb{C}[z^{-1}, z]$ and

$$h(z/p) = th(z) + g(z)$$

then $h \in \mathbb{C}[z^{-1}, z]$ as well. □

We can now prove the theorem.

Proof. Let f be as in the theorem, Let $M \subset \widehat{K}$ be the K -subspace spanned by $\sigma^i \tau^j f$. By the assumption that f satisfies, it is a finite dimensional space, and in fact

$$r = \dim_K M \leq nm.$$

This M is clearly invariant under the group Γ generated by $\Phi_\sigma = \sigma$ and $\Phi_\tau = \tau$ and is therefore an elliptic (p, q) -difference module. Let g_1, \dots, g_r be a basis of M over K with respect to which Φ_σ acts like A^{-1} and Φ_τ acts like B^{-1} , where A and B are as in Theorem 35, namely

$$A(z) = U(z/p)TU(z)^{-1}, \quad B(z) = U(z/q)SU(z)^{-1}$$

with $T, S \in G(\mathbb{C})$. Use the matrix $U(z) = (u_{ij})$ to transform the vector $g = {}^t(g_1, \dots, g_r)$ to a vector $h = {}^t(h_1, \dots, h_r)$

$$h_j = \sum_{i=1}^r u_{ij} g_i \in \widehat{K}$$

on which σ acts via the scalar matrix T^{-1} , and τ via S^{-1} (the h_i need not be in M). By the Lemma, the h_i are Laurent polynomials. Since the entries u_{ij} of $U(z)$ are in R , so are the g_i , and hence also f . \square

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