$\mathcal{L}$-INVARIANTS AND p-ADIC SPECIAL SERIES (D'APRES BREUIL)

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This is a survey of [Br04], with some background material. Our approach is sometimes different than Breuil's, but none of the results is new.

1. Notation

1.1. **General.** Let K be a finite extension of $\mathbb{Q}_p$, and denote by $|.|_K$ the normalized absolute value on K. Let $\mathbb{C}_p$ be the completion of a fixed algebraic closure of K, and $|.|$ the extension of $|.|_K$ to $\mathbb{C}_p$. Let E (the "field of coefficients") be a finite extension of K in $\mathbb{C}_p$. Fix $\mathcal{L} \in E$ and let $\log_{\mathcal{L}}$ be the unique homomorphism from $K^\times$ to E which is given by the usual power series for $\log(1+x)$ if $|x| < 1$, and satisfies $\log_{\mathcal{L}}(p) = \mathcal{L}$. The Iwasawa logarithm is $\log_0$, and for any other $\mathcal{L}$ we have $\log_{\mathcal{L}}(x) = \log_0(x) + ord_p(x) \cdot \mathcal{L}$, where $ord_p : K^\times \to \mathbb{Q}$ is the valuation, normalized by $ord_p(p) = 1$. The symbol π_K will stand for a uniformizer of K, if we have to choose one.

1.2. **The Drinfel'd upper half plane and the tree.** We denote by $\mathfrak{X} \subset \mathbb{P}^1$ the Drinfel'd upper half plane for K, i.e. the complement of the compact set $\mathbb{P}^1(K)$ in $\mathbb{P}^1$, viewed as a rigid analytic space over E via base change (not to be confused with the Drinfel'd upper half plane for E). The group $G = GL_2(K)$ acts on it on the left as usual.

We denote by $\mathcal{T}$ the Bruhat-Tits tree of $PGL_2(K)$, by $\mathcal{T}_0$ its vertices and by $\mathcal{T}_1$ its *oriented* edges. The group G acts on $\mathcal{T}$. The ends of $\mathcal{T}$ are canonically and G-equivariantly identified with $\mathbb{P}^1(K)$, and if $\varepsilon \in \mathcal{T}_1$ we denote by K_ε the disk in $\mathbb{P}^1(K)$ corresponding under this identification to the ends passing through ε (recall that an end is an equivalence class of geodesics, and we say that an end passes through ε if it contains a representative passing through ε).

There is a reduction map r from $\mathfrak{X}(\mathbb{C}_p)$ to the real realization $|\mathcal{T}|$ of the tree. If v, ε and T are a vertex, an (oriented) edge or a finite subtree, we denote by $\mathfrak{X}_v$, $\mathfrak{X}_\varepsilon$ and $\mathfrak{X}_T$ the preimages under r of $|v|$, $|\varepsilon|$ and $|T|$ respectively. The first and the last are affinoids, the middle one an oriented open annulus (by $|\varepsilon|$ we mean the *open* interval connecting the end points of ε in $|\mathcal{T}|$). If $\varepsilon = (v, v') \in \mathcal{T}_1$, then $\mathcal{T} - \{\varepsilon\}$ is the union of two trees, $\mathcal{T}_v$ and $\mathcal{T}_{v'}$, rooted in the two vertices. We denote by U_ε the open disk in $\mathbb{P}^1$ which is the union of K_ε and the part of $\mathfrak{X}$ undergoing reduction to $|\varepsilon| \cup |\mathcal{T}_{v'}|$.

If T is a finite subtree of $\mathcal{T}$ we let $\mathcal{E}(T)$, the *ends of* T, stand for the collection of oriented edges $\varepsilon = (v, v')$ where $v \in T$ and $v' \notin T$. We let

$$U_T = \bigcup_{\varepsilon \in \mathcal{E}(T)} U_\varepsilon = \mathbb{P}^1 - \mathfrak{X}_T. \tag{1}$$

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If $T \subset T'$ then $U_{T'} \subset U_T$. We write $T \subset\subset T'$ if T is contained in the interior of T' (relative to $\mathcal{T}$).

We denote by C^1_{har} the G-module of harmonic E-valued 1-cochains on $\mathcal{T}$. This is the space of alternating functions c from $\mathcal{T}_1$ to E satisfying, for every vertex $v \in \mathcal{T}_0$,

$$\sum_{\varepsilon \in \mathcal{E}(v)} c(\varepsilon) = 0. \tag{2}$$

If $c \in C^1_{har}$ then the associated distribution μ_c is the finitely additive function on the Boolean algebra of compact open subsets of $\mathbb{P}^1(K)$ satisfying

$$\mu_c(K_\varepsilon) = c(\varepsilon). \tag{3}$$

It has total mass 0, and any finitely additive distribution of total mass 0 is of this form for a unique c. Alternatively, one may think of μ_c as a linear functional on the locally constant functions annihilating the constants.

We denote by $\mathcal{O}$ the sheaf of rigid analytic functions on $\mathfrak{X}$, and by Ω the sheaf of rigid analytic differential forms.

1.3. **Residues.** We shall use $res_\zeta\omega$ to denote the residue of a differential form ω at a point ζ where it is meromorphic. We shall use $Res_\varepsilon\omega$ to denote the residue of ω over an oriented annulus where it is analytic. This is the coefficient a_{-1} in the Laurent expansion

$$\omega = \sum_{n=-\infty}^{\infty} a_n z^n dz \tag{4}$$

of ω with respect to a uniformizing parameter z in the annulus. It is well known that this coefficient is independent of the parameter.

2. Some p-adic functional analysis

2.1. **Topological vector spaces.** The standard reference to all that will be mentioned in this section is [NFA]. The field E is locally compact, hence also spherically complete. All topological vector spaces will be over E, and will be assumed to be Hausdorff and *locally convex.* If V is such a topological vector space, V' will always denote its *strong dual* (denoted V'_b in [NFA]). We recall the definition. A set S in a topological vector space is *bounded* if every open neighborhood of the origin in V contains a scalar multiple of S (in a normed space this is equivalent to being bounded in the norm). The strong dual consists of the continuous linear functionals on V, endowed with the strong topology. A basis of neighborhoods at the origin for the strong topology is given by

$$U(S, \delta) = \{\lambda \in V' || \lambda(u)| < \delta \text{ for all } u \in S\} \tag{5}$$

for all bounded sets S, and $\delta > 0$. Clearly V' is again Hausdorff and locally convex. If V is a normed linear space, then this is the topology on V' induced by the dual norm, and V' is then Banach (a complete normed linear space).

For every topological vector space V there is a canonical continuous map $\iota : V \to (V')'$. We call V *reflexive* if this map is a topological isomorphism. In sharp contrast to the classical case, p-adic Banach spaces are reflexive if and only if they are finite dimensional.

A continuous linear map $i : V \to W$ between two Banach spaces is called *compact* if for every bounded $S \subset V$, $\overline{i(S)}$ is compact in W (the bar denotes, as usual, closure). If i is compact so is the dual map $i' : W' \to V'$. We remark that the treatment of compact operators is greatly simplified by our assumption that the field E is locally compact. If this is not the case, there are various equivalent definitions of what it means for i to be compact, but they are all more complicated.

If $(V_n, i_{n,m})_{n<m}$ is an inductive system of topological vector spaces, we topologize $V = \lim_{\to} V_n$ by endowing it with the finest locally convex topology with repsect to which all the maps $i_n : V_n \to V$ are continuous. We caution that in general V may not be Hausdorff, but if V_n are Banach and all the maps $i_{n,m}$ are compact and injective (as in the case considered below), V is also Hausdorff.

If $(V_n, \pi_{n,m})_{n>m}$ is a projective system, we topologize $V = \lim_{\leftarrow} V_n$ by endowing it with the coarsest topology with respect to which all the projections $\pi_n : V \to V_n$ are continuous. This topology is automatically locally convex and Hausdorff. It is the topology induced on V from the embedding as a closed subspace of the product $\prod V_n$, endowed with the product topology.

2.2. **Spaces of type (C) and (F).** We shall call a topological vector space W over E *of type (C)* if it is an inductive limit of Banach spaces under injective compact transition maps. We shall call a topological vector space V over E *of type (F)* if it is a projective limit of Banach spaces under compact transition maps (which may be taken to have a dense image).

Let $W = \lim_{\to} W_n$ where W_n are Banach and the transition maps $i_{n,m} : W_n \to W_m$ are injective compact maps. Then W is reflexive and complete, and the canonical map

$$\left(\lim_{\to} W_n\right)' \to \lim_{\leftarrow} W_n' \tag{6}$$

is a topological isomorphism ([NFA], 16.10).

Let $V = \lim_{\leftarrow} V_n$ where V_n are Banach and the transition maps are compact. Without changing V we may assume that the projections from V to V_n have dense images, in which case the projections $\pi_{n,m}$ also have dense images, hence their duals are injective compact maps as above. Then V is reflexive and complete and the canonical map

$$\lim_{\to} V_n' \to \left(\lim_{\leftarrow} V_n\right)' \tag{7}$$

is a topological isomorphism ([NFA], 16.5).

Duality therefore exchanges the two types of spaces and is an involution, despite the fact that the individual Banach spaces appearing in the limits do not have to be reflexive.

The typical example to bear in mind is

$$W = c_c(\mathbb{Z}), \; V = l(\mathbb{Z}). \tag{8}$$

Here W is the space of all maps from $\mathbb{Z}$ to E with finite support, and V is the space of all the maps from $\mathbb{Z}$ to E without any restriction. The pairing between W and V is the usual "inner product".

2.3. **The dual category of the category of Banach spaces.** The reference to this subsection is [ST02]. As infinite dimensional Banach spaces are never reflexive, the category of Banach spaces is not self-dual. It is therefore desirable to describe a category which is dual to it.

Consider the category whose objects are compact torsion-free topological $\mathcal{O}_E$-modules M with

$$Hom(M,N) = Hom_{\mathcal{O}_E,cont}(M,N) \otimes_{\mathcal{O}_E} E. \tag{9}$$

If B is a Banach space we let $\mathbf{D}(B)$ be the unit ball of its Banach dual B', equipped with the weak* topology. By the Banach Alaoglu theorem, it is a compact torsion-free module. Every $f \in Hom(B_1, B_2)$ induces by duality a map $\mathbf{D}(f) \in Hom(\mathbf{D}(B_2), \mathbf{D}(B_1))$.

Conversely, if M is a compact torsion-free $\mathcal{O}_E$-module we let

$$\mathbf{D}(M) = Hom_{\mathcal{O}_E,cont}(M,E). \tag{10}$$

In the sup norm it becomes a Banach space, and every $\phi \in Hom(M_1, M_2)$ induces by duality a map $\mathbf{D}(\phi) \in Hom(\mathbf{D}(M_2), \mathbf{D}(M_1))$.

Schneider and Teitelbaum prove that the natural maps $M \to \mathbf{D}(\mathbf{D}(M))$ and $B \to \mathbf{D}(\mathbf{D}(B))$ are topological isomorphisms and that the functors $\mathbf{D}$ induce an anti-equivalence between the two categories.

The typical example to bear in mind here is

$$B = c_0(\mathbb{Z}), \quad M = \mathcal{O}_E^{\mathbb{Z}}, \tag{11}$$

(B equipped with the sup norm, M with the product topology). Another important example that will show up later is $B = C(\Gamma, E)$ the space of continuous functions from a compact p-adic Lie group Γ to E, and $M = \mathcal{O}_E[[\Gamma]]$, the Iwasawa algebra of integral measures on Γ.

3. Morita duality (trivial coefficients)

3.1. **Certain G-modules of type (C).** Denote by $\mathcal{C}^{sm}$ the G-module of smooth (locally constant) E-valued functions on $\mathbb{P}^1(K)$. The group G acts through $gf(x) = f(g^{-1}x)$. The Steinberg representation

$$St = \mathcal{C}^{sm}/E \tag{12}$$

is smooth and irreducible.

Denote by $\mathcal{C}^{an}$ the G-module of locally analytic E-valued functions on $\mathbb{P}^1(K)$ (with the same G-action) and

$$\Sigma = \mathcal{C}^{an}/E, \tag{13}$$

the "locally analytic Steinberg representation".

We wish to express $\mathcal{C}^{sm}$ and $\mathcal{C}^{an}$ as spaces of type (C). To this end, let $Z_\varepsilon = U_\varepsilon - \mathfrak{X}_\varepsilon$ and

$$Z_T = \bigcup_{\varepsilon \in \mathcal{E}(T)} Z_\varepsilon, \tag{14}$$

a disjoint union of closed affinoid disks, contained in U_T, and containing $\mathbb{P}^1(K)$. If we denote by $\mathcal{C}^{sm}(Z_T)$ the locally constant rigid analytic functions on Z_T then

$$\mathcal{C}^{sm} = \lim_{\to} \mathcal{C}^{sm}(Z_T), \quad \mathcal{C}^{an} = \lim_{\to} \mathcal{O}(Z_T). \tag{15}$$

If $T \subset\subset T'$ then the transition maps in these limits are compact. This is clear for $\mathcal{C}^{sm}(Z_T)$, which are finite dimensional spaces. For $\mathcal{O}(Z_T)$, which are Banach under the sup norm on Z_T, it follows from the fact that if $R \in |E^\times| < 1$ and D_R is the closed affinoid disk of radius R, the restriction map

$$\mathcal{O}(D_1) \to \mathcal{O}(D_R) \tag{16}$$

is compact: every sequence of rigid analytic functions which are bounded by 1 on D_1, has a subsequence which is uniformly convergent on D_R. Once we divide out the constants, the spaces St and Σ inherit a structure of topological vector spaces of the same type.

Denote by $\mathcal{C}$ the G-module of *locally meromorphic* E-valued functions on $\mathbb{P}^1(K)$. For each $f \in \mathcal{C}$ there is a finite covering $\mathcal{U} = \{U_i\}$ of $\mathbb{P}^1(K)$ by disjoint disks, and points $\zeta_i \in U_i$, such that $f|U_i$ is given by a Laurent expansion around ζ_i with coefficients in E.

The E-submodule $\mathcal{R} \subset E(z)$ of rational functions defined over E, all of whose poles lie in $\mathbb{P}^1(K)$, is contained in $\mathcal{C}$ and stable under G. The *Theorem on Principal Parts* says that

$$\mathcal{C}^{an} \cap \mathcal{R} = E, \ \ \mathcal{C}^{an} + \mathcal{R} = \mathcal{C}, \tag{17}$$

in other words, any assignment of principal parts with coefficients form E at finitely many $\zeta_i \in K$ can be realized by a function from $\mathcal{R}$, uniquely up to a constant. We therefore have for the "locally analytic Steinberg"

$$\Sigma = \mathcal{C}^{an}/E = \mathcal{C}/\mathcal{R}. \tag{18}$$

Denote by $\Omega_{\mathcal{C}}$ the G-module of locally meromorphic E-valued differential forms on $\mathbb{P}^1(K)$.

Denote by $\Omega_{\mathcal{R}}$ the subspace of $\Omega_{\mathcal{C}}$ spanned by all fdg with $f, g \in \mathcal{R}$.

Given $\eta \in \Omega_{\mathcal{C}}$ we denote by $res(\eta)$ the sum of its residues at $\zeta \in \mathbb{P}^1(K)$.

3.2. **Certain G-modules of type (F).** Denote by $\mathcal{O}(\mathfrak{X})$ the G-module of rigid analytic functions on $\mathfrak{X}$ which are defined over E. The group G acts through $gf = f \circ g^{-1}$.

Denote by $\Omega(\mathfrak{X})$ the G-module of rigid analytic differential forms on $\mathfrak{X}$ defined over E, with the same G-action.

The spaces $\mathcal{O}(\mathfrak{X})$, $\Omega(\mathfrak{X})$ and C^1_{har} are all of type (F). For example,

$$\mathcal{O}(\mathfrak{X}) = \lim_{\leftarrow} \mathcal{O}(\mathfrak{X}_T) \tag{19}$$

($\mathcal{O}(\mathfrak{X}_T)$ a Banach space under the sup norm). The transition maps are compact whenever $T \subset\subset T'$ for the same reason as before. Note that the projection from $\mathcal{O}(\mathfrak{X})$ to $\mathcal{O}(\mathfrak{X}_T)$ has a dense image.

3.3. **Exact sequences and duality.** Given $\eta \in \Omega_{\mathcal{C}}$ and $f \in \mathcal{O}(\mathfrak{X})$ we define the pairing

$$\langle f, \eta \rangle = \sum_{\varepsilon \in \mathcal{E}(T)} Res_\varepsilon(f\eta) \tag{20}$$

where T is taken large enough so that on every U_ε, $\varepsilon \in \mathcal{E}(T)$, η is given by a convergent Laurent expansion around some $\zeta \in K_\varepsilon$. Since this is the case, $f\eta$ is analytic on the (oriented) annulus $\mathfrak{X}_\varepsilon$, and by Res_ε we mean its residue there. Cauchy's theorem in the affinoid $\mathfrak{X}_{v'}$ implies that if we replace T by the tree $T \cup [v, v']$ (for $v \in T$, $v' \notin T$), the sum is unchanged. The same theorem in the affinoid $\mathfrak{X}_T$ implies that if $\eta \in \Omega_{\mathcal{R}}$ then $\langle f, \eta \rangle = 0$. Indeed, $f\eta$ extends then analytically to $\mathfrak{X}_T$.

Given $h \in \mathcal{C}$ and $\omega \in \Omega(\mathfrak{X})$ we similarly define

$$\langle \omega, h \rangle = - \sum_{\varepsilon \in \mathcal{E}(T)} Res_\varepsilon(\omega h) \tag{21}$$

(note the minus sign!) for a large enough T. The same argument implies that this is well defined and that it vanishes for $h \in \mathcal{R}$.

If $f \in \mathcal{O}(\mathfrak{X})$ and $h \in \mathcal{C}$ then $\langle f, dh \rangle = \langle df, h \rangle$ because on any annulus $Res_\varepsilon d(fh) = 0$.

3.4. Morita duality.

Theorem 3.1. *The following two sequences are exact, and the above pairings set them up in duality, which is a perfect topological pairing of spaces of type (F) with spaces of type (C):*

$$0 \to E \ \to \ \mathcal{O}(\mathfrak{X}) \ \xrightarrow{d} \ \Omega(\mathfrak{X}) \ \to \ C^1_{har} \ \to 0 \tag{22}$$

and

$$0 \leftarrow E \xleftarrow{-res} \Omega_{\mathcal{C}}/\Omega_{\mathcal{R}} \xleftarrow{d} \mathcal{C}/\mathcal{R} \leftarrow \mathcal{C}^{sm}/E \ \leftarrow 0. \tag{23}$$

Proof. Exactness and commutativity are easy to check. The classical theorem of Morita is the assertion that the pairing between $\Omega(\mathfrak{X})$ and $\Sigma = \mathcal{C}^{an}/E = \mathcal{C}/\mathcal{R}$ identifies each of them topologically with the strong dual of the other. It is easily checked that it induces a duality between C^1_{har} and St realizing the harmonic cochains as the full (i.e. strong) dual of Steinberg. The remaining assertions follow from there. ∎

4. Breuil's duality (trivial coefficients)

4.1. Coleman primitives. Let $\mathcal{O}_{\mathcal{L}}(\mathfrak{X})$ be the space of functions F on $\mathfrak{X}$ which, on every affinoid $\mathfrak{X}_T$, are of the form

$$F|_{\mathfrak{X}_T} = f + \sum a_i \log_{\mathcal{L}}(z - \zeta_i) \tag{24}$$

with $f \in \mathcal{O}(\mathfrak{X}_T)$, $a_i \in E$ and $\zeta_i \in K$. (It is enough to take, for every $\varepsilon \in \mathcal{E}(T)$ such that U_ε is bounded, one $\zeta_i \in K_\varepsilon$.) This space is of type (F), and is stable under G. For every $F \in \mathcal{O}_{\mathcal{L}}(\mathfrak{X})$, $dF \in \Omega(\mathfrak{X})$, and if $\omega \in \Omega(\mathfrak{X})$, there is an $F_\omega \in \mathcal{O}_{\mathcal{L}}(\mathfrak{X})$, unique up to a constant from E, such that $dF_\omega = \omega$. Such an F_ω is called a (global) Coleman primitive of ω. We have a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \to & E & \to & \mathcal{O}(\mathfrak{X}) & \to & \Omega(\mathfrak{X}) & \to & C^1_{har} & \to & 0 \\ & & || & & \cap & & || & & & & \\ 0 & \to & E & \to & \mathcal{O}_{\mathcal{L}}(\mathfrak{X}) & \xrightarrow{d} & \Omega(\mathfrak{X}) & \to & 0 & & \end{array} \tag{25}$$

with exact rows. The snake lemma implies that $\mathcal{O}_{\mathcal{L}}(\mathfrak{X})/\mathcal{O}(\mathfrak{X}) \simeq C^1_{har}$.

Similarly we let $\mathcal{C}_{\mathcal{L}}$ be the functions on $\mathbb{P}^1(K)$ which are locally, in the neighborhood of any $\zeta \in \mathbb{P}^1(K)$, of the form $h + b \log_{\mathcal{L}}(z - \zeta)$ (or $h + b \log_{\mathcal{L}} z$ near ∞), with h meromorphic at ζ. This space is stable under G, for every $H \in \mathcal{C}_{\mathcal{L}}$, $dH \in \Omega_{\mathcal{C}}$, and every $\eta \in \Omega_{\mathcal{C}}$ has a (local) Coleman primitive $H_\eta \in \mathcal{C}_{\mathcal{L}}$, unique up to a function from $\mathcal{C}^{sm}$, such that $dH_\eta = \eta$.

We denote by

$$\mathcal{R}_{\mathcal{L}} = \{F_\omega | \omega \in \Omega_{\mathcal{R}}\} \tag{26}$$

the space of (global) Coleman primitives of $\omega \in \Omega_{\mathcal{R}}$ (F_ω is only determined up to a constant, and our space by definition contains E). We let

$$\Sigma_{\mathcal{L}} = \mathcal{C}_{\mathcal{L}}/\mathcal{R}_{\mathcal{L}}. \tag{27}$$

Note that Σ is embedded in $\Sigma_{\mathcal{L}}$, and the latter is again a space of type (C). If $H \in \mathcal{C}_{\mathcal{L}}$ we let its *residual divisor*

$$rdiv(H) = \sum_{\zeta \in \mathbb{P}^1(K)} res_\zeta(dH)[\zeta], \tag{28}$$

so that $res(dH) = \deg(rdiv(H))$. Since a residual divisor is realized by the residues of some $\omega \in \Omega_{\mathcal{R}}$ precisely when its degree is 0, an element of $\Sigma_{\mathcal{L}}$ belongs to Σ if and only if the degree of its residual divisor is 0. This proves that we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \leftarrow & E & \overset{-res}{\leftarrow} & \Omega_{\mathcal{C}}/\Omega_{\mathcal{R}} & \overset{d}{\leftarrow} & \Sigma & \leftarrow & St & \leftarrow & 0 \\ & & || & & \uparrow d & & || & & & & \\ 0 & \leftarrow & E & \leftarrow & \Sigma_{\mathcal{L}} & \leftarrow & \Sigma & \leftarrow & 0 & & \end{array} . \tag{29}$$

The snake lemma implies that the kernel of $d : \Sigma_{\mathcal{L}} \to \Omega_{\mathcal{C}}/\Omega_{\mathcal{R}}$ is simply St.

4.2. **A pairing between $\mathcal{O}_{\mathcal{L}}(\mathfrak{X})$ and $\Sigma_{\mathcal{L}}$.** Let $F \in \mathcal{O}_{\mathcal{L}}(\mathfrak{X})$, and $H \in \Sigma_{\mathcal{L}}$. We shall now define a pairing $[F, H]$ extending the pairing $\langle f, \eta \rangle$ between $\mathcal{O}(\mathfrak{X})$ and $\Omega_{\mathcal{C}}/\Omega_{\mathcal{R}}$, and also the pairing $\langle \omega, h \rangle$ between $\Omega(\mathfrak{X})$ and Σ, namely we shall have

$$[F, h] = \langle dF, h \rangle \tag{30}$$

if $h \in \Sigma$, and

$$[f, H] = \langle f, dH \rangle \tag{31}$$

if $f \in \mathcal{O}(\mathfrak{X})$.

By a slight abuse of language denote by H also a function from $\mathcal{C}_{\mathcal{L}}$ (in the specified class modulo $\mathcal{R}_{\mathcal{L}}$). Assume first that for any ζ in the support of $rdiv(H)$, there is an open disk U around ζ such that F is rigid analytic in $U \cap \mathfrak{X}$. In such a case, for a large enough finite subtree T, for every $\varepsilon \in \mathcal{E}(T)$, F or H must be analytic in $U_\varepsilon \cap \mathfrak{X}$. If F is analytic there we put $[F, H]_\varepsilon = Res_\varepsilon(F \cdot dH)$, and if H is analytic there we put $[F, H]_\varepsilon = -Res_\varepsilon(dF \cdot H)$. If both are analytic in $U_\varepsilon \cap \mathfrak{X}$, the two definitions agree, as we have seen. Put

$$[F, H] = \sum_{\varepsilon \in \mathcal{E}(T)} [F, H]_\varepsilon. \tag{32}$$

Once again, Cauchy's theorem implies that this is independent of T. To show that the pairing is well-defined we must check that if $H = F_\omega \in \mathcal{R}_{\mathcal{L}}$ is a global Coleman primitive of some $\omega \in \Omega_{\mathcal{R}}$, the pairing with any $F \in \mathcal{O}_{\mathcal{L}}(\mathfrak{X})$ which satisfies the assumption on supports, vanishes. If ω is exact, so that H extends analytically to $\mathfrak{X}$, then $[F, H] = \langle dF, H \rangle = 0$ as we have already noticed. It remains to check (after a shift in the variable) the case $H = \log_{\mathcal{L}} z$. In this case $rdiv(H) = [0] - [\infty]$, and by our assumption F is analytic near 0 and ∞. If $F \in \mathcal{O}(\mathfrak{X})$, then $[F, H] = \langle F, dz/z \rangle = 0$. Thus we only have to verify $[F, H] = 0$ for $F = \log_{\mathcal{L}}((z-\zeta_1)/(z-\zeta_2))$, $H = \log_{\mathcal{L}}(z)$, where $\zeta_i \neq 0$. This is done by direct computation.

To extend the pairing to an arbitrary pair F, H note that we can always move the support of $rdiv(H)$ away from a given number of points by adding a suitable $F_\omega \in \mathcal{R}_{\mathcal{L}}$ (in fact, we may assume that the residual divisor of H is supported on any *one* pre-given point). We may also assume that $F = f + \sum a_i \log_{\mathcal{L}}(z - \zeta_i)$ for $f \in \mathcal{O}(\mathfrak{X})$ and *finitely many* ζ_i (the F's of this sort being dense in $\mathcal{O}_{\mathcal{L}}(\mathfrak{X})$). Thus, having checked that the pairing factors through $\Sigma_{\mathcal{L}}$, we can always find a representative H for our class in $\Sigma_{\mathcal{L}}$ which satisfies the assumption on supports with respect to F.

4.3. **Breuil's duality theorem.**

Theorem 4.1. *The above pairing is a perfect pairing between $\mathcal{O}_{\mathcal{L}}(\mathfrak{X})$ and $\Sigma_{\mathcal{L}}$.*

Proof. The pairing induces a pairing between the two exact sequences

$$0 \to E \to \mathcal{O}_{\mathcal{L}}(\mathfrak{X}) \to \Omega(\mathfrak{X}) \to 0 \tag{33}$$

and

$$0 \leftarrow E \leftarrow \Sigma_{\mathcal{L}} \leftarrow \Sigma \leftarrow 0\ . \tag{34}$$

Since the terms at the ends are paired perfectly, so are the terms in the middle. ∎

5. Morita and Breuil duality with coefficients

5.1. **Coefficients and filtrations.** Let M be an irreducible algebraic representation of G over the field E. Then $M = Sym^n(E^2) \otimes \det^m$ for uniquely determined integers $n \geq 0$ and m. The central character of M is $t \mapsto t^{n+2m}$. The dual representation is $M^\vee = Sym^n(E^2) \otimes \det^{-n-m}$. From the previous duality theorems we get new ones by tensoring one exact sequence with M and its dual with $M^\vee$. Following an old tradition, we write $V(M)$ for $V \otimes_E M$. We also write, for any representation V of G, $V\{m\}$ for $V \otimes (\det)^m$. We omit the reference to $\mathfrak{X}$ in the notation when there is no risk of confusion between a sheaf and its module of global sections. Thus $\mathcal{O}(M)$ is really a short-hand for $\mathcal{O}(\mathfrak{X}) \otimes_E M$ etc.

While $St(M)$ and $C^1_{har}(M^\vee)$ are irreducible, this is not so for $\Sigma(M)$ or for $\Omega(M^\vee)$. The reason is this: St is a smooth representation, so $Lie(G)$ acts on it trivially, while M is "rigid" - whatever happens is completely determined by the infinitesimal action of $Lie(G)$. This remark can be translated to an easy proof of the irreducibility of $St(M)$ (an observation due to Prasad). On the other hand Σ is a quotient of the locally analytic induction of the trivial character from the Borel subgroup B, and M is the algebraic induction from B of a character χ (depending on n, m). Locally analytic functions, unlike the smooth ones, have enough room in them to accomodate the algebraic functions. Hence, by a variant of the *projection formula*

$$loc.an.ind._B^G 1 \otimes Ind_B^G \chi = loc.an.ind._B^G (1 \otimes Res_B^G Ind_B^G \chi). \tag{35}$$

The weight filtration on $Res_B^G Ind_B^G \chi$ therefore induces a filtration on the left hand side, and on $\Sigma(M)$. Morita duality induces a corresponding filtration on $\Omega(M^\vee)$. The same remarks apply to GL_{d+1} for $d > 1$ and to any prinicipal series representations. All that remains is to compute these filtrations.

We shall describe the filtration by G-invariant subspaces for

$$M_n = Sym^n(E^2)\{-n\} \tag{36}$$

and leave the modifications needed to treat the general case to the reader. Our representation can be realized as

$$M_n = E[u, v]_{\deg = n}, \tag{37}$$

the space of all homogoneous polynomials P of degree n in the variables u and v, with $g \in G$ acting by $gP(u,v) = P((u,v)\,{}^t g^{-1})$. For $0 \le k \le n$ let

$$F^k\mathcal{O}(M_n) = Span_{\mathcal{O}}\left\{(u-zv)^{n-l}v^l \,|\, 0 \le l \le n-k\right\}. \tag{38}$$

This is a free $\mathcal{O}$-module of rank $n+1-k$, invariant under G. It can be characterized as *global sections of $\mathcal{O}(M_n)$ having a zero of order $\ge k$ along $u - zv = 0$.*

The map

$$f \mapsto f(z)(u-zv)^k v^{n-k} \bmod F^{k+1} \tag{39}$$

is an $\mathcal{O}$-linear isomorphism

$$\Theta_k : \mathcal{O}(n-2k)\{-k\} \simeq F^k/F^{k+1}$$

of G-modules. Here $\mathcal{O}(m)$ is the vector space $\mathcal{O}$ with the twisted G-action

$$g \cdot f(z) = j(g^{-1}, z)^m f(g^{-1}z), \tag{40}$$

where

$$j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = cz + d. \tag{41}$$

In particular, the last non-zero piece in the filtration is $\mathcal{O}\cdot(u-zv)^n$, and the map $f(z) \mapsto f(z)(u-zv)^n$ is an isomorphism

$$\mathcal{O}(-n)\{-n\} \simeq F^n\mathcal{O}(M_n). \tag{42}$$

Since $\Omega \simeq \mathcal{O}(-2)\{-1\}$ (Kodaira-Spencer!), a similar filtration is induced on $\Omega(M_n)$ and the map $f(z) \mapsto f(z)(u-zv)^n dz$ is an isomorphism

$$\mathcal{O}(-n-2)\{-n-1\} \simeq F^n\Omega(M_n). \tag{43}$$

There are plenty more G-submodules of $\mathcal{O}(M_n)$ besides $F^k\mathcal{O}(M_n)$. For example, using the G-homomorphism $d : \mathcal{O}(M_n) \to \Omega(M_n)$ we can form new G-submodules of $\mathcal{O}(M_n)$ by taking $d^{-1}\left(F^k\Omega(M_n)\right)$. In particular, the next theorem shows that when we take the last step in the filtration,

$$d^{-1}(F^n\Omega(M_n)) \simeq \mathcal{O}(n). \tag{44}$$

Theorem 5.1. *There is a commutative diagram of G-modules with exact rows*

(45)

$$\begin{array}{ccccccccccc} 0 & \to & P_n(n) & \to & \mathcal{O}(n) & \xrightarrow{\delta_n} & \mathcal{O}(-n-2)\{-n-1\} & \to & C^1_{har}(M_n) & \to & 0 \\ & & \|\ \alpha_n & & \downarrow \beta_n & & \downarrow \gamma_n & & \| & & \\ 0 & \to & M_n & \to & \mathcal{O}(M_n) & \xrightarrow{d} & \Omega(M_n) & \to & C^1_{har}(M_n) & \to & 0 \end{array}.$$

The space $P_n(n)$ is the space of polynomials of degree n in z, with the twisted action induced from its inclusion in $\mathcal{O}(n)$. The map δ_n is given by

$$\delta_n = \frac{1}{n!}\left(\frac{d}{dz}\right)^{n+1}. \tag{46}$$

The vertical maps are injective. They are given by the following formulae:

$$\alpha_n(P) = v^n P(u/v) \tag{47}$$

$$\beta_n(f) = \sum_{k=0}^{n} \frac{f^{(k)}(z)}{k!}(u-zv)^k v^{n-k} \tag{48}$$

$$\gamma_n(h) = h(z)(u-zv)^n dz. \tag{49}$$

Remark 5.1. *The maps β_n and δ_n are not $\mathcal{O}$-linear, only E-linear.*

Proof. See [SS91]or [dS09], where a similar theorem is proved for SL_2 instead of GL_2 (the assertions are classical over the complex numbers, and the proofs are simple adaptatios of that case). The exactness of the bottom row, and the commutativity of the diagram, as well as the injectivity of the vertical arrows, are straightforward. The exactness of the top row and the fact that δ_n is a G-homomorphism are proved in [dS09], 2.1. From here we can deduce that β_n is a G-homomorphism as follows. Let $f \in \mathcal{O}(n)$ and $g \in G$. Then $\beta_n(gf) - g\beta_n(f) \in M_n$ because δ_n, γ_n and d are G-homomorphisms. On the other hand it is easy to check that $\beta_n(gf) - g\beta_n(f) \in F^1\mathcal{O}(n)$. Since $M_n \cap F^1\mathcal{O}(n) = 0$, we must have $\beta_n(gf) = g\beta_n(f)$.

Alternatively, one can prove *first* that β_n is a G-homomorphism, and use this fact, together with the injectivity of γ_n, to conclude that δ_n commutes with G. To prove that β_n commutes with G write $w = u/v$, and

$$\tilde{\beta}_n(f) = v^{-n}\beta_n(f) = \sum_{k=0}^{n} \frac{f^{(k)}(z)}{k!}(w-z)^k. \tag{50}$$

Here the G-action is $g\tilde{\beta}_n(f)(z,w) = \tilde{\beta}_n(f)(g^{-1}z, g^{-1}w)$. Now $\tilde{\beta}_n(f)$ may be uniquely characterized as the function on $\mathfrak{X} \times \mathfrak{X}$ which (a) is polynomial in w of degree n and (b) approximates $f(w)$ to order n along the diagonal $w = z$. The compatability $\beta_n(gf) = g\beta_n(f)$ translates to

$$\tilde{\beta}_n(j(g^{-1},z)^n f \circ g^{-1})(z,w) = j(g^{-1},w)^n(\tilde{\beta}_n f)(g^{-1}z, g^{-1}w). \tag{51}$$

But the right hand side is clearly a polynomial in w of degree n, which approximates $j(g^{-1},w)^n f \circ g^{-1}(w)$ to order n near $w = z$. Hence it is equal to the left hand side. ∎

5.2. **Morita duality (with coefficients).** Consider the diagram

$$\begin{array}{ccccccc} 0 & \leftarrow & Q_{-n-1}(-n-2)\{-1\} & \xleftarrow{p_\infty} & \mathcal{C}/\mathcal{R}(-n-2)\{-1\} & \xleftarrow{\delta_n^\vee} \\ & & \| \ \alpha_n^\vee & & \uparrow \beta_n^\vee & \\ 0 & \leftarrow & M_n^\vee & \xleftarrow{-res} & \Omega_{\mathcal{C}}/\Omega_{\mathcal{R}} \otimes M_n^\vee & \xleftarrow{d} \end{array} \tag{52}$$

$$\begin{array}{cccccc} \xleftarrow{\delta_n^\vee} & \mathcal{C}/\mathcal{R}(n)\{n\} = (\mathcal{C}^{an}[n\infty]/P_n)(n)\{n\} & \leftarrow & \left(\mathcal{C}^{pol.n}/P_n\right)(n)\{n\} & \leftarrow & 0 \\ & \uparrow \gamma_n^\vee = \mathrm{mod}\,\Phi^1 & & \| \,\mathrm{mod}\,\Phi^1 & & \\ \xleftarrow{d} & \mathcal{C}/\mathcal{R}(M_n^\vee) & \leftarrow & \mathcal{C}^{sm}(M_n^\vee)/M_n^\vee & \leftarrow & 0 \end{array}.$$

The bottom row is (23) tensored with $M_n^\vee$. In the top row, $\mathcal{C}^{pol.n}$ is the space of functions on $\mathbb{P}^1(K)$ which are locally polynomial in z of degree at most n (including at ∞), and the (n) indicates that the G-action is twisted by $j(g^{-1},z)^n$:

$$gf(z) = j(g^{-1},z)^n \cdot f(g^{-1}z). \tag{53}$$

The G-action preserves the space, as well as the subspace $P_n(n)$. The filtration Φ^k on $\mathcal{C}/\mathcal{R}\otimes M_n^\vee$ is defined in a way similar to the filtration F^k on $\mathcal{O}(M_n)$ (see [dS09], 2.3). First note that $M_n^\vee = M_n\{n\}$. Then put

$$\Phi^k\mathcal{C}(M_n) = Span_{\mathcal{C}}\left\{(u-zv)^{n-l}v^l \,|\, 0 \le l \le n-k\right\} \tag{54}$$

(functions vanishing to order at least k "along the line $u = zv$"). Let

$$\Phi^k\mathcal{R}(M_n) = \mathcal{R}(M_n) \cap \Phi^k\mathcal{C}(M_n), \tag{55}$$

and twist by $(\det)^n$ to get the filtration on $\mathcal{C}(M_n^\vee)$ or $\mathcal{R}(M_n^\vee)$. Denoting $\mathcal{C}/\mathcal{R}$ by Σ, a computation shows (as before) that

$$\Phi^k\Sigma(M_n^\vee)/\Phi^{k+1}\Sigma(M_n^\vee) \simeq \Sigma(n-2k)\,\{n-k\}\,. \tag{56}$$

In particular, $gr^0_\Phi = \Phi^0/\Phi^1$ is $\Sigma(n)\,\{n\}$, which yields the arrow labeled $\gamma_n^\vee$. The submodule $St(M_n^\vee)$ reduces injectively modulo Φ^1.

Denote by $\mathcal{C}^{an}[n\infty]$ the space of locally analytic functions on K, which have a pole at ∞ of order at most n. This space is invariant under the action of G twisted by $j(g^{-1},z)^n$. Since

$$\mathcal{C} = \mathcal{R} + \mathcal{C}^{an}[n\infty], \quad P_n = \mathcal{R}\cap\mathcal{C}^{an}[n\infty], \tag{57}$$

$\Sigma(n)$ can also be realized as $(\mathcal{C}^{an}[n\infty]/P_n)(n)$. Under this identification the image of $St(M_n) \bmod \Phi^1$ is just $\left(\mathcal{C}^{pol.n}/P_n\right)(n)$. Indeed, if f is locally constant and $P\in E[u,v]_{\deg=n}$, then $f\otimes P \bmod \Phi^1 = f(z)P(z,1)v^n \bmod \Phi^1$, corresponds to the locally polynomial function $f(z)P(z,1)$ with the G-action twisted (n). When M_n is replaced by $M_n^\vee$ we have to further twist by $(\det)^n$. This explains the two right-most columns of the diagram.

The map

$$\delta_n^\vee = \frac{(-1)^n}{n!}\left(\frac{d}{dz}\right)^{n+1} : \Sigma\to\Sigma, \tag{58}$$

whose kernel is just $\mathcal{C}^{pol.n}/P_n$, preserves local analyticity on K. When restricted to $\mathcal{C}^{an}[n\infty]$, it kills the polar part at ∞. The image $\delta_n^\vee(\mathcal{C}^{an}[n\infty])$ is therefore contained in $\mathcal{C}^{an}$, the space of functions which are locally analytic on all of $\mathbb{P}^1(K)$. Moreover, the functions in the image vanish to order $n+2$ at ∞, since their Laurent expansions have terms z^k for $k\le -n-2$ only. It is clear that these are the only restrictions. The image of $\delta_n^\vee$ therefore coincides with the subspace

$$\mathcal{C}^{an}[(-n-2)\infty] \tag{59}$$

of locally analytic functions on $\mathbb{P}^1(K)$ vanishing to order $n+2$ at ∞, which is stabilized by the G-action twisted $(-n-2)$. The $(n+1)$-dimensional quotient

$$\Sigma/\mathcal{C}^{an}[(-n-2)\infty] \tag{60}$$

we denote by Q_{-n-1}, and the projection onto it by p_∞. The space $Q_{-n-1}(-n-2)\,\{-1\}$ is in a natural way the dual of $P_n(n)$: represent a class in Q_{-n-1} by a locally analytic function Q, and pair it with a polynomial $P\in P_n$ to give

$$res_\infty P(z)Q(z)dz = \sum_{\zeta\in\mathbb{P}^1(K)} res_\zeta P(z)Q(z)dz. \tag{61}$$

Having identified M_n with $P_n(n)$ via the map α_n, we may therefore identify $M_n^\vee$ with $Q_{-n-1}(-n-2)\,\{-1\}\,.$

Theorem 5.2. *The above diagram is commutative with exact rows, and is dual to the diagram in Theorem 5.1.*

Proof. (See [dS09], 2.3) Exactness and duality for the bottom row follow at once from Morita duality without coefficients. The exactness of the top row follows from the preceding discussion, and it is then easy to verify that we get a sequence dual to the top row in 5.1, with the correct sign in $\delta_n^\vee$. Since the vertical maps are defined as the duals of the vertical maps in 5.1, the commutativity is tautological. That the dual of γ_n is also the reduction modulo Φ^1 follows from the fact that the filtrations

F^k and Φ^k are orthogonal to each other, namely the exact annihilator of $F^k\Omega(M_n)$ is $\Phi^{n+1-k}\Sigma(M_n^\vee)$ (see [dS09], Corollary 2.6). ∎

5.3. **A commutative diagram with logarithms.** Let $\mathcal{O}(n,\mathcal{L})$ denote the space of functions F on $\mathfrak{X}$ which, on every affinoid $\mathfrak{X}_T$ are of the form

$$F|_{\mathfrak{X}_T} = f + \sum p_i \log_{\mathcal{L}}(z-\zeta_i) \tag{62}$$

where $f \in \mathcal{O}(\mathfrak{X}_T)$ and $p_i \in P_n$ is a polynomial of degree at most n. (It is enough to take, for every $\varepsilon \in \mathcal{E}(T)$ such that U_ε is bounded, one $\zeta_i \in K_\varepsilon$.) The action of G is twisted (n):

$$gF(z) = j(g^{-1},z)^n F(g^{-1}z). \tag{63}$$

In analogy with theorem 5.1 we have the following.

Theorem 5.3. *There is a commutative diagram of G-modules with exact rows*

$$\begin{array}{ccccccccc} 0 & \to & P_n(n) & \to & \mathcal{O}(n,\mathcal{L}) & \overset{\delta_n}{\to} & \mathcal{O}(-n-2)\{-n-1\} & \to & 0 \\ & & \downarrow \alpha_n & & \downarrow \beta_n & & \downarrow \gamma_n & & \\ 0 & \to & M_n & \to & \mathcal{O}_{\mathcal{L}}(M_n) & \overset{d}{\to} & \Omega(M_n) & \to & 0 \end{array}. \tag{64}$$

The vertical maps are given by the same formulas as in 5.1. The map α_n is an isomorphism, β_n and γ_n are injective, and

$$\delta_n = \frac{1}{n!}\left(\frac{d}{dz}\right)^{n+1}. \tag{65}$$

Proof. First note that since each p_i is of degree n and δ_n invokes differentiation $n+1$ times, the logarithms in F are bound to be differentiated, so the image of δ_n consists of rigid analytic functions. Exactness of the bottom row follows immediately from the exactness of the same sequence with trivial coefficients. For the exactness of the top row, the only question is the surjectivity of δ_n. It is enough to show surjectivity of the same map when the functions are restricted to an affinoid $\mathfrak{X}_T$. Given a rigid analytic function in $\mathfrak{X}_T$, use the Mittag-Leffler decomposition to write it as a sum of power series in $(z-\zeta_\varepsilon)^{-1}$, one for each $\varepsilon \in \mathcal{E}(T)$ such that U_ε is bounded ($\zeta_\varepsilon \in K_\varepsilon$) and a power series in z. The power series in z can be integrated $n+1$ times with no difficulty, and the power series in $(z-\zeta_\varepsilon)^{-1}$ can be integrated once we remove the terms with $(z-\zeta_\varepsilon)^{-k}$ for $k \le n+1$. But these polynomials in $(z-\zeta_\varepsilon)^{-1}$ that we have removed are easily obtained as $\delta_n(p\log(z-\zeta_\varepsilon))$ for appropriate polynomials p of degree at most n.

The fact that β_n is a G-homomorphism can be deduced as follows. We have already seen that

$$\beta_{n,T} : \mathcal{O}(n)(\mathfrak{X}_T) \to \mathcal{O}(M_n)(\mathfrak{X}_T) \tag{66}$$

satisfies $\beta_{n,gT}\circ g = g\circ\beta_{n,T}$ ($g \in G$), where $g : \mathcal{O}(n)(\mathfrak{X}_T) \to \mathcal{O}(n)(\mathfrak{X}_{gT})$ etc. (Indeed, we have seen it on global sections of $\mathcal{O}(n)$, but these are dense in $\mathcal{O}(n)(\mathfrak{X}_T)$ and β_n as well as g are continuous.) Let $F \in \mathcal{O}(n,\mathcal{L})$ and assume, without loss of generality, that F can be *globally* written as $f + \sum p_i \log_{\mathcal{L}}(z-\zeta_i)$ with finitely many ζ_i. Then there is a $v \in \mathcal{T}_0$ such that on the affinoid $\mathfrak{X}_v$, F is rigid analytic, hence $g(\beta_n F) - \beta_n(gF)$ vanishes on $\mathfrak{X}_{gv}$. But for a function in $\mathcal{O}_{\mathcal{L}}(M_n)$ this is enough to guarantee that it vanishes identically.

The commutativity of the left square was already proved. The commutativity of the right square and the G-equivariance of δ_n can be deduced by the same argument as above from the corresponding assertions in Theorem 5.1. ∎

Corollary 5.4. *We have an exact sequence*

$$0 \to \mathcal{O}(n) \to \mathcal{O}(n, \mathcal{L}) \to C^1_{har}(M_n) \to 0. \tag{67}$$

Proof. This follows from the snake lemma upon comparison of the top row in Theorem 5.3 with the top row of Thoerem 5.1. The map to $C^1_{har}(M_n)$ takes $F \in \mathcal{O}(n, \mathcal{L})$ to the cochain of residues of $d(\beta_n F) = \gamma_n(\delta_n F)$. ∎

5.4. **Breuil duality (with coefficients).** We now turn to the dual diagram. Let $\mathcal{C}(n, \mathcal{L})$ be the space of functions on $\mathbb{P}^1(K)$ which, locally near any finite ζ, are of the form

$$h + p \log_{\mathcal{L}}(z - \zeta) \tag{68}$$

with h analytic and $p \in P_n$, and near ∞, they are of the form $h + p\log_{\mathcal{L}}(z)$ with h meromorphic with a pole of order at most n and p as before. A $g \in G$ acts on $H \in \mathcal{C}(n, \mathcal{L})$ via

$$gH(z) = j(g^{-1}, z)^n H(g^{-1}z). \tag{69}$$

Let $P(n, \mathcal{L})$ be the subspace of those functions which are of the form $p + \sum p_i \log(z - \zeta_i)$ where p and p_i are from P_n. This subspace is G-invariant, and its intersection with $C^{an}[n\infty]$ is P_n. Let

$$\Sigma(n, \mathcal{L}) = \mathcal{C}(n, \mathcal{L}) / P(n, \mathcal{L}). \tag{70}$$

Then $\Sigma(n) = C^{an}[n\infty]/P_n \subset \Sigma(n, \mathcal{L})$. Since $\delta_n^\vee$ involves $n+1$ differentiations, it maps $\mathcal{C}(n, \mathcal{L})$ to $\mathcal{C}$, and $P(n, \mathcal{L})$ to $\mathcal{R}$. Recall that $\Sigma = \mathcal{C}/\mathcal{R}$ by the theorem on principal parts. Paying attention to the G-action we see that we have an exact sequence

$$0 \to St(M_n^\vee) \to \Sigma(n, \mathcal{L})\,\{n\} \xrightarrow{\delta_n^\vee} \Sigma(-n-2)\,\{-1\} \to 0. \tag{71}$$

Here the first arrow is the inclusion of $St(M_n^\vee)$ in $\Sigma(n)\,\{n\}$, whose image is

$$\left(\mathcal{C}^{pol.n}/P_n\right)(n)\,\{n\} \tag{72}$$

(see 5.2 above). Exactness in the middle is obvious, and surjectivity of $\delta_n^\vee$ can be proved by looking at local Laurent expansions.

Theorem 5.5. *There is a commutative diagram of G-modules with exact rows*

$$\begin{array}{ccccccccc} 0 & \leftarrow & Q_{-n-1}(-n-2)\,\{-1\} & \overset{p_\infty \circ \delta_n^\vee}{\longleftarrow} & \Sigma(n, \mathcal{L})\,\{n\} & \leftarrow & \Sigma(n)\{n\} = (\mathcal{C}^{an}[n\infty]/P_n)(n)\,\{n\} & \leftarrow & 0 \\ & & \uparrow \alpha_n^\vee & & \uparrow \beta_n^\vee = \mathrm{mod}\,\Phi^1 & & \uparrow \gamma_n^\vee & & \\ 0 & \leftarrow & M_n^\vee & \leftarrow & \Sigma_{\mathcal{L}}(M_n^\vee) & \leftarrow & \Sigma(M_n^\vee) & \leftarrow & 0 \end{array} . \tag{73}$$

The diagram is naturally dual to the diagram that appeared in 5.3.

Proof. The exactness of the bottom row was already established. The maps $\alpha_n^\vee$ and $\gamma_n^\vee$ have already been discussed and shown to be the duals of α_n and γ_n, in Section 5.2 (on Morita duality with coefficients). The inclusion of $\Sigma(n)$ in $\Sigma(n,\mathcal{L})$ is obvious. The map p_∞ was defined before. Since both $\delta_n^\vee$ and p_∞ are surjective, their compositum is also surjective. Finally, as

$$\ker(p_\infty) = \delta_n^\vee \Sigma(n), \tag{74}$$

$$\ker(p_\infty \circ \delta_n^\vee) = \left(\delta_n^\vee\right)^{-1} \delta_n^\vee \Sigma(n) = \Sigma(n) \tag{75}$$

because $\Sigma(n)$ contains $St(M_n)$ (we have suppressed the twist by $\det^n$ in the notation). Thus the top row is exact. The map $\beta_n^\vee$ is obtained, like $\gamma_n^\vee$, from the filtration Φ^k. This filtration extends from $\Sigma(M_n)$ to $\Sigma_\mathcal{L}(M_n)$ with the same definitions, and

$$gr_\Phi^0 \Sigma_\mathcal{L}(M_n) = \Sigma(n,\mathcal{L}). \tag{76}$$

The pairing between $\mathcal{O}_\mathcal{L}$ and $\Sigma_\mathcal{L}$ yields of course a pairing between $\mathcal{O}_\mathcal{L}(M_n)$ and $\Sigma_\mathcal{L}(M_n^\vee)$. That the filtrations F^k and Φ^k are orthogonal under this pairing is formal, see [dS09], Lemma 2.4. This gives the duality between the middle columns of the diagrams under consideration, and completes the proof of the theorem. (Commutativity is evident once we know that the diagram is dual to a commutative diagram.) ∎

5.5. **Summary.** For convenience we summarize Morita and Breuil duality by looking at the following two commutative diagrams, which are dual to each other. In each diagram the row, the column and the 4-term sequence obtained by following the diagonal arrows, are exact.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & P_n(n) \arrow[d] \arrow[dl] & & \\
0 \arrow[r] & \mathcal{O}(n) \arrow[r] \arrow[dr] & \mathcal{O}(n,\mathcal{L}) \arrow[r] \arrow[d, "\delta_n"] & C^1_{har}(M_n) \arrow[r] & 0 \\
 & & \mathcal{O}(-n-2)\{-n-1\} \arrow[d] \arrow[ur, "c\circ\gamma_n"'] & & \\
 & & 0 & &
\end{tikzcd} \tag{77}$$

is dual to

(78)

$$\begin{tikzcd}
 & & 0 & & \\
 & & Q_{-n-1}(-n-2)\{n\} \arrow[u] & & \\
0 & \Sigma(-n-2)\{-1\} \arrow[l] \arrow[ur, "p_\infty"] & \Sigma(n,\mathcal{L})\{n\} \arrow[l, "\delta_n^\vee"'] \arrow[u] & St(M_n^\vee) \arrow[l] \arrow[dl] & 0 \arrow[l] \\
 & & \Sigma(n)\{n\} \arrow[u] \arrow[ul] & & \\
 & & 0 \arrow[u] & &
\end{tikzcd} \cdot$$

5.6. **Jordan Hölder components and intertwining operators.** The graded pieces in the 3-step filtration

$$0 \subset St(M_n^\vee) \subset \Sigma(n)\,\{n\} \subset \Sigma(n,\mathcal{L})\,\{n\} \tag{79}$$

are (from the top down) $M_n^\vee$, $\ker(p_\infty) = \delta_n^\vee(\Sigma(n)\,\{n\})$ and $St(M_n^\vee)$. The first and the last have already been observed to be (topologically) irreducible. This is also true of the middle graded piece

$$\delta_n^\vee(\Sigma(n)\,\{n\}) = \mathcal{C}^{an}[(-n-2)\infty](-n-2)\,\{-1\} \tag{80}$$

by a result of Schneider and Teitelbaum which we now explain. By duality, the same will hold true for the graded pieces of $\mathcal{O}(n,\mathcal{L})$.

The middle graded piece (after a twist by the determinant) satisfies

$$\mathcal{C}^{an}[(-n-2)\infty](-n-2) \simeq loc.an.Ind_B^G(n+2,0). \tag{81}$$

Indeed, take a locally analytic $\Phi : G \to E$ from the representation on the right. Then

$$\Phi\left(\begin{pmatrix} t & * \\ 0 & * \end{pmatrix} h\right) = t^{n+2}\Phi(h) \tag{82}$$

and $g\Phi(h) = \Phi(hg)$. Attach to it the locally analytic function $\phi(z)$ defined by

$$\phi(z) = \Phi\left(\begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix}\right). \tag{83}$$

An easy computation shows that $g\Phi$ corresponds to the function

$$g\phi(z) = j(g^{-1}, z)^{-n-2}\phi(g^{-1}z). \tag{84}$$

To check the behavior of ϕ at infinity note that

$$\phi(z^{-1}) = \Phi\left(\begin{pmatrix} z & 1 \\ 0 & z^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}\right) = z^{n+2}\Phi\left(\begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}\right) \tag{85}$$

must have a zero of order $\geq n+2$ at 0, hence ϕ has a zero of the same order at ∞. Conversely, to such a ϕ one can attach a Φ which is locally analytic on G.

Now the result of Schneider and Teitelbaum quoted above implies that this locally analytic principal series representation is irreducible.

The extensions providing the filtration of $\Sigma(n,\mathcal{L})$ are non-split. It is not difficult to obtain from here that $\Sigma(n,\mathcal{L}) \simeq \Sigma(n',\mathcal{L}')$ if and only if $n = n'$, $\mathcal{L} = \mathcal{L}'$, and that $End(\Sigma(n,\mathcal{L})) = E$ ([Br04], Lemme 2.4.1).

6. Integral structures and completions

6.1. **Bounded vectors in the representations of type (F).** From now on assume that n is even and consider

$$M = Sym^n(E^2)\left\{-\frac{n}{2}\right\} = M_n\left\{\frac{n}{2}\right\}. \tag{86}$$

The reason for this particular twist is that unlike M_n, M has a trivial central character. It is also self dual. Twisting all the modules in (74) by $(\det)^{n/2}$ we get a diagram upon which the center of G acts trivially.

Let V be any of the G-modules of type (F) appearing in (74), with this new twist by $(\det)^{n/2}$. As a topological vector space, forgetting the G-action,

$$V = \varprojlim V_T \tag{87}$$

where T ranges over finite subtrees of $\mathcal{T}$ and V_T is an E-Banach space. We fix a Banach norm on each V_T and denote it by $||\cdot||_T$ (they are all equivalent). For example, if V is the space $\mathcal{O}$, then $V_T = \mathcal{O}(\mathfrak{X}_T)$ and the norm can be taken to be the usual sup norm. If V is $\mathcal{O}(n, \mathcal{L})$ then V_T consists of all the functions as on the right hand side of (59), hence contains $\mathcal{O}(\mathfrak{X}_T)$ as a closed subspace of codimension $(n+1)\,(\#\mathcal{E}(T)-1)$, and any extension of the sup norm of $\mathcal{O}(\mathfrak{X}_T)$ to V_T will do. If V is $C^1_{har}(M)$ then V_T is a finite dimensional space.

In all of the examples, if $T \subset\subset T'$, the restriction map $V_{T'} \to V_T$ is *compact.*

Letting T be any finite tree which is not reduced to a single vertex, so that its G-translates cover $\mathcal{T}$, define the submodule of bounded vectors to be

(88) $$V^{bnd} = \left\{ f \in V| \; ||f||_{\mathfrak{X}} = \sup_{g\in G} ||gf||_T < \infty \right\}.$$

This space is evidently stable under G, although it may well be 0. We denote the topology induced on V^{bnd} from V by τ, and call it the weak topology. Thus the space V^{bnd} is equipped with two topologies: the one coming from the norm, and τ, and to distinguish the two we shall use the notation

(89) $$(V^{bnd}, ||\cdot||_{\mathfrak{X}}) \text{ or } (V^{bnd}, \tau).$$

Proposition 6.1. *(i) The norms $||\cdot||_{\mathfrak{X}}$ may depend on the choices of T and $||\cdot||_T$, but they are all equivalent, hence the space V^{bnd} is well defined. In fact, $f \in V^{bnd}$ if and only if for every continuous functional $\phi \in V'$ the matrix coefficient*

(90) $$g \mapsto \phi(gf)$$

is a bounded function on G.

(ii) $(V^{bnd}, ||\cdot||_{\mathfrak{X}})$ is an E-Banach space (inseparable, in general).

(iii) The unit ball V_1^{bnd} of $(V^{bnd}, ||\cdot||_{\mathfrak{X}})$ is τ-compact (but not τ-open).

Proof. (i) We do the case of $V = \mathcal{O}(-n-2)\left\{-\frac{n}{2}-1\right\}$ and leave the rest to the reader. Fix T and let $||\cdot||_T$ be the sup norm on $\mathfrak{X}_T$. Assume that $||f||_{\mathfrak{X}} \le 1$. Let $g_1 \in G$, and consider the inequality

(91) $$|g_1gf(z)| = |j(g_1^{-1}, z)^{-n-2} \det(g_1)^{-\frac{n}{2}-1} gf(g_1^{-1}z)| \le 1$$

which should hold for all $g \in G$ and $z \in \mathfrak{X}_T$. It implies that there exists a constant $C(g_1)$ such that

(92) $$|gf(z)| \le C(g_1)$$

for all $z \in \mathfrak{X}_{g_1^{-1}T}$, independently of g. If T' is another tree, it is covered by finitely many translates $g_1^{-1}T$, so it follows that the norms $||\cdot||_{\mathfrak{X}}$ are all equivalent.

Suppose $f \in V^{bnd}$ and $\phi \in V_T' \subset V'$. Since

(93) $$|\phi(gf)| \le ||\phi||_{V_T'} ||gf||_T$$

the function $\phi(gf)$ is bounded on G. Conversely, suppose that $g \mapsto \phi(gf)$ is bounded for every $\phi \in V'$. Fix T. The collection $\{gf\}_{g\in G}$ is a collection of linear functionals on $V_T' \subset V'$ which is pointwise bounded at every ϕ. By the Banach-Steinhaus theorem it is equicontinuous, the norms $||gf||_T$ are bounded, so $f \in V^{bnd}$.

(ii) Clear.

(iii) Let $\{f_n\}$ be a sequence from V_1^{bnd}. Let T be a finite tree and select a tree T' such that $T \subset\subset T'$. The sequence is bounded in the norm $||\cdot||_{T'}$, hence projects to a bounded sequence in $V_{T'}$. Since $V_{T'} \to V_T$ is compact, it has a subsequence that converges in $||\cdot||_T$. By the usual diagonal argument, one finds a subsequence that converges in the τ-topology. ∎

Lemma 6.2. *The morphisms in (74) respect boundedness.*

Proof. Let, for the moment,

$$\mathcal{F} = \mathcal{O}(n,\mathcal{L})\left\{\frac{n}{2}\right\},\ \mathcal{G} = \mathcal{O}(-n-2)\left\{-\frac{n}{2}-1\right\} \tag{94}$$

and consider the map $\delta_n : \mathcal{F} \to \mathcal{G}$. Fix Banach norms on $\mathcal{F}(\mathfrak{X}_T)$ and $\mathcal{G}(\mathfrak{X}_T)$ for every finite tree T. Fix T as above. The continuity of δ_n implies that there exists a constant $c > 0$ and a tree T' such that for every $f \in \mathcal{F}(\mathfrak{X})$

$$||\delta_n(f)||_T \leq c||f||_{T'}. \tag{95}$$

But now, for every $g \in G$

$$||g\delta_n(f)||_T = ||\delta_n(gf)||_T \leq c||gf||_{T'}$$

so $||\delta_n f||_{\mathfrak{X}} \leq c||f||_{\mathfrak{X}}$ (where we have used T and T' with their respective norms to define the global norms), and δ_n maps $\mathcal{F}^{bnd}$ to $\mathcal{G}^{bnd}$. The same applies to all the other maps in the diagram. ∎

Theorem 6.3. *Let $n > 0$. (i) We have*

$$P_n(n)\left\{\frac{n}{2}\right\}^{bnd} = \mathcal{O}(n)\left\{\frac{n}{2}\right\}^{bnd} = 0. \tag{96}$$

(ii) $\mathcal{O}(n,\mathcal{L})\left\{\frac{n}{2}\right\}^{bnd}$ is mapped injectively into $\mathcal{O}(-n-2)\left\{-\frac{n}{2}-1\right\}^{bnd}$.
(iii) The map $c\circ\gamma_n$ induces an isomorphism

$$\mathcal{O}(-n-2)\left\{-\frac{n}{2}-1\right\}^{bnd} \simeq C^1_{har}(M)^{bnd}. \tag{97}$$

Proof. For part (iii), see Theorem 3.2 of [dS09]. Neither the injectivity nor the surjectivity (which is a consequence of the Amice-Velu-Vishik theorem) are trivial. For part (i), an algebraic representation can not have bounded vectors, since otherwise G would be compact. Thus $P_n(n)\left\{\frac{n}{2}\right\}^{bnd} = 0$ and therefore $\mathcal{O}(n)\left\{\frac{n}{2}\right\}^{bnd}$ gets mapped injectively into $\mathcal{O}(-n-2)\left\{-\frac{n}{2}-1\right\}^{bnd}$ by δ_n. Part (iii) implies that $\delta_n(\mathcal{O}(n)\left\{\frac{n}{2}\right\})$ does not meet $\mathcal{O}(-n-2)\left\{-\frac{n}{2}-1\right\}^{bnd}$, hence $\mathcal{O}(n)\left\{\frac{n}{2}\right\}^{bnd} = 0$ as well. This concludes the proof of (i), and (ii) follows at once since

$$\mathcal{O}(n)\left\{\frac{n}{2}\right\} \cap \mathcal{O}(n,\mathcal{L})\left\{\frac{n}{2}\right\}^{bnd} = \mathcal{O}(n)\left\{\frac{n}{2}\right\}^{bnd}. \tag{98}$$

∎

Remark 6.1. *The notion of boundedness could be similarly defined for $\mathcal{O}(M)$, $\mathcal{O}_{\mathcal{L}}(M)$ and $\Omega(M)$, and then, via β_n and γ_n*

$$\mathcal{O}(n)\left\{\frac{n}{2}\right\}^{bnd} = \mathcal{O}(n)\left\{\frac{n}{2}\right\} \cap \mathcal{O}(M)^{bnd}, \tag{99}$$

$$\mathcal{O}(n,\mathcal{L})\left\{\frac{n}{2}\right\}^{bnd} = \mathcal{O}(n,\mathcal{L})\left\{\frac{n}{2}\right\} \cap \mathcal{O}_{\mathcal{L}}(M)^{bnd}, \tag{100}$$

and

$$\mathcal{O}(-n-2)\left\{-\frac{n}{2}-1\right\}^{bnd} = \mathcal{O}(-n-2)\left\{-\frac{n}{2}-1\right\} \cap \Omega(M)^{bnd}. \tag{101}$$

This follows, in the first case, for example, from the equivalence of the Banach norm on $\mathcal{O}(n)\left\{\frac{n}{2}\right\}(\mathfrak{X}_T)$ with the norm induced on the same space via β_n from the inclusion in $\mathcal{O}(M)(\mathfrak{X}_T)$. The other two cases are similar.

Remark 6.2. *Were we dealing with $SL_2(K)$ instead of $GL_2(K)$ (as in* [dS09]*), we would not have to worry about determinant twists, and no parity assumption on n would be necessary. Even with our G, the parity assumption on n could be avoided if we allowed twisting by a smooth character, not only by a power of the determinant. This is because all that is required from the central character, to make room for bounded elements, is to be* unitary.

6.2. **The duals of the bounded submodules.** Fix a norm $||\cdot||_{\mathfrak{x}}$ on any of the spaces $W = V^{bnd}$ as above, and let W_1 be the unit ball in that norm, equipped with the topology induced from τ, in which, as we have seen, it is compact. Then W_1 is a compact torsion-free $\mathcal{O}_E$-module. The space

$$\mathbf{D}(W_1) = Hom_{\mathcal{O}_E,cont}(W_1, E) \tag{102}$$

is a Banach space under the sup norm. Up to isomorphism, it is independent of the choice of $||\cdot||_{\mathfrak{x}}$.

We emphasize that $\mathbf{D}(W_1)$ is *neither* the Banach dual of $(W, ||\cdot||_{\mathfrak{x}})$ (which would consist of *all* the bounded linear maps on W_1), *nor* the topological dual of (W,τ). Indeed, the linear extension of an $\mathcal{O}_E$-linear continuous functional on (W_1,τ) will not be in general continuous on (W,τ), as W_1 is not open there. Moreover, since (W,τ) is dense in V the topological dual of (W,τ) is just V'.

For example, consider $V = C^1_{har}(M)$. Its topological dual V' is $St(M)$. The Banach dual of $\left(V^{bnd}, ||\cdot||_{\mathfrak{x}}\right)$ is huge. Our space is an intermediate one

$$St(M) \subset \mathbf{D}(C^1_{har}(M)^{bnd}_1) \subset Hom(C^1_{har}(M)^{bnd}_1, \mathcal{O}_E) \otimes E. \tag{103}$$

We shall soon see that $\mathbf{D}(C^1_{har}(M)^{bnd}_1)$ is the completion of $St(M)$ with respect to a certain integral structure.

6.3. **Integral structures in representations of type (C).** Consider the unit ball W_1 in any of the spaces $W = V^{bnd}$ where

$$V = \mathcal{O}(n,\mathcal{L})\left\{\frac{n}{2}\right\},\ V = \mathcal{O}(-n-2)\left\{-\frac{n}{2}-1\right\} \text{ or } V = C^1_{har}(M). \tag{104}$$

The $\mathcal{O}_E$-module of all the elements in $V' = \Sigma(n,\mathcal{L})\left\{\frac{n}{2}\right\}$, $\Sigma(n)\left\{\frac{n}{2}\right\}$ or $St(M)$, that are bounded by 1 on W_1,

$$(V')^{int} = \{\phi \in V' \,|\, |\phi(f)| \le 1 \text{ for all } f \in W_1\}$$

is an *integral structure* in V': it spans V' over E, and is G stable. Indeed, if $\phi \in V'_T$ and $f \in W_1$ we have

$$|\phi(f)| \le ||\phi||_{V'_T} ||f||_T \le ||\phi||_{V'_T} \tag{105}$$

so a certain multiple of ϕ lies in $(V')^{int}$.

On $V' = St(M)$ or $\Sigma(n)\left\{\frac{n}{2}\right\}$ we now get *two* integral structures: the first is $St(M)^{int}$ or $\Sigma(n)\left\{\frac{n}{2}\right\}^{int}$ defined above, and the second is induced by the unit ball in $\mathcal{O}(n,\mathcal{L})\left\{\frac{n}{2}\right\}^{bnd}$, i.e. it is the intersection of $St(M)$ or $\Sigma(n)\left\{\frac{n}{2}\right\}$ with $\Sigma(n,\mathcal{L})\left\{\frac{n}{2}\right\}^{int}$. We denote this second integral structure by

$$St(M)^{\mathcal{L}} \text{ or } \Sigma(n)\left\{\frac{n}{2}\right\}^{\mathcal{L}}. \tag{106}$$

Needless to say, all our integral structures are well defined only up to commensurability, since the norms on W are only defined up to equivalence. Note that $St(M)^{int} \subset St(M)^{\mathcal{L}}$ and likewise for the twisted analytic Steinberg representation, because

$$\mathcal{O}(n,\mathcal{L})\left\{\frac{n}{2}\right\}^{bnd} \subset C^1_{har}(M)^{bnd}. \tag{107}$$

Since *we do not know yet that the space of bounded elements is nonzero,* it is a priori possible for the integral structure to be the whole space. However, since $St(M)$ is an irreducible G-module, if any of the two integral structures is not the whole space, it contains no E-line, in which case we call it a *genuine* integral structure.

Proposition 6.4. *$C^1_{har}(M)^{bnd}$ is non-zero, hence $St(M)^{int}$ is a genuine integral structure. It is the minimal integral structure in $St(M)$, in the sense that it is commensurable with $\mathcal{O}[G]v$ for any $v \in St(M)$.*

Proof. There are several proofs that $C^1_{har}(M)^{bnd} \neq 0$, but none of them is self-evident. One way is to look at a discrete cocompact subgroup Γ of G, and show that

$$0 \neq C^1_{har}(M)^{\Gamma} \subset C^1_{har}(M)^{bnd}. \tag{108}$$

For another approach, that does not use Γ, see [V08]. Once we know that there is *some* genuine integral structure, the minimal integral structure (well defined up to commensurability because of the irreducibility of the representation) is clearly genuine. We must show that $St(M)^{int}$ *is indeed* the minimal integral structure. This is a consequence of the Hahn-Banach theorem as we now show.

We work more generally and let π be a smooth irreducible representation of G over E and ρ an algebraic irreducible E-representation. Let

$$\Pi = \pi \otimes \rho \tag{109}$$

and $V = Hom(\Pi, E)$, the full algebraic dual. We topologize Π as a direct limit of finite dimensional spaces, and V as the inverse limit of their duals. Let V^{bnd} be the submodule of all $v \in V$ for which the "matrix coefficient" $\langle v, gu\rangle$ is a bounded function of $g \in G$ for every $u \in \Pi$. Since Π is irreducible, it is enough to require that this function is bounded for *one* nonzero u. We assume $V^{bnd} \neq 0$.

Fix $u \neq 0$, let $||v|| = \sup|\langle v, gu\rangle|$, and let V_1^{bnd} be the unit ball in that norm. Our goal is to show that Π^{int}, the submodule of Π consisting of all u' such that

$$|\left\langle V_1^{bnd}, u'\right\rangle| \leq 1 \tag{110}$$

coincides with $\mathcal{O}_E[G]u$. That it contains it, is clear. Suppose that $u' \in \Pi^{int}$, but $u' \notin \mathcal{O}_E[G]u$. Let $x \in E^{\times}$ be such that $xu' \in \mathcal{O}_E[G]u$ but $yu' \notin \mathcal{O}_E[G]u$ for $|y| > |x|$. By Hahn-Banach there exists a $v \in V$ such that $|\langle v, xu'\rangle| = 1$ and $|\langle v, \mathcal{O}_E[G]u\rangle| \leq 1$, i.e. $v \in V_1^{bnd}$. This contradicts $|\langle v, u'\rangle| \leq 1$. ∎

6.4. **Trivial coefficients.** Before we go on we dispense with the case $n = 0$. In this case $P_n(n)^{bnd} = \mathcal{O}(n)^{bnd} = E$ by Liouville's theorem, and

$$\Omega(\mathfrak{X})^{bnd} \simeq C^{1,bnd}_{har}. \tag{111}$$

If c is a bounded harmonic cochain, the associated distribution μ_c is a *bounded measure* and the bounded 1-form whose residues are given by c is obtained from μ_c

via the Poisson integral:

$$\omega_c = \int_{\mathbb{P}^1(K)} \frac{d\mu_c(\zeta)}{z-\zeta} dz. \tag{112}$$

The space $\mathcal{O}_{\mathcal{L}}^{bnd}$ is the space of Coleman primitives of these ω's, and is an extension of $C_{har}^{1,bnd}$ by E. The extension class depends on $\mathcal{L}$.

Dually, St^{int} may be identified with the standard integral structure in the Steinberg representation, Σ^{int} with the $\mathcal{O}_E$-valued locally analytic functions on $\mathbb{P}^1(K)$ modulo constants, and $\Sigma_{\mathcal{L}}^{int}$ with the $\mathcal{O}_E$-module generated by Σ^{int} and the images of $\log_{\mathcal{L}}(z-\zeta)\cdot\chi_\varepsilon$ in $\Sigma_{\mathcal{L}}$ (χ_ε is the characteristic function of K_ε, $\zeta \in K_\varepsilon$; check that the module is G-stable). The map $H \mapsto -res(dH)$ gives the extension

$$0 \to \Sigma^{int} \to \Sigma_{\mathcal{L}}^{int} \to \mathcal{O}_E \to 0, \tag{113}$$

which depends on $\mathcal{L}$ (compare (34)).

6.5. **Completions and Banach space representations.** Assume from now on that $n > 0$. We let $B(n)$ be the completion of $St(M)$ with respect to the integral structure $St(M)^{int}$, namely

$$B(n) = \varprojlim \left(St(M)^{int}/\pi_E^k St(M)^{int}\right) \otimes_{\mathcal{O}_E} E. \tag{114}$$

Likewise we denote by $B(n,\mathcal{L})$ the completion of $St(M)$ with respect to $St(M)^{\mathcal{L}}$:

$$B(n,\mathcal{L}) = \varprojlim \left(St(M)^{\mathcal{L}}/\pi_E^k St(M)^{\mathcal{L}}\right) \otimes_{\mathcal{O}_E} E. \tag{115}$$

The integral structures are well defined only up to commensurability, but the completions are independent of the choice of a lattice. Both $B(n)$ and $B(n,\mathcal{L})$ are separable Banach space representations of G, and $St(M)$ densely embeds in $B(n)$. It is conjectured, but not known in general, that $B(n,\mathcal{L})$ is non-zero. This is equivalent to $St(M)^{\mathcal{L}}$ being a genuine integral structure. If correct, $St(M)$ embeds densely in $B(n,\mathcal{L})$ as well.

In any case, the inclusion $St(M)^{int} \subset St(M)^{\mathcal{L}}$ induces a continuous map with dense image

$$B(n) \to B(n,\mathcal{L}). \tag{116}$$

Proposition 6.5. *There are canonical isomorphisms*

$$B(n) \simeq \mathbf{D}(C_{har}^1(M)_1^{bnd}), \tag{117}$$

and

$$B(n,\mathcal{L}) \simeq \mathbf{D}\left(\mathcal{O}(n,\mathcal{L})\left\{\frac{n}{2}\right\}_1^{bnd}\right). \tag{118}$$

Note that $C_{har}^1(M)_1^{bnd} = \mathcal{O}(-n-2)\left\{-\frac{n}{2}-1\right\}_1^{bnd}$.

Proof. Since

$$C_{har}^1(M) \supset Hom_{\mathcal{O}_E}(St(M)^{int}, \mathcal{O}_E) = Hom(B(n)_1, \mathcal{O}_E) = \mathbf{D}(B(n)) \tag{119}$$

and the weak* topology on $\mathbf{D}(B(n))$ (the topology of pointwise convergence) is the topology induced from $C_{har}^1(M)$, it is enough to show that

$$C_{har}^1(M)_1^{bnd} = Hom_{\mathcal{O}_E}(St(M)^{int}, \mathcal{O}_E). \tag{120}$$

The desired result will follow then by applying the duality functor $\mathbf{D}$ to the equality $C_{har}^1(M)_1^{bnd} = \mathbf{D}(B(n))$. Now the inclusion $C_{har}^1(M)_1^{bnd} \subset Hom_{\mathcal{O}_E}(St(M)^{int}, \mathcal{O}_E)$

follows from the definition of $St(M)^{int}$, and the inverse inclusion follows from the Hahn-Banach theorem (see [NFA], Corollary 13.5).

Similarly, since

$$C^1_{har}(M) \supset Hom_{\mathcal{O}_E}(St(M)^{\mathcal{L}}, \mathcal{O}_E) = Hom(B(n,\mathcal{L})_1, \mathcal{O}_E) = \mathbf{D}(B(n,\mathcal{L})), \tag{121}$$

it is enough to show (assuming $St(M)^{\mathcal{L}}$ is a genuine integral structure, so that $B(n,\mathcal{L}) \neq 0$) that

$$\mathcal{O}(n,\mathcal{L}) \left\{ \frac{n}{2} \right\}_1^{bnd} = Hom_{\mathcal{O}_E}(St(M)^{\mathcal{L}}, \mathcal{O}_E). \tag{122}$$

Again the inclusion $\mathcal{O}(n,\mathcal{L}) \left\{ \frac{n}{2} \right\}_1^{bnd} \subset Hom_{\mathcal{O}_E}(St(M)^{\mathcal{L}}, \mathcal{O}_E)$ follows from the definition of $St(M)^{\mathcal{L}}$ and the inverse inclusion form Hahn-Banach. ∎

6.6. **Admissibility.** Let B be an E-Banach space unitary representation of G, and B_1 its unit ball. Let $H = GL_2(\mathcal{O}_K)$ (or any other open compact subgroup of G) and consider the Iwasawa algebra

$$\Lambda = \mathcal{O}_E[[H]] \tag{123}$$

of integral E-valued measures on H. If $\lambda \in \Lambda$ and $\phi \in \mathbf{D}(B) = Hom(B_1, \mathcal{O}_E)$ (viewed as a compact torsion-free $\mathcal{O}_E$ module) we let $\lambda\phi \in \mathbf{D}(B)$ be defined by

$$\lambda\phi(v) = \int_H \phi(h^{-1}v)d\lambda(h). \tag{124}$$

This turns $\mathbf{D}(B)$ into a Λ-module, and one calls B admissible if $\mathbf{D}(B)$ is finitely generated over Λ.

In the case of $B(n)$, the question is that of the structure of $C^1_{har}(M)^{bnd}_1 = \mathcal{O}(-n-2)\left\{-\frac{n}{2}-1\right\}_1^{bnd}$ as a Λ-module. Breuil ([Br04], 4.4.5) shows that this module is not finitely generated over Λ, hence $B(n)$ is not admissible. On the other hand, the smaller module $\mathcal{O}(n,\mathcal{L})^{bnd}_1$ is expected to be finitely generated over Λ, hence $B(n,\mathcal{L})$ is expected to be admissible.

7. Global considerations and non-vanishing of $B(n,\mathcal{L})$

7.1. **Coleman integration vs. Schneider integration and a cirterion for the non-vanishing of $\mathcal{O}(n,\mathcal{L})$.** Let Γ be a discrete cocompact subgroup of $SL_2(K) \subset G$. Let

$$I_{\mathcal{L}} : H^0(\Gamma, \mathcal{O}(-n-2)) \to H^1(\Gamma, M) \tag{125}$$

be the connecting homomorphism in cohomology associated with the short exact sequence (61). (Since Γ consists of unimodular matrices we have ignored twists by the determinant). Theorem 5.3 supplies the following interpretation of $I_{\mathcal{L}}(f)$. Consider

$$\omega_f = f(z)(u - zv)^n dz \in \Omega(M) \tag{126}$$

and let $F_{\omega_f} \in \beta_n(\mathcal{O}(n,\mathcal{L})) \subset \mathcal{O}_{\mathcal{L}}(M)$ be a Coleman primitive of ω_f in $\mathfrak{X}$, i.e.

$$dF_{\omega_f} = \omega_f. \tag{127}$$

Then $I_{\mathcal{L}}(f)$ is the cohomology class of the 1-cocycle

$$\gamma \mapsto \gamma(F_{\omega_f}) - F_{\omega_f}. \tag{128}$$

Let $\tilde{C}^0_{har}(M)$ denote the space of M-valued 0-cycles c on $\mathcal{T}$ satisfying the "harmonicity" condition

$$(q+1)c(v) = \sum_{(v,v')\in\mathcal{T}_1} c(v') \tag{129}$$

for every vertex v. The exact sequence

$$0 \to M \to \tilde{C}^0_{har}(M) \to C^1_{har}(M) \to 0 \tag{130}$$

gives rise to a connecting homomorphism

$$I_{Sch} : H^0(\Gamma, C^1_{har}(M)) \to H^1(\Gamma, M). \tag{131}$$

The following fundamental results hold.

Proposition 7.1. *(*[dS89] *Prop. 4.2,* [Br04] *5.2.2) Let $\mathcal{L}$ and $\mathcal{L}'$ belong to E, and denote by e the absolute index of ramification of E. Let c_{ω_f} be the harmonic cochain associated to (the cohomology class of) ω_f. Then*

$$I_{\mathcal{L}}(f) - I_{\mathcal{L}'}(f) = \frac{\mathcal{L}-\mathcal{L}'}{e} I_{Sch}(c_{\omega_f}). \tag{132}$$

Proposition 7.2. *(*[dS89] *6.6, 6.10 and Theorem 3.9) Suppose Γ is an arithmetic subgroup of G. Then I_{Sch} is an isomorphism.*

Our main interest in $I_{\mathcal{L}}(f)$ comes from the following simple observation. Since Γ is cocompact in G,

$$H^0(\Gamma, \mathcal{O}(n,\mathcal{L})) \subset \mathcal{O}(n,\mathcal{L})^{bnd}. \tag{133}$$

Indeed, if $f \in H^0(\Gamma, \mathcal{O}(n,\mathcal{L}))$ and $\phi \in \mathcal{O}(n,\mathcal{L})' = \Sigma(n,\mathcal{L})\{n\}$ then the matrix coefficient $g \mapsto \phi(gf)$ is a continuous function on G and is Γ-invariant, hence it is bounded.

Suppose $I_{\mathcal{L}}(f) = 0$. Then $f = \delta_n(g)$ for some $g \in H^0(\Gamma, \mathcal{O}(n,\mathcal{L}))$, hence the module $\mathcal{O}(n,\mathcal{L})^{bnd}$ does not vanish, $St(M)^{\mathcal{L}}$ is a genuine integral structure, and $B(n,\mathcal{L}) \neq 0$.

7.2. **A special choice of Γ.** For simplicity we take from now on $K = \mathbb{Q}_p$. Let R be an Eichler $\mathbb{Z}[1/p]$-order (of some prime-to-p level N^-) in a quaternion algebra B over $\mathbb{Q}$ which is definite, but split at p (of some prime-to-p discriminant N^+). Let Γ be the group of norm-1 elements in $R^\times$, embedded in $SL_2(\mathbb{Q}_p)$ via a fixed isomorphism of B_p with $M_2(\mathbb{Q}_p)$. Then Γ is discrete and cocompact in $SL_2(\mathbb{Q}_p)$. See [Br04], 5.3 and [dS89], Section 6.

In this case it is well known that there is an algebra of Hecke operators acting on $H^0(\Gamma, \mathcal{O}(-n-2))$, $H^0(\Gamma, C^1_{har}(M))$ and $H^1(\Gamma, M)$, and these actions are compatible under the maps $I_{\mathcal{L}}$ and I_{Sch}. The three spaces have the same dimension, and by "multiplicity one", each system of Hecke eigenvalues appears in them at most once: any two eigenvectors with the same Hecke eigenvalues are proportional. It follows that if $f \in H^0(\Gamma, \mathcal{O}(-n-2))$ is a Hecke eigenform, $I_{\mathcal{L}}(f)$ and $I_{Sch}(c_{\omega_f})$ belong to the same 1-dimensional space in $H^1(\Gamma, M)$. It follows from Propositions 7.1 and 7.2 that there exists a unique choice of $\mathcal{L}$ making $I_{\mathcal{L}}(f) = 0$. For this $\mathcal{L}$ we conclude that $\mathcal{O}(n,\mathcal{L})^{bnd} \neq 0$.

Iovita and Spiess [IS03] have proved that the uniqe $\mathcal{L}$ making $I_{\mathcal{L}}(f) = 0$ is the same as the $\mathcal{L}$-invariant attached by Mazur and Fontaine to the p-adic local Galois representation which corresponds to the modular form f (under an appropriate sign normalization).

Remark 7.1. *It would be interesting to examine the space*

$$\sum_{\Gamma} H^0(\Gamma, \mathcal{O}(n, \mathcal{L})) \tag{134}$$

where the sum ranges (1) over all Γ *conjugate in* G *to a fixed* Γ_0 *(in this case only finitely many* $\mathcal{L}$ *give rise to nonzero spaces), (2) over all congruence subgroups, or more generally, over all the subgroups of finite index of a given* Γ_0*, and their conjugates, or (3) over all discrete cocompact* Γ *(one may restrict to arithmetic ones, or even to congruence subgroups of arithmetic ones). All these spaces are contained in* $\mathcal{O}(n, \mathcal{L})^{bnd}$*, and are invariant under* G*.*

References

[Br04] C. Breuil: Invariant L et série spéciale p-adique, Ann. Scient. de l'E.N.S. **37** (2004), 559-610.

[dS89] E. de Shalit: Eichler cohomology and periods of modular forms on p-adic Schottky groups, J. reine angew. Math. **400** (1989), 3-31.

[dS09] E. de Shalit: Bounded cohomology of the p-adic upper half plane, in *Symmetries in Algebra and Number Theory,* I. Kersten, R.Meyer eds., Universitätsverlag Göttingen 2009, 27-48.

[IS03] A. Iovita, M. Spiess: Derivatives of p-adic L-functions, Heegner cycles and monodromy modules attached to modular forms, Invent. Math. **154** (2003), 333-384.

[NFA] P. Schneider: *Nonarchimedean Functional Analysis*, Springer 2002.

[SS91] P. Schneider, U. Stuhler: The cohomology of p-adic symmetric spaces, Invent.Math. **105** (1991), 47-122.

[ST02] P. Schneider, J. Teitelbaum: Banach space representations and Iwasawa theory, Israel J. Math. **127** (2002), 359-380.

[V08] M.-F. Vigneras: A criterion for integral structures and coefficient systems on the tree of $PGL(2, F)$, Pure and Applied Mathematics Quarterly, **4** (2008), 1-29.

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