INTRODUCTORY ALGEBRAIC GEOMETRY

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1. Affine algebraic sets

Classical algebraic geometry deals, initially, with *affine algebraic sets*, i.e. subsets of affine space defined by polynomial equations. We shall learn how to deal with these objects intrinsically, that is independently of the ambient space, and how to glue them along open subsets. For that purpose it will be necessary to introduce as early as possible a topology on our algebraic sets, the *Zariski topology.*

A word of warning is necessary before we embark on such a project. The proposed path sounds similar to the study of many other geometric objects, for example differentiable manifolds. Starting with manifolds embedded in $\mathbb{R}^n$, one formalizes various attributes intrinsically, and uses gluing to construct the most general manifolds out of the initial examples. However, while differentiable manifolds locally look all alike - as d-dimensional open balls, local algebraic geometry encompasses the whole subject of commutative algebra. This is only one difference. More subtle differences will emerge in due course.

1.1. **Algebraic sets and the Zariski topology.**

Let k be an algebraically closed field, and

$$k[X] = k[X_1, \ldots, X_n] \tag{1}$$

the polynomial ring in n variables over k. The *affine n-space over k* is the set

$$\mathbb{A}^n = \{(x_1, \ldots, x_n) |\, x_i \in k\}\,. \tag{2}$$

A subset $Z \subset \mathbb{A}^n$ is called an *affine algebraic set* (in short: an algebraic set) if it is the set of common zeros of some collection of polynomials,

$$Z = \{x \in \mathbb{A}^n |\, f_\alpha(x) = 0\, \forall \alpha \in A\} \tag{3}$$

where each $f_\alpha \in k[X]$. We write $Z = Z(\{f_\alpha\}_{\alpha \in A})$. For example, a *hypersurface* in $\mathbb{A}^n$ is $Z(f)$, for a single nonconstant $f \in k[X]$.

Proposition 1.1. *(i) $\mathbb{A}^n$ and the empty set are algebraic sets.*
(ii) The union of two algebraic sets is algebraic.
(iii) The intersection of any family of algebraic sets is algebraic.

Exercise 1.1. *Prove the proposition. Hint:*

$$Z\left(\{f_\alpha\}_{\alpha \in A}\right) \cup Z(\{g_\beta\}_{\beta \in B}) = Z(\{f_\alpha g_\beta\}_{(\alpha,\beta) \in A \times B}). \tag{4}$$

The meaning of the proposition is that if we take the algebraic sets as closed sets we have defined a topology on $\mathbb{A}^n$, called the *Zariski topology.* If Z is any algebraic set, the Zariski topology on Z is the topology induced on it from $\mathbb{A}^n$. Closed/open sets in Z are intersections of Z with closed/open sets in $\mathbb{A}^n$.

Exercise 1.2. *Show that the closed sets in $\mathbb{A}^1$ are the whole affine line, and finite sets. In particular, the Zariski topology is not Hausdorff.*

The open sets

$$D(f) = \mathbb{A}^n - Z(f) \tag{5}$$

are called *principal open sets*. For any algebraic set Z we let $D_Z(f) = Z \cap D(f)$ and call it a principal open set in Z.

If $\{f_\alpha\}_{\alpha \in A}$ is a collection of polynomials, we write $I = (f_\alpha)_{\alpha \in A}$ for the ideal that they generate in $k[X]$ (in this course the whole ring will also be considered an ideal; an ideal strictly smaller than the whole ring will be called a *proper ideal*).

Exercise 1.3. *Prove that $Z(\{f_\alpha\}_{\alpha \in A}) = Z((f_\alpha)_{\alpha \in A})$.*

For this reason it is enough to look at $Z(I)$ where I is an ideal.

1.2. **The Hilbert basis theorem.** It is a great relief to know that every algebraic set can be defined by a *finite number* of equations. This is a consequence of the following famous theorem of Hilbert. Recall that a commutative ring is called Noetherian if every ideal in it is finitely generated, or equivalently, if every ascending sequence $I_1 \subset I_2 \subset \cdots$ of ideals, stabilizes: from a certain m on, $I_m = I_{m+1} = \cdots$.

Exercise 1.4. *Prove the equivalence of the two definitions.*

Theorem 1.2. *The ring $k[X]$ is Noetherian. More generally, if R is Noetherian, so is $R[T]$.*

Proof. Let $I \subset R[T]$ be an ideal, and let $J_d \subset R$ be the collection of leading coefficients of polynomials of degree d from I. Clearly J_d is an ideal in R and $J_d \subset J_{d+1}$. If R is Noetherian, there is an m such that $J_m = J_{m+1} = \cdots$. For $0 \le i \le m$ let $f_{i,j} \in I$ be a finite list of polynomials of degree i whose leading coefficients generate J_i (once again we use Noetherianity of R). Let I' be the ideal generated by all the $f_{i,j}$. We claim that $I' = I$. If not, let $f \in I - I'$ be a polynomial of minimal degree d. If $d \le m$, subtract from f a scalar linear combination of $f_{d,j}$ to kill the leading term. Since the resulting polynomial lies in I but has a degree smaller than d, it must belong to I'. But then $f \in I'$, contradiction. If $m < d$, subtract a suitable scalar linear combination of $T^{d-m} f_{m,j}$ (note $J_d = J_m$) and argue as before. □

Corollary 1.3. *(i) Every algebraic set is $Z(g_1, \ldots, g_r)$ for some r. (ii) The principal open sets form a basis for the Zariski topology.*

Exercise 1.5. *Prove the two corollaries.*

A natural question arises: what is the minimal number of equations needed to define an algebraic set Z? Our intuition says that this has to be related to the dimension (or better, the codimension, both concepts yet to be defined) of the algebraic set: a single equation should suffice to define a surface in $\mathbb{A}^3$, a second one will be needed to cut off a curve in the surface, and three equations to define a finite collection of points. While this has a grain of truth in it, the answer is not as easy. An algebraic subset Z of $\mathbb{A}^n$ of dimension d is called an *(affine) set-theoretic complete intersection* if it is the zero locus of $n - d$ polynomials. It is called a *scheme-theoretic complete intersection* if its ideal $I(Z)$ (see below) can be generated by $n-d$ polynomials. A famous example of a non-scheme-theoretic-complete-intersection is the "twisted cubic"

$$Z = Z(X_0X_2 - X_1^2, X_1X_3 - X_2^2, X_0X_3 - X_1X_2) \subset \mathbb{A}^4, \tag{6}$$

which is a surface, but whose ideal can not be generated by any two polynomials.

Exercise 1.6. *Show that $Z = \{(s^3, s^2t, st^2, t^3) | \, s, t \in k\}$. Show that Z is a set-theoretic complete intersection. In higher dimensions there are examples of non set-theoretic complete intersections too.*

1.3. **The correspondence between algebraic subsets and ideals.** If $I_0 \subset k[X]$ is any subset we let $Z(I_0)$ be, as before, the set of its common zeroes, an algebraic set. If $Z_0 \subset \mathbb{A}^n$ we let $I(Z_0)$ be the ideal of polynomials that vanish on Z_0.

Proposition 1.4. *(i) The correspondences Z and I reverse inclusions.*
(ii) $I(\emptyset) = k[X]$, $I(\mathbb{A}^n) = (0)$, $Z(k[X]) = \emptyset$, $Z(0) = \mathbb{A}^n$.
(iii) $I_0 \subset I(Z(I_0))$, $Z_0 \subset Z(I(Z_0))$
(iv) $I(Z_0) = I(Z(I(Z_0)))$, $Z(I_0) = Z(I(Z(I_0)))$
(v) $Z(I(Z_0))$ is the Zariski closure of Z_0.

Exercise 1.7. *Prove the assertions.*

There is a striking lack of symmetry: $I(Z(I_0))$ contains, but is *not equal*, in general, to the ideal generated by I_0. In fact, even if I_0 was an ideal to begin with, it is clear that if $f^m \in I_0$ for some $m \geq 1$, then $f \in I(Z(I_0))$. That this is the only condition one has to impose on f is the celebrated Hilbert's *nullstellensatz.*

1.4. **Hilbert's nullstellensatz.** If I is an ideal in a commutative ring R we let

$$\sqrt{I} = \{a \in R | \, \exists m \geq 1, \, f^m \in I\} \, . \tag{7}$$

Exercise 1.8. *Prove that $J = \sqrt{I}$ is an ideal containing I, and that $\sqrt{J} = J$. If $R = \mathbb{Z}$, which are the ideals satisfying $\sqrt{I} = I$? Prove that in every commutative ring, prime ideals P satisfy $\sqrt{P} = P$.*

Theorem 1.5. *For any ideal $I_0 \subset k[X]$, $I(Z(I_0)) = \sqrt{I_0}$.*

Exercise 1.9. *Prove that this is false if $k = \mathbb{R}$.*

There are many proofs for the Nullstellensatz. The one we give below is traditional, and not the shortest, but it relies on two other important theorems, whose geometric meaning will become clear in a short while, so it is a good opportunity to introduce them now. First a definition.

Let $A \to B$ be a homomorphism of commutative rings. We say that B is *finite* (or of finite type as a module) over A is there are finitely many $b_1, \ldots, b_m$ such that

$$B = \bar{A}b_1 + \cdots + \bar{A}b_m, \tag{8}$$

where $\bar{A}$ denotes the image of A in B. We say that B is *finitely generated* (or of finite type as a ring) over A if the homomorphism can be extended to a surjective homomorphism $A[X_1, \ldots, X_m] \to B$, or, equivalently, if for some b_i

$$B = \bar{A}[b_1, \ldots, b_m]. \tag{9}$$

Exercise 1.10. *Of the rings $k[X]$, $k[X,Y]$, $k[X,Y]/(Y^2 - X^3)$, $k[X,Y]/(XY-1)$ (with the obvious maps between them), which is finite over which?*

Exercise 1.11. *If $A \to B \to C$ are homomorphisms, C is finite over B, and B is finite over A, then C is finite over A. Same with "finitely generated".*

Theorem 1.6. *(Noether normalization). If B is finitely generated over a field k, then there are $x_1, \dots, x_d \in B$, algebraically independent over k, such that B is finite over $A = k[x_1, \dots, x_d]$.*

Lemma 1.7. *If $f \in k[X]$, there is a linear change of variables*

$$X_i = \sum a_{ij} Y_j, \quad \det(a_{ij}) \neq 0 \tag{10}$$

such that f is monic with respect to Y_n, i.e.

$$f = \sum_{r=0}^{d} f_r(Y_1, \dots, Y_{n-1}) Y_n^{d-r} \tag{11}$$

and $f_0 = 1$.

Exercise 1.12. *Prove the lemma as follows. Let d be the (total) degree of f. We may assume that f is homogeneous of degree d, because the homogenous parts of smaller degrees will not change the term $f_0(Y_1, \dots, Y_{n-1}) Y_n^d$. Substitute $X_n = \lambda_n Y_n$ and $X_i = Y_i + \lambda_i Y_n$. Show that one can choose $\lambda_1, \dots, \lambda_n$ so that this change of variables is invertible and $f_0 = 1$.*

Proof. (of Noether's normalization theorem). We prove it by induction on the minimal number of generators of B. If this number is 0, $B = k$ and there is nothing to prove. Let

$$\varphi : k[X_1, \dots, X_n] \to B \tag{12}$$

be a surjective homomorphism. If $\ker(\varphi) = 0$, $A = B$ and we are done. Otherwise let f be a nonzero polynomial in the kernel. We may assume, by the lemma, that f is monic with respect to X_n. The ring $k[X_1, \dots, X_n]/(f)$ is finite over $k[X_1, \dots, X_{n-1}]$ (in fact, even finite and free with $1, X_n, \dots, X_n^{\deg(f)-1}$ as a basis). Let $B_0 = \varphi(k[X_1, \dots, X_{n-1}])$. Then B is finite over B_0 because it is spanned as a module over B_0 by $\varphi(X_n^i)$ for $0 \leq i \leq \deg(f) - 1$. But B_0 is generated by $n-1$ elements over k, so by the induction hypothesis there is a subring $A \subset B_0$ which is a polynomial ring over k and over which B_0 is finite. Now "finite over finite is finite" implies that B is finite over A. $\square$

Theorem 1.8. *(The "going up" theorem) Let B be finite over a* sub*ring A and M_A a maximal ideal in A. Then there is a maximal ideal M_B in B such that $M_B \cap A = M_A$.*

Proof. It is enough to show that $B \neq M_A B$, because then $M_A B$ is contained in a maximal ideal M_B and $M_A \subset M_B \cap A \subset A$, so from the maximality of M_A and the fact that $1 \notin M_B$ we conclude $M_A = M_B \cap A$. Let $B = \sum_{i=1}^m A b_i$. If $B = M_A B$ then

$$b_i = \sum x_{ij} b_j \tag{13}$$

with $x_{ij} \in M_A$. Viewing this as a matrix equation

$$(I - X)\underline{b} = 0 \tag{14}$$

where $X = (x_{ij})$ and $\underline{b}$ is the column vector ${}^t(b_1, \dots, b_m)$. Multiplying on the left by $adj(I - X)$ we get that $\delta = \det(I - X)$ annihilates every b_i, hence annihilates B, so $\delta = 0$. But $\delta \equiv 1 mod M_A$, contradiction. $\square$

Proof. (of the nullstellensatz) Let $f \in I(Z(I_0))$. We must show that a certain power of f lies in I_0 (the opposite inclusion being obvious). Suppose, to the contrary, that I_0 does not intersect the set $\{1, f, f^2, \dots\}$. Let P be an ideal containing I_0 which is maximal among all the ideals not containing any f^m. Such a P exists by Zorn's lemma (or, without Zorn, because $k[X]$ is Noetherian). We claim that P is a prime ideal. Indeed, if g and h are not in P, then both $P+(g)$ and $P+(h)$ contain powers f^i and f^j. But then

$$f^{i+j} \in [P+(g)][P+(h)] \subset P+(gh). \tag{15}$$

If $gh \in P$, $f^{i+j} \in P$, contradicting the way P was constructed. Hence $gh \notin P$, and P is prime.

Consider now the integral domain $k[X]/P$. Let $\bar{f}$ be the image of f in it and let

$$B = (k[X]/P)[\frac{1}{\bar{f}}] \tag{16}$$

be the subring of the field of fractions of $k[X]/P$ obtained by inverting $\bar{f}$. By Noether's normalization theorem there exists a polynomial ring $A = k[y_1, \dots, y_r] \subset B$, over which B is finite. Let M_A be the ideal $(y_1, \dots, y_r)$ in A and let M_B be a maximal ideal in B whose intersection with A is M_A. Then B/M_B is a field which is finite over $A/M_A = k$. Since k is algebraically closed

$$B/M_B = A/M_A = k. \tag{17}$$

Let $\varphi : B \to k$ be the homomorphism whose kernel is M_B. Let $x_i = \varphi(\bar{X}_i)$. Since $I_0 \subset P$, $x = (x_1, \dots, x_n) \in Z(I_0)$. Since $\varphi(\bar{f}) \neq 0$, $f(x) \neq 0$, so we have shown that $f \notin I(Z(I_0))$. □

1.5. Consequences of the nullstellensatz.

Exercise 1.13. *Prove the following:*

(i) Z and I put the algebraic sets in $\mathbb{A}^n$ in one-to-one inclusion reversing correspondence with the ideals of $k[X]$ satisfying $\sqrt{I} = I$.

(ii) The maximal ideals of $k[X]$ are the ideals $(X_1 - x_1, \dots, X_n - x_n)$.

(iii) $Z(I) = 0$ if and only if $I = k[X]$.

(iv) $Z(I)$ is covered by the principal open sets $D(f_\alpha)$ if and only if there is a finite collection among the f_α, denoted f_i, and $h_i \in k[X]$ such that

$$1 \equiv \sum h_i f_i mod I. \tag{18}$$

(v) Every decreasing sequence $Z_1 \supset Z_2 \supset \cdots$ of algebraic sets stabilizes.

(vi) Every algebraic set is quasi-compact (from every open cover one can extract a finite subcover).

1.6. The ring of regular functions.

If Z is an algebraic set we call

$$k[Z] = k[X]/I(Z) \tag{19}$$

the ring of regular functions on Z. Note that this is indeed a ring of functions on Z, i.e. every $f \in k[Z]$ defines a well-defined function $Z \to k$, and the ring homomorphism

$$k[Z] \to k^Z \tag{20}$$

obtained in this way is injective: f is determined by its values at all points of Z.

Let $z \in Z$ be a point. The homomorphism of *evaluation at* z

$$ev_z : f \mapsto f(z) \tag{21}$$

is a surjective homomorphism $k[Z] \to k$, and we denote by M_z its kernel. It is a maximal ideal, generated by the images of $X_i - x_i$ modulo $I(Z)$, if $z = (x_1, \dots, x_n)$.

Exercise 1.14. *Use the nullstellensatz to show that every maximal ideal of Z is an M_z for a unique point $z \in Z$. Thus the points of Z may be identified with the maximal ideals of $k[Z]$.*

1.7. **Irreducibility.** We call Z *irreducible* if whenever $Z = Z_1 \cup Z_2$ with Z_i algebraic sets, then $Z = Z_1$ or $Z = Z_2$. Otherwise, it is called reducible. For example, $Z(fg) = Z(f) \cup Z(g)$ will in general be reducible.

Proposition 1.9. *Z is irreducible if and only if $I(Z)$ is a prime ideal, if and only if $k[Z]$ is an integral domain.*

Proof. Suppose $Z = Z_1 \cup Z_2$ and $Z_i \neq Z$. Then there are $f_i \in I(Z_i) - I(Z)$, so $f_1 f_2 \in I(Z)$ and $I(Z)$ can not be prime. □

Exercise 1.15. *Prove the converse direction.*

Exercise 1.16. *Prove that Z is irreducible if and only if every non-empty open set in Z is dense.*

If Z is irreducible we call the field of fractions

$$k(Z) = frac(k[Z]) \tag{22}$$

the field of rational functions on Z. If $f = g/h \in k(Z)$ with g and h from $k[Z]$ then f is defined at all the points of $D_Z(h)$. Note however that we may have another representation $f = g_1/h_1$ so that f will also be defined at the points of $D_Z(h_1)$. The union of the $D_Z(h)$, for all the possible denominators h of f, is called the *domain of definition* of f. It is open, but need not be a principal open set.

Exercise 1.17. *Show that the possible denominators for a given $f \in k(Z)$ form an ideal I in $k[Z]$, and that the domain of definition of f is the complement of its set of zeros.*

Proposition 1.10. *Every algebraic set is a union of finitely many irreducible algebraic sets.*

Proof. If this is not true, let Z be a smallest algebraic set which is *not* the union of finitely many irreducibles (Z exists by part (v) of one of the exercises above). In particular, Z itself is reducible, so $Z = Z_1 \cup Z_2$ with Z_i strictly smaller than Z. This implies that Z_i is each a union of finitely many irreducibles, hence so is Z. Contradiction. □

Exercise 1.18. *Suppose $Z = \bigcup Z_i$ (a finite union of irreducibles) and no Z_i is contained in any Z_j if $i \neq j$. Suppose also that $Z = \bigcup Z'_i$ and the same holds with the Z'_i. Show that up to a permutation, $Z_i = Z'_i$ (hint: consider the decomposition*

$$Z'_i = \bigcup_j Z'_i \cap Z_j.) \tag{23}$$

The Z_i are called the irreducible components *of Z.*

2. Morphisms and affine varieties

It is now time to talk about maps between algebraic sets. We would like to single out those maps which are themselves algebraic, and call them "morphisms". Note that we already have defined the regular functions on Z, and these should coincide with the morphisms

$$Z \to \mathbb{A}^1. \tag{24}$$

2.1. **Morphisms between algebraic sets.** Let $Z \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ be two algebraic sets. We use $X_1, \ldots, X_n$ as coordinates in the first affine space and $Y_1, \ldots, Y_m$ in the second. Any map $\varphi : Z \to W$ induces, by pull back, a ring homomorphism $\varphi^* : k^W \to k^Z$, namely $\varphi^*(g) = g \circ \varphi$. We say that φ is a morphism of algebraic sets if $\varphi^*(k[W]) \subset k[Z]$. Explicitly, if we let $x_i = X_i mod I(Z)$ and $y_j = Y_j mod I(W)$, we should have m polynomials φ_j in the variables X_i, uniquely determined modulo $I(Z)$, "yielding the set theoretic function φ":

$$\varphi^*(y_j) = \varphi_j(x_1, \ldots, x_n). \tag{25}$$

The only condition on the φ_j is that if $g \in I(W)$,

$$0 = \varphi^*(\bar{g}) = g(\varphi_1(x_1, \ldots, x_n), \ldots, \varphi_m(x_1, \ldots, x_n)), \tag{26}$$

i.e., $g \circ \varphi \in I(Z)$.

Proposition 2.1. *A morphism is automatically continuous in the Zariski topology.*

Proof. If φ is a morphism as above, then $\varphi^{-1}(D_W(g)) = D_Z(\varphi^*(g))$. Since principal open sets form a basis to the Zariski topology, the inverse image of any open set is open. □

A morphism is *open* or *closed* if it is so as a continuous map in the Zariski topology. However, the more subtle properties of morphisms can not be read off from their topological attributes, and an arbitrary continuous map in the Zariski topology need not be a morphism. Thus topological considerations alone are insufficient for most purposes.

Clearly the composition of morphisms is a morphism. An isomorphism is a morphism φ as above for which there exists another morphism $\psi : W \to Z$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity maps on W and Z respectively. Two algebraic sets are called isomorphic if there is an isomorphism between them. Proving that two algebraic sets are isomorphic may not be easy. The simplest way to prove that two algebraic sets are *not* isomorphic would be to find invariants of algebraic sets that are preserved under isomorphism, and show that the two sets have different invariants.

Proposition 2.2. *There is a one-to one correspondence between ring homomorphisms $k[W] \to k[Z]$ and morphisms $Z \to W$. The category of affine algebraic sets with morphisms as defined above is anti-equivalent to the category of finitely generated reduced*[1] *k-algebras.*

Exercise 2.1. *Check the proposition (it only summarizes, in a different language, the above discussion). Prove that $\mathbb{A}^1$ and the hyperbola $Z(X_1X_2 - 1) \subset \mathbb{A}^2$ are not isomorphic.*

[1] reduced means having no nilpotents

Exercise 2.2. *Show that the product $Z \times W \subset \mathbb{A}^{n+m}$ of two algebraic sets is an algebraic set.*

Exercise 2.3. *Let $\varphi : Z \to W \subset \mathbb{A}^m$ be a map between algebraic sets. Show that φ is a morphism if and only if the induced map $\varphi : Z \to \mathbb{A}^m$ is a morphism. Show that the graph of φ is an algebraic set in $\mathbb{A}^{n+m}$ isomorphic to Z.*

Exercise 2.4. *Show that if $I = \ker \varphi^*$ then $Z_W(I) = \overline{\varphi(Z)}$.*

Exercise 2.5. *Show that $\varphi(z) = w$ if and only if $M_z \supset \varphi^* M_w$, if and only if $(\varphi^*)^{-1}(M_z) = M_w$.*

Exercise 2.6. *If $f : Z \to W$ is a morphism of algebraic sets, show that for every irreducible component Z' of Z there is an irreducible component W' of W such that $f(Z') \subset W'$.*

An *affine variety* is an isomorphism class of affine algebraic sets[2]. In other words, while we reserve the terminology "algebraic set" to denote a concrete subset of affine space, the concept of an affine variety is intrinsic and does not carry any information about the ambient space. Attached to an affine variety Z is its ring of regular functions $k[Z]$, which is well defined only up to isomorphism, and conversely, $k[Z]$ determines Z as a variety. Giving an embedding of Z as an algebraic subset of some $\mathbb{A}^n$ is equivalent to giving a surjective ring homomorphism $k[X_1, \ldots, X_n] \to k[Z]$.

2.2. **Properties of varieties and of morphisms between varieties.** The equivalence of categories between finitely generated reduced k-algebras and affine varieties over k allows one to set up a dictionary between algebraic properties of $k[Z]$ and geometric properties of Z (and similarly for ring homomorphisms and morphisms). It should be understood that this dictionary is tautological, and one is left with the task of developing geometric intuition for algebraic concepts. We list a few examples. The first two we have already encountered.

- A variety Z is called *irreducible* if $k[Z]$ is an integral domain.
- A morphism $f : Z \to W$ is called *finite* (W is finite over Z) if $f^* : k[W] \to k[Z]$ is finite.
- A morphism $f : Z \to W$ is called *dominant* if f^* is injective. In other words, $f(Z)$ is dense in W. For example, if W is irreducible and f is an open morphism, then it is dominant, but dominant morphisms need not be open.
- Two irreducible varieties Z and W are called *rationally equivalent* if their fields of rational functions $k(Z)$ and $k(W)$ are isomorphic over k. This is a much weaker notion than being isomorphic: for example, $\mathbb{A}^1$ and the hyperbola in $\mathbb{A}^2$ are rationally equivalent, although, as we have seen, they are not isomorphic. A morphism $f : Z \to W$ is *birational*[3] if f^* induces an isomorphism between $k(W)$ and $k(Z)$.
- A variety Z is called *factorial* if $k[Z]$ is a unique factorization domain. The geometric meaning of this concept is not self-evident.

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[2]Some authors reserve the term *variety* only for *irreducible* algebraic sets.

[3]Not to be confused with a birational *map*, or more generally a rational *map*, which need not be a morphism. A rational map is only defined on an open dense subset of Z, while a morphism must be defined everywhere.

Exercise 2.7. *Let Z and W be irreducible. Show that a morphism $f : Z \to W$ is birational if and only if there exists a $g \in k[W]$ such that f^* induces an isomorphism of $k[W][1/g]$ and $k[Z][1/f^*g]$. Conclude that a birational f induces a bijection between $D_Z(f^*g) = f^{-1}D_W(g)$ and $D_W(g)$, and is dominant.*

Exercise 2.8. *Consider the morphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ given by $f(x, y) = (x, xy)$. Is it finite? Is it dominant? Is it open? Is it closed? Describe the fibers of f. (This exercise becomes easier if you use some of the facts proven in the next paragraph, but try to do it straight form the definitions, and come back to it later when you have more tools at hand.)*

2.3. **Finite morphisms and Noether normalization revisited.** We can give now a geometric meaning to Noether normalization and the going up theorem. First note that if $f : Z \to W$ is a finite morphism, then for any point $w \in W$, $k[Z]/f^*M_wk[Z]$ is finite over $k[W]/M_w = k$, i.e. is a finite dimensional k-algebra.

Exercise 2.9. *Prove that a finite dimensional k-algebra B has only finitely many homomorphisms $B \to k$.*

From the exercise we conclude that there are only finitely many maximal ideals M_z containing f^*M_w, or, what is the same, points z in the fiber $f^{-1}(w)$. If f is also dominant, so that $k[W]$ is a subring of $k[Z]$ (via f^*), the going up theorem says that this fiber is not empty, in other words, *a finite dominant morphism is surjective.*

A morphism $f : Z \to W$ all of whose fibers are finite is called *quasi-finite,* and we have just seen that finite morphisms are quasi-finite, but the opposite need not be true. The projection of the hyperbola $Z(X_1X_2 - 1) \subset \mathbb{A}^2$ to the X_1-line $\mathbb{A}^1 = Z(X_2)$ is quasi-finite and dominant, but not surjective, hence not finite.

Exercise 2.10. *Don't get fooled to think that quasi-finite* plus *surjective is finite. Show that $(x_1, x_2) \mapsto (x_1-1)^2$ is a quasi finite, surjective morphism of $Z(X_1X_2-1)$ to $\mathbb{A}^1$ which is not finite.*

Exercise 2.11. *Prove that a finite morphism is closed: the image of a closed set is closed. Hint: reduce to showing that if $f : Z \to W$ is finite, then $f(Z)$ is closed, and then reduce to the finite and dominant case.*

The geometric meaning of Noether normalization is now clear. Every affine variety is a finite cover of some affine space (i.e. admits a finite dominant map to $\mathbb{A}^d$). Moreover, a close examination of the proof reveals that if Z is an algebraic set in $\mathbb{A}^n$, after a linear change of coordinates in $\mathbb{A}^n$ we may assume that the projection of Z to the first d coordinates is a finite morphism.

2.4. **Closed embeddings.** Closed embeddings are special cases of finite morphisms. A morphism $f : Z \to W$ is called a *closed embedding* if f^* is surjective. If we denote by I the kernel of f^*, then f^* identifies $k[Z]$ with $k[W]/I$ and f induces therefore an isomorphism between Z and the closed subvariety $Z_W(I)$ of W.

2.5. **Open immersions.** A morphism between irreducible varieties $f : Z \to W$ is called an *open immersion* if for every $z \in Z$ there exists a $g \in k[W]$ such that f^* induces an isomorphism

$$f^* : k[W][1/g] \simeq k[Z][1/f^*g], \tag{27}$$

and $(f^*g)(z) \neq 0$. For example, the projection of the hyperbola $Z(X_1X_2 - 1) \subset \mathbb{A}^2$ to the X_1-line $\mathbb{A}^1 = Z(X_2)$ is an open immersion.

Open immersions have the following properties: they are birational, open (hence clearly dominant), and injective.

Exercise 2.12. *(i) Let k have characteristic p. Show that the map of raising to power p is a bijective open morphism from $\mathbb{A}^1$ to itself, which is nevertheless not birational, hence not an open embedding.*

(ii) Let $W = Z(X_2^2 - X_1^3) \subset \mathbb{A}^2$. Show that the morphism $f : \mathbb{A}^1 \to W$ given by $f(t) = (t^2, t^3)$ is bijective, open and *birational, but not an open immersion. (Hint: the point z where the condition defining an open immersion is violated is the origin.)*

Using the notion of an open immersion we can answer the question left hanging in the air before, about the precise relation between quasi finite and finite morphisms.

Theorem 2.3. *(Zariski's Main Theorem, Grothendieck's version) A quasi finite morphism $f : Z \to W$ between two irreducible varieties factors as $f' \circ i$ where i is an open immersion $Z \to Z'$ (for an appropriate Z') and $f' : Z' \to W$ is finite.*

There are many proofs of this celebrated theorem, but they are all beyond the scope of these notes. For a transparent, purely algebraic (but lengthy) proof see chapter 13 of Peskine's book *An algebraic introduction to complex projective geometry,* vol. I.

2.6. **Normal varieties.** Let Z be an irreducible variety. We want to study finite birational morphisms $f : Z' \to Z$. Such a morphism is surjective (since it is finite and dominant) and generically one-to-one, i.e. there exists an open set $D_Z(g) \subset Z$ such that f restricts to a bijection between $f^{-1}(D_Z(g)) = D_{Z'}(f^*g)$ and $D_Z(g)$. See exercise 2.7. Typical examples are the morphism f in Exercise 2.12(ii), which is finite, birational and bijective, yet not an isomorphism, or the morphism

$$f : \mathbb{A}^1 \to Z = Z(X_2^2 - X_1^2(1 - X_1)) \subset \mathbb{A}^2 \tag{28}$$

given by $f(t) = (1 - t^2, t(1 - t^2))$, which is finite birational and one-to-one everywhere except above $(0, 0)$, where the fiber contains two points. Both examples are instances of *blow-up* maps, used to resolve singularities, which will be studied later on in depth.

Exercise 2.13. *Check all the assertions made above.*

By the correspondence between affine varieties and rings, what we have to study is finitely generated k-algebras B

$$k[Z] \subset B \subset k(Z) \tag{29}$$

which are finite over $k[Z]$. It turns out that there exists a maximal such B that contains all the others.

Definition 2.1. *Let A be a subring of a field L. An element $x \in L$ is* integral over *A if it satisfies a monic algebraic equation with coefficients from A. The set of all the elements of L which are integral over A is called the* integral closure *of A in L. We call A* integrally closed *if it is equal to its integral closure in its field of fractions (note that in the above definition, L may be larger than the field of fractions of A, and that if A is a subfield of L, then being integral over A is the same as being algebraic over A).*

Lemma 2.4. *The following are equivalent:*
(i) x is integral over A
(ii) $A[x]$ is finite over A
(iii) There is a finite A-submodule $M \subset L$ closed under multiplication by x.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are clear. Assume (iii) and let $m_1, \dots, m_r$ be generators of M over A. Then

$$xm_i = \sum_{j=1}^{r} a_{ij} m_j \tag{30}$$

for some $a_{ij} \in A$. As in the proof of the going up theorem, $\delta = \det(xI - (a_{ij})) = 0$, exhibiting a monic polynomial in $A[X]$ satisfied by x. □

If $A \subset B \subset L$ are two subrings in a field L, we say that B is integral over A if every element of B is integral over A.

Lemma 2.5. *If $A \subset B \subset C \subset L$, B is integral over A, and C is integral over B, then C is integral over A.*

Proof. Let $x \in C$ and let $b_1, \dots, b_d$ be the coefficients of a monic polynomial from $B[X]$ satisfied by x. Since every b_i is integral over A, $B' = A[b_1, \dots, b_d]$ is a subring of B which is finite over A (use induction on d and Exercise 1.11). Since $B'[x]$ is also finite over B', $B'[x]$ is finite over A. Since it is clearly stable under multiplication by x, x is finite over A. □

Proposition 2.6. *The integral closure of A in L is a subring of L, which is integrally closed.*

Proof. If x and y are integral over A, consider the subring $A[x, y]$, which is finite over A. Since it is stable under multiplication by $x \pm y$ and xy, both these elements are also integral over A. Hence the integral closure is a ring, and from the last lemma it is integrally closed. □

Coming back to geometry, let B be the integral closure of $k[Z]$ in $k(Z)$. It contains $k[Z']$, for every morphism $f : Z' \to Z$ which is finite and birational. Suppose we showed B is finitely generated over k, so that it is of the form $k[\tilde{Z}]$ for a variety $\tilde{Z}$. It is then generated by finitely many elements, each of which is integral over $k[Z]$. It follows that B is also finite over $k[Z]$,i.e. the morphism $\tilde{Z} \to Z$ is birational and finite.

Lemma 2.7. *The ring B is finitely generated over k.*

Proof. Let us start by examining a special case, when $A = k[X]$ is the polynomial ring in n variables, which is a unique factorization domain (Gauss' Lemma). In this case A is already integrally closed. In fact, if $x \in L$ satisfies a monic polynomial

$$x^d + a_1 x^{d-1} + \cdots + a_d = 0 \tag{31}$$

with $a_i \in A$, write $x = u/v$ where $u, v \in A$ are relatively prime, arriving at the equation

$$u^d + a_1 u^{d-1} v + \cdots + a_d v^d = 0. \tag{32}$$

If v is not a unit we get a contradiction, because v must divide u^d, while it was assumed to be relatively prime to u. Thus v is a unit, and $x \in A$.

The general case is proven with the help of Noether's normalization theorem. Let A_0 be a polynomial ring over which A is finite. Let L_0 be the field of fractions of A_0, a purely transcendental extension of k. The field of fractions L of A is a finite extension of L_0 because A is finite over A_0, so L is generated by finitely many elements which are algebraic over L_0. The integral closure B of A in L is the same as the integral closure of A_0, because the elements of A are anyhow integral over A_0. Thus we may forget A, and prove that the integral closure of A_0 in a finite extension L of L_0 is finitely generated over k.

Let us assume that L is a separable extension of L_0 (which is always the case if the characteristic is 0). Let $\langle .,. \rangle$ be the trace form

$$\langle x, y\rangle = Tr_{L/L_0}(xy). \tag{33}$$

Since L/L_0 is separable, this is a *non-degenerate* bilinear form. Let $\omega_1, \ldots, \omega_r$ ($r = [L : L_0]$) be a basis of L over L_0 consisting of elements which are integral over A_0. (Exercise: prove that for every $\omega \in L$ there exists an $a \in A_0$ such that $a\omega$ is integral over A_0, hence such a basis exists).) Let $\{\omega_i'\}$ be the dual basis w.r.t. the trace form. If $x \in B$ then $\langle x, \omega_i\rangle \in A_0$ because, being the trace of the integral element $x\omega_i$ (the sum of Galois conjugates, all of which are integral over A_0), it is also integral over A_0. But $\langle x, \omega_i\rangle \in L_0$, and A_0 is integrally closed in L_0. As $\langle x, \omega_i\rangle$ is the coefficient of ω_i' in the expansion of x, this means that

$$B \subset \sum_{i=1}^{r} A_0\omega_i'. \tag{34}$$

Finally, A_0 is noetherian, so every submodule of a finitely generated module is finitely generated. Hence B is finite over A_0, and therefore finitely generated over k.

Without assuming sepearability of L over L_0 one has to work a little harder. We omit the details and refer the reader to the literature. □

We summarize the discussion in geometric terms.

Theorem 2.8. *Let Z be an irreducible affine variety. There exists a finite birational morphism $\tilde{f} : \tilde{Z} \to Z$ which is maximal in the sense that for any other finite birational morphism $f : Z' \to Z$ there exists a unique finite birational morphism $h : \tilde{Z} \to Z'$ such that $\tilde{f} = f \circ h$.*

The variety $\tilde{Z}$ and the morphism $\tilde{f}$ are unique up to isomorphism.

An irreducible variety Z is called *normal* if $k[Z]$ is integrally closed. The variety $\tilde{Z}$ constructed above is called the *normalization* of Z. When we study later on smooth varieties, we shall see that smooth implies normal. However, for many purposes normal is a good enough substitute to smooth, and it is much easier to construct the normalization than to resolve singularities (in particular, resolution of singularities in characteristic p is still an open problem).

2.7. **Dimension.** We give three equivalent definitions of the *dimension of an irreducible variety Z*. The dimension of a reducible variety is defined to be the maximum of the dimensions of its irreducible components.

First definition: $\dim Z$ is the transcendence degree of the field of rational functions $k(Z)$ over k (i.e. the maximal number of algebraically independent elements).

Second definition: $\dim Z$ is the maximal length d of a chain

$$Z = Z_0 \supset Z_1 \supset \cdots \supset Z_d \tag{35}$$

of non-empty irreducible varieties strictly contained in each other.

Third definition: $\dim Z = d$ if there is a finite map from Z to $\mathbb{A}^d$.

Noether normalization guarantees that such a map exists, but the fact that the d it yields is well-defined only follows from the equivalence of the third definition with the first two. The first definition has the corollary that $\dim Z$ is a birational invariant: if Z and Z' are birationally equivalent, they have the same dimension. The second definition may be phrased entirely in terms of the ring $k[Z]$: it is the maximal length d of a chain

$$0 = P_0 \subset P_1 \subset \cdots \subset P_d \tag{36}$$

of prime ideals in $k[Z]$ properly contained in each other. This definition makes sense in any commutative ring R (if R is not a domain the chain will not start with $P_0 = 0$) and leads to the notion of the *Krull dimension* of R.

Theorem 2.9. *The three definitions are equivalent.*

Proof. Let $Z \subset \mathbb{A}^n$ and let $x_i = X_i mod I(Z)$ be the ith coordinate function. If $k[Z]$ is a finite extension of $k[y_1, \dots, y_d]$ and the y_i are algebraically independent over k then $k(Z)$ is generated over $K = k(y_1, \dots, y_d)$ by the x_i which belong to the subring $K \cdot k[Z]$ of $k(Z)$. Since this subring is a finite dimensional vector space over K, the x_i are algebraic over K, hence the transcendence degree of $k(Z)$ is d (and $K \cdot k[Z] = k(Z)$). This shows the equivalence of the first and the third definitions.

Maintaining the same notation let $W_1, \dots, W_r$ be the irreducible components of the closed subset W of Z defined by the equation $y_d = 0$. If $y_1, \dots, y_{d-1}$ are algebraically dependent on each W_i, then they satisfy there an equation

$$p_i(y_1, \dots, y_{d-1}) = 0 \tag{37}$$

and the polynomial which is the product of the p_i vanishes identically on W. But W maps under $(y_1, \dots, y_{d-1})$ surjectively onto $\mathbb{A}^{d-1}$, so this is impossible. It follows that at least on one of the W_i, which we call Z_1, the functions $y_1, \dots, y_{d-1}$ are algebraically independent, so the transcendence degree of $k(Z_1)$ is at least $d - 1$. We may then continue by induction to construct a chain of length d of irreducible varieties strictly contained in each other, so, denoting the dimensions according to the various definitions by d_I, d_{II} and d_{III} we have the inequality $d_{II} \geq d_I$.

Suppose on the other hand that we start with a chain of length d as in the second definition. Let f_i be a function in $k[Z]$ which vanishes identically on Z_i but not on Z_{i-1}. We claim that $f_1, \dots, f_d$ are algebraically independent, hence $d_I \geq d_{II}$. By induction we may assume that $f_2, \dots, f_d$ are algebraically independent in $k[Z_1]$. Suppose the equation

$$\sum_{i=0}^{m} G_i(f_2, \dots, f_d) f_1^i = 0 \tag{38}$$

held in Z. Since Z is irreducible, and f_1 does not vanish on it identically, we may assume that G_0 is not the zero polynomial. But then $G_0(f_2, \dots, f_d) = 0$ holds identically on Z_1, contradicting the induction hypothesis. □

Exercise 2.14. *Prove that if $f : Z \to W$ is a finite morphism,* $\dim(Z) \leq \dim(W)$, *and that if $Z \to W$ is dominant* $\dim(Z) \geq \dim(W)$.

Exercise 2.15. *Prove that if $f \in k[X_1 \dots, X_n]$ is nonconstant, the hypersurface $Z(f)$ in $\mathbb{A}^n$ has dimension $n-1$.*

In fact, every irreducible component of $Z(f)$ has dimensiuon $n-1$, and the same is true for hypersurfaces in any irreducible variety.

Theorem 2.10. *Let W be an irreducible affine variety. Let $f \in k[W]$ be a nonzero element. Then if f is not a unit in $k[W]$, the dimension of any irreducible component Z of $Z_W(f)$ satisfies* $\dim(Z) = \dim(W) - 1$.

Proof. It is clear that $\dim(Z) \leq \dim(W) - 1$ because any chain of irreducible varieties in Z yields a chain one longer in W. It is not easy however to prove the inverse inequality, because it is not clear that we can force a chain of maximal length to "pass through Z". Translating into commutative algebra gives the equivalent statement that if P is a minimal prime containing f, then the Krull dimension of $k[W]/P$ is one smaller than the Krull dimension of $k[W]$. This translation does not circumvent the difficulty. In fact, the theorem is non-trivial, and is known as *Krull's Hauptidealsatz.*

Let $f \in k[W]$ and assume first that $Z = Z_W(f)$ is itself irreducible. Let $\varphi : W \to \mathbb{A}^d$ be a finite surjective map as in Noether's normalization theorem. The map $F = (\varphi, f) : W \to \mathbb{A}^{d+1}$ is also finite. Since finite morphisms are closed and since W was irreducible, $F(W)$ is an irreducible algebraic set. Since finite dominant maps preserve the dimension, $F(W)$ is of dimension d. Let h be an irreducible polynomial in $X_1, \dots, X_{d+1}$ vanishing on $F(W)$. The ring $k[X_1, \dots, X_{d+1}]$ is a unique factorization domain (Gauss' lemma) and h being irreducible implies that (h) is a prime ideal, hence $Z(h)$ is irreducible. Then $F(W) \subset Z(h)$ and both are irreducible of dimension d, so they must coincide. Write

$$h = \sum_{i=0}^{m} h_i(X_1, \dots, X_d) X_{d+1}^i. \tag{39}$$

The finite map $F : W \to Z(h)$ restricts to a finite map of Z to $Z(h, X_{d+1})$. But $Z_{\mathbb{A}^{d+1}}(h, X_{d+1}) = Z_{\mathbb{A}^d}(h_0)$ is $d-1$ dimensional because h_0 is neither zero (or h would be reducible) nor a unit (or f would be a unit). Since a finite dominant map preserves dimension, Z is $d-1$ dimensional as well.

In the general case, let $g \in k[W]$ vanish on all the irreducible components of $Z_W(f)$ other than Z. Let W' be the affine variety whose coordinate ring is $k[W'] = k[W][1/g]$. The function fields $k(W) = k(W')$ coincide, so $\dim(W') = \dim(W) = d$. The inclusion $k[W] \subset k[W']$ corresponds to an injective morphism $W' \to W$ whose image is $D_W(g)$. It follows that

$$Z_{W'}(f) = W' \cap Z_W(f) = W' \cap Z \tag{40}$$

is irreducible, so by what we have shown already $d-1 = \dim(W' \cap Z) \leq \dim(Z)$. $\square$

3. Sheaves

3.1. **Presheaves.** Sheaves are the main tool that is used in geometry to pass from local to global. They are also indispensible for the study of cohomology later on.

Let X be a topological space. A *presheaf* $\mathcal{A}$ of abelian groups on X is the assignment of an abelian group $\mathcal{A}(U)$ (called the group of *sections* of $\mathcal{A}$ over U) to every open set $U \subset X$, and homomorphisms of abelian groups $r_V^U : \mathcal{A}(U) \to \mathcal{A}(V)$ whenever $V \subset U$ (called the *restriction* homomorphisms) satisfying (i) $r_U^U = id_{\mathcal{A}(U)}$, and $r^V_W \circ r^U_V = r^U_W$ whenever $W \subset V \subset U$ are three open sets. We shall also use the notation $s|_V$ for $r^U_V(s)$.

A homomorphism of presheaves $f : \mathcal{A} \to \mathcal{B}$ is a collection of homomorphisms $f_U : \mathcal{A}(U) \to \mathcal{B}(U)$ for every open set U satisfying

$$f_V \circ r^U_V = r^U_V \circ f_U \tag{41}$$

whenever $V \subset U$ (where we used the same notation for the restriction maps in $\mathcal{A}$ and $\mathcal{B}$). A sub-presheaf $\mathcal{A} \subset \mathcal{B}$ is a presheaf in which every $\mathcal{A}(U)$ is a subgroup of $\mathcal{B}(U)$ and f_U, the inclusion, commutes with the restriction maps. The quotient presheaf $\mathcal{A}/\mathcal{B}$ is then defined as usual.

Exercise 3.1. *Let τ_X be the category whose objects are the open sets of X, where $Mor(V, U)$ is empty unless $V \subset U$, in which case it consists of a single element i^U_V (called the* inclusion *of V in U). Define the composition law in τ_X and show that a presheaf of abelian groups on X is the same as a contravariant* functor *from the category τ_X to the category* Ab *of abelian groups. Define a presheaf of sets, or of commutative rings, in the same way.*

Example 3.1. *Let A be an abelian group. The presheaf of A-valued functions on X associates to U the group A^U. The maps r^U_V are the ordinary restriction maps. If A is a topological group, the presheaf of continuous A-valued functions associated to U the continuous functions from U to A. If $A = \mathbb{R}$ this is a presheaf of rings. If A is equipped with the discrete topology, then this becomes the presheaf of locally constant A-valued functions. If $A = \mathbb{R}$ and X is a smooth differentiable manifold we can talk about the presheaf of smooth real-valued functions. We can also talk about the presheaf of differential p-forms for any $0 \le p \le \dim X$ (here we assume that the reader is familiar with the notion of a differential form on a smooth manifold).*

A presheaf $\mathcal{S}$ is a *sheaf* if it satisfies *the sheaf axiom:* $\mathcal{S}(\emptyset) = 0$, and for any open set U and any covering $\{U_\alpha\}$ of U by open sets, given a collection of elements $s_\alpha \in \mathcal{S}(U_\alpha)$ such that

$$r^{U_\alpha}_{U_\alpha \cap U_\beta} s_\alpha = r^{U_\beta}_{U_\alpha \cap U_\beta} s_\beta, \tag{42}$$

then there exists a unique $s \in \mathcal{S}(U)$ such that $r^U_{U_\alpha} s = s_\alpha$ for every α . This axiom guarantees the existence and uniqueness of gluing. The homomorphisms between two sheaves are the homomorphisms between them as presheaves. Thus the category of sheaves on X is a full subcategory of the category of presheaves.

Exercise 3.2. *Check that in all of the above examples, the presheaves are in fact sheaves. If $f : \mathcal{A} \to \mathcal{B}$ is a homomorphism between two sheaves, define the* kernel *presheaf $\mathcal{K}$ by $\mathcal{K}(U) = \ker f_U$, and prove that it is a sheaf too. However, the* cokernel *presheaf will not be a sheaf in general!*

3.2. **Stalks.** Let $\mathcal{A}$ be a presheaf of abelian groups on X. The *stalk* of $\mathcal{A}$ at $x \in X$ is the group

$$\mathcal{A}_x = \varinjlim \mathcal{A}(U) \tag{43}$$

where U runs over all open neighborhoods of x. For example, if X is the complex plane and $\mathcal{O}$ is the sheaf of holomorphic functions, $\mathcal{O}_x$ is the ring of all power series centered at x with positive radius of convergence. In this case the maps $\mathcal{O}(U) \to \mathcal{O}(V)$ for connected neighborhoods $x \in V \subset U$ are injective. In other examples, such as the sheaf of continuous functions, the maps are not injective, and an element of $\mathcal{A}_x$ may not be identified with a function in any neighborhood of x. It is rather a *germ* of a function, two functions (defined in some neighborhood) defining the same germ if there is an even smaller neighborhood where they coincide.

A morphism $f : \mathcal{A} \to \mathcal{B}$ of presheaves defines homomorphisms f_x between their stalks at every $x \in X$.

Exercise 3.3. *Let $\mathcal{A}$ be the presheaf of sets on the complex plane assigning to every open set U the set of holomorphic functions f on U satisfying there $zf'(z) = 1$. Show that this is a sheaf. Show that its stalk at every point other than 0 is non-canonically identified with $\mathbb{C}$, while the stalk at 0 is empty. What is $\mathcal{A}(U)$ if U is the punctured unit disk? If U is the union of 2 disjoint disks not containing* 0?

3.3. **Sheaves.** The *presheaf of discontinuous sections* $\mathcal{A}^\dagger$ attached to $\mathcal{A}$ is defined by

$$\mathcal{A}^\dagger(U) = \prod_{x \in U} \mathcal{A}_x \tag{44}$$

with the obvious restriction maps. A homomorphism $f : \mathcal{A} \to \mathcal{B}$ of presheaves defines $f^\dagger : \mathcal{A}^\dagger \to \mathcal{B}^\dagger$. Furthermore, there is a canonical map

$$\iota : \mathcal{A} \to \mathcal{A}^\dagger \tag{45}$$

assigning to $s \in \mathcal{A}(U)$ the section $s^\dagger \in \mathcal{A}^\dagger(U)$ where $s^\dagger(x) = s_x$. A section $\sigma \in \mathcal{A}^\dagger(U)$ is called *continuous* if there is an open covering $\{U_\alpha\}$ of U such that $\sigma|_{U_\alpha} = s_\alpha^\dagger$ for some $s_\alpha \in \mathcal{A}(U_\alpha)$. We let $\widetilde{\mathcal{A}}$ denote the sub-presheaf of $\mathcal{A}^\dagger$ consisting of continuous sections.

Exercise 3.4. *(i) Show that $\widetilde{\mathcal{A}}$ is a sheaf, and that $\mathcal{A}_x = \widetilde{\mathcal{A}}_x = \mathcal{A}^\dagger_x$.*

(ii) Show that the image of ι lies in $\widetilde{\mathcal{A}}$.

(iii) Show that $\mathcal{A}$ is a sheaf if and only if ι is an isomorphism of $\mathcal{A}$ onto $\widetilde{\mathcal{A}}$.

(iv) Show that $\widetilde{\mathcal{A}}$ has the following universal property: for every homomorphism $f : \mathcal{A} \to \mathcal{S}$ where $\mathcal{S}$ is a sheaf, there is a unique homomorphism $\tilde{f} : \widetilde{\mathcal{A}} \to \mathcal{S}$ such that $f = \tilde{f} \circ \iota$.

Exercise 3.5. *Let A be an abelian group and $\mathcal{A}$ the constant presheaf with group A. Show that $\widetilde{\mathcal{A}}$ can be identified with the locally constant functions from X to A.*

The sheaf $\widetilde{\mathcal{A}}$ is called the *sheafification* of the presheaf $\mathcal{A}$. Sometimes it is convenient to introduce a special terminology for ι being injective. Such presheaves are called decent. A presheaf is decent if and only if it is a subpresheaf of a sheaf.

3.4. **The category of sheaves; quotients.** Let X be a topological space. The category of abelian sheaves on X is *additive*: $Hom(\mathcal{A}, \mathcal{B})$ is an abelian group, composition of morphisms is bilinear, and finite direct sums of sheaves serve as categorical products and coproducts. We define the kernel of a morphism of sheaves as the kernel in the category of presheaves. However, to define the cokernel we take the presheaf cokernel and sheafify it. With these definitions it is not difficult to verify that the category of sheaves on X is an *abelian* category. A short exact sequence

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0 \tag{46}$$

is a sequence of sheaves such that $\mathcal{A}$ is a subsheaf of $\mathcal{B}$ and $\mathcal{C}$ is the sheafification of the presheaf $\mathcal{B}/\mathcal{A}$. For every open U,

$$0 \to \mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U) \tag{47}$$

is then exact. We say that the functor of global sections (on U) is left exact.

Exercise 3.6. *Show that a short sequence of sheaves is exact if and only if for every $x \in X$ the corresponding sequence of stalks is a short exact sequence of abelian groups.*

Exercise 3.7. *Let $X = \mathbb{C} \setminus \{0\}$ (with the usual topology). Consider the short exact sequence of sheaves*

$$0 \to \mathbb{C} \to \mathcal{O} \to \Omega^1 \to 0 \tag{48}$$

where $\mathbb{C}$ is the constant sheaf, $\mathcal{O}$ is the sheaf of holomorphic functions and Ω^1 the sheaf of holomorphic 1-forms. Check that the sequence is exact. Prove that the sequence of global sections

$$0 \to \mathbb{C} \to \mathcal{O}(X) \to \Omega^1(X) \tag{49}$$

is not right exact. Hint: dz/z has no primitive on X. We say that the functor of global sections is not right-exact. Conclude that the presheaf *$\mathcal{O}/\mathbb{C}$ is not a sheaf.*

As a matter of notation, from now on, if $\mathcal{A}$ is a subsheaf of a sheaf $\mathcal{B}$, we denote by $\mathcal{B}/\mathcal{A}$ the *sheaf* quotient, i.e. the sheafification of the quotient presheaf.

A sheaf is *flabby* if the restriction maps r^U_V are surjective. For example, the sheaf of discontinuous sections of a presheaf is always flabby.

Exercise 3.8. *Show that if*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0 \tag{50}$$

is an exact sequence of sheaves and $\mathcal{A}$ is flabby, then for every open set U, $\mathcal{B}(U) \to \mathcal{C}(U)$ is surjective.

Exercise 3.9. *Show that if $\mathcal{A}$ and $\mathcal{B}$ are flabby, so is $\mathcal{C}$.*

3.5. **Maps between spaces and operations on sheaves.** Let $f : X \to Y$ be a continuous map between topological spaces. We want to transfer sheaves from one space to the other. If $\mathcal{A}$ is a sheaf (of abelian groups) on X, its push-forward (or *direct image*) $f_*\mathcal{A}$ is defined by

$$f_*\mathcal{A}(V) = \mathcal{A}(f^{-1}V). \tag{51}$$

Exercise 3.10. *(i) Check that $f_*\mathcal{A}$ is a sheaf.*

(ii) If $\alpha : \mathcal{A} \to \mathcal{B}$ is a sheaf homomorphism on X, define in a functorial way $f_*\alpha : f_*\mathcal{A} \to f_*\mathcal{B}$.

(iii) Show that if $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is a short exact sequence of sheaves on X,

$$0 \to f_*\mathcal{A} \to f_*\mathcal{B} \to f_*\mathcal{C} \tag{52}$$

is exact.

(iv) If $Y = \{y\}$ is a one-point space, sheaves over it are just abelian groups, and if $f : X \to \{y\}$ is the constant map $f(x) = y$, then $f_*\mathcal{A} = \mathcal{A}(X)$. Deduce that f_* is not right exact in general.

In the opposite direction the definition is more involved, but the result has better properties. Let $\mathcal{A}$ be a sheaf on Y this time. Define

$$f^{-1}\mathcal{A}(U) = \lim_{\rightarrow} \mathcal{A}(V) \tag{53}$$

where the limit is over the directed set of all open sets V in Y containing $f(U)$. Note that f need not be open, but if it is, this is just $\mathcal{A}(f(U))$.

Exercise 3.11. *Show that $f^{-1}\mathcal{A}$ is a sheaf, and that if $\alpha : \mathcal{A} \to \mathcal{B}$ is a homomorphism, then $f^{-1}\alpha : f^{-1}\mathcal{A} \to f^{-1}\mathcal{B}$ is functorially defined. Show that the stalk of $f^{-1}\mathcal{A}$ at x is canonically identified with the stalk of $\mathcal{A}$ at $f(x)$. Using this show that f^{-1} is an exact functor form sheaves on Y to sheaves on X.*

Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{F}$ be an abelian sheaf on X and $\mathcal{G}$ an abelian sheaf on Y. An f-homomorphism ϕ from $\mathcal{G}$ to $\mathcal{F}$ is a collection of homomorphisms

$$\phi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V) \tag{54}$$

for every open $V \subset Y$ commuting with the restriction maps. By definition, this is the same as a sheaf homomorphism from $\mathcal{G}$ to $f_*\mathcal{F}$ (over Y). Now if U is open in X and $V \supset f(U)$, we may follow ϕ_V by the restriction from $f^{-1}V$ to U, to get a map

$$\phi_{V,U} : \mathcal{G}(V) \to \mathcal{F}(U). \tag{55}$$

If $V \supset V' \supset f(U)$, then $\phi_{V',U} \circ r_{V,V'} = \phi_{V,U}$. In this way we get from ϕ also a homomorphism from $f^{-1}\mathcal{G}$ to $\mathcal{F}$ (over X).

Exercise 3.12. *Prove that*

$$Hom_X(f^{-1}\mathcal{G}, \mathcal{F}) = Hom_Y(\mathcal{G}, f_*\mathcal{F}). \tag{56}$$

One says that f^{-1} is left adjoint to f_* and f_* is right adjoint to f^{-1}. Describe the canonical maps $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ and $\mathcal{G} \to f_*f^{-1}\mathcal{G}$.

Exercise 3.13. *Describe the sheaves $f^{-1}\mathcal{G}$ and $f_*\mathcal{F}$ when $f : X \to Y$ is the inclusion of a closed subset.*

4. General varieties

We shall use the language of sheaves to define the general notion of a variety. We start by localizing the notion of a regular function on an affine variety, and we shall then globalize by gluing to get the most general varieties.

4.1. **The structure sheaf $\mathcal{O}$ on an affine variety.** Let Z be an affine variety. We define a sheaf of rings $\mathcal{O}$ in the Zariski topology on Z (called the *sheaf of regular functions*, or the structure sheaf). Let U be Zariski open. We let $\mathcal{O}(U)$ be the ring of functions $f : U \to k$ for which, for every $x \in U$, there exist $g, h \in k[Z]$, $g(x) \neq 0$, such that $gf = h$ on U.

We remark that if there were g, h as above such that $gf = h$ held in a neighborhood $x \in U_x \subset U$ only, then choosing $u \in k[Z]$ vanishing on $U - U'$ but not at x, and replacing g and h by gu and hu, we could get the identity to hold in all of U. From this remark it follows at once that $\mathcal{O}$ satisfies the sheaf axiom: Suppose $\{U_\alpha\}$ is a covering of U and $f|_{U_\alpha}$ is of the prescribed form for each α. Given x, x belongs to some U_α, hence there exist g and h satisfying $gf = h$ in U_α, $g(x) \neq 0$.

But then f satisfies the same condition (with different g, h) in all of U. Since this holds for every x, $f \in \mathcal{O}(U)$.

Lemma 4.1. *We have $\mathcal{O}(Z) = k[Z]$.*

Proof. Let $f \in \mathcal{O}(Z)$ and let I be the collection of all the $g \in k[Z]$ such that $gf \in k[Z]$. Clearly I is an ideal in $k[Z]$, and by definition, for every $x \in Z$ there is a $g \in I$ such that $g(x) \neq 0$. By the Nullstellensatz, $I = k[Z]$, so $1 \in I$ and $f \in k[Z]$. $\square$

Exercise 4.1. *Suppose that Z is irreducible. Modify the proof to show that*

$$\mathcal{O}(D_Z(g)) = k[Z][1/g]. \tag{57}$$

Exercise 4.2. *Let $Z = \mathbb{A}^2$ and $U = \mathbb{A}^2 - \{(0,0)\}$. Show that $\mathcal{O}(U) = k[X_1, X_2]$.* Hint: $k[X_1, X_2]$ *is a unique factorization domain.*

The stalk of $\mathcal{O}$ at $x \in Z$ is called the *local ring at x*. Recall that a commutative ring R is called local if it has a unique maximal ideal M, i.e. $R \backslash M = R^\times$. To justify the terminology we show now that $\mathcal{O}_x$ is local. We can clearly associate to every $\phi \in \mathcal{O}_x$ its value $\phi(x)$ at x (why?), and we let $\mathfrak{m}_x \subset \mathcal{O}_x$ be the kernel of evaluation at x. If $\phi \notin \mathfrak{m}_x$, let $f \in \mathcal{O}(U)$ (for some $x \in U$) represent ϕ. Let $g, h \in k[Z]$ be such that $gf|_U = h|_U$, and $g(x) \neq 0$. Then $h(x) \neq 0$ as well. The open set $V = U \cap D_Z(hg)$ is a neighborhood of x and $1/f \in \mathcal{O}(V)$ because $h(1/f)|_V = g|_V$. It follows that $1/f \in \mathcal{O}(V)$ represents $1/\phi$ so ϕ is invertible, and $\mathfrak{m}_x$ is the unique maximal ideal.

Moreover, if $f : Z \to W$ is a morphism, there is a pull-back ring homomorphism for every open $U \subset W$

$$f_U^{\#} : \mathcal{O}_W(U) \to \mathcal{O}_Z(f^{-1}U). \tag{58}$$

This homomorphism induces homomorphisms $f_x^{\#} : \mathcal{O}_{W,f(x)} \to \mathcal{O}_{Z,x}$ for every $x \in Z$, which are *local*: they send $\mathfrak{m}_{W,f(x)}$ to $\mathfrak{m}_{Z,x}$. The collection $\left\{ f_U^{\#} \right\}$ constitute a f-homomorphism from $\mathcal{O}_W$ to $\mathcal{O}_Z$.

4.2. **General varieties.** We are now ready to define the abstract notion of a variety.

Definition 4.1. *A* variety over k *is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of functions $\mathcal{O}_X$ from X to k, for which there exist: a finite open cover*

$$X = \bigcup_{i=1}^{r} U_i, \tag{59}$$

affine varieties Z_i, and homeomorphisms

$$\phi_i : U_i \approx Z_i \tag{60}$$

(U_i with the induced topology, Z_i with the Zariski topology), so that the pull back of functions $\phi_i^{\#}$ identifies the sheaf $\mathcal{O}_{Z_i}$ with the sheaf $\mathcal{O}_X|_{U_i}$ (i.e. $\phi_i^{\#}$ is an isomorphism of $\mathcal{O}_{Z_i}(V)$ with $\mathcal{O}_X(\phi_i^{-1}V)$ for every open $V \subset Z_i$).

Here the restriction of a sheaf to an open subspace of X is defined in the obvious way. An open subset U of X for which such an isomorphism can be found is called an *affine open subset*.

Example 4.1. *Every affine variety Z is in an obvious way a variety, and every principal open set in it is affine. Indeed, if $g \in k[Z]$ consider*

$$Z_g = \left\{(x,t) \in Z \times \mathbb{A}^1;\ g(x)t = 1\right\}. \tag{61}$$

This is an affine variety and $\phi : D_Z(g) \approx Z_g$, $\phi(x) = (x, g(x)^{-1})$ is a homeomorphism carrying $\mathcal{O}_{Z_g}$ to $\mathcal{O}_Z|_{D_Z(g)}$. Every open subset of a variety is a variety, but an open subset of an affine variety need not be affine. Open subsets of affine varieties are called quasi-affine.

A *morphism* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ between two varieties is a continuous map $f : X \to Y$ such that for every V open in Y and every $s \in \mathcal{O}_Y(V)$, $f^{\#}(s) = s \circ f \in \mathcal{O}_X(f^{-1}V) = f_*\mathcal{O}_X(V)$. In other words, pull back by f,

$$f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X \tag{62}$$

is a sheaf homomorphism.

Let X be covered by open affines U_i, and Y by open affines V_j, and let ϕ_i and ψ_j be isomorphisms of U_i and V_j with affine varieties Z_i and W_j respectively, as in the definition of a variety above. Then in these charts $f : X \to Y$ is given by a collection of maps

$$f_{ij} : f^{-1}V_j \cap U_i \to V_j. \tag{63}$$

Since principal open subsets are affine and they form a basis for the topology, we may refine the affine cover of X so that for each i there is a j such that $U_i \subset f^{-1}V_j$. In such a case $f_{ij} : U_i \to V_j$, and f is a morphism if and only if $\psi_j \circ f_{ij} \circ \phi_i^{-1}$ is a morphism from Z_i to W_j for every pair of indices such that $U_i \subset f^{-1}V_j$.

The local rings $\mathcal{O}_{X,x}$ are called the local rings of X.

The notions of irreducibility and irreducible components generalize to arbitrary varieties without any difficulties. If X is an irreducible variety, every open subset in it is dense. If U and V are two affine open subsets of X, and W is an open affine contained in both, then the function fields $k(U)$, $k(V)$ and $k(W)$ all coincide. We call this the function field of X, and denote it by $k(X)$. We let $\dim X$ be the dimension of any open affine subset of it, which, as we have seen, is the transcendence degree of $k(X)$ over k. Recall that the Krull dimension of a commutative ring is the maximal length of a chain of prime ideals in the ring, a chain of $d+1$ ideals with d inclusions counting as having length d. It can be shown that for any $x \in X$,

$$\text{Krull-}\dim \mathcal{O}_{X,x} = \dim X. \tag{64}$$

Exercise 4.3. *If X is irreducible, show that $k(X) = \varinjlim \mathcal{O}(U)$ where the limit is over all the non-empty open subsets of X.*

4.3. **Products.** Products of affine varieties are again affine varieties in an obvious way (why?). Note however that the Zariski topology on the product is *not* the product topology (think of $\mathbb{A}^1 \times \mathbb{A}^1$).

Exercise 4.4. *Prove that if X and Y are affine,*

$$k[X \times Y] \simeq k[X] \otimes k[Y]. \tag{65}$$

Hint: If X and Y are affine spaces, this is easy. In general, let $I(X)$ and $I(Y)$ be their ideals. Define maps

$$k[\mathbb{A}^n]/I(X) \otimes k[\mathbb{A}^m]/I(Y) \to k[\mathbb{A}^{n+m}]/I(X \times Y) \tag{66}$$

and, in the converse direction, a map from $k[\mathbb{A}^{n+m}]$. To prove that the converse map factors through $I(X \times Y)$ you will have to show that

$$I(X \times Y) = I(X) \otimes k[\mathbb{A}^m] + k[\mathbb{A}^n] \otimes I(Y). \tag{67}$$

We now define the product of two abstract varieties. As a set $X \times Y$ is the cartesian product. As a basis of the topology we take U open in X, V open in Y, $f_i \in \mathcal{O}_X(U)$, $g_i \in \mathcal{O}_Y(V)$, and consider sets of the form

$$\left\{(x, y) \in U \times V; \sum f_i(x)g_i(y) \neq 0\right\}. \tag{68}$$

If X and Y are affine, we recover the Zariski topology on $X \times Y$. If W is open in $X \times Y$, a function $f \in \mathcal{O}_{X\times Y}(W)$ if W can be covered by open sets W_α of the above form, and for each α there are $u_i \in \mathcal{O}_X(U)$, $v_i \in \mathcal{O}_Y(V)$ such that

$$f|_{W_\alpha} = \frac{\sum u_i(x)v_i(y)}{\left(\sum f_i(x)g_i(y)\right)^m} \tag{69}$$

for some m.

With this definition of $\mathcal{O}_{X\times Y}$ it is clear that $X \times Y$ is a variety. The universal property of the product is the following.

Proposition 4.2. *Let Z be an arbitrary variety. Let π_X and π_Y denote the projections to X an Y. Then*

$$\phi \to (\pi_X\phi, \pi_Y\phi) \tag{70}$$

is a bijection (of sets) between $Mor(Z, X \times Y)$ and $Mor(Z, X) \times Mor(Z, Y)$.

Exercise 4.5. *(i) Prove that $X \times Y$ is irreducible if both X and Y are. (ii) Prove that* $\dim(X \times Y) = \dim X + \dim Y$.

4.4. **Projective varieties.** Besides affine varieties, a very important class is that of *projective varieties.* Recall that $\mathbb{P}^N$ is the collection of lines through the origin in $\mathbb{A}^{N+1}$. If $a = (a_0, \ldots, a_N)$ is not the origin, we denote by $[a] = (a_0 : \cdots : a_N)$ the unique line passing through a. Thus $[a] = [b]$ if and only if there exists a non-zero λ such that $b = \lambda a$.

An ideal $I \subset k[X_0, \ldots, X_N]$ is called *homogenous* if whenever $f = \sum f_i \in I$ and f_i is homogenous of degree i, then $f_i \in I$ for every i. Let I be a homogenous ideal. By Hilbert's basis theorem we may assume that I is generated by finitely many homogenous polynomials. The converse is also true (and easy): an ideal generated by homogenous polynomials, is homogenous. Now homogenous polynomials have the property that $f(\lambda a) = 0$ whenever $f(a) = 0$. Thus the zero set of I in $\mathbb{A}^{N+1}$ is a union of lines, and defines a subset

$$Z = Z_{\mathbb{P}^N}(I) \subset \mathbb{P}^N. \tag{71}$$

A projective algebraic set is such a Z. The projective algebraic sets are the closed sets for the Zariski topology on $\mathbb{P}^N$, and the Zariski topology on a given Z is the relative topology. We define a sheaf of rings $\mathcal{O}_Z$ as follows. If U is Zariski open, we let $\mathcal{O}_Z(U)$ consist of all the functions $f : Z \to k$ for which there exists an open cover $\{U_\alpha\}$ of U, and homogenous polynomials *of the same degree* h_α, g_α such that $g_\alpha(x) \neq 0$ for every $x \in U_\alpha$, and $f|_{U_\alpha} = h_\alpha/g_\alpha$. Note that h_α/g_α is a well-defined function on U_α.

We now show that $(Z, \mathcal{O}_Z)$ is an abstract algebraic variety. To that end we cover $\mathbb{P}^N$ by $N+1$ affine sets $\mathbb{A}^N_i$ $(0 \le i \le N)$, the set of points a where $a_i \neq 0$. We employ

$x_j = X_j/X_i$ ($j \neq i$) as affine coordinates on $\mathbb{A}_i^N$. If $f_1, \ldots, f_k$ are homogenous polynomials defining Z, the polynomials $f_l^{(i)}$ obtainded by substituting 1 for X_i and x_j for X_j if $j \neq i$, define $Z_i = Z \cap \mathbb{A}_i^N$. Thus Z is obtained by gluing the affine algebraic sets Z_i along their intersections. Moreover, $\mathcal{O}_Z|_{Z_i} = \mathcal{O}_{Z_i}$, so $(Z, \mathcal{O}_Z)$ is an abstract algebraic variety.

Exercise 4.6. *(Segre) Consider the map* $\iota : \mathbb{P}^N \times \mathbb{P}^M \to \mathbb{P}^{(N+1)(M+1)-1}$ *defined by*

$$\iota([a],[b]) = (a_0 b_0 : \cdots : a_N b_M). \tag{72}$$

Show that its image is the closed projective subset defined by the equations $Z_{ij}Z_{kl} = Z_{il}Z_{kj}$ *for all choices of indices. Here we have used the variables* Z_{ij} *(*$0 \leq i \leq N$, $0 \leq j \leq M$*) on* $\mathbb{P}^{(N+1)(M+1)-1}$*. Let* $X \subset \mathbb{P}^N$ *and* $Y \subset \mathbb{P}^M$ *be closed sets defined by the homogenous equations* $f(X_0, \ldots, X_N) = 0$ *and* $g(Y_0, \ldots, Y_M) = 0$ *(for simplicity of notation, assume these are hypersurfaces). Show that* $\iota(X \times Y)$ *is the closed set defined by* $Z_{ij}Z_{kl} = Z_{il}Z_{kj}$*, and by the equations* $f(Z_{0j}, \ldots, Z_{Nj}) = 0$ *(*$0 \leq j \leq M$*),* $g(Z_{i0}, \ldots, Z_{iM}) = 0$ *(*$0 \leq i \leq N$*).*

Exercise 4.7. *Let* $Z \subset \mathbb{A}^N$ *be a closed affine set and* $I = I(Z)$*. If* $f(X_1, \ldots, X_N) \in I$ *and* $d = \deg(f)$ *let*

$$f^*(X_0, \ldots, X_N) = X_0^d f(X_1/X_0, \ldots, X_N/X_0). \tag{73}$$

Let I^* *be the homogenous ideal in* $k[X_0, \ldots, X_N]$ *generated by* f^* *for* $f \in I$*. Let* $Z^* \subset \mathbb{P}^N$ *be the closed projective set defined by* I^**. Show that* $Z^* \cap \mathbb{A}_0^N = Z$*, and that* Z^* *is the closure of* Z *in* $\mathbb{P}^N$*. Study the case* $N = 2$*, where* Z *is defined by a single equation* $f = 0$*, and show that* $Z^* - Z$ *may be identified with the roots in* $\mathbb{P}^1$ *of the highest homogenous part of* f*.*

Open subsets of projective varieties are called quasi-projective. Affine and quasi-affine varieties are quasi-projective.

4.5. **Separatedness.** Products of affine varieties are affine. Products of projective varieties are projective (as the Segre embedding shows). However, products of general varieties may turn out to be pretty nasty. Consider the variety X obtained by gluing two copies of $\mathbb{A}^1$ along the complement of the origin, the so called "affine line with double origin". The two origins can not be separated by regular functions: every regular function on an open set U containing both origins that vanishes at one of them, vanishes at the other. The product $X \times X$ has two x-axis, two y-axis and four origins. Let Δ be the diagonal in $X \times X$. Then Δ is *not closed*: let o_1 and o_2 be the two origins of X. Suppose there were a neighborhood of $(o_1, o_2) \in X \times X$ disjoint from Δ. By the definition of the topology of the product, there would be open sets U and V containing o_1 and o_2 respectively, and functions f_i and g_i on them such that $\sum f_i(o_1)g_i(o_2) \neq 0$, but $\sum f_i(x)g_i(y) = 0$ on $\Delta \cap (U \times V)$. But this is impossible: since $g_i(o_1) = g_i(o_2)$ we would have $\sum f_i(o_1)g_i(o_1) \neq 0$, hence the rational function $\sum f_i(x)g_i(x)$ would vanish at only finitely many $x's$ and in particular will not vanish on $\Delta \cap (U \times V)$.

A variety X is called *separated* if the diagonal Δ is closed in $X \times X$. For example, if for every two distinct points o_1 and o_2 we can find an open U containing both and $f \in \mathcal{O}_X(U)$ separating the points, then Δ is closed. Indeed, the subset of $U \times U$ where $f(x) - f(y) \neq 0$ is a neighborhood of the point (o_1, o_2) not meeting Δ.

Proposition 4.3. *Let $f : X \to Y$ be a morphism, and assume that Y is separated. Then*
(i) the graph $\Gamma_f \subset X \times Y$ is closed
(ii) any closed subvariety of Y is separated
(iii) the product of two separated varieties is separated
(iv) affine and projective varieties are separated.

Proof. For (i), $\Gamma_f = (f \times 1)^{-1}(\Delta_Y)$ is closed if Δ_Y is. For (ii), if f is the inclusion of X in Y, then $\Delta_X = \Gamma_f$ is closed in $X \times Y$, hence clearly in $X \times X$. Part (iii) is left as an exercise, and part (iv) follows, for example, because in these cases points can be separated by functions as in the discussion above. $\square$

A nice feature of separated varieties is that if U and V are affine open subsets, $U \cap V$ is again open affine. Indeed,

$$U \cap V = \Delta \cap (U \times V). \tag{74}$$

Clearly, $U \times V$ is affine, and a closed subset of an affine variety is affine, so if Δ is closed, $U \cap V$ is affine.

4.6. **Complete varieties.** Projective varieties have remarkable properties that make them very different from their affine counterparts. For example, the only everywhere regular functions on an irreducible projective variety are the constants. This property (and others) are shared, more generally, by complete varieties.

Definition 4.2. *A variety Z is complete if it is separated, and if for every variety Y, the projection $Z \times Y \to Y$ is closed (maps closed sets to closed sets).*

The affine line is not complete: take $Y = \mathbb{A}^1$ also, and consider the hyperbola $xy = 1$, which is closed in $\mathbb{A}^1 \times \mathbb{A}^1$, but gets mapped to the non-closed $\mathbb{A}^1 - \{0\}$ under the projection to the second variable. If we were dealing with the classical topology, and Z were compact, then the projection $\pi : Z \times Y \to Y$ would be closed: let $W \subset Z \times Y$ be closed, and suppose $y_i \in \pi(W)$ converged to y. Let $(z_i, y_i) \in W$. Passing to a subsequence we could assume, from the compactness of Z, that z_i converge to z. But then $(z, y) \in W$ so $y \in \pi(W)$. In fact, it is easy to see that in the classical topology, our property characterizes compactness. Since in the Zariski topology every variety is quasi-compact, completeness is the right substitute for compactness. If $k = \mathbb{C}$, Z is complete if and only if $Z(\mathbb{C})$ is compact in the classical topology.

Proposition 4.4. *(i) Let Z be complete and irreducible. Then $\mathcal{O}_Z(Z) = k$. (ii) A closed subset of a complete variety is complete.*

Proof. (i) Let $0 \neq f \in \mathcal{O}_Z(Z)$ so that $D_Z(f)$ is open and dense. The set

$$\big\{(z, t) \in Z \times \mathbb{A}^1 ;\, tf(z) = 1\big\} \tag{75}$$

is closed and irreducible, so its projection to $\mathbb{A}^1$ must be the whole $\mathbb{A}^1$ or a point. Since the projection does not contain 0, it must be a single point. This means that f is constant. Part (ii) is clear because if $W \subset Z' \times Y$ is closed and Z' is closed in Z, then W is also closed in $Z \times Y$. $\square$

Theorem 4.5. *Let Z be a projective variety in $\mathbb{P}^N$. Then Z is complete.*

Proof. By (ii) of the proposition it is enough to prove that $\mathbb{P}^N$ itself is complete. It is enough to show that if Y is affine, the projection

$$\pi : \mathbb{P}^N \times Y \to Y \tag{76}$$

is closed, because every Y is a union of finitely many open affines Y_i, and if $W_i = W \cap (\mathbb{P}^N \times Y_i)$, clearly $\pi(W) \cap Y_i = \pi(W_i)$. If every $\pi(W_i)$ is closed in Y_i, $\pi(W)$ is closed. As a last reduction, we can now replace Y by $\mathbb{A}^M$, since if W is closed in $\mathbb{P}^N \times Y$ and Y is closed in $\mathbb{A}^M$, then W is closed in $\mathbb{P}^N \times \mathbb{A}^M$ and our goal is to prove that its projection to $\mathbb{A}^M$ is closed.

Let $I(W) \subset k[X_0, \dots, X_N, Y_1, \dots, Y_M]$ be the ideal of polynomials vanishing on W. The affine algebraic set $\tilde{W} \subset \mathbb{A}^{N+1} \times \mathbb{A}^M$ defined by $I(W)$ is the cone over W in the X-variables: if $(x, y) \in \tilde{W}$, then $(\lambda x, y) \in \tilde{W}$. This implies (as in the case where there are no Y's) that $I(W)$ is homogenous in the X-variables, and therefore is generated by finitely many polynomials f_l $(0 \le l \le r)$ which are homogenous, say of degree $d_l \ge 0$, in the X-variables. For $n \ge \max d_l$ consider the homomorphism

$$\beta_n : \sum_{l=1}^{r} P_{n-d_l} \otimes k[Y] \to P_n \otimes k[Y] \tag{77}$$

where P_n is the vector space of homogenous polynomials of degree n in the X-variables, obtained by multiplying the l-th summand by f_l. In terms of a basis of the P_n's (to fix ideas use the lexicographic ordering of monomials) β_n is given by a matrix B_n with coefficients in $k[Y]$. If $y \in \mathbb{A}^M$ we let $\beta_n(y)$ be the homomorphism obtained by specializing Y_j to y_j, and $B_n(y)$ the corresponding matrix.

Now $y \notin \pi(W)$ if and only if the ideal $(f_l(y)) \subset k[X]$ defines the empty set *or* the origin. In the first case it is the whole ring $k[X]$. In the second case, by the Nullstellensatz, it contains P_n for large enough n. Thus $y \notin \pi(W)$ if and only if $\beta_n(y)$ is surjective for some n. However, $\beta_n(y)$ is surjective if and only if $rank B_n(y)$ is maximal (and equal to $\dim P_n$), a condition which is tested by the non-vanishing of one of the maximal minors of B_n. Conversely, $y \in \pi(W)$ if for every n, all the maximal minors of B_n vanish. This is the intersection of closed conditions, hence $\pi(W)$ is closed. We remark that by Noetherianity, finitely many such conditions suffice to characterize $\pi(W)$, and this can be made effective: a bound on the n for which one has to test the minors of B_n can be computed. Kempf quotes $n = (\max d_l - 1)(N + 1) + 1$. □

As a corollary, if $f : X \to \mathbb{P}^N$ is a morphism, and X is complete, $f(X)$ must be closed (projective). There do not exist any non-constant morphisms from a complete variety to $\mathbb{A}^N$.

4.7. **Intersections in projective space.** Another useful property of projective varieties concerns intersections of subvarieties. In affine varieties, two rather large subvarieties may not intersect: for example, parallel hyperplanes in affine space. In projective varieties this can not happen, and moreover there is a lower bound on the dimension of the components of the intersection.

Theorem 4.6. *(i) Let X and Y be irreducible closed subvarieties of $\mathbb{A}^N$. Then every component of $X \cap Y$ has dimension which is at least* $\dim X + \dim Y - N$.

(ii) Let X and Y be irreducible closed subvarieties of $\mathbb{P}^N$. Then the same inequality holds, and if $\dim X + \dim Y \ge N$, *the intersection is non-empty.*

Proof. (i) If Δ is the diagonal in $\mathbb{A}^N \times \mathbb{A}^N$, then

$$X \cap Y = \Delta \cap (X \times Y). \tag{78}$$

Since $\dim(X \times Y) = \dim X + \dim Y$ and Δ is defined by the N equations $x_i = y_i$, a repeated application of the Hauptidealsatz proves (i). For (ii) consider the cones $\tilde{X}$ and $\tilde{Y}$ over X and Y as closed irreducible varieties of $\mathbb{A}^{N+1}$. Their dimensions are one bigger (prove it!) so under our assumption, (i) implies that every component of $\tilde{X} \cap \tilde{Y}$ has dimension at least $\dim X + \dim Y + 1 - N \geq 1$, form which we get the desired inequality for $X \cap Y$. But $\tilde{X} \cap \tilde{Y}$ contains the origin, hence contains at least one component of positive dimension, which implies that $X \cap Y$ is non-empty. □

4.8. **Hyperplane sections and a taste of intersection theory.** A particular case of the above is when X is a projective variety of dimension d embedded in $\mathbb{P}^N$, and H is a hyperplane, isomorphic to $\mathbb{P}^{N-1}$. If X is not contained in H (in which case we could have replaced N by $N-1$ from the beginning), then all the components of $X \cap H$ have dimension $d-1$, and are embedded in $\mathbb{P}^{N-1}$. We can now study X inductively by the method of hyperplane sections, inducting on the dimension of X or of the ambient projective space. This method is very useful for several reasons.

(i) The hyperplanes in $\mathbb{P}^N$ are labeled by the dual projective space $(\mathbb{P}^N)^*$, and in fact form an algebraic family $\mathbb{H} \subset \mathbb{P}^N \times (\mathbb{P}^N)^*$. The intersection of $\mathbb{H}$ with $X \times (\mathbb{P}^N)^*$ is an *algebraic family* of *hyperplane sections*. Thus hyperplane sections come naturally in families (although they are far from disjoint from each other, so this is not a fibration). This leads to the important notion of *linear systems*.

(ii) If X is irreducible and $d \geq 2$, the generic hyperplane section is again irreducible.

(iii) If X is smooth (a notion not yet defined) the generic hyperplane section of X is smooth again (Bertini's theorem).

Exercise 4.8. *Assume that X is an irreducible curve in $\mathbb{P}^2$ defined by a homogenous polynomial $f(X,Y,Z)$ of degree $d \geq 1$. Prove that for a dense open subset of lines $H_a \subset \mathbb{P}^2$ (the lines are labeled by $a \in \mathbb{P}^2$) the intersection $H_a \cap X$ consists of d distinct points.*

This exercise can be generalized as follows. Let $X \subset \mathbb{P}^N$ be irreducible projective of dimension n. Then there exists a number d such that for a dense open subset of n-tuples of hyperplanes

$$(H_1, \dots, H_n) \in \left\{(\mathbb{P}^N)^*\right\}^n, \tag{79}$$

$X \cap H_1 \cap \cdots \cap H_n$ consists of d distinct points. This d is called the *degree* of X.

Another direction in which the exercise can be generalizd is to consider the intersection of X with another curve $Y \subset \mathbb{P}^2$, of degree e. Under appropriate assumptions on the relative position of X and Y, the intersection will consist of de distinct points (Bezout's theorem).

Counting intersections becomes very complicated in higher dimensions, when the varieties are arbitrary. The right framework to deal with such questions is homological algebra, and we shall not say anything more in this course on the subject.

4.9. **Blow-ups.** A third very important way projective varieties show up is in the resolution of singularities. Roughly speaking, at a singular point $x \in X$ we would like to separate the directions along which we approach x. If $X \subset \mathbb{A}^N$ (which we can always assume by replacing X by some affine open enighborhood of x) and x is the origin, the totality of directions in $\mathbb{A}^N$ form a $\mathbb{P}^{N-1}$ and we need to select those directions which are tangent to X at x.

Formally, the blow-up of $\mathbb{A}^N$ at the origin is the closed subvariety

$$Bl_0\mathbb{A}^N \subset \mathbb{A}^N \times \mathbb{P}^{N-1} \tag{80}$$

defined as the zero-set of the polynomials $X_iY_j - X_jY_i$ $(1 \le i, j \le N)$. Here we use the X's as coordinates on $\mathbb{A}^N$ and the Y's on $\mathbb{P}^N$. The projection $\pi : Bl_0\mathbb{A}^N \to \mathbb{A}^N$ is one-to-one outside 0, but the fiber above 0 is the whole $\mathbb{P}^{N-1}$. You should think of $Bl_0\mathbb{A}^N$ as affine space from which the origin has been removed, and replaced by a miniature $\mathbb{P}^{N-1}$.

If, on the other hand, you look at the projection π' to $\mathbb{P}^{N-1}$, you see that $Bl_0\mathbb{A}^N$ forms a family of lines fibered over $\mathbb{P}^{N-1}$. In fact, it is just the *tautological family*, the fiber over $y \in \mathbb{P}^{N-1}$ being the line represented by that point.

Consider now a closed affine $X \subset \mathbb{A}^N$ passing through the origin. The Zariski closure of $\pi^{-1}(X - \{0\})$ in $Bl_0\mathbb{A}^N$ will be called the blow-up of X at 0, and denoted Bl_0X. Of course, in the way we have defined it, Bl_0X depends a priori on the embedding of X in $\mathbb{A}^N$, but in fact it can be given an intrinsic definition depending only on X. Clearly, Bl_0X maps to X under π. The part over $X - \{0\}$ maps isomorphically to $X - \{0\}$. The part over 0, contained in $\{0\} \times \mathbb{P}^{N-1}$, has dimension $\dim X - 1$ (this is non trivial), and is called the *exceptional divisor* E in Bl_0X.

Exercise 4.9. *Consider* $X = Z(X_1^2 - X_0^2(1 - X_0)) \subset \mathbb{A}^2$. *Describe its blow-up* Bl_0X, *and its exceptional divisor. Prove that the blow-up is isomorphic to* $\mathbb{A}^1$.

Exercise 4.10. *More generally, let* $f(X_0, X_1) = \sum_{i=2}^{d} f_i$ *where* f_i *is homogeneous of degree* i, *and assume that* f_2 *is the product of two distinct linear factors. Describe the blow-up of* $Z(f) \subset \mathbb{A}^2$ *at the origin.*

4.10. **Chow's lemma.** Affine varieties are the building blocks of general varieties. But projective varieties are special, and it is not clear how general they are. Indeed, there are complete varieties that are not projective, hence do not admit an embedding in projective space. However, every complete variety is birationally dominated by a projective variety. In spite of the usefulness of projective methods briefly indicated in the preceding sections, this is a very good fact to know.

Lemma 4.7. *(Chow's lemma) Let* Y *be a complete irreducible variety. Then there exists a projective variety* X *and a birational morphism* $f : X \to Y$.

Proof. Let $Y = \bigcup_{i=1}^r Y_i$ be an open covering of Y by affine subsets. Embed each Y_i in some $\mathbb{A}^{n_i}$ and let $\bar{Y}_i$ be its projective closure in $\mathbb{P}^{n_i}$. Let $O = \bigcap_{i=1}^r Y_i$ (dense open, in fact affine too) and let X be the Zariski closure of O (embedded diagonally) in $\bar{Y} = \prod_{i=1}^r \bar{Y}_i$. Then X is projective. Let $W \subset X \times Y$ be the closure of O (embedded diagonally). Since both X and Y are complete, $pr_X(W)$ and $pr_Y(W)$ are both closed. Since they contain a dense subset (namely, O) they coincide with X and Y respectively.

Let $W_i = W \cap (X \times Y_i)$ and

$$X_i = \left\{(y_j) \in X \subset \bar{Y} |\ y_i \in Y_i\right\}. \tag{81}$$

This is an open subset of X. If $(y_1, \ldots, y_i, \ldots, y_r, y) \in W_i$ then $y_i = y$. Indeed, this equation holds in $O \subset X \times Y_i$ and W_i is its closure in $X \times Y_i$, so the equation continues to hold there. It follows that $\psi_i : X_i \to W_i$, given by $\psi_i((y_j)) = ((y_j), y_i)$ is an isomorphism (its inverse being the projection to X_i). Since the W_i cover W (as the Y_i cover Y), the $X_i = pr_X(W_i)$ cover $X = pr_X(W)$ and the ψ_i glue to give an inverse to $pr_X : W \to X$. We now set $f = pr_Y \circ \psi$. This morphism is onto and on the open set O it's an isomorphism. Hence it is birational. □

5. Sheaves of modules

5.1. **Modules.** Let $(X, \mathcal{O}_X)$ be a variety over k. A sheaf of $\mathcal{O}_X$-modules on X is a sheaf $\mathcal{F}$ together with a structure of an $\mathcal{O}_X(U)$-module on $\mathcal{F}(U)$ for each open set U such that the restriction maps $r_{U,V}$ are compatible with the module structure. A homomorphism of sheaves of modules is defined in the obvious way, as a homomorphism of sheaves, respecting the module structure. If U is an open subset of X and $\mathcal{F}$ a sheaf of modules on X, then the restriction of $\mathcal{F}$ to (open subsets) of U, denoted $\mathcal{F}|_U$, is in an obvious way a sheaf of $\mathcal{O}_U = \mathcal{O}_X|_U$-modules.

Example 5.1. *(i) $\mathcal{O}_X^n$, the free module of rank n. More generally a* free *module is $\mathcal{O}_X^I$ where I is any set (direct* sum*). A sheaf of modules $\mathcal{F}$ is* locally free *if there is an open covering of X by sets U_i such that $\mathcal{F}|_{U_i}$ is free. If X is* connected *(in the Zariski topology), the* rank *of a locally free sheaf is a well-defined number. In other words, the sheaf can not be at the same time locally isomorphic to $\mathcal{O}^n$ on some open set and to $\mathcal{O}^m$ $(m \neq n)$ on another. This follows from the fact that the rank of a free module over a commutative ring is well-defined (if $R^n \simeq R^m$ as R-modules, choose any maximal ideal$\mathfrak{m}$ and reduce modulo $\mathfrak{m}$ to get $k^n \simeq k^m$ as k-vector spaces where $k = R/\mathfrak{m}$).*

A locally free sheaf of rank 1 is called an invertible sheaf*. Thus an invertible sheaf is locally isomorphic to $\mathcal{O}_X$.*

(ii) The sheaf $\mathcal{O}(m)$ on $\mathbb{P}^N$. If U is Zariski open, we let

$$\mathcal{O}(m)(U) = \left\{ \begin{array}{c} f/g \mid f, g \in k[X_0, \ldots, X_N] \text{ homogenous}, \\ \deg(f) = \deg(g) + m,\ g \neq 0 \text{ in } U \end{array} \right\}. \tag{82}$$

When $m = 0$ we recover the definition of $\mathcal{O}$. The $\mathcal{O}$-module structure is clear. This is an invertible sheaf, which is of primary importance in projective geometry. To see that it is invertible, we claim that

$$\mathcal{O}(m)|_{\mathbb{A}_0^N} \simeq \mathcal{O}|_{\mathbb{A}_0^N} \tag{83}$$

(and similarly when restricted to any of the open sets $\mathbb{A}_i^N$). To prove this last isomorphism we have to present, for all open $V \subset \mathbb{A}_0^N$, isomorphisms of $\mathcal{O}(V)$-modules

$$\alpha_V : \mathcal{O}(m)(V) \simeq \mathcal{O}(V) \tag{84}$$

which are compatible with restriction. Note that the isomorphisms will only be defined for $V \subset \mathbb{A}_0^N$ and will depend *on the choice of the open piece $\mathbb{A}_0^N$. Isomorphisms for different pieces will not be compatible over their intersection. To define α_V simply let*

$$\alpha_V(\frac{f}{g}) = \frac{f}{X_0^m g}. \tag{85}$$

(What is the inverse isomorphism? why are they well-defined?).

The sheaf $\mathcal{O}(m)$ for $m \neq 0$ is not *free. Were it free, then since its rank is 1, it would have been (globally) isomorphic to $\mathcal{O}$, and then for* every *open set we should have had isomorphisms $\mathcal{O}(m)(U) \simeq \mathcal{O}(U)$ as $\mathcal{O}(U)$-modules. In particular, for $U = \mathbb{P}^N$ we have seen that $\mathcal{O}(\mathbb{P}^N) = k$ (the only globally regular functions on a complete variety are the constants, and $\mathbb{P}^N$ is complete), while if $m > 0$, $\mathcal{O}(m)(\mathbb{P}^N)$ contains the linearly independent global sections represented by the various monomials of degree m, while if $m < 0$, the global sections vanish, so in any case, when $m \neq 0$, its dimension as a vector space over k is not* 1.

(iii) Let X be affine. If M is a $k[X]$-module, we define a sheaf of $\mathcal{O}_X$-modules $\tilde{M}$ by sheafifying the presheaf

$$U \mapsto M \otimes_{k[X]} \mathcal{O}_X(U). \tag{86}$$

We have the following properties:

$(M \oplus N)\tilde{} = \tilde{M} \oplus \tilde{N}$, $(M/N)\tilde{} = \tilde{M}/\tilde{N}$ etc. A homomorphism $M \to N$ induces in an obvious way a sheaf homomorphism $\tilde{M} \to \tilde{N}$.

(iv) Let $Z \subset X$ be a closed subset. If $X = \bigcup X_i$ is a (finite) union of affine open subsets, then each $Z_i = Z \cap X_i$ is closed in X_i, hence affine, and also open in Z. Thus $Z = \bigcup Z_i$ is an open affine cover of Z. For every open $U \subset X$ let

$$\mathcal{I}_Z(U) = \{f \in \mathcal{O}_X(U) |\ f|_{Z\cap U} = 0\}. \tag{87}$$

Then $\mathcal{I}_Z$ is a sheaf of ideals in $\mathcal{O}_X$. Restricted to X_i, this is of course the sheaf $\mathcal{I}_{Z_i}$. If X is affine, and $I_Z \subset k[X]$ is the ideal of regular functions vanishing on Z, then $\mathcal{I}_Z = \widetilde{I_Z}$. To prove it consider the tautological morphism of presheaves

$$I_Z \otimes_{k[X]} \mathcal{O}_X \to \mathcal{I}_Z. \tag{88}$$

Since $\mathcal{I}_Z$ is a sheaf, it factors through a morphism of sheaves $\widetilde{I_Z} \to \mathcal{I}_Z$. Let $f \in \mathcal{I}_Z(U)$ for U open. Cover U by principal open $D(g_i)$, $g_i \in k[X]$. For some n, fg_i^n extends to all of X, and then fg_i^{n+1} vanishes outside U. Replacing g_i by g_i^{n+1} we may assume that fg_i extends to a function h_i in $k[X]$ which vanishes outside U (the extension is unique only if X is irreducible, but this is not important for our purpose). This h_i vanishes on Z, so lies in I_Z. Thus $f|_{D(g_i)}$ is the image of $f_i = h_i \otimes (1/g_i)$. Next we examine the compatability of the f_i over the intersections $D(g_ig_j)$. Over such an intersection $h_ig_j = g_ih_j$, so

$$f_i|_{D(g_ig_j)} = h_ig_j \otimes (1/g_ig_j) = h_jg_i \otimes (1/g_ig_j) = f_j|_{D(g_ig_j)}. \tag{89}$$

This means that the f_i glue to a section of $\widetilde{I_Z}(U)$, mapping to f, so our map is surjective.

We prove the injectivity in the same way. Suppose $f_i \in I_Z$, $h_i \in \mathcal{O}_X(U)$ and $\sum f_ih_i = 0$ as an element of $\mathcal{I}_Z(U)$. Cover U by principal opens $D(g)$. On each such $D(g)$ we have that $g^{n_i}h_i \in k[X]$ for some large enough n_i. Replacing g by a suitable power we may assume that $gh_i \in k[X]$ for all i. But then

$$\sum f_i \otimes h_i|_{D(g)} = \sum f_i(h_ig) \otimes (1/g)|_{D(g)} = 0. \tag{90}$$

Thus the image of $\sum f_i \otimes h_i$ in $\widetilde{I_Z}$, the sheafification of $I_Z \otimes \mathcal{O}_X$, is 0.

(v) Continuing the previous example, let

$$\mathcal{O}'_Z = \mathcal{O}_X/\mathcal{I}_Z. \tag{91}$$

If X is affine, this is just $(k[X]/I_Z)\tilde{} = k[Z]\tilde{}$, so we have recovered the structure sheaf of Z, as a sheaf on X. More precisely, $\mathcal{O}'_Z(U) = \mathcal{O}_Z(U \cap Z)$ for every U

open in X. We say that the sheaf $\mathcal{O}'_Z$ is supported *on Z. From now on we shall not distinguish between $\mathcal{O}_Z$ and $\mathcal{O}'_Z$. In general, $(Z, \mathcal{O}_Z)$ is called a* closed subvariety *of X.*

(vi) If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of $\mathcal{O}_X$-modules, $\mathcal{F} \otimes \mathcal{G}$, $Hom(\mathcal{F}, \mathcal{G})$ or $Sym^n\mathcal{F}$ are defined as the sheaves associated to the corresponding presheaves. The same applies to any other linear algebra operation.

Exercise 5.1. *Prove that on $\mathbb{P}^N$, $\mathcal{O}(m) \otimes \mathcal{O}(n) \simeq \mathcal{O}(m+n)$, and $Hom(\mathcal{O}(m), \mathcal{O}) \simeq \mathcal{O}(-m)$. Deduce that the sheaves $\mathcal{O}(m)$ are pairwise non-isomorphic.*

Exercise 5.2. *Let $X \subset \mathbb{P}^N$ be a projective variety. Let $\mathcal{I}_X$ be its ideal sheaf as in (iv) above, and $\mathcal{O}_X(m) = \mathcal{O}_{\mathbb{P}^N}(m) \otimes_{\mathcal{O}_{\mathbb{P}^N}} \mathcal{O}_{\mathbb{P}^N}/\mathcal{I}_X$. Prove that this is an invertible sheaf on X. We shall later see that $\mathcal{O}_X(1)$ "remembers" the embedding of X in $\mathbb{P}^N$, i.e. given $(X, \mathcal{O}_X(1))$ we can reproduce the dimension N and the projective embedding in $\mathbb{P}^N$ up to a projective change of coordinates.*

The *stalk* of a sheaf of modules at a point $x \in X$ is a module over the local ring $\mathcal{O}_{X,x}$. The support of a sheaf of modules $\mathcal{F}$ is the set of $x \in X$ for which $\mathcal{F}_x \neq 0$. If $\mathcal{F}$ is locally free of rank n, then its stalks are isomorphic to $\mathcal{O}^n_{X,x}$. The usefulness of stalks comes when one discusses exact sequences, because exactness can be tested stalk-wise.

Exercise 5.3. *A* skyscraper *sheaf is a sheaf that is concentrated at a finite collection of points: all its stalks vanish except at those points. Let $x \in X$ and let M be a k-vector space. Define $\mathcal{F}(U) = 0$ if $x \notin U$ and $\mathcal{F}(U) = M$ if $x \in U$. Put on $\mathcal{F}$ a structure of a sheaf of modules. Show that any skyscraper sheaf is a direct sum of finitely many sheaves of this sort.*

Sheaves that are locally (on open affines) of the form $\tilde{M}$ as in (iii) above are nice, but not every sheaf of modules is of this nature. Consider for example the affine line $\mathbb{A}^1_{\mathbb{C}}$. The sheaf $\mathcal{O}^{an}$ of complex analytic functions (in the Zariski topology) is clearly a sheaf of $\mathcal{O}_X$-modules, which is not of the form $\tilde{M}$. Nevertheless, such sheaves are of little use in algebraic geometry.

5.2. **Quasi-coherent and coherent sheaves.** A sheaf of modules $\mathcal{M}$ is *quasi-coherent* if it locally fits into an exact sequence

$$\mathcal{O}_U^I \to \mathcal{O}_U^J \to \mathcal{M}|_U \to 0. \tag{92}$$

In other words, $\mathcal{M}$ is locally given by generators and relations. Let m_j ($j \in J$) be the images of the basis elements in $\mathcal{O}_U(U)^J$. These are elements of $\mathcal{M}(U)$, and $m_{j,x}$ (their images in $\mathcal{M}_x$) generate $\mathcal{M}_x$ over $\mathcal{O}_{X,x}$.

If U is affine and M is a $k[U]$ module, consider a presentation

$$k[U]^I \to k[U]^J \to M \to 0. \tag{93}$$

Since tensor product is right-exact, we get for every V open in U an exact sequence

$$\mathcal{O}_U(V)^I \to \mathcal{O}_U(V)^J \to M \otimes_{k[U]} \mathcal{O}_U(V) \to 0. \tag{94}$$

This implies that the *sheaf* $\tilde{M}$ is quasi-coherent, as the cokernel of a homomorphism between free sheaves. Conversely, if $\mathcal{M}$ is quasi-coherent and is given on an affine U as above, let $M = \mathcal{M}(U)$. The presheaf map

$$M \otimes_{k[U]} \mathcal{O}_U \to \mathcal{M} \tag{95}$$

defines a map $\tilde{M} \to \mathcal{M}$. We also get a short exact sequence

$$\mathcal{O}_U^I \to \mathcal{O}_U^J \to \tilde{M} \to 0 \tag{96}$$

which maps to the same sequence, ending with $\mathcal{M}$. This shows that $\tilde{M} \to \mathcal{M}$ is bijective.

Thus quasi-coherent sheaves are just the sheaves which are locally of the form $\tilde{M}$.

A quasi coherent sheaf is *coherent* if locally I and J can be chosen finite. Since $k[U]$ is Noetherian, and a submodule of a finitely generated module over a Noetherian ring is finitely generated, it is enough to require that J be finite. Alternatively, the module M should be finite over $k[U]$.

For example, the module $\mathcal{I}_Z$ is coherent.

5.3. **Quasi-coherent sheaves on affine varieties. Localizations**. Let R be a commutative ring with 1. Recall that the localization M_S of an R-module M in a multiplicative set $S \subset R$ is the set of equivalence classes of pairs (m, s) where $m \in M$ and $s \in S$, and $(m, s) \sim (m', s')$ if there exists $s'' \in S$ such that $s''(s'm - sm') = 0$. It has a natural module structure (think of $(m, s) = m/s$ and add like adding fractions). If $S = \{f^i\}$ this is the same as

$$\lim_{\to}(M, \times f). \tag{97}$$

(The limit is the set of equivalence classes of all pairs (m, i) where $m \in M$ and $i \in \mathbb{N}$ under $(m_1, i_1) \sim (m_2, i_2)$ if there exists a $j \geq \max(i_1, i_2)$ such that $f^{j-i_1}m_1 = f^{j-i_2}m_2$.) The localization in $\{f^i\}$ is denoted by M_f. The localization in the *complement* of a prime ideal P of R is denoted by M_P. If the multiplicative set S contains 0, then $M_S = 0$. Thus when we localize *the ring* R itself, to gurantee that in R_S $1 \neq 0$, we sometimes insist that $0 \notin S$. Finally, there is a natural identification $M \otimes R_S \simeq M_S$.

Proposition 5.1. *If X is affine, and M is a $k[X]$-module, then $\tilde{M}(X) = M$, and $\tilde{M}(D(f)) = M_f$. The stalk of $\tilde{M}$ at x is $M_{\mathfrak{m}_x}$ where $\mathfrak{m}_x \subset k[X]$ is the maximal ideal corresponding to $x \in X$.*

Proof. For the last assertion, $\tilde{M}_x = \lim_{\to} M \otimes_{k[X]} \mathcal{O}_X(U)$, the limit taken over the neighborhoods of x in X. We may confine ourselves to $U = D(g)$, $g(x) \neq 0$, the principal open sets being cofinal. But then $M \otimes_{k[X]} \mathcal{O}_X(D(g)) = M \otimes_{k[X]} k[X]_g = M_g$ and when we take the limit over $D(g)$ we get $M_{\mathfrak{m}_x}$.

The second claim follow from the first because

$$\tilde{M}|_{D(f)} = M \otimes_{k[X]} \mathcal{O}_X|_{D(f)} = M \otimes_{k[X]} k[X]_f \otimes_{k[X]_f} \mathcal{O}_{D(f)} = M_f \otimes_{k[X]_f} \mathcal{O}_{D(f)}. \tag{98}$$

To prove the first claim, map M to $\tilde{M}(X)$. If m becomes 0, for every $x \in X$ there is $g_x \in k[X]$, $g_x(x) \neq 0$, such that $g_x m = 0$. Since the g_x span an ideal that has no zeroes, they span (by the Nullstellensatz) the whole ring, so also $m = 1m = 0$. This shows the map from M to $\tilde{M}(X)$ is injective. Let $\alpha \in \tilde{M}(X)$ be gotten from pasting sections α_i from $M_{f_i} = M \otimes_{k[X]} \mathcal{O}(D(f_i))$, which agree on $D(f_i f_j)$. We assume that the $D(f_i)$ cover X. Replacing f_i by a power we may assume that $f_i \alpha_i = m_i \in M$. The patching condition means that

$$(f_i f_j)^n (f_i m_j - f_j m_i) = 0 \tag{99}$$

for some high enough n. Since the cover may be assumed to be finite, we may take one n that works for all, and replacing f_i by f_i^{n+1}, and m_i by $f_i^n m_i$, we may now assume that $f_i m_j = f_j m_i$ for every i and j. Let $a_j \in k[X]$ be such that $\sum a_j f_j = 1$. Let $m = \sum a_j m_j$. Then

$$f_i m = \sum_j a_j f_i m_j = \sum_j a_j f_j m_i = m_i. \tag{100}$$

This shows that m represents α_i on $D(f_i)$, hence $m = \alpha$. □

Let X be affine and $\mathcal{M}$ a quasi-coherent sheaf of modules on X.

Proposition 5.2. *Let $M = \mathcal{M}(X)$. Then $\mathcal{M} = \tilde{M}$.*

Proof. We know that such a result holds locally, on a fine enough (finite) covering of X by open affines. We must show that it holds on all of X. Note that the proposition has already been proved in one special case: for the sheaf $\mathcal{O}_X$.

(i) Claim: The (presheaf) direct limit of a directed system of sheaves on X is already a sheaf.

Let $\{\mathcal{F}_i\}$ be such a directed system, and

$$\mathcal{F}(V) = (\varinjlim \mathcal{F}_i)(V) = \varinjlim \mathcal{F}_i(V). \tag{101}$$

This presheaf satisfies the patching condition for the union of two sets: the exactness of

$$0 \to \mathcal{F}(U \cup V) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V). \tag{102}$$

By induction it satisfies the patching condition for the union of finitely many open sets. But since X is a noetherian space, every union of open sets is a finite union of a subcollection.

(ii) Let $f \in k[X]$. Define two sheaves $\mathcal{M}_f$ and $\mathcal{M}'_f$ on X as follows:

$$\mathcal{M}_f(U) = \varinjlim (\mathcal{M}(U), f\times) \tag{103}$$

$$\mathcal{M}'_f(U) = \mathcal{M}(D(f) \cap U). \tag{104}$$

There is a homomorphism $\mathcal{M}_f \to \mathcal{M}'_f$ sending $[(m,i)]$ to m/f^i. We claim that this is an isomorphism of sheaves. Since this can be checked *locally*, we may assume that $\mathcal{M} = \tilde{M}$ on X, and take $U = D(g)$. Then $\mathcal{M}(D(g)) = M_g = \varinjlim (M, g\times)$ so $\mathcal{M}_f(D(g)) = M_{fg}$. But also $\mathcal{M}'_f(D(g)) = \mathcal{M}(D(f) \cap D(g)) = \mathcal{M}(D(fg)) = M_{fg}$.

(iii) Let $M = \mathcal{M}(X)$ and consider the map $\psi : \tilde{M} \to \mathcal{M}$. We want to show that ψ is an isomorphism. On any $D(g)$, $\tilde{M}(D(g)) = M_g$. On the other hand $\mathcal{M}(D(g)) = \mathcal{M}'_g(X) = \mathcal{M}_g(X) = M_g$ too. Thus ψ is an isomorphism. □

Theorem 5.3. *Let X be affine. The maps $M \mapsto \tilde{M}$, $\mathcal{M} \to \mathcal{M}(X)$ are an equivalence of categories between the category of $k[X]$-modules and the category of quasi-coherent sheaves on X. They carry exact sequences to exact sequences.*

Proof. We have seen that they are inverse to each other. If

$$0 \to M_1 \to M_2 \to M_3 \to 0 \tag{105}$$

is an exact sequence of modules, then localizing at every $\mathfrak{m}_x$ and using $\tilde{M}_x = M_{\mathfrak{m}_x}$, we see that the sequence of stalks of the $\tilde{M}_i$ at every x is exact, hence the exactness of

$$0 \to \tilde{M}_1 \to \tilde{M}_2 \to \tilde{M}_3 \to 0. \tag{106}$$

Conversely, suppose

$$0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0 \tag{107}$$

is exact and let $M_i = \mathcal{M}_i(X)$. The only problem is to show that $M_2 \to M_3$ is surjective. If not, let

$$0 \to M_1 \to M_2 \to M_3 \to N \to 0 \tag{108}$$

be exact, with $N \neq 0$. Since we have already proved that $M \mapsto \tilde{M}$ is exact, and $\tilde{M}_i = \mathcal{M}_i$, we get $\tilde{N} = 0$. But then $N = 0$ too, contradiction. □

When we learn about cohomology, we shall see that the exactness of the functor of global sections, special to affine varieties, means that the higher cohomologies of quasi coherent sheaves on affine varieties, vanish.

5.4. **Coherent sheaves on affine varieties.** If X is affine, and M is a *finite* $k[X]$ modules, then $\tilde{M}$ is clearly coherent. The converse is alos true.

Proposition 5.4. *If $\mathcal{M}$ is coherent, $M = \mathcal{M}(X)$ is a finite $k[X]$ module.*

Proof. Cover X by finitely many principal open sets $D(g_i)$ such that on each $D(g_i)$ we have a presentation $\mathcal{M}|_{D(g_i)} = \tilde{M}_i$ with M_i a finite $k[X]_{g_i}$-module. Let m_{ij} be finitely many generators of M_i over $k[X]_{g_i}$. Multiplying them by appropriate powers of g_i we may assume that $m_{ij} \in M$. Let N be the finite $k[X]$-submodule of M generated by all the m_{ij}. Then $\tilde{N}$ is a subsheaf of $\tilde{M}$, which agrees with it on every $D(g_i)$. Hence $\tilde{N} = \tilde{M}$ and $N = M$, so M is a finite $k[X]$-module. □

Exercise 5.4. *A quasi-coherent subsheaf of a coherent sheaf is coherent.*

Recall that if $\iota : X \subset Y$ is a closed embedding, we have defined the ideal $\mathcal{I}_X$ as a subsheaf of $\mathcal{O}_Y$. Quasi-coherent sheaves $\mathcal{M}$ on X correspond bijectively to quasicoherent sheaves $\mathcal{M}'$ on Y which are annihilated by $\mathcal{I}_X$. Such sheaves are supported on X, but not every sheaf on Y which is supported on X is annihilated by $\mathcal{I}_X$, for example the sheaf $\mathcal{O}_Y/\mathcal{I}_X^2$. To make the correspondence, to $\mathcal{M}$ we associate $\mathcal{M}'$ defined by

$$\mathcal{M}'(U) = \mathcal{M}(X \cap U) \tag{109}$$

and to $\mathcal{M}'$ we attach $\mathcal{M} = \iota^{-1}\mathcal{M}'$, namely

$$\mathcal{M}(V) = \lim_{U} \mathcal{M}'(U), \tag{110}$$

the limit taken over open $U \subset Y$ such that $U \cap X = V$. Since $\mathcal{M}'$ was assumed to be supported on X, $\mathcal{M}'(U)$ is independent of U, and the limit is over a constant sequence of modules! Since $\mathcal{M}'$ is annihilated by $\mathcal{I}_X$, $\mathcal{M}$ is an $\mathcal{O}_X$-module, and it is easily checked that it is quasi-coherent. In what follows we shall omit the $'$ and simply write $\mathcal{M}$ for both sheaves, on X and on Y. Clearly $\mathcal{M}$ is coherent as an $\mathcal{O}_X$-module if and only if it is coherent as an $\mathcal{O}_Y$-module.

If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module which is not necessarily annihilated by $\mathcal{I}_X$ we can define

$$\mathcal{F}|_X = \mathcal{F}/\mathcal{I}_X\mathcal{F}, \tag{111}$$

a sheaf which is annihilated by $\mathcal{I}_X$. We denote it also by $\iota^*\mathcal{F}$.

Exercise 5.5. *Prove that*

$$\iota^*\mathcal{F} \simeq \iota^{-1}\mathcal{F} \otimes_{\iota^{-1}\mathcal{O}_Y} \mathcal{O}_X. \tag{112}$$

The right hand side defines the sheaf $\iota^*\mathcal{F}$ for any morphism $\iota : X \to Y$, not necessarily a closed embedding. For example, what is it if ι is an open *immersion? Let $f : \mathbb{A}^1 \to Z$ be the normalization of the node given in Section 2.6 (28). Study the difference between $f^{-1}\mathcal{O}_Z$ and $f^*\mathcal{O}_Z = \mathcal{O}_{\mathbb{A}^1}$. Do the same for the normalization of the cusp $y^2 = x^3$.*

5.5. **Differentials.** The sheaf of Kahler differentials is a very important sheaf on a variety X. Let $\Delta : X \to X \times X$ be the diagonal (closed embedding). Let $\mathcal{I}_X$ be the ideal of functions on $X \times X$ vanishing on the diagonal. If X is affine, $\mathcal{I}_X = \tilde{I}_X$ where $\tilde{I}_X \subset k[X] \otimes k[X]$ is the ideal generated by all the functions of the form $\delta(f) = f \otimes 1 - 1 \otimes f$ (prove it!). The sheaf of differentials is $\Omega^1_X = \mathcal{I}_X/\mathcal{I}^2_X$. If X is affine we denote the class of $\delta(f)$ modulo $\tilde{I}^2_X$ by df. You should think of df as "$f(x) - f(y) mod (x-y)^2$", or as "$f(x+h) - f(x) modo(h^2)$".

The sheaf Ω^1_X is clearly quasi-coherent, supported on X. We may therefore multiply (locally) differentials by functions. A section of Ω^1_X is locally expressible as $\sum g_i df_i$.

Exercise 5.6. *Show that*

$$d(fg) = fd(g) + gd(f). \tag{113}$$

5.6. **Nakayama's lemma.** If $\mathcal{M}$ is a quasi-coherent sheaf, we may study, in addition to the stalk $\mathcal{M}_x$, its *fiber* $\mathcal{M}(x)$, which is by definition the vector space

$$\mathcal{M}(x) = \mathcal{M}_x/\mathfrak{m}_{X,x}\mathcal{M}_x = \mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}. \tag{114}$$

If $\mathcal{M}$ is locally free of rank r, its fibers are r-dimensional vector spaces over k. If m is a section of $\mathcal{M}$ in a neighborhood of x we denote by $m(x)$ its value at x, which is simply its image in the fiber $\mathcal{M}(x)$.

Lemma 5.5. *Let $\mathcal{M}$ be a coherent sheaf. Then $\mathcal{M}(x) = 0$ if and only if $\mathcal{M} = 0$ in an open neighborhood of x.*

Proof. One direction is obvious. Assume that $\mathcal{M}(x) = 0$. We may assume that X is affine and $\mathcal{M} = \tilde{M}$ for a finite $k[X]$-module M. Let m_i be generators of M. Then in $M_{\mathfrak{m}_x}$ for every i

$$m_i = \sum \lambda_{ij} m_j \tag{115}$$

with $\lambda_{ij} \in \mathfrak{m}_x$. Since there are only finitely many λ_{ij} involved, this means that m_i are annihilated by

$$g = \det(1 - (\lambda_{ij})) \tag{116}$$

which is a function from $k[X]$ not vanishing at x. Thus $M_g = 0$ and $\mathcal{M}|_{D(g)} = 0$. $\square$

Corollary 5.6. *(i) Let $\mathcal{M}$ be a coherent sheaf and σ_i $(1 \leq i \leq n)$ sections over X. Then the homomorphism*

$$\mathcal{O}^n_X \to \mathcal{M} \tag{117}$$

sending the standard basis to σ_i is surjective in a neighborhood of $x \in X$ if and only if the $\sigma_i(x)$ span $\mathcal{M}(x)$.

(ii) The function $\dim \mathcal{M}(x)$ *is upper semi-continuous:* $\{x | \dim \mathcal{M}(x) \geq m\}$ *is closed.*

(iii) $\mathcal{M}$ *is locally free if and only if this function is continuous (locally constant).*

Proof. Let $\mathcal{N}$ be the subsheaf spanned by the σ_i. Then $\mathcal{M}/\mathcal{N}(x) = 0$, so by the previous lemma, $\mathcal{M} = \mathcal{N}$ in a neighborhood of x. This gives (i). For (ii) we show that $\{x | \dim \mathcal{M}(x) \leq n\}$ is open. Assume $\sigma_1(x), \ldots, \sigma_n(x)$ span $\mathcal{M}(x)$. By (i), the same sections span $\mathcal{M}$ in a neighborhood of x, so in that neighborhood $\dim \mathcal{M}(y) \leq n$ for all y. For (iii) suppose the dimension of the fiber is constant and equal to m. Let $x \in X$ and let σ_i be sections near x s.t. $\sigma_i(x)$ form a basis for $\mathcal{M}(x)$. Localizing, we may assume that we have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_X^n \xrightarrow{\psi} \mathcal{M} \to 0 \tag{118}$$

with $\sigma_i(x)$ being a basis for *every* $\mathcal{M}(x)$. This means that $\psi(x)$ is an isomorphism for all x. Let $(f_1, \ldots, f_m)$ be a section of $\mathcal{K} = \ker(\psi)$, so that $\sum f_i \sigma_i = 0$. Then $f_i(x) = 0$ for all x, so $f_i = 0$. This means that $\mathcal{K} = 0$. □

5.7. **Quasi-coherent sheaves on projective varieties.** Let $X \subset \mathbb{P}^N$ be projective, and let $CX \subset \mathbb{A}^{N+1}$ be the cone over X. Then $k[CX] = k[\mathbb{A}^{N+1}]/I(X)$ is a graded ring. Let M be a *graded module* over $k[CX]$:

$$M = \bigoplus_{n=0}^{\infty} M_n, \quad k[CX]_m M_n \subset M_{m+n}. \tag{119}$$

We associate to M a sheaf $\tilde{M}$ over X as follows. Consider the old $\tilde{M}$ over CX and restrict it to the open set $CX - \{0\}$:

$$M^{\#} = (\text{old } \tilde{M})|_{CX-\{0\}}. \tag{120}$$

This $M^{\#}$ is graded by $\mathbb{Z}$, and we let $\tilde{M} = (M^{\#})_{\deg 0}$. It is quasi-coherent, and coherent if M is finitely generated. We caution that we may have $M \subset L$ and not equal, yet $\tilde{M} = \tilde{L}$ (this happens if for all n large enough, $M_n = L_n$).

Theorem 5.7. *Every quasi-coherent sheaf on* X *is of the form* $\tilde{M}$, *and if it is coherent* M *is finitely generated.*

Proof. We use the notation (which should have been introduced earlier!)

$$\Gamma(U, \mathcal{F}) = \mathcal{F}(U) \tag{121}$$

for the sections of a sheaf $\mathcal{F}$ over U. Let $\mathcal{M}$ be quasi-coherent and

$$M = \bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{M}(n)) \tag{122}$$

where $\mathcal{M}(n) = \mathcal{M} \bigotimes \mathcal{O}_X(n)$. There is a graded ring homomorphism

$$k[CX] \to \bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{O}_X(n)), \tag{123}$$

and an action $\mathcal{O}_X(m) \times \mathcal{M}(n) \to \mathcal{M}(m+n)$, so this gives M a graded $k[CX]$ structure. We have a homomorphism of sheaves

$$\tilde{M} \to \mathcal{M}. \tag{124}$$

If f is a homogenous degree d regular function on CX, then $\tilde{M}|_{D(f)}$ is the sheaf associated on the open affine $D(f)$ to the module $M_{f,\deg 0}$. But

$$\begin{aligned} M_{f,\deg 0} &= \lim_{\to}\left(\Gamma(X,\mathcal{M}) \stackrel{\times f}{\to} \Gamma(X,\mathcal{M}(d)) \stackrel{\times f}{\to} \cdots\right) \\ &= \Gamma(X, \lim_{\to}(\mathcal{M} \stackrel{\times f}{\to} \mathcal{M}(d) \stackrel{\times f}{\to} \cdots)) \\ &= \Gamma(X, \mathcal{M}(D(f)\cap -)) = \Gamma(D(f),\mathcal{M}). \end{aligned} \tag{125}$$

Since $D(f)$ is affine, $\mathcal{M}|_{D(f)} = \tilde{M}_{|D(f)}$, their global sections being mapped isomorphically onto each other. Since this holds for every open $D(f)$ and they form a basis, $\mathcal{M} = \tilde{M}$.

Now if $\mathcal{M}$ is coherent, let M_i be an increasing sequence of finite submodules of M whose union is M. Since $\mathcal{M} = \tilde{M} = \bigcup \tilde{M}_i$, and it is coherent, it is equal to $\tilde{M}_i$ for some i. □

Recall that a sheaf of modules $\mathcal{F}$ on X is *generated by global sections* if there are global sections $\sigma_i \in \Gamma(X,\mathcal{F})$ such that for every open U, and every $s \in \Gamma(U,\mathcal{F})$ there are functions $f_i \in \mathcal{O}_X(U)$ such that $s = \sum f_i\sigma_i|_U$. Equivalently, the $\sigma_{i,x}$ span $\mathcal{F}_x$ over $\mathcal{O}_{X,x}$ for every x.

Corollary 5.8. *If $\mathcal{M}$ is coherent on a projective variety X, there is an m_0 such that $\mathcal{M}(m)$ is generated by global sections for all $m \geq m_0$.*

Proof. Take m_0 to be the maximal degree of generators of M. □

Exercise 5.7. *Prove the last corollary directly. You will need the fact that if X is a variety, $f \in \mathcal{O}_X(U)$, $\mathcal{M}$ a quasi-coherent sheaf, and $s \in \Gamma(D_U(f),\mathcal{M})$, then for some $n \geq 1$ there exists $s' \in \Gamma(U,\mathcal{M})$ with $s'|_{D_U(f)} = f^n s$. If $\mathcal{M}$ is coherent the same n can be chosen to work for all s. Where is this fact "hidden" in the proof of the theorem (point to the exact step where it is used)?*

5.8. **Invertible sheaves and divisors.** In this section we assume that X is an *irreducible* variety. Let $Pic(X)$ be the group of isomorphism classes of invertible sheaves on X with tensor product as the group operation (the tensor product of two invertible sheaves is again an invertible sheaf because

$$\mathcal{O}_X \otimes \mathcal{O}_X = \mathcal{O}_X \tag{126}$$

(the tensor is *over* $\mathcal{O}_X$!). The inverse of $\mathcal{L}$ is its dual $Hom(\mathcal{L},\mathcal{O}_X)$. Indeed, consider the morphism

$$\mathcal{L} \otimes Hom(\mathcal{L},\mathcal{O}_X) \to \mathcal{O}_X \tag{127}$$

sending the section $\sigma \bigotimes \lambda$ to $\lambda(\sigma)$ (the pairing being $\mathcal{O}_X$-linear). Then this is an isomorphism, because it is locally an iosmorphism.

Let $K = k(X)$ and let $\mathcal{K}$ be the constant sheaf $\mathcal{K}(U) = K$. Note that this is a quasi-coherent sheaf of $\mathcal{O}_X$-modules (in fact, algebras), containing the coherent sheaf $\mathcal{O}_X$ as a subsheaf.

We may also view $\mathcal{K}^\times$ as a sheaf of multiplicative groups. It contains the abelian sheaf $\mathcal{O}_X^\times$.

A *Cartier divisor on* X is a global section of the quotient sheaf $\mathcal{D} = \mathcal{K}^\times/\mathcal{O}_X^\times$. In down to earth terms, a Cartier divisor is given by a finite open covering $X = \bigcup U_i$, and on each U_i a rational function f_i such that $f_i f_j^{-1}$ is invertible on the intersection

$U_i \cap U_j$. Two such collections $\{U_i, f_i\}$ and $\{U'_i, f'_i\}$ represent the same Cartier divisor, if $f_i f'^{-1}_j$ is invertible on $U_i \cap U'_j$.

The collection of Cartier divisors is a group, Div_X.

A *principal* Cartier divisor is one given by

$$div(f) = \{X, f\}\,. \tag{128}$$

The group of principal Cartier divisors is isomorphic to $\mathcal{K}^\times/\mathcal{O}_X(X)^\times$, and is denoted P_X. The divisor class group of X is the group

$$Cl_X = Div_X/P_X. \tag{129}$$

Exercise 5.8. *Prove that $Cl_{\mathbb{P}^1} = \mathbb{Z}$ while $Cl_U = 0$ for any proper open $U \subset \mathbb{P}^1$.*

Exercise 5.9. *(This exercise requires acquaintance with the notion of a Dedekind domain). Suppose X is affine and $k[X]$ is a Dedekind domain (every nonzero prime ideal is maximal, and $k[X]$ is integrally closed, equivalently, all the localizations $k[X]_\mathfrak{p}$ at non-zero primes are DVR's). Then*

$$Cl_X = Cl_{k[X]}, \tag{130}$$

and in particular if $k[X]$ is not a PID Cl_X is non-trivial.

An *invertible (fractional) ideal* is an invertible subsheaf of $\mathcal{K}$: i.e. a submodule sheaf $\mathcal{I} \subset \mathcal{K}$ which is locally generated by one equation.

Proposition 5.9. *The following three objects are the same: (i) isomorphism classes of invertible sheaves (ii) isomorphism classes of invertible ideals (iii) classes of Cartier divisors modulo principal divisors.*

Proof. An invertible ideal is in particular an invertible sheaf. Conversely, let $\mathcal{I}$ be an invertible sheaf on X and σ a section of $\mathcal{I}$ over some open dense U. We attach to $\mathcal{I}$ the invertible ideal $\mathcal{J}$ defined by

$$\mathcal{J}(V) = \{f \in \mathcal{K} |\, f\sigma \,\text{extends to a section of}\, \mathcal{I} \,\text{over}\, V\}\,. \tag{131}$$

This is an invertible ideal (check!). Multiplication by σ (and unique extension!) provides the isomorphism with $\mathcal{I}$. This shows (i) and (ii) are the same. Given a Cartier divisor D defined by the data $\{U_i, f_i\}$ we define an invertible ideal $\mathcal{O}(D)$ as follows

$$\mathcal{O}(D)(V) = \{f \in \mathcal{K} |\, \forall i, f f_i \in \mathcal{O}_X(V \cap U_i)\}\,. \tag{132}$$

It is easy to check (a) that this is independent of the data defining D, (b) that every invertible ideal is of this form and (c) $\mathcal{O}(D) \simeq \mathcal{O}(D')$ if and only if $D' = D + div(g)$ for some g.

As a corollary the groups Cl_X and Pic_X are naturally isomorphic. $\square$

A *Weil divisor* on X is a formal linear combination of closed irreducible subvarieties of codimension 1. It can be shown that if all the local rings $\mathcal{O}_{X,x}$ are UFD's then the groups of Cartier divisors is isomorphic to the group of Weil divisors. The isomorphism attaches to an irreducible Weil divisor Z its local defining equations $\{U_i, f_i\}$ (which exist by our assumption). The most common example when all the local rings $\mathcal{O}_{X,x}$ are UFD's is when X is smooth (to be defined later), so for smooth X the two notions of divisors coincide.

5.9. **The Picard group of $\mathbb{P}^N$.** We shall show now that every invertible sheaf on $X = \mathbb{P}^N$ $(N > 0)$ is isomorphic to a unique $\mathcal{O}_X(m)$, hence $Pic_X = \mathbb{Z}$. For that pick an irreducible Weil divisor D on $\mathbb{P}^N$. The divisor D is defined (globally!) by some homogenous polynomial σ of degree m. We show $\mathcal{O}(D) \simeq \mathcal{O}(m)$. Let U_i be the standard ith affine piece denoted earlier $\mathbb{A}_i^N$. If $f \in \mathcal{O}(D)(U_i)$, then $f\sigma X_i^{-m}$ is a regular function over U_i (σX_i^{-m} is a defining equation for $D \cap U_i$), but then $f\sigma \in \mathcal{O}(m)(U_i)$, so multiplication by σ is the desired isomorphism.

5.10. **Morphisms to projective space.** If $f : X \to \mathbb{P}^N$ is a morphism, the global sections $\sigma_0, \dots, \sigma_N$ corresponding to the coordinates generate $\mathcal{O}_{\mathbb{P}^N}(1)$ everywhere, so $f^*\sigma_i$ generate $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^N}(1)$. Conversely, given an invertible sheaf $\mathcal{L}$ on X as $\sigma_i \in \Gamma(X, \mathcal{L})$ generating $\mathcal{L}$ everywhere, we can define a map

$$f : X \to \mathbb{P}^N \tag{133}$$

by

$$f(x) = [\sigma_0(x) : \dots : \sigma_N(x)] \tag{134}$$

(trivialize $\mathcal{L}$ at x by σ and replace every σ_i by $(\sigma_i/\sigma)(x)$, one of them at least being non-zero).

The invertible sheaf $\mathcal{L}$ is called *very ample* if the map f thus constructed is an isomorphism of X onto its image. It is called *ample* if some power of it is very ample. For example, $\mathcal{O}_X$ is very ample if X is affine!

Exercise 5.10. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are ample on X_i respectively, then $pr_1^*\mathcal{L}_1 \otimes pr_2^*\mathcal{L}_2$ is ample on $X_1 \times X_2$.*

Exercise 5.11. *Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on X. If $\mathcal{L}$ is generated by global sections and $\mathcal{M}$ is ample, $\mathcal{L} \otimes \mathcal{M}^n$ is ample for some $n \geq 0$.*

Exercise 5.12. *Let $\mathcal{L}$ be an invertible sheaf on a projective variety $X \subset \mathbb{P}^N$. Then for some m, $\mathcal{L}(m)$ is very ample.*

6. Smoothness

6.1. **The cotangent space.** One nice feature of algebraic geometry is that it allows one to do differential calculus in a purely formal manner, without introducing the concept of limit, ε or δ.

Let X be a variety and $x \in X$. The *cotangent space* at x is

$$Cot_x(X) = m_{X,x}/m_{X,x}^2, \tag{135}$$

a vector space over $k = \mathcal{O}_{X,x}/m_{X,x}$. It is finite dimensional because $\mathcal{O}_{X,x}$ is noetherian. If $f \in \mathcal{O}_{X,x}$ then $f - f(x) \in m_{X,x}$ and we denote its class in $Cot_x(X)$ by $df|_x$.

Lemma 6.1. *The following hold:*

(i) if c is a constant, $dc|_x = 0$

(ii) $d(f+g)|_x = df|_x + dg|_x$

(iii) $d(fg)|_x = f(x)dg|_x + g(x)df|_x$.

Furthermore, if $\delta : \mathcal{O}_{X,x} \to W$ is a map of $\mathcal{O}_{X,x}$ to a vector space over k satisfying (i)-(iii) (such a map is called a derivation centered at x*) then it is of the form $T \circ d|_x$ for a unique linear transformation T of $Cot_x(X)$ to W.*

By Nakayama's lemma $df_1|_x, \dots, df_n|_x$ span $Cot_x(X)$ over k if and only if $f_1, \dots, f_n$ span $m_{X,x}$ as an $\mathcal{O}_{X,x}$ module. (Apply Nakayama's lemma to $m_{X,x}/(f_1, \dots, f_n)$ to deduce that the latter is 0). The dimension of $Cot_x(X)$ is therefore the minimal number of generators of $m_{X,x}$ as an $\mathcal{O}_{X,x}$ module.

Proposition 6.2. *(i) Let* $\dim_x X$ *be the maximal dimension of an irreducible component of* X *passing through* x*. Then*

$$\dim Cot_x(X) \geq \dim_x X. \tag{136}$$

(ii) $Cot_{(x,y)}(X \times Y) = Cot_x(X) \oplus Cot_y(Y)$

(iii) If X *is affine and* I_x *is the ideal of functions in* $k[X]$ *vanishing at* x*, then* $Cot_x(X) = I_x/I_x^2$.

Exercise 6.1. *Let* $X \subset \mathbb{A}^2$ *be defined by a single equation* $f = 0$ *without constant term. Show that* $\dim Cot_0(X) = 1$ *if* f *has a non-zero linear term and is* 2 *otherwise.*

Definition 6.1. *The point* $x \in X$ *is called* smooth (non-singular) *if equality holds in (i).*

Proof. We may assume that X is affine and $\dim_x X = \dim X$ (otherwise remove the irreducible components not passing through x). Let I_x be the ideal of x, and consider $(f_1, \dots, f_n)$ where $df_i|_x$ is a basis for $Cot_x(X)$. Then by Nakayama's lemma, as we have observed, the two ideals give, when localized at x, the same ideal $m_{X,x}$. This means that they are equal when localized to a smaller affine open $x \in U \subset X$. Since $\{x\}$ is given in U by n equations, by the Hauptidealsatz

$$n \geq \dim X - \dim \{x\} = \dim X. \tag{137}$$

This gives (i). For (ii) note that if X and Y are affine

$$I_{(x,y)} = I_x \otimes k[Y] + k[X] \otimes I_y \tag{138}$$

and the same relation holds after we localize at (x, y), so

$$m_{(x,y)}/m_{(x,y)}^2 = m_x/m_x^2 \otimes k + k \otimes m_y/m_y^2 \tag{139}$$

but the latter sum is clearly a direct sum. For (iii) note that if $S = k[X] - I_x$,

$$m_x/m_x^2 = S^{-1}I_x/S^{-1}I_x^2 = S^{-1}(I_x/I_x^2) = I_x/I_x^2 \tag{140}$$

the last equality holding because everey $s \in S$ already acts invertibly on the module I_x/I_x^2. □

The *tangent space* T_xX is defined as the k-dual of $Cot_x(X)$. Once this definition is made, we may denote $Cot_x(X)$ by the more common T_x^*X. Recalling the universal property of $Cot_x(X)$ we see that for any k-vector space W, $T_xX \otimes W$ can be identified with the space of derivations on X, centered at x, with values in W. If $f \in \mathcal{O}_{X,x}$ and $v \in T_xX$ is a tangent vector at x, *the derivative of* f at x *in the direction* v is

$$v(f) = \langle v, df|_x \rangle . \tag{141}$$

It depends only on $f mod m_{X,x}^2$, as it should be!

If $\phi : X \to Y$ is a morphism and $y = \phi(y)$ then $\phi^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ induces $\phi^* : T_y^*Y \to T_x^*X$ and $\phi_* : T_xX \to T_yY$ (differentials, like functions, are pulled back, and tangent vectors are pushed forward) and these two maps are dual to each other.

Exercise 6.2. *Show that $T_x^*\mathbb{A}^N$ is N-dimensional, and $dX_i|_x$ (X_i are the coordinates) is a basis. The dual basis is denoted $\partial/\partial X_i|_x$. Note that dX_i has an intrinsic meaning, but $\partial/\partial X_i$ becomes meaningful only after we specify* all *the coordinates.*

6.2. **Simple finite maps.** We shall be sketchy, and send the reader to the literature for complete details.

Assume for simplicity that *char.k* $= 0$. Assume that $\phi : X \to Y$ is a finite dominant map between affine irreducible varieties and that furthermore

$$k[X] = k[Y][h] \tag{142}$$

where h satisfies the irreducible monic equation $F(h) = 0$, and

$$F(T) = T^d + a_1 T^{d-1} + \cdots + a_d \tag{143}$$

($a_i \in k[Y]$). The point $y \in Y$ is said to be *unramified* for ϕ (or ϕ unramified above y) if F has d *distinct roots* when we evaluate the a_i at y. Let $R(F, F')$ be the *resultant* of F and F' - this is a polynomial in the a_i, i.e. a regular function on Y, that vanishes exactly where F has multiple roots, i.e. common roots of F an F'. Since F is irreducible, the resultant is not identically zero (non-zero as an element of $k[Y]$ or $k(Y)$) hence outside its zero locus, in a dense open set, the map ϕ is unramified.

Let $x \in \phi^{-1}(y)$ where ϕ is unramified above y. We claim that $\phi^* : T_y^*Y \to T_x^*X$ is an isomorphism. In fact, more is true:

$$\phi^* : \widehat{\mathcal{O}}_{Y,y} \simeq \widehat{\mathcal{O}}_{X,x} \tag{144}$$

where $\widehat{\mathcal{O}}$ is the (separated) completion of $\mathcal{O}$ in the topology given by powers of the maximal ideal. This implies the assertion on cotangent spaces because

$$T_x^*X = m_x/m_x^2 = \hat{m}_x/\hat{m}_x^2. \tag{145}$$

Now the assertion on the completions of the local rings is a consequence of *Hensel's lemma*: If R is a complete local ring with residue field k and $F \in R[T]$ a monic polynomial of degree d whose reduction $\bar{F}$ has d distinct roots in k, then F is the product of d linear factors over R, and so every root of $\bar{F}$ can be lifted to a *unique* root in R. It follows from our assumptions and Hensel's lemma that $\widehat{\mathcal{O}}_{Y,y}[T]/(F(T))$ is isomorphic, as a ring, to the product of d copies of $\widehat{\mathcal{O}}_{Y,y}$, and that $h \in \widehat{\mathcal{O}}_{Y,y}$, although it is of degree d over $\mathcal{O}_{Y,y}$. Thus

$$\widehat{\mathcal{O}}_{X,x} = \widehat{\mathcal{O}}_{Y,y}[h] = \widehat{\mathcal{O}}_{Y,y}. \tag{146}$$

6.3. **Most points are non-singular.** Let us consider now a general finite dominant morphism $\phi : X \to Y$. Replacing Y by an open affine subset, we may assume that ϕ is obtained by a succession of finite, simple, everywhere unramified maps. These maps are isomorphisms on cotangent spaces, hence we have proved that for every $\phi : X \to Y$ finite and dominant, there is an open nonempty $U \subset Y$ over which ϕ is unramified, and for every $y \in U$ and $x \in \phi^{-1}(y)$, $\phi^* : T_y^*Y \simeq T_x^*X$.

Invoking Noether's normalization theorem, which says that any d-dimensional affine X is a finite dominant cover of $\mathbb{A}^d$, we see that X has an open dense subset U where $\dim T_x^*X = \dim X$. In other words, the set of non-singular points contains a dense open set, and therefore the singular locus in X is contained in a proper closed subvariety.

6.4. **Relation with the module of differentials.** We have defined a sheaf of differentials Ω_X. We have also defined the cotangent space at a point, T_x^*X. We now want to show how the two are related.

Lemma 6.3. *There is a canonical isomorphism*

$$\Omega_X(x) \simeq T_x^*X. \tag{147}$$

Proof. We may assume that X is affine. We let I_x be the ideal of x in $k[X]$ and I the ideal of the diagonal $\Delta \subset X \times X$, namely the kernel of

$$k[X] \otimes k[X] \to k[X]. \tag{148}$$

We have to find a natural isomorphism of modules between $\Omega_X(x) = I/I^2 \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/m_{X,x}$ and I_x/I_x^2. We use the universal property of the cotangent space and get a homomorphism

$$I_x/I_x^2 \to \Omega_X(x) \tag{149}$$

sending $df|_x$ to $df \otimes 1$. On the other hand, consider the projection $k[X] \otimes k[X] \to k[X]$ associated with

$$X \simeq X \times \{x\} \hookrightarrow X \times X. \tag{150}$$

The function $\delta f = f \otimes 1 - 1 \otimes f$ gets mapped to $f - f(x)$ and so I gets mapps to I_x and I/I^2 to I_x/I_x^2, mapping $df = [\delta f]$ to $df|_x$. This gives the map in the opposite direction, so the two vector spaces are isomorphic. □

Corollary 6.4. *The set of singular points of X is a proper* closed *set, and if it is empty, i.e. if X is non-singular, Ω_X is a locally free sheaf of rank $d = \dim X$.*

Proof. This is the set of points where $\dim \Omega_X(x) > d$, so the corollary follows from the semi-continuity theorem. □

6.5. **Bertini's theorem.** Let $X \subset \mathbb{P}^N$ be an irreducible smooth projective variety. The set of hyperplanes $H \subset \mathbb{P}^N$ is parametrized by the dual projective space, as we have seen. If $a \in (\mathbb{P}^N)^\vee$, we let H_a be the corresponding hyperplane.

Theorem 6.5. *For a dense open set of a's we have that $H \cap X$ is an* irreducible smooth *variety of dimension* $\dim X - 1$.