A note on the noise sensitivity of weighted majority

Noam Berger
University of California, Los Angeles
berger@math.ucla.edu

March 22, 2007

Abstract

We consider the weighted majority of n i.i.d. unbiased bits and show that the set of weights most sensitive under the flip of two randomly chosen bits is $\{0, 1, 2, \ldots, n-1\}$. This stands in contrast to the common belief that in most cases the most sensitive weights are uniform.

Let $\mathcal{X} = \{-1, 1\}^n$ and let $a_1, \ldots, a_n$ be nonnegative. The **weighted majority** with weights $a_1, \ldots, a_n$ is defined to be

$$f_{a_1,\ldots,a_n}(x_1, \ldots, x_n) = \text{sign} \sum_{i=1}^{n} a_i x_i.$$

We apply some arbitrary balanced monotone tie breaker in case the sum is zero.

We assume the uniform distribution on $\mathcal{X}$ and let φ be a random function from $\mathcal{X}$ to itself. We are interested in the probability that $f(\varphi(x)) \neq f(x)$. In particular, for a given φ we want to find the set of weights that maximizes

$$T(\varphi, a_1, a_2, \ldots, a_n) = \mathbf{P}(f(\varphi(x)) \neq f(x))$$

i.e. the most sensitive weights under the random function φ.

We restrict ourselves to functions φ of the following type: Let A be a randomly chosen subset of $\{1, \ldots, n\}$, independent of x, then φ flips all the bits within A, i.e.

$$(\varphi(x))_i = \begin{cases} x_i & i \notin A \\ -x_i & i \in A \end{cases}$$

We also assume that the distribution of A is exchangeable, i.e all sets of the same size have the same probability.

It is widely believed that when the size of A is small with respect to n, the most sensitive weights are (approximately) uniform. Indeed, it is well known that if A contains exactly one element (i.e. φ is the flip of one uniformly chosen bit) then the most sensitive set of weights is the uniform. See [2] for theorems and conjectures regarding the behavior when $|A|$ is a small fraction of n and [1] for a much wider background on the subject.

In this short note we consider the case $|A| = 2$ i.e. when φ is the flip of two bits chosen uniformly among all pairs. Our main result is the following:

Theorem 1. *Let $|A| = 2$. Then the set of weights most sensitive under φ is $a_i = i - 1$.*

Proof. Let $a_i = i-1,\ i = 1, \ldots, n$ and let $b_1, \ldots, b_n$ be some arbitrary set of weights. We want to show that

$$T(\varphi, \bar{a}) \geq T(\varphi, \bar{b}).$$

Assume w.l.o.g. that $b_1 < b_2 < \ldots < b_n$. (we may assume strict inequalities because the only effect of a small enough change of a weight is, possibly, changing the value of the sum from zero to non-zero, in which case we may adapt the tie-breaking mechanism accordingly without influencing the sensitivity)

We assume uniform distribution on $\mathcal{X} = \{-1, 1\}^n$ and define the random set B_b as follow:

$$B_b := \{\langle i, j\rangle : \text{sign}(b_i x_i + b_j x_j) = f_{\bar{b}}(\bar{x})\}.$$

B_a is defined accordingly.

For a pair $i < j$, let $\varphi_{i,j} : \mathcal{X} \to \mathcal{X}$ denote the flip of the bits x_i and x_j. We note that

$$\mathbf{P}\left[\langle i, j\rangle \in B_b\right] = 0.5 + \frac{1}{2}\mathbf{P}\left[f(x) \neq f(\varphi_{i,j}(x))\right].$$

Therefore,

$$\begin{aligned}
\mathbf{E}(|B_b|) &= \sum_{1 \leq i < j \leq n} \mathbf{P}\left[\langle i, j\rangle \in B_b\right] \\
&= \frac{1}{2}\binom{n}{2} + \frac{1}{2}\sum_{1 \leq i < j \leq n} \mathbf{P}\left[f(x) \neq f(\varphi_{i,j}(x))\right] \\
&= \frac{1}{2}\binom{n}{2} + \frac{1}{2}\binom{n}{2} T(\varphi, \bar{b}) \qquad (1)
\end{aligned}$$

In view of (1), all we need to show is that

$$\mathbf{E}(|B_a|) \geq \mathbf{E}(|B_b|).$$

To this end, we will show that for every choice of $x \in \mathcal{X}$,

$$|B_a(x)| \geq |B_b(x)|.$$

For every x and every $j > i$,

$$\text{sign}(b_i x_i + b_j x_j) = \text{sign}(a_i x_i + a_j x_j) = x_j.$$

Therefore, either $B_b(x) = B_a(x)$ or $B_b(x) = B_a(x)^c$. Therefore we only need to show that $|B_a(x)| \geq |B_a(x)^c|$. Let

$$B^+ = \{\langle i, j\rangle : \text{sign}(a_i x_i + a_j x_j) > 0\}$$

and let B^- be defined accordingly. We need to show that

$$|B^+| > |B^-| \implies B_a = B^+ \tag{2}$$

and

$$|B^-| > |B^+| \implies B_a = B^-. \tag{3}$$

For every pair $j > i$, we know that $\text{sign}(a_i x_i + a_j x_j) = x_j$. Therefore

$$|B^+| = \sum_{j=1}^{n} \mathbf{1}_{\{x_j=1\}} \#\{i : i < j\} = \sum_{j=1}^{n} (j-1)\mathbf{1}_{\{x_j=1\}}$$

and

$$|B^-| = \sum_{j=1}^{n} \mathbf{1}_{\{x_j=1\}} \#\{i : i < j\} = \sum_{j=1}^{n} (j-1)\mathbf{1}_{\{x_j=-1\}}.$$

Therefore,

$$|B^+| - |B^-| = \sum_{j=1}^{n} (j-1) \cdot x_j = \sum_{j=1}^{n} a_j x_j$$

so

$$f_{\bar{a}}(x) = \text{sign}\left(\sum_{j=1}^{n} a_j x_j\right) = \text{sign}(|B^+| - |B^-|)$$

yielding (2) and (3). □

We conclude with an open problem:

Problem 1. *What is the most sensitive set of weights under the flip of* 3 *bits?*

Acknowledgment. I thank Y. Peres for useful discussions

References

[1] I. Benjamini, G. Kalai O. Schramm (2001), Noise sensitivity of Boolean functions and applications to percolation. *Inst. Hautes Etudes Sci. Publ. Math.*, **90**, 5–43.

[2] Y. Peres (2004), Noise Stability of Weighted Majority. *preprint, available at* `http://front.math.ucdavis.edu/math.PR/0412377`.