AVRONFEST: CONGRATULATING YOSI
Invariants for J-unitaries on Real Krein spaces and classification of transfer operators

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Harper operator

On $\ell^2(\mathbb{Z}^2)$

$$H \;=\; U_1^* + U_1 + U_2 + U_2^*$$

where $U_1 = e^{i\varphi X_2} S_1$ and $U_2 = S_2$ with $\varphi \in \mathbb{R}$ and $S_{1,2}$ shifts

Jacobi operator with operator coefficients on $\mathcal{H} = \ell^2(\mathbb{Z})$

$$H \;=\; e^{-i\varphi X_2}\, S_1^* + (S_2 + S_2^*) + e^{i\varphi X_2}\, S_1$$

Transfer operators on $\mathcal{H} \oplus \mathcal{H}$ at energy $E \in \mathbb{R}$:

$$T^E \;=\; \begin{pmatrix} (E\,\mathbf{1} - S_2 - S_2^*)e^{i\varphi X_2} & -\, e^{i\varphi X_2} \\ e^{i\varphi X_2} & 0 \end{pmatrix}$$

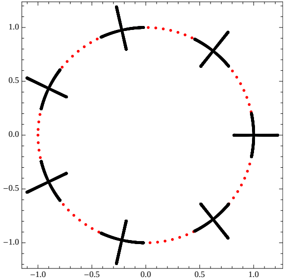
For $\psi = (\psi_n)_{n\in\mathbb{Z}}$ with $\psi_n \in \ell^2(\mathbb{Z})$ and $\Psi_n = \binom{\psi_{n+1}}{\psi_n}$

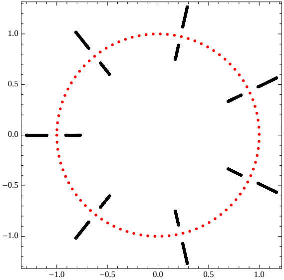
$$H\psi = E\psi \qquad \Longleftrightarrow \qquad T^E\Psi_n = \Psi_{n+1}$$

Spectra of transfer operators

Proposition: $E \notin \sigma(H) \iff \sigma(T^E) \cap \mathbb{S}^1 = \emptyset$

Example: With flux $\varphi = 2\pi\,\frac{3}{7}$ and $E = 2.2$ as well as $E = 1.9$





The T^E are J-unitary on $\mathcal{H} \oplus \mathcal{H}$:

$$(T^E)^* J T^E = J \qquad J = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

Half-space restrictions

$\widehat{H} = H$ with Dirichlet conditions on $\ell^2(\mathbb{Z} \times \mathbb{N})$

$\widehat{H} = e^{-i\varphi X_2}\,\widehat{S}_1^* + (\widehat{S}_2 + \widehat{S}_2^*) + e^{i\varphi X_2}\,\widehat{S}_1$

with partial isometry $\widehat{S}_2^*\widehat{S}_2 = \mathbf{1} - |0\rangle\langle 0|$

Discrete Fourier decomposition in 1-direction $\widehat{H} \cong \int_{-\pi}^{\pi} dk_1\ \widehat{H}(k_1)$

where $\widehat{H}(k_1) = \widehat{S}_2 + \widehat{S}_2^* + 2\cos(k_1 + \varphi X_2)$ half-sided Jacobi matrix

$\widehat{H}(k_1) \oplus \widehat{H}_l(k_1)$ compact perturbation of periodic $H(k_1)$ on $\ell^2(\mathbb{Z})$

Definition: edge spectrum of $\widehat{H} \;=\; \bigcup_{k_1 \in [-\pi,\pi]}\ \sigma_{\rm dis}(\widehat{H}(k_1))$

J-unitary transfer operators $\widehat{T}^E$ on $\widehat{\mathcal{H}} \oplus \widehat{\mathcal{H}}$ where $\widehat{\mathcal{H}} = \ell^2(\mathbb{N})$

$\widehat{T}^E \oplus \widehat{T}_l^E$ compact perturbation of T^E

Proposition: $\widehat{T}^E$ has unit eigenvalue $\Longleftrightarrow$ E in edge spect. of $\widehat{H}$

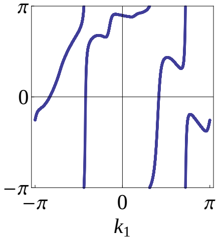
Edge state calculation

$T_2^E(k_1)$ transfer matrices of $H(k_1)$ in 2-direction, J-unitary

$$\theta(k_1) \;=\; \text{angle between the contracting direction of } T_2^E(k_1) \text{ and the Dirichlet boundary condition}$$

$$E \in \sigma_{\text{dis}}(\widehat{H}(k_1)) \Longleftrightarrow \theta(k_1) \;=\; 0$$

Example: Harper flux $\varphi = 2\pi\,\frac{3}{7}$ and $E = 1.9$



Resumé: J-unitary transfer operators $\widehat{T}^E$ with eigenvalues on $\mathbb{S}^1$ linked to edge states

Krein stability theory

Definition: Krein space $(\mathcal{K}, J)$ is a complex Hilbert space $\mathcal{K}$ with fundamental symmetry $J = \overline{J}$, $J^* = J^{-1}$, $J^2 = \eta\,\mathbf{1}$ with $\eta = \pm 1$

Normal form: $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}'$ and $J = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$ or $J = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$

Definition: $T \in \mathcal{B}(\mathcal{K})$ J-unitary $\iff T^* J T = J$

Example: $\mathcal{K} = \mathbb{C}^n \oplus \mathbb{C}^m$, $J = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \implies \{J\text{-unitaries}\} = \mathrm{U}(n, m)$

Proposition: Then $\sigma(T) = \left(\overline{\sigma(T)}\right)^{-1}$ reflection on $\mathbb{S}^1$

Proof: $J^*(T - \lambda\mathbf{1})J = (T^*)^{-1} - \lambda\,\mathbf{1}$ and spectral mapping

Krein stability analysis: Given a (continuous) path $t \mapsto T_t$ of J-unitaries, discrete eigenvalues can leave $\mathbb{S}^1$ only during collisions through eigenvalues with inertia of indefinite sign.

Krein inertia

For $\lambda \in \sigma_{\mathrm{dis}}(T)$, generalized eigenspace

$$\mathcal{E}_\lambda = \mathrm{span} \oint_{\partial B_\epsilon(\lambda)} \frac{dz}{2\pi i} (z\,\mathbf{1} - T)^{-1}$$

and Krein inertia for $\lambda \in \mathbb{S}^1$

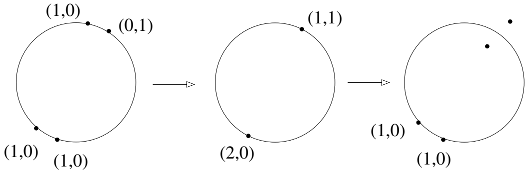
$$\nu(\lambda) = (\nu_+(\lambda), \nu_-(\lambda)) = \#\text{ pos./neg. eigenvalues of } \sqrt{\eta}\, J|_{\mathcal{E}_\lambda}$$

and signature $\mathrm{Sig}(\lambda) = \nu_+(\lambda) - \nu_-(\lambda)$

Definite sign $\Longleftrightarrow$ $\nu_+(\lambda) = 0$ or $\nu_-(\lambda) = 0$. Otherwise indefinite.

Facts: For $\lambda \notin \mathbb{S}^1$, inertia on $\mathcal{E}_\lambda \oplus \mathcal{E}_{(\overline{\lambda})^{-1}}$ is $(\dim(\mathcal{E}_\lambda), \dim(\mathcal{E}_\lambda))$

Sum of inertia is continuous at eigenvalue collisions



Global signature

Definition: Essentially $\mathbb{S}^1$-gapped J-unitaries

$$\mathbb{G}(\mathcal{K}) \;=\; \{T \;J\text{-unitary}\,|\,\sigma_{\mathrm{ess}}(T)\cap\mathbb{S}^1=\emptyset\}$$

with $\sigma_{\mathrm{ess}}(T)=\sigma(T)\setminus\sigma_{\mathrm{dis}}(T)$. Then

$$\mathrm{Sig}(T) \;=\; \sum_{\lambda\in\sigma(T)\cap\mathbb{S}^1}\mathrm{Sig}(\lambda)$$

Theorem: $\mathbb{G}(\mathcal{K})$ open and Sig homotopy invariant

Remarks: Similar to Fredholm index, each component non-trivial

Theorem: $T\in\mathbb{G}(\mathcal{K})$ has path in resolvent set $\rho(T)$ from ∞ to $\mathbb{S}^1$

$K=-J^*KJ$ compact $\Longrightarrow$ $Te^K\in\mathbb{G}(\mathcal{K})$

Proof: analytic Fredholm theory

$\mathbb{S}^1$-Fredholm operators

Example: $\mathcal{K} = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$, $J = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$. For $r < 1$ and shift S

$$T_t = \begin{pmatrix} r\,S & 0 \\ 0 & r^{-1}\,S \end{pmatrix} \exp t \begin{pmatrix} 0 & |0\rangle\langle 0| \\ -|0\rangle\langle 0| & 0 \end{pmatrix}$$

Then

$$\sigma(T_t) = \begin{cases} \text{filled ring}\,, & t = \frac{\pi}{2}, \frac{3\pi}{2}\,, \\ r\,\mathbb{S}^1 \cup r^{-1}\mathbb{S}^1\,, & \text{otherwise}\,. \end{cases}$$

Definition: With $\sigma'_{\mathrm{ess}}(T) = \{\lambda \in \mathbb{C} \,|\, T - \lambda\mathbf{1} \text{ not Fredholm}\}$

$$\mathbb{F}(\mathcal{K}) = \{T \ J\text{-unitary} \,|\, \sigma'_{\mathrm{ess}}(T) \cap \mathbb{S}^1 = \emptyset\}$$

Remarks: $\mathbb{F}(\mathcal{K})$ open and stable under compact perturbations,

$\mathbb{G}(\mathcal{K}) \subset \mathbb{F}(\mathcal{K})$ but not equal, $\mathrm{Ind}(T - \lambda\mathbf{1}) = 0$ for $\lambda \in \mathbb{S}^1$

Theorem: $\pi_1(\mathbb{F}(\mathcal{K})) \supset \mathbb{Z}$, given by Conley-Zehnder index

Spectral flow and calculation of signature

Theorem: $J = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$ and $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ J-unitary. Then

$$V(T) \;=\; \begin{pmatrix} (a^*)^{-1} & bd^{-1} \\ -d^{-1}c & d^{-1} \end{pmatrix}$$

is unitary on $\mathcal{K}$ and

(i) geom. mult. of 1 as EV of T $=$ mult. of 1 as EV of $V(T)$

(ii) $T \in \mathbb{F}(\mathcal{K}) \iff 1 \notin \sigma_{\mathrm{ess}}(V(T))$

Spectral flow of $t \mapsto V(T_t)$ by 1 $=$ Conley-Zehnder index

(iii) For $T \in \mathbb{G}(\mathcal{K})$,

$\mathrm{Sig}(T) =$ spectral flow of $t \in [0, 2\pi) \mapsto V(e^{-it}T)$ through 1

Real symmetries on Krein space

Fundamental symmetry $J_{\rm F}$ real unitary with $J_{\rm F}^2 = \eta_{\rm F}\,\mathbf{1}$

Real symmetry $J_{\rm R}$ real unitary with $J_{\rm R}^2 = \eta_{\rm R}\,\mathbf{1}$ and $J_{\rm F} J_{\rm R} = \eta_{\rm FR} J_{\rm R} J_{\rm F}$

kind $(\eta_{\rm F}, \eta_{\rm R}, \eta_{\rm FR}) \in \{-1, 1\}^3$

connection to Clifford groups

Fact: After real unitary basis change, normal forms (real Pauli)

Definition: $J_{\rm F}$-unitaries with Real symmetry $J_{\rm R}$

$$\mathbb{U}(\mathcal{K}, J_{\rm F}, J_{\rm R}) \;=\; \big\{\, T \ J_{\rm F}\text{-unitary} \;\big|\; J_{\rm R}^*\,\overline{T}\,J_{\rm R} = T \,\big\}$$

$$\mathbb{G}(\mathcal{K}, J_{\rm F}, J_{\rm R}) \;=\; \big\{\, T \in \mathbb{G}(\mathcal{K}, J_{\rm F}) \;\big|\; J_{\rm R}^*\,\overline{T}\,J_{\rm R} = T \,\big\}$$

Invariants with Real symmetries

η_{F}	η_{R}	η_{FR}	Class. Group	$\pi_0 \supset$	Invariant
1	1	1	$\mathrm{O}(N, M)$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathrm{Sig} \times \mathrm{Sec}$
-1	1	-1		$\mathbb{Z} \times \mathbb{Z}_2$	$\mathrm{Sig} \times \mathrm{Sec}$
-1	1	1	$\mathrm{SP}(2N, \mathbb{R})$	1	
1	1	-1		1	
-1	-1	1	$\mathrm{SO}^*(2N)$	$\mathbb{Z}_2$	Sig_2
1	-1	-1		$\mathbb{Z}_2$	Sig_2
1	-1	1	$\mathrm{SP}(2N, 2N)$	$\mathbb{Z}$	$\frac{1}{2}$-Sig
-1	-1	-1		$\mathbb{Z}$	$\frac{1}{2}$-Sig

Theorem: For $T \in \mathbb{U}(\mathcal{K}, J_{\mathrm{F}}, J_{\mathrm{R}})$.

(i) $\sigma(T) = \overline{\sigma(T)}$ spectral quadrouples

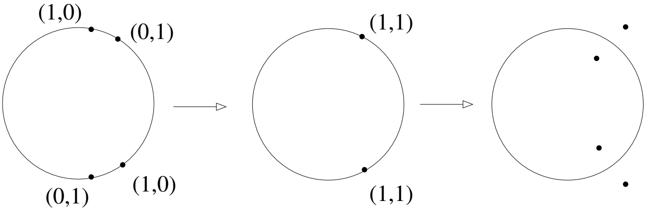
(ii) $\nu_\pm(\lambda) = \nu_{\pm\eta_{\mathrm{F}}\eta_{\mathrm{FR}}}(\overline{\lambda})$ and $\mathrm{Sig}(\lambda) = \eta_{\mathrm{F}}\eta_{\mathrm{FR}}\,\mathrm{Sig}(\overline{\lambda})$

(iii) $\eta_{\mathrm{R}} = -1 \implies$ Kramers degeneracy for real eigenvalues

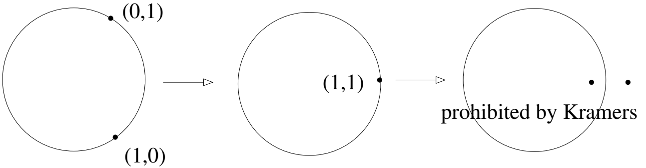
(iv) Invariants labelling $\pi_0 = \pi_0(\mathbb{G}(\mathcal{K}, J_{\mathrm{F}}, J_{\mathrm{R}}))$

Invariants for $\eta_{\mathrm{F}}\eta_{\mathrm{FR}} = -1$

Krein collisions



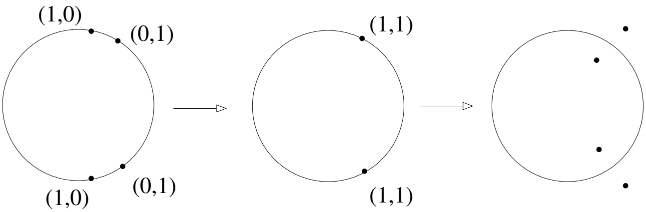
Tangent bifurcation prohibited for $\eta_{\mathrm{R}} = -1$



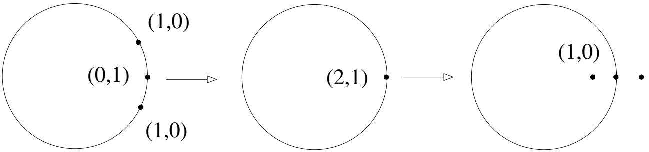
$$\mathrm{Sig}_2(T) \;=\; \sum_{\lambda\in\mathbb{S}^1} \nu_+(\lambda) \bmod 2 \;\in\; \mathbb{Z}_2\;.$$

Invariants for $\eta_{\rm F}\eta_{\rm FR} = 1$

Krein collisions



Mediated tangent bifurcation for kind $\eta_{\rm R} = 1$



$$\mathrm{Sec}(T) \;=\; \mathrm{Sig}(1) \bmod 2 \;\in\; \mathbb{Z}_2 \,.$$

Back to discrete Schrödinger operators

Next-nearest hopping and fiber $\mathbb{C}^L$ (spin, isospin, particle-hole)

$$H \;=\; \sum_{i=1}^{4} (W_i^* U_i + W_i U_i^*) + V \qquad \text{on } \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^L$$

with $U_3 = U_1^* U_2$ and $U_4 = U_1 U_2$, further W_i and $V = V^*$ matrices

Jacobi operator with operator coefficients A, B on $\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^L$

$$H \;=\; A S_1^* + B + A^* S_1$$

If A invertible, transfer operators on $\mathcal{H} \oplus \mathcal{H}$ at energy $E \in \mathbb{R}$:

$$T^E \;=\; \begin{pmatrix} (E\,\mathbf{1} - B)A^{-1} & -A^* \\ A^{-1} & 0 \end{pmatrix} \in \mathbb{G}(\mathcal{H} \oplus \mathcal{H}) \qquad \text{for } E \notin \sigma(H)$$

Half-space restrictions: $\widehat{H}$ and $\widehat{T}^E$

$\widehat{T}^E \notin \mathbb{G}(\widehat{\mathcal{H}} \oplus \widehat{\mathcal{H}}) \iff \mathbb{S}^1 \subset \sigma_p(\widehat{T}^E) \iff$ flat band of edge states

Calculation of unit eigenvalues of $\widehat{T}^E$

$\widehat{H} = \int_{-\pi}^{\pi} dk_1\, \widehat{H}(k_1)$ with matrix-valued Jacobi operators

$T_2^E(k_1)$ transfer matrices of $\widehat{H}(k_1)$ in 2-direction, J-unitary

$\Phi^E(k_1)$ contracting directions of $T_2^E(k_1)$, J-Lagrangian in $\mathbb{C}^{2L}$

$E \in \sigma_{\rm dis}(\widehat{H}(k_1)) \iff$ intersect. $\Phi^E(k_1) \cap$ bound. cond. non-trivial

Bott-Maslov intersection theory for Lagrangian planes in $\mathbb{C}^{2L}$:

$$U^E(k_1) \;=\; \begin{pmatrix} \mathbf{1} \\ -\imath\mathbf{1} \end{pmatrix}^* \Phi^E(k_1) \left(\begin{pmatrix} \mathbf{1} \\ \imath\mathbf{1} \end{pmatrix}^* \Phi^E(k_1) \right)^{-1} \qquad L \times L \;\text{ unitary}$$

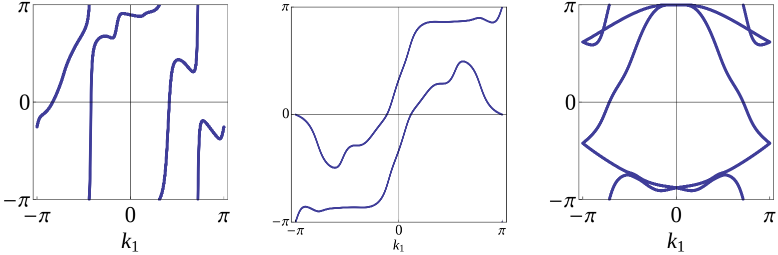
Proposition: e^{ik_1} eigenvalue of $\widehat{T}^E \iff 1$ eigenvalue of $U^E(k_1)$

Proposition: Krein inertia of $e^{ik_1} = \text{sign}(\partial_{k_1}\theta(k_1)|_0)$

New technique for calculating the Chern numbers

Theorem: $\widehat{T}^E$ essentially gapped $\Longrightarrow \mathrm{Sig}(\widehat{T}^E) = \mathrm{Ch}(P_E)$

Ex: Harper model, $p + ip$ wave supercond, Kane-Mele model



Implementing symmetries

Time reversal symmetry:

even: $\overline{H} = H \implies \overline{T^E} = T^E$

odd: $I_{\mathrm{s}}^* \overline{H} I_{\mathrm{s}} = H$ with $I_{\mathrm{s}} = e^{i\pi s^y} \implies (\mathbf{1} \otimes I_{\mathrm{s}})^* \, \overline{T^E} \, (\mathbf{1} \otimes I_{\mathrm{s}}) = T^E$

Example: Kane-Mele ($\mathbb{Z}_2$-topological insulator, quantum spin Hall)

Particle-hole symmetry ($K_{\mathrm{ph}}^2 = \pm\mathbf{1}$ even or odd):

$$K_{\mathrm{ph}}^* \, \overline{H} \, K_{\mathrm{ph}} \; = \; -H$$

$$\implies (J \otimes K_{\mathrm{ph}})^* \, \overline{T^E} \, (J \otimes K_{\mathrm{ph}}) = T^E \quad \text{with } J = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

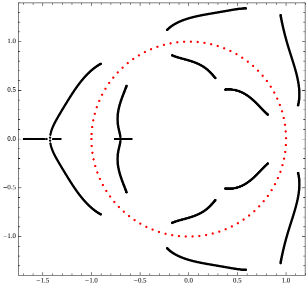
Fundamental symmetry: $(T^E)^*(I \otimes \mathbf{1}) T^E = (I \otimes \mathbf{1})$ with $I = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$

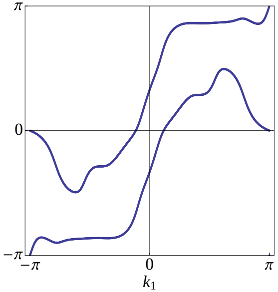
Majorana fermions at the edge

$(p+ip)$-wave superconductor in (Hartree-Fock) BdG-description:

$$H \;=\; \begin{pmatrix} U_1+U_1^*+U_2+U_2^*-\mu & \delta_p\,(S_1-S_1^*\pm\imath(S_2-S_2^*)) \\ \delta_p\,(S_1^*-S_1\pm\imath(S_2-S_2^*)) & -\overline{U}_1-\overline{U}_1^*-\overline{U}_2-\overline{U}_2^*+\mu \end{pmatrix}$$

Even particle-hole $K_{\rm ph}=\left(\begin{smallmatrix}0 & \mathbf{1}\\ \mathbf{1} & 0\end{smallmatrix}\right)$. For $\mu=\delta_p=0.2$ and $\varphi=2\pi\,\frac{1}{3}$





Eigenvector at ± 1 is real $\Longrightarrow$ self-adjoint creation operators

Resumé

1) J-unitaries on Krein spaces with Real symmetries
2) Krein signatures lead to new homotopy invariants
3) Applied to transfer operators allow to distinguish different phases of topological insulators

References

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