Unique continuation principle for spectral projections of Schrödinger operators and optimal Wegner estimates for non-ergodic random Schrödinger operators

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Schrödinger operators

We consider a Schrödinger operator

$$H = -\Delta + V \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d),$$

where Δ is the Laplacian operator and V is a bounded potential.

- We define balls and boxes:

$$B(x,\delta) := \left\{ y \in \mathbb{R}^d ; |y - x| < \delta \right\}, \quad \text{with} \quad |x| := |x|_2 = \left(\sum_{j=1}^{d} |x_j|^2 \right)^{\frac{1}{2}};$$

$$\Lambda_L(x) := \left\{ y \in \mathbb{R}^d ; |y - x|_\infty < \tfrac{L}{2} \right\}, \quad \text{with} \quad |x|_\infty := \max_{j=1,2,\ldots,d} |x_j|.$$

- H_Λ denotes the restriction of H to the the box $\Lambda \subset \mathbb{R}^d$:

$$H_\Lambda = -\Delta_\Lambda + V_\Lambda \quad \text{on} \quad \mathrm{L}^2(\Lambda).$$

  - Δ_Λ is the Laplacian on Λ with either Dirichlet or periodic boundary condition.
  - V_Λ is the restriction of V to Λ..

Unique continuation principle for spectral projections

A UCPSP on a box Λ is an estimate of the form

$$\chi_I(H_\Lambda) W_\Lambda \chi_I(H_\Lambda) \geq \kappa \chi_I(H_\Lambda) \quad \text{on} \quad \mathrm{L}^2(\Lambda),$$

where χ_I is the characteristic function of the interval $I \subset \mathbb{R}$, $W \geq 0$ is a potential, and $\kappa > 0$ is a constant.

- If $W \geq \kappa > 0$ (covering condition) the UCPSP is trivial.
- If V and W are bounded $\mathbb{Z}^d$-periodic potentials, $W \geq 0$ with $W > 0$ on an open set, Combes, Hislop and Klopp (2003) proved the UCPSP for H_Λ with periodic boundary condition, for boxes $\Lambda = \Lambda_L(x_0) \subset \mathbb{R}^d$ with $L \in \mathbb{N}$ and arbitrary bounded intervals I, with a constant $\kappa > 0$ depending on $\sup I$ (and d, V, W), but not on the box Λ. Their proof uses the unique continuation principle and Floquet theory.
- Germinet and Klein (2013) proved a modified version of the CHK UCPSP, using Bourgain and Kenig's quantitative unique continuation principle and (some) Floquet theory, obtaining control of the constant κ in terms of the relevant parameters.

Theorem (UCPSP)

There exists a constant $M_d > 0$, depending only on d, such that:

- Let $H = -\Delta + V$ be a Schrödinger operator on $\mathrm{L}^2(\mathbb{R}^d)$.
- Given an energy $E_0 > 0$ and $\delta \in]0, \frac{1}{2}]$, define $\gamma = \gamma(d, K, \delta) > 0$ by

$$\gamma^2 = \tfrac{1}{2}\delta^{M_d\left(1+K^{\frac{2}{3}}\right)}, \quad \text{where} \quad K = K(V, E_0) = 2\|V\|_\infty + E_0.$$

Then, given

- $\{y_k\}_{k\in\mathbb{Z}^d} \subset \mathbb{R}^d$ with $B(y_k, \delta) \subset \Lambda_1(k)$ for all $k \in \mathbb{Z}^d$,
- a closed interval $I \subset]-\infty, E_0]$ with $|I| \le \frac{2}{5}\gamma$,
- a box $\Lambda = \Lambda_L(x_0)$ with $x_0 \in \mathbb{R}^d$ and $L \ge 150\sqrt{d}$,

and setting

$$W^{(\Lambda)} = \sum_{k\in\mathbb{Z}^d, \Lambda_1(k)\subset\Lambda} \chi_{B(y_k,\delta)},$$

we have

$$\chi_I(H_\Lambda) W^{(\Lambda)} \chi_I(H_\Lambda) \ge \gamma^2 \chi_I(H_\Lambda) \quad \text{on} \quad \mathrm{L}^2(\Lambda).$$

Comments on the UCPSP

- Rojas-Molina and Veselić (2013) proved, under the hypotheses of the Theorem, that for boxes $\Lambda = \Lambda_L(x_0)$ with $x_0 \in \mathbb{Z}^d$ and $L \in \mathbb{N}_{\text{odd}}$, if ψ is an eigenfunction of H_Λ with eigenvalue $E \in]-\infty, E_0]$, then
$$\left\| W^{(\Lambda)} \psi \right\|_2^2 \geq \kappa_{E_0} \left\| \psi \right\|_2^2 \quad \text{with} \quad \kappa_{E_0} > 0.$$
This is just the UCPSP when $I = \{E\}$.Their proof uses the quantitative unique continuation principle (Bourgain and Kenig).
- Our Theorem is derived from the quantitative unique continuation principle as in Bourgain and Klein using the "dominant boxes" introduced by Rojas-Molina and Veselić.
- The UCPSP is a crucial ingredient for proving Wegner estimates for for Anderson Hamiltonians, random Schrödinger operators on $\mathrm{L}^2(\mathbb{R}^d)$ with $q\mathbb{Z}^d$-periodic background potential ($q \in \mathbb{N}$) and alloy-type random potentials located in the lattice $\mathbb{Z}^d$. The UCPSP replaces the covering condition.

Quantitative unique continuation principle (Bourgain-Klein)

Let $\Omega \subset \mathbb{R}^d$ open. Let $\psi \in \mathrm{H}^2(\Omega)$ and let $\zeta \in \mathrm{L}^2(\Omega)$ be defined by

$$-\Delta\psi + V\psi = \zeta \quad \text{a.e. on} \quad \Omega,$$

where V is a bounded real measurable function on Ω, $\|V\|_\infty \le K < \infty$. Let $\Theta \subset \Omega$ be a bounded measurable set where $\|\psi\chi_\Theta\|_2 > 0$.

$$\text{Set} \quad Q(x,\Theta) := \sup_{y\in\Theta} |y - x| \quad \text{for} \quad x \in \Omega.$$

$$\text{Let} \quad x_0 \in \Omega \setminus \overline{\Theta} \quad \text{satisfy} \quad Q = Q(x_0,\Theta) \ge 1 \quad \text{and} \quad B(x_0, 6Q+2) \subset \Omega.$$

Then, given

$$0 < \delta \le \min\left\{2\,\mathrm{dist}\,(x_0,\Theta), \tfrac{1}{300}\right\},$$

we have

$$\left(\frac{\delta}{Q}\right)^{m_d\left(1+K^{\frac{2}{3}}\right)\left(Q^{\frac{4}{3}}+\log\frac{\|\psi\chi_\Omega\|_2}{\|\psi\chi_\Theta\|_2}\right)} \|\psi\chi_\Theta\|_2^2 \le \left\|\psi\chi_{B(x_0,\delta)}\right\|_2^2 + \|\zeta\chi_\Omega\|_2^2,$$

where $m_d > 0$ is a constant depending only on d.

A corollary to the quantitative unique continuation principle

Corollary

There exists a constant $M_d > 0$*, depending only on* d*, such that:*

- *Let* $H = -\Delta + V$ *be a Schrödinger operator on* $\mathrm{L}^2(\mathbb{R}^d)$*, where* V *is a bounded potential with* $\|V\|_\infty \leq K$*.*
- *Fix* $\delta \in]0, \frac{1}{2}]$ *and sites* $\{y_k\}_{k\in\mathbb{Z}^d} \subset \mathbb{R}^d$ *with* $B(y_k, \delta) \subset \Lambda_1(k)$ *for all* $k \in \mathbb{Z}^d$*.*
- *Consider a box* $\Lambda = \Lambda_L(x_0)$*, where* $x_0 \in \mathbb{Z}^d$ *and* $L \in \mathbb{N}_{\mathrm{odd}}$*,* $L \geq 72\sqrt{d}$*.*

Then for all real-valued $\psi \in \mathscr{D}(\Delta_\Lambda) = \mathscr{D}(H_\Lambda)$ *we have (on* $\mathrm{L}^2(\Lambda)$*)*

$$\delta^{M_d\left(1+K^{\frac{2}{3}}\right)} \|\psi\|_2^2 \leq \sum_{k\in\Lambda\cap\mathbb{Z}^d} \left\|\psi\chi_{B(y_k,\delta)}\right\|_2^2 + \delta^2 \|H_\Lambda\psi\|_2^2$$

$$= \left\|W^{(\Lambda)}\psi\right\|_2^2 + \delta^2 \|H_\Lambda\psi\|_2^2.$$

Proof of the Corollary

Take $\Lambda = \Lambda_L(0)$ with $L \in \mathbb{N}_{\mathrm{odd}}$. We extend functions φ on Λ to functions $\widehat{V}$ and $\widetilde{\varphi}$ on $\mathbb{R}^d$ and V to a potential $\widehat{V}$ on $\mathbb{R}^d$ so

$$(\widetilde{-\Delta + V})\psi = (-\Delta + \widehat{V})\widetilde{\psi}.$$

Take $Y \in \mathbb{N}_{\mathrm{odd}}$, $9 \le Y < \frac{L}{2}$. Since L is odd, we have $\overline{\Lambda} = \bigcup_{k \in \Lambda \cap \mathbb{Z}^d} \overline{\Lambda_1(k)}$. It follows that for all $\varphi \in \mathrm{L}^2(\Lambda)$ we have

$$\sum_{k \in \Lambda \cap \mathbb{Z}^d} \left\| \widetilde{\varphi}_{\Lambda_Y(k)} \right\|_2^2 \le (2Y)^d \left\| \varphi_\Lambda \right\|_2^2 .$$

We now fix $\psi \in \mathscr{D}(\Delta_\Lambda)$. Following Rojas-Molina and Veselić, we call a site $k \in \widehat{\Lambda}$ *dominating* (for ψ) if

$$\left\| \psi_{\Lambda_1(k)} \right\|_2^2 \ge \frac{1}{2(2Y)^d} \left\| \widetilde{\psi}_{\Lambda_Y(k)} \right\|_2^2 ,$$

and note that, letting $\widehat{D} \subset \Lambda \cap \mathbb{Z}^d$ denote the collection of dominating sites,

$$\sum_{k \in \widehat{D}} \left\| \psi_{\Lambda_1(k)} \right\|_2^2 \ge \frac{1}{2} \left\| \psi_\Lambda \right\|_2^2 .$$

Proof of the Corollary-continued

If $k \in \widehat{D}$ we apply the QUCP with $\Omega = \Lambda_Y(k)$ and $\Theta = \Lambda_1(k)$, obtaining

$$\delta^{m'_d\left(1+K^{\frac{2}{3}}\right)} \left\|\psi_{\Lambda_1(k)}\right\|_2^2 \leq \left\|\psi_{B(y_{J(k)},\delta)}\right\|_2^2 + \delta^2 \left\|\widetilde{\zeta}_{\Lambda_Y(k)}\right\|_2^2,$$

where $\zeta = (-\Delta + V)\psi$, Y is appropriately chosen, $Y \leq 40\sqrt{d} < \frac{L}{2}$, and the map $J\colon \widehat{D} \to \Lambda \cap \mathbb{Z}^d$ is defined appropriately so

$J(k) \in \Lambda_Y(k)$ and $\# J^{-1}(\{j\}) \leq 2$ for all j.

Summing over $k \in \widehat{D}$ and using $\sum_{k\in\widehat{D}} \left\|\psi_{\Lambda_1(k)}\right\|_2^2 \geq \frac{1}{2}\left\|\psi_\Lambda\right\|_2^2$, we get

$$\begin{aligned}\tfrac{1}{2}\delta^{m'_d\left(1+K^{\frac{2}{3}}\right)} \left\|\psi_\Lambda\right\|_2^2 &\leq 2\sum_{k\in\Lambda\cap\mathbb{Z}^d} \left\|\psi_{B(y_k,\delta)}\right\|_2^2 + (2Y)^d\delta^2\left\|\zeta_\Lambda\right\|_2^2 \\ &\leq 2\sum_{k\in\Lambda\cap\mathbb{Z}^d} \left\|\psi_{B(y_k,\delta)}\right\|_2^2 + (80\sqrt{d})^d\delta^2\left\|\zeta_\Lambda\right\|_2^2,\end{aligned}$$

which implies (with a different constant $M_d > 0$)

$$\delta^{M_d\left(1+K^{\frac{2}{3}}\right)} \left\|\psi_\Lambda\right\|_2^2 \leq \sum_{k\in\Lambda\cap\mathbb{Z}^d} \left\|\psi\chi_{B(y_k,\delta)}\right\|_2^2 + \delta^2\left\|\zeta_\Lambda\right\|_2^2.$$

Proof of the UCSP Theorem

Let $E_0 > 0$ and $I \subset]-\infty, E_0]$ a closed interval; set $\beta = \frac{1}{2}|I|$. Since $H_\Lambda \geq -\|V\|_\infty$ for any box Λ, without loss of generality we assume $I = [E-\beta, E+\beta]$ with $E \in [-\|V\|_\infty, E_0]$, so

$$\|V - E\|_\infty \leq \|V\|_\infty + \max\{E_0, \|V\|_\infty\} \leq K = 2\|V\|_\infty + E_0.$$

Moreover, for any box Λ we have

$$\|(H_\Lambda - E)\psi\|_2 \leq \beta \|\psi\|_2 \quad \text{for} \quad \psi = \chi_I(H_\Lambda)\psi.$$

Let Λ be a box as in the Corollary and $\psi = \chi_I(H_\Lambda)\psi$ real-valued. It follows from the Corollary applied to $H - E$ that

$$\delta^{M_d\left(1+K^{\frac{2}{3}}\right)}\|\psi\|_2^2 \leq \left\|W^{(\Lambda)}\psi\right\|_2^2 + \delta^2\|(H_\Lambda - E)\psi\|_2^2 \leq \left\|W^{(\Lambda)}\psi\right\|_2^2 + \beta^2\|\psi\|_2^2.$$

If $\beta^2 \leq \gamma^2 := \frac{1}{2}\delta^{M_d\left(1+K^{\frac{2}{3}}\right)}$, i.e., $|I| \leq 2\gamma$, we get

$$\gamma^2\|\psi\|_2^2 \leq \left\|W^{(\Lambda)}\psi\right\|_2^2, \quad \text{i.e.,} \quad \gamma^2\chi_I(H_\Lambda) \leq \chi_I(H_\Lambda)W^{(\Lambda)}\chi_I(H_\Lambda).$$

Crooked Anderson Hamiltonians

A crooked Anderson Hamiltonian is the random Schrödinger operator

$$H_\omega := H_0 + V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d)$$

1. $H_0 = -\Delta + V^{(0)}$, with $V^{(0)}$ a bounded potential and $\inf \sigma(H_0) = 0$.
2. V_ω is a crooked alloy-type random potential:

$$V_\omega(x) := \sum_{j \in \mathbb{Z}^d} \omega_j u_j(x), \quad \text{with} \quad u_j(x) = v_j(x - y_j),$$

   where, for some $\delta_- \in]0, \frac{1}{2}]$ and $u_-, \delta_+, M \in]0, \infty[$:

   1. $\{y_j\}_{j \in \mathbb{Z}^d}$ are sites in $\mathbb{R}^d$ with $B(y_j, \delta_-) \subset \Lambda_1(j)$ for all $j \in \mathbb{Z}^d$;
   2. the single site potentials $\{v_j\}_{j \in \mathbb{Z}^d}$ are measurable functions on $\mathbb{R}^d$ with
   $$u_- \chi_{B(0,\delta_-)} \le v_j \le \chi_{\Lambda_{\delta_+}(0)} \quad \text{for all} \quad j \in \mathbb{Z}^d;$$
   3. $\omega = \{\omega_j\}_{j \in \mathbb{Z}^d}$ is a family of independent random variables whose probability distributions $\{\mu_j\}_{j \in \mathbb{Z}^d}$ are non-degenerate with
   $$\operatorname{supp} \mu_j \subset [0, M] \quad \text{for all} \quad j \in \mathbb{Z}^d.$$

Remark: If $V^{(0)}$ is $q\mathbb{Z}^d$-periodic with $q \in \mathbb{N}$, and $y_j = j$, $v_j = v_0$, $\mu_j = \mu_0$ for all $j \in \mathbb{Z}^d$, then H_ω is the ergodic (usual) Anderson Hamiltonian.

Finite volume crooked Anderson Hamiltonians

We define finite volume crooked Anderson Hamiltonians on a box $\Lambda = \Lambda_L(x_0)$, $x_0 \in \mathbb{R}^d$ and $L > 0$, with either Dirichlet or periodic boundary condition, by

$$H_{\omega,\Lambda} = H_{0,\Lambda} + V_\omega^{(\Lambda)} \quad \text{on} \quad \mathrm{L}^2(\Lambda),$$

where

- $H_{0,\Lambda} = (H_0)_\Lambda$ is the restriction of H_0 to Λ with the specified boundary condition,
- $$V_\omega^{(\Lambda)}(x) := \sum_{j \in \Lambda \cap \mathbb{Z}^d} \omega_j u_j(x) \quad \text{for} \quad x \in \Lambda.$$

We also set

$$U(x) := \textstyle\sum_{j \in \mathbb{Z}^d} u_j(x) \quad \text{and} \quad U^{(\Lambda)}(x) := \displaystyle\sum_{j \in \Lambda \cap \mathbb{Z}^d} u_j(x),$$

$$W(x) := \sum_{j \in \mathbb{Z}^d} \chi_{B(y_j,\delta_-)}(x) \quad \text{and} \quad W^{(\Lambda)}(x) := \sum_{j \in \Lambda \cap \mathbb{Z}^d} \chi_{B(y_j,\delta_-)}(x).$$

Remark and notation

Note that

$$0 \leq W_\Lambda \leq \tfrac{1}{u_-} U_\Lambda.$$

We will use the following notation:

- $P_{\omega,\Lambda}(B) := \chi_B(H_{\omega,\Lambda})$ for a Borel set $B \subset \mathbb{R}^d$.
- The concentration function of the probability measure μ is defined by

$$S_\mu(t) := \sup_{a\in\mathbb{R}} \mu([a, a+t]) \quad \text{for} \quad t \geq 0.$$

-

$$S_\Lambda(t) := \max_{j\in\Lambda\cap\mathbb{Z}^d} S_{\mu_j}(t).$$

Optimal Wegner estimates

An optimal Wegner estimate for Anderson Hamiltonians is an estimate of the form

$$\mathbb{E}\left\{\operatorname{tr} P_{\omega,\Lambda}(I)\right\} \leq C\, S_\Lambda(|I|)\, |\Lambda|\,.$$

- Combes, Hislop (1994) proved optimal Wegner estimates for ergodic Anderson Hamiltonians with a covering condition.
- Combes, Hislop, Klopp (2007) proved optimal Wegner estimates for ergodic Anderson Hamiltonians with periodic boundary condition and boxes $\Lambda = \Lambda_L(x_0)$ with L a multiple of the period.
  Their proof uses the UCSP for the (nonrandom) periodic operator H_0.
- Rojas-Molina and Veselić (2013) proved Wegner estimates for Delone-Anderson models, optimal up to an additional factor:
  $$\mathbb{E}\left\{\operatorname{tr} P_{\omega,\Lambda}(I)\right\} \leq C\, |\log|I||^d\, S_\Lambda(|I|)\, |\Lambda|.$$
  They used their single energy UCSP for the (nonrandom) operator H_0.
- Wegner estimates for crooked Anderson Hamiltonians imply corresponding Wegner estimates for Delone-Anderson models.

Optimal Wegner estimate for crooked Anderson Hamilts.

Using the UCPSP for the full random operator H_ω, we prove

Theorem

Let H_ω be a crooked Anderson Hamiltonian. Given $E_0 > 0$, define $\gamma > 0$ by

$$\gamma^2 = \tfrac{1}{2}\delta_-^{M_d\left(1+K^{\frac{2}{3}}\right)}, \quad \textit{where} \quad K = E_0 + 2\left(\|V^{(0)}\|_\infty + M\|U\|_\infty\right).$$

and $M_d > 0$ is the constant in the UCPSP Theorem.
Then for any closed interval $I \subset]-\infty, E_0]$ with $|I| \leq \frac{2}{5}\gamma$ and any box $\Lambda = \Lambda_L(x_0)$ with $x_0 \in \mathbb{R}^d$ and $L \geq 150\sqrt{d} + \delta_+$, we have

$$\mathbb{E}\left\{\operatorname{tr} P_{\omega,\Lambda}(I)\right\} \leq C_{d,\delta_+,\|V^{(0)}\|_\infty}\left(u_-^{-2}\gamma^{-4}(1+E_0)\right)^{2^{1+\frac{\log d}{\log 2}}} S_\Lambda(|I|)\,|\Lambda|.$$

UCPSP $\Longrightarrow$ Optimal Wegner estimate

The theorem (optimal Wegner estimates) follows from the UCPSP theorem and the following lemma.

Lemma

Let H_ω be a crooked Anderson Hamiltonian.
Let $I \subset]-\infty, E_0]$ be a closed interval and $\Lambda = \Lambda_L(x_0)$ a box centered at $x_0 \in \mathbb{R}^d$ with $L \geq 2 + \delta_+$.

Suppose there exists a constant $\kappa > 0$ such that

$$P_{\omega,\Lambda}(I) U^{(\Lambda)} P_{\omega,\Lambda}(I) \geq \kappa P_{\omega,\Lambda}(I) \quad \textit{with probability one.}$$

Then

$$\mathbb{E}\left\{ \operatorname{tr} P_{\omega,\Lambda}(I) \right\} \leq C_{d,\delta_+,\|V^{(0)}\|_\infty} \left(\kappa^{-2}(1+E_0) \right)^{2^{1+\frac{\log d}{\log 2}}} S_\Lambda(|I|)\,|\Lambda| .$$

Proof of Lemma

We fix Λ and $I \subset]-\infty, E_0]$, let $P = P_{\omega,\Lambda}(I)$ $U = U^{(\Lambda)}$. Then (Dirichlet bc)

$$\begin{aligned}
\operatorname{tr} P &\le \kappa^{-1} \operatorname{tr} PUP = \kappa^{-1} \operatorname{tr} \sqrt{U} P \sqrt{U} \le \kappa^{-2} \operatorname{tr} \sqrt{U} PUP \sqrt{U} = \kappa^{-2} \operatorname{tr} PUPU \\
&= \kappa^{-2} \operatorname{tr} PUPUP \le \kappa^{-2}(1+E_0) \operatorname{tr} PU(H_{\omega,\Lambda}+1)^{-1} UP \\
&\le \kappa^{-2}(1+E_0) \operatorname{tr} PU(H_{0,\Lambda}+1)^{-1} UP \\
&= \kappa^{-2}(1+E_0) \operatorname{tr} UPU(H_{0,\Lambda}+1)^{-1} \\
&= \kappa^{-2}(1+E_0) \sum_{i,j \in \Lambda \cap \mathbb{Z}^d} \operatorname{tr} \sqrt{u_j} P \sqrt{u_i}\, T_{ij},
\end{aligned}$$

$$\text{where} \quad T_{ij} = \sqrt{u_i}(H_{0,\Lambda}+1)^{-1}\sqrt{u_j} \quad \text{for} \quad i,j \in \Lambda \cap \mathbb{Z}^d.$$

We may now adapt an argument in in Combes, Hislop, Klopp obtaining

$$\mathbb{E} \operatorname{tr} P \le C_{d,\delta_+,V_\infty^{(0)}} \left(\kappa^{-2}(1+E_0)\right)^{2^{1+\frac{\log d}{\log 2}}} S_\Lambda(|I|)\,|\Lambda|.$$

Wegner estimates at high disorder

Let $H_{\omega,\lambda} = H_0 + \lambda V_\omega$ be a crooked Anderson Hamiltonian, where $\lambda > 0$ is the disorder parameter.We want make explicit the dependence on λ in the Wegner estimate.

If we have the covering condition $U^{(\Lambda)} \geq \alpha \chi_\Lambda$ with $\alpha > 0$, we get, following Combes-Hislop or the Lemma,

$$\mathbb{E}\left\{\operatorname{tr} P_{\omega,\lambda,\Lambda}(I)\right\} \leq C_{d,\delta_+,\alpha,\|V^{(0)}\|_\infty,E_0}\, S_\Lambda(\lambda^{-1}\,|I|)\,|\Lambda|\,,$$

a Wegner estimate that gets better as the disorder increases.

Without the covering condition, we get, using the UCPSP,

$$\mathbb{E}\left\{\operatorname{tr} P_{\omega,\lambda,\Lambda}(I)\right\} \leq C_{E_0} \mathrm{e}^{c_{E_0}\left(1+\lambda^{\frac{2}{3}}\right)} S_\Lambda(\lambda^{-1}\,|I|)\,|\Lambda|\,.$$

If we use the UCPSP for H_0, as in Combes, Hislop and Klopp, we get

$$\mathbb{E}\left\{\operatorname{tr} P_{\omega,\lambda,\Lambda}(I)\right\} \leq C_{E_0}\left(1+\lambda^{2^{2+\frac{\log d}{\log 2}}}\right) S_\Lambda(\lambda^{-1}\,|I|)\,|\Lambda|\,.$$

These Wegner estimates get worse as the disorder increases.

Optimal Wegner estimate at the bottom of the spectrum at high disorder

Theorem

Let $H_{\omega,\lambda}$ *be a crooked Anderson Hamiltonian with disorder* $\lambda > 0$*. Then*

$$E(\infty) := \lim_{t\to\infty} E(t) = \sup_{t\geq 0} E(t) > 0, \quad \text{where} \quad E(t) := \inf \sigma(H_0 + tu_- W).$$

Moreover, for each $E_1 \in]0, E(\infty)[$ *there exists* $\kappa = \kappa(E_1) > 0$*, independent of* λ*, such that the following holds for all* $\lambda > 0$*:*
Given a box $\Lambda = \Lambda_L(x_0)$ *with* $x_0 \in \mathbb{R}^d$ *and* $L \geq 2 + \delta_+$*, we have*

$$P^{(D)}_{\omega,\lambda,\Lambda}(]-\infty, E_1])\, U^{(\Lambda)}\, P^{(D)}_{\omega,\lambda,\Lambda}(]-\infty, E_1]) \geq \kappa\, P^{(D)}_{\omega,\lambda,\Lambda}(]-\infty, E_1]),$$

and, for any interval $I \subset]-\infty, E_1]$*,*

$$\mathbb{E}\left\{\operatorname{tr} P^{(D)}_{\omega,\lambda,\Lambda}(I)\right\} \leq C_{d,\delta_+,V^{(0)}_\infty} \left(\kappa^{-2}(1+E_1)\right)^{2^{1+\frac{\log d}{\log 2}}} S_\Lambda(\lambda^{-1}|I|)\, |\Lambda|\,.$$

A lower bound on $E(\infty)$

Lemma

Let H_0, u_-, W be as in a crooked Anderson Hamiltonian, set $H(t) = H_0 + tu_- W$ for $t \geq 0$, and let $E(t) = \inf \sigma(H(t))$, $E(\infty) = \lim_{t\to\infty} E(t) = \sup_{t\geq 0} E(t)$. Then

$$E(t) \geq tu_- \delta_-^{M_d\left(1+\left(2V_\infty^{(0)}+2tu_-\right)^{\frac{2}{3}}\right)} \quad \textit{for all} \quad t \geq 0,$$

so we conclude that

$$E(\infty) \geq \sup_{t\in[0,\infty[} t\delta_-^{M_d\left(1+\left(2V_\infty^{(0)}+2t\right)^{\frac{2}{3}}\right)} > 0.$$

This lemma is proven from the Corollary to the QUCP.

An abstract UCSP

The Theorem now follows using an extension of an abstract UCPSP due to Boutet de Monvel, Lenz, and Stollmann (2011).

Lemma

Let H_0 be a self-adjoint operator on a Hilbert space $\mathscr{H}$, bounded from below, and let $Y \geq 0$ be a bounded operator on $\mathscr{H}$.
Let $H(t) = H_0 + tY$ for $t \geq 0$, and set $E(t) = \inf \sigma(H(t))$.
Let $E(\infty) = \lim_{t\to\infty} E(t) = \sup_{t\geq 0} E(t)$.
Suppose $E(\infty) > E(0)$. Given $E_1 \in]E(0), E(\infty)[$, let

$$\kappa = \kappa(H_0, Y, E_1) = \sup_{s>0;\ E(s)>E_1} \frac{E(s) - E_1}{s} > 0.$$

Then for all bounded operators $V \geq 0$ on $\mathscr{H}$ and Borel sets $B \subset]-\infty, E_1]$ we have

$$\chi_B(H_0 + V)\, Y\, \chi_B(H_0 + V) \geq \kappa \chi_B(H_0 + V).$$

Proof of the abstract UCPSP

Fix $E_1 \in]E(0), E(\infty)[$. For all Borel sets $B \subset]-\infty, E_1]$ we have, writing $P_V(B) = \chi_B(H_0 + V)$,

$$P_V(B)(H_0 + V)P_V(B) \leq E_1 P_V(B).$$

Since $E_1 \in]E(0), E(\infty)[$, there is $s > 0$ such that $E(s) > E_1$. Then,

$$P_V(B)(H(s) + V - sY - E_1)P_V(B) = P_V(B)(H_0 + V - E_1)P_V(B) \leq 0,$$

and hence, using $V \geq 0$,

$$\begin{aligned} sP_V(B)YP_V(B) &\geq P_V(B)(H(s) + V - E_1)P_V(B) \\ &\geq P_V(B)(H(s) - E_1)P_V(B) \geq (E(s) - E_1)P_V(B). \end{aligned}$$

We conclude that

$$\chi_B(H_0 + V)\, Y \,\chi_B(H_0 + V) \geq \kappa \chi_B(H_0 + V).$$

Localization in a fixed interval at high disorder

Theorem

Let $H_{\omega,\lambda}$ be an ergodic Anderson Hamiltonian with disorder $\lambda > 0$, and suppose the single-site probability distribution μ has a bounded density (or is uniformly Hölder continuous).

Then, given $E_1 \in]0, E(\infty)[$, there exists $\lambda(E_1) < \infty$, such that $H_{\omega,\lambda}$ exhibits complete localization on the interval $[0, E_1[$ for all $\lambda \geq \lambda(E_1)$.

By complete localization on an interval I we mean that for all $E \in I$ there exists $\delta(E) > 0$ such that we can perform the bootstrap multiscale analysis on the interval $(E - \delta(E), E + \delta(E))$, obtaining Anderson and dynamical localization.

This theorem was previously known only with a covering condition $U^{(\Lambda)} \geq \alpha\chi_\Lambda$, $\alpha > 0$, in which case $E(\infty) = \infty$.

This theorem holds for crooked Anderson Hamiltonians with appropriate hypotheses on the single site probability distributions μ_j.