Inverted oscillators & collapse of wave packets

Italo Guarneri

Avronfest, Jerusalem
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Center for Nonlinear and Complex Systems - Universita' dell'Insubria a Como

"The world is full of interesting operators"
Y. Avron (2004) , at the $\hbar$ Cafe'

Table of contents

Smilansky's "irreversible" model.

A particle in a line (coordinate x) coupled to a linear harmonic oscillator (coordinate q) by point interaction:

$$\hat{\mathcal{H}} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{1}{2}\omega^2 q^2 + + \alpha\, q\, \delta(x - x_0)\,. \qquad (1)$$

$\alpha > 0$ a parameter, x_0 a fixed point. [1]

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If $\alpha > \omega$, then : no point spectrum , and the ac spectrum acquires an additional component with multiplicity 1 that coincides with $\mathbb{R}$.

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Boxing the particle

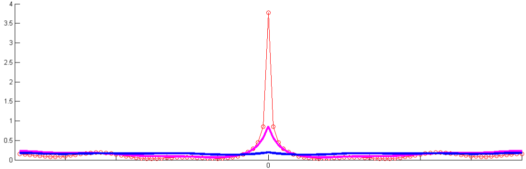
Let the particle be confined within a box, with either hard-wall or periodic boundary conditions.

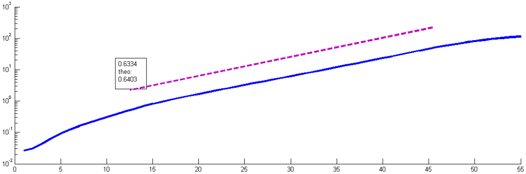
Theorem

If $\alpha < \omega$, the spectrum of $\hat{\mathcal{H}}$ is pure point. If $\alpha > \omega$, generically no pp spectrum, and absolutely continuous component of multiplicity 1, that coincides with $\mathbb{R}$.

Solomyak&Naboko's proof still works over the threshold, with minor adaptations .

Dynamics of Smilansky's model I





Dynamics of Smilansky's model II

Theorem

The time-averaged energy of the oscillator grows in time, at least exponentially fast: i.e.,

$$\liminf_{T\to+\infty} \frac{\ln(E_{\rm osc}(T))}{T} > 0 . \qquad (2)$$

The probability distribution of the position x of the particle weakly converges to $\delta(x - x_0)$ in the limit $t \to +\infty$.

$$E_{\rm osc}(T)) = \tfrac{1}{T} \int_0^T dt\, (\psi(t), \hat{H}_{\rm osc} \otimes \mathbb{I}\psi(t)) .$$

Band formalism.

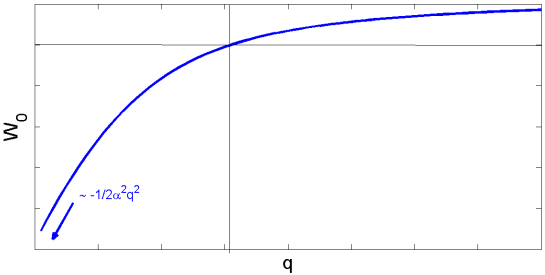
The particle is moving in a circle $\mathbb{S}$ parametrized by $x \in [-\pi, +\pi]$ with a distinguished point O $(x = 0)$.

$$\mathcal{H} \;=\; \int_{\mathbb{R}}^{\oplus} dq\; H_\alpha(q) \;+\; \mathbb{I} \otimes H^{\rm osc} \;,$$

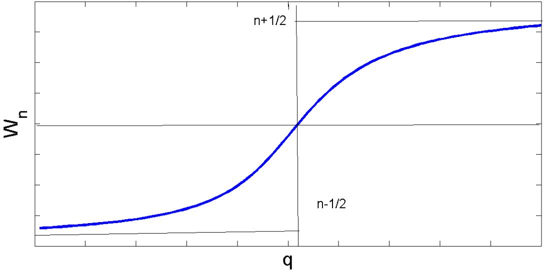
$$H_\alpha(q) \;=\; -\frac{1}{2}\frac{d^2}{dx^2} \;+\; \alpha q\; \delta(x) \;.$$

$$\psi(x,q) \;=\; \sum_{n=0}^{+\infty} Q_n(q)\; \phi_{q,n}(x) \;,$$

$$H_\alpha(q)\phi_{q,n} \;=\; W_n(q)\phi_{q,n} \;, \quad \phi_{q,n} \in L^2(\mathbb{S}) \;. \tag{3}$$



W0
~ -1/2α2q2
q



Wn
n+1/2
n-1/2
q

Band formalism, II

$$
\begin{aligned}
(\psi,\ \mathcal{H}\psi) \;=\; & (\psi,\ \mathbb{I} \otimes H_\omega^{(\mathrm{osc})}\ \psi) + \\
& + \sum_{n=0}^{+\infty} \int_{\mathbb{R}} dq\ W_n(q) |Q_n(q)|^2 \\
\geq\; & (\psi,\ \mathbb{I} \otimes \tilde{H}_{\alpha,\omega}\ \psi) + \\
& + \sum_{n=1}^{+\infty} \int_{\mathbb{R}} dq\ (n - 1/2) |Q_n(q)|^2 \,.
\end{aligned}
$$

$$
\tilde{H}_{\alpha,\omega} \;=\; -\frac{1}{2}\frac{d^2}{dq^2} \;+\; \frac{1}{2}\omega^2 q^2 \;+\; W_0^-(q)
$$

$$
W_0^-(q) = \tfrac{1}{2}(1 - \mathrm{sign}(q)) W_0(q)
$$

A diversion on inverted oscillators

$$H_{i\omega}^{(\mathrm{osc})} \;=\; \frac{p^2}{2} \;-\; \frac{1}{2}\omega^2 q^2 \;=\; H_{\omega}^{(\mathrm{osc})} \;-\; \omega^2 q^2 \tag{4}$$

on the $H_{\omega}^{(\mathrm{osc})}$ eigenbasis, matrix elements $(H_{i\omega}^{(\mathrm{osc})})_{nm} \neq 0$ only if $m - n = \pm 2$.

$$2\omega H_{n,n+2} \;=\; -\sqrt{(n+1)(n+2)}\,, \qquad 2\omega H_{n,n-2} \;=\; -\sqrt{n(n-1)}.$$

$u(k, E)$: amplitude of the (even) formal eigenfunction of energy E at $n = 2k$

$$\begin{aligned} u(k+2, E) \;+\; p(k, E)u(k+1) \;+ q(k)u(k, E) \;&=\; 0 \\ p(k, E) \sim \tfrac{E}{\omega}\tfrac{1}{k}\,, \quad q(k) \sim 1 \;-\; \tfrac{1}{k} & \end{aligned} \tag{5}$$

1

BA&WL Theory

Birkhoff (1911), Adams (1928), Wong and Li (1993). Provides asymptotic ($n \to \infty$) approximations for solutions of 2nd order difference equations of the form

$$C(n+2) \; + \; p(n)C(n+1) \; + \; q(n)C(n) \; = \; 0$$

whenever the coefficients have asymptotic expansions $p(n) \sim \sum_r a(r)n^{-r}$ and $q(n) \sim \sum_r b(r)n^{-r}$.

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whenever the coefficients have asymptotic expansions $p(n) \sim \sum_r a(r) n^{-r}$ and $q(n) \sim \sum_r b(r) n^{-r}$.
"Normal", linearly independent solutions exist, which have the asymptotic form:

$$C^{\star}_{\pm}(n) \; \sim \; {\sigma_{\pm}}^n n^{\alpha_{\pm}} \sum_{s=0}^{\infty} c_{\pm}(s) n^{-s} \; ,$$

$${\sigma_{\pm}}^2 + a(0)\sigma_{\pm} + b(0) = 0 \; , \quad \alpha_{\pm} = \frac{a(1)\sigma_{\pm} + b(1)}{a(0)\sigma_{\pm} + 2b(0)} \; .$$

(assuming $\sigma_+ \neq \sigma_-$)

Application to the inverted oscillator

$$\phi(2n, E) \sim \frac{1}{\sqrt{n}} \cos\Big(E \log(n)/2\omega \ + \ n\frac{\pi}{4} \ + \beta(E)\Big)$$

consistent with $q \to +\infty$ asymptotics : G.Barton, Ann. Phys. 166 (1986)

$\Rightarrow$ ac spectrum.

ac spectrum + $n^{-1/2}$ eigfs decay $\Rightarrow$ exponential growth of energy.

Spectral Analysis over threshold.

Expansion in normalized Hermite functions (HO eigenfunctions):

$$\psi(x,q) \;=\; \sum_{n=0}^{\infty} \psi_n(x)\; h_n(q)\;,$$

identifies the Hilbert space of the system as $\mathfrak{H} = \ell^2(\mathbb{N}_0) \otimes L^2_+(\mathbb{S})$ of sequences $\{\psi_n(x)\}_{n\in\mathbb{N}_0}$, $\psi_n(x) \in L^2(\mathbb{S})$, ψ even, $\sum_n \|\psi_n\|^2 < +\infty$.

Hamiltonian

$\{\psi_n(x)\} \mapsto \{L_n\psi_n(x)\}$, $L_n = -\frac{1}{2}\frac{d^2}{dx^2} + (n+\frac{1}{2})\omega$ with periodic bdary conditions and matching conditions at $x = 0$:

$$\psi_n'(0+) - \psi_n'(0-) = 2\alpha(2\omega)^{-1}\left(\sqrt{n+1}\psi_{n+1}(0) + \sqrt{n}\psi_{n-1}(0)\right)$$

Formal Eigenfunctions

Sequences $\{u_n(x,E)\}$ that solve $L_n u_n(x,E) = E u_n(x,E)$ for all integer $n \geq 0$ and satisfy the matching conditions. May be written as

$$u_n(x,E) \;=\; C(n,E)\, v_n(x,E)$$

where $v_n(x) =$ normalized sol. of $L_n v_n = E v_n$, and constants $C(n,E)$ satisfy:

2nd order Difference Equation

$$h_2(n,E)C(n+2,E) + h_1(n,E)C(n+1,E) + h_0(n,E)C(n,E) \;=\; 0\;, \quad (n$$

with "initial" condition:
$h_2(-1,E)C(1,E) \;=\; -h_1(-1,E)C(0,E)\;.$

details

$$
\begin{aligned}
h_2(n, E) &= \alpha\sqrt{n+2}\rho_{n+2}(E)\cos(k_{n+2}(E)\pi) \\
h_1(n, E) &= \sqrt{2\omega}k_{n+1}(E)\rho_{n+1}(E)\sin(k_{n+1}(E)\pi) \\
h_0(n, E) &= \alpha\sqrt{n+1}\rho_n(E)\cos(k_n(E)\pi) \\
\rho_n(E) &= (\pi + k_n(E)^{-1}\sin(2k_n(E)\pi))^{-1/2} , \\
k_n(E) &= \sqrt{2E - (2n+1)\omega} .
\end{aligned}
\tag{3}
$$

BA&WL again

'Exceptional' energies E : those which are either branch points or zeros for some coefficient $h_2(n, E)$, $h_1(n, E)$.

Theorem

If $\alpha > \omega$ then for all non-exceptional E, a formal eigenfunction exists, and has the $n \to \infty$ asymptotics

$$C(n, E) \sim \frac{1}{\sqrt{\pi n}} \cos(n\theta - \lambda E \log(n) + \zeta(E)) + O(n^{-3/2})$$

$$\theta = \arccos(\omega/\alpha) , \quad \lambda = \tfrac{1}{2}(\alpha^2 - \omega^2)^{-1/2} .$$

The phase $\zeta(E)$ is a C^1 function of E in any interval containing no exceptional energies.

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$\sum_n |C(n, E)|^2$ diverges logarithmically

Spectral Expansion

Theorem

For $\Psi(E)$ a function on $\mathbb{R}$ compactly supported away from exceptional points, and for any $n \in \mathbb{N}_0$, define $\psi_n(x) = \int dE\, \Psi(E) u_n(x, E)$. Then:
1) $\{\psi_n(x)\}_{n\in\mathbb{N}_0} \in \mathfrak{H}$,
2) the map $\imath : \Psi \mapsto \{\psi_n(x)\}_{n\in\mathbb{N}_0}$ extends to a unitary isomorphism of $L^2(\mathbb{R})$ onto an absolutely continuous subspace of the Hamiltonian $\hat{\mathcal{H}}$,
3)$\forall t \in \mathbb{R}$, $\imath\left(e^{-iEt}\Psi\right) = e^{-i\hat{\mathcal{H}}t}\imath(\Psi)$.

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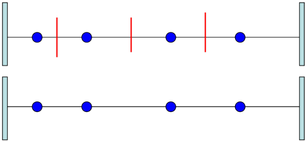
Multi-Oscillator case: Wave operators



Above: $\hat{\mathcal{H}}^{(b)}$. Below: $\hat{\mathcal{H}}$.

(Evans, Solomyak, 2005) The wave operators:

$$\Omega^{\pm} = \lim_{t\to\pm\infty} e^{i\mathcal{H}t} e^{-i\hat{\mathcal{H}}^{(b)}t}$$

exist and are complete.

Collapse

If $\alpha > \omega$ then, for any ψ in the absolutely continuous subspace of $\mathcal{H}$, the probability distribution of the particle converges weakly as $t \to \pm\infty$ to a superposition of δ functions supported in the interaction points:

$$\int_{\mathbb{R}^N} \cdots \int dq_1 \dots dq_N \, |\psi(x, q_1, \dots, q_N, t)|^2 \quad \xrightarrow[t\to\pm\infty]{} \quad \sum_{j=1}^{N} \gamma_j^{\pm} \, \delta(x - x_j) \; ,$$

where:

$$\gamma_j^{\pm} \; = \; \| P_j \, \Omega_{\pm}(\mathcal{H}^{(b)}, \mathcal{H}) \psi \|^2 \; , \tag{6}$$

and P_j denotes projection onto $\mathfrak{H}_j = L^2(B_j) \otimes L^2(\mathbb{R}^N)$, ($B_j$: j-th box).

Decoherence

Corollary

For all initial ψ *in the absolutely continuous subspace of* $\mathcal{H}$*, the reduced density matrix* $\mathrm{Tr}\big(\hat{B}\hat{S}_\psi(t)\big) = \big(\psi(t),\ \hat{B}\otimes\hat{\mathbb{I}}\psi(t)\big)$ *for all bounded* $\hat{B}$*, satisfies:*

$$\lim_{t\to\pm\infty}\big(\phi\ ,\ \hat{S}_\psi(t)\varphi\big)\ =\ 0$$

for all $\phi, \varphi \in L^2(\mathbb{S})$.

Band formalism

Born-Oppenheimer-like description: oscillator dynamics described by the "ground-band Hamiltonian"

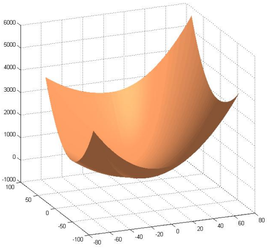
$$-\frac{1}{2}\frac{\partial^2}{\partial q_1^2} + \ldots - \frac{1}{2}\frac{\partial^2}{\partial q_N^2} + \frac{1}{2}\omega^2(q_1^2 + \ldots + q_N^2) + W(q_1, \ldots, q_N)$$

where $W(q_1, \ldots)$ is the ground state energy of the particle Hamiltonian

$$-\frac{1}{2}\frac{d^2}{dx^2} + \alpha \sum_1^N q_j \delta(x - x_j)$$

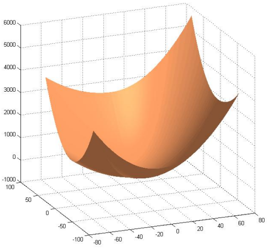
parametrically dependent on $q_1, \ldots, q_N$.

"Phase transition"

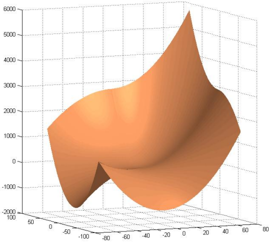


$$\alpha \ < \ \omega$$

"Phase transition"

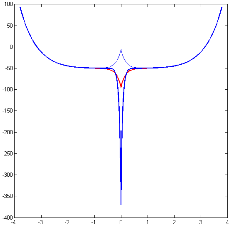


$\alpha < \omega$

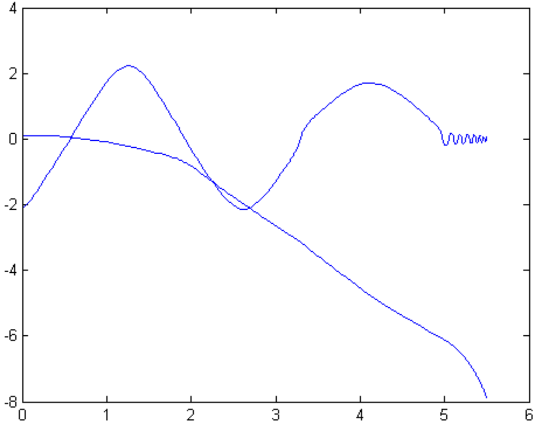


$\alpha > \omega$

Classical version



$$V(x,q) = (1 - ax^2)\exp(ax^2) + bq^2\,\mathrm{sign}(q)\exp(-c|qx|) \quad (7)$$



4
2
0
-2
-4
-6
-8
0
1
2
3
4
5
6