Atomic clocks

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M DOKTORANDSKYCH A MANAGERSKYCH STUD

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Relativistic Invariance and Quantum Phenomena*

EUGENE P. WIGNER

" For example, a clock, with a running time of a day and an accuracy of 10^{-8} *second, must weigh almost a gram—for reasons stemming solely from uncertainty principles and similar considerations."*

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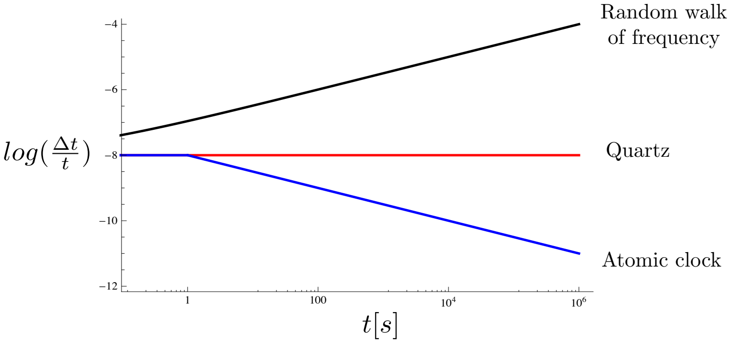
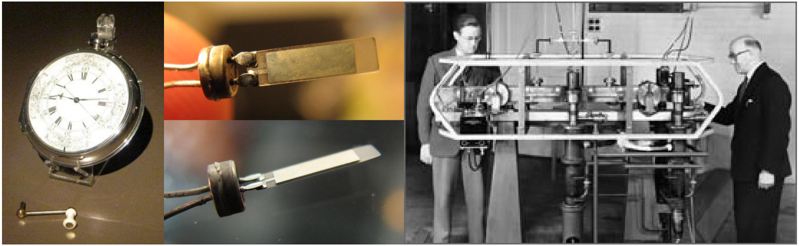


Brief history of man made clocks



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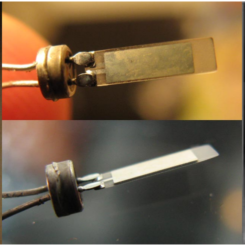
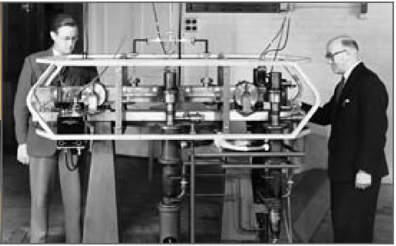

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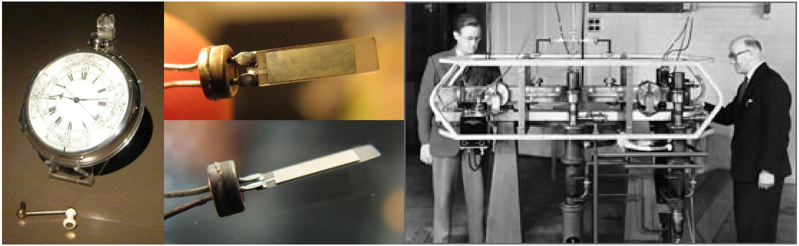

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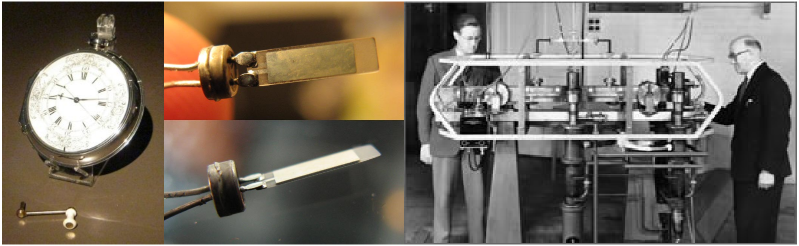

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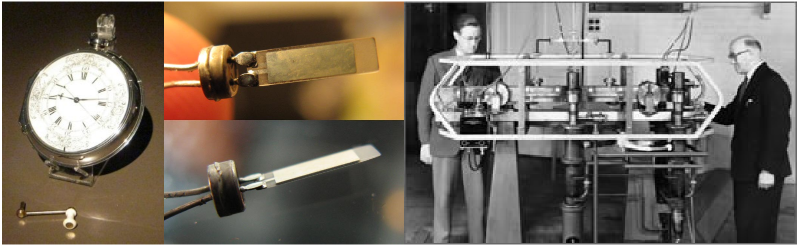

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2010	AI^+ Optical Clock	10^{-6} μs

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- Limits on size, mass, power, etc.

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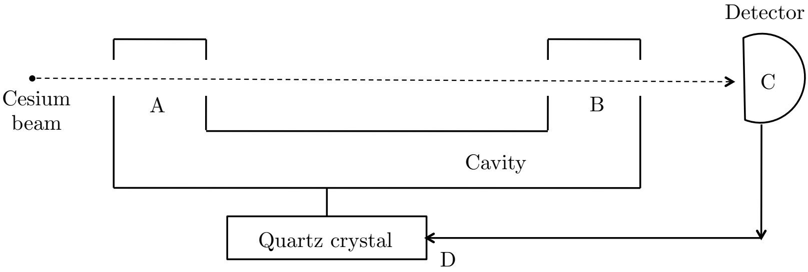

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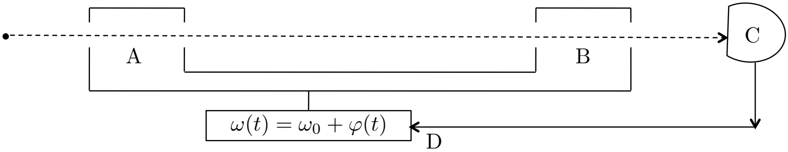

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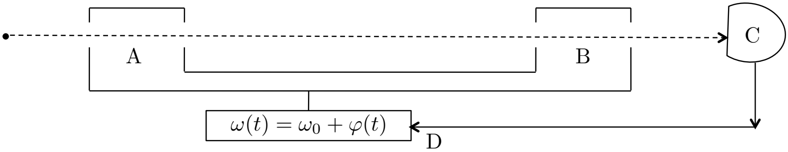

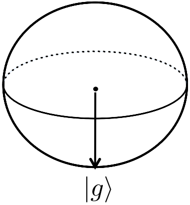

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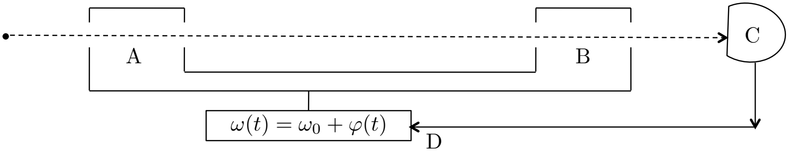

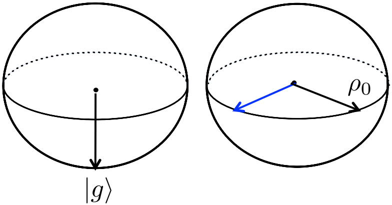

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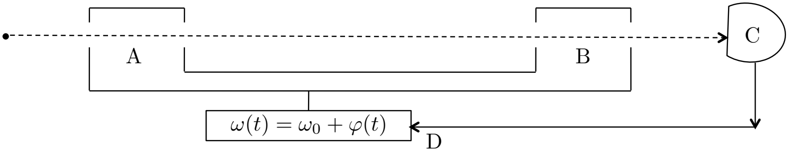

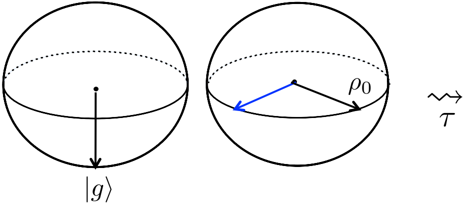

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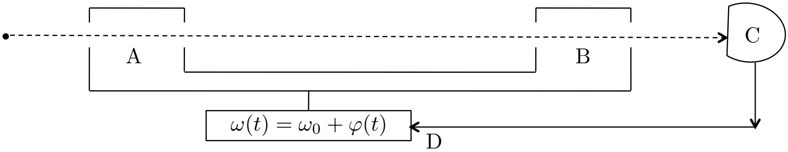

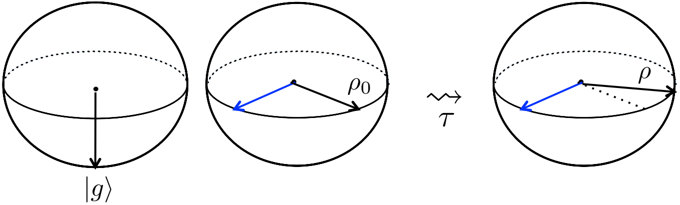

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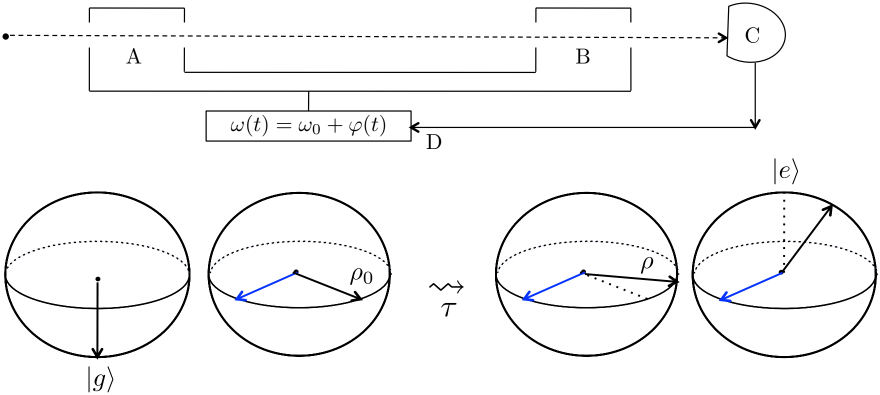

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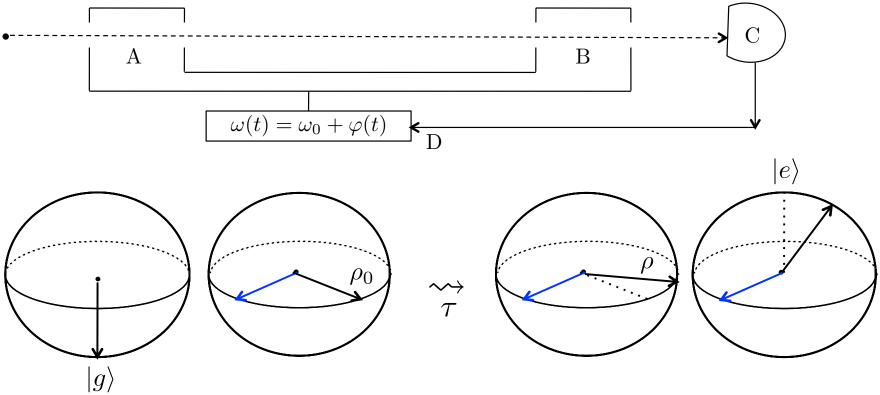

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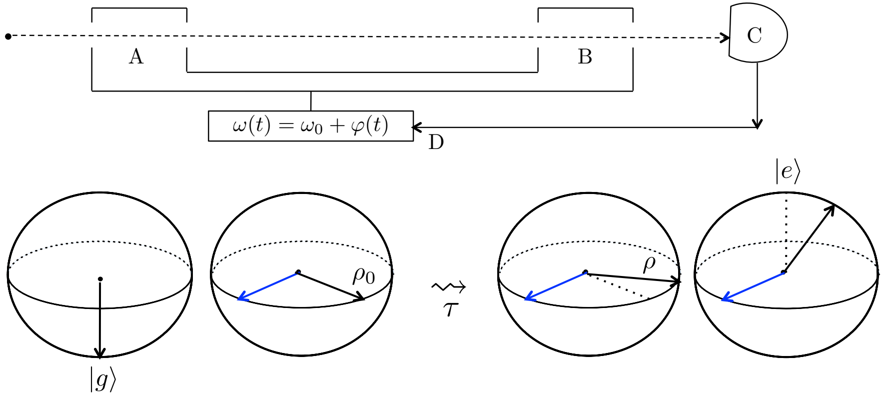

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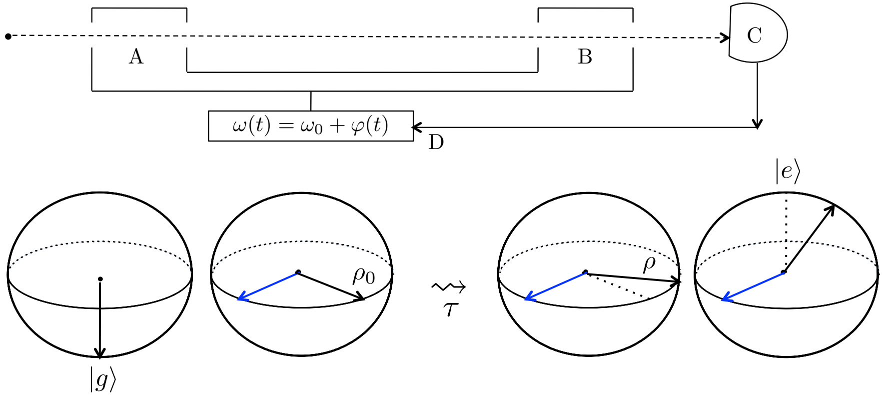

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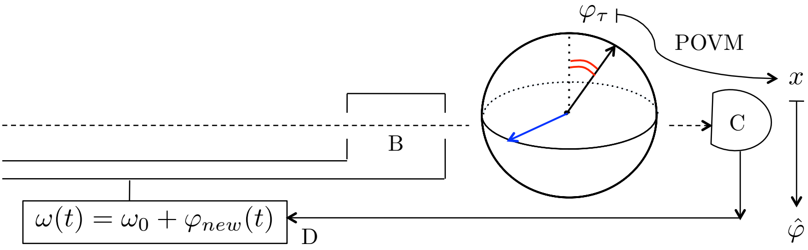

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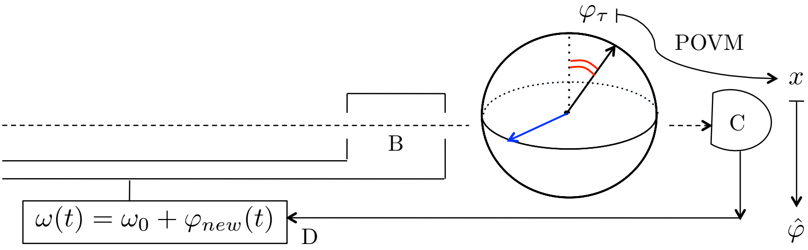

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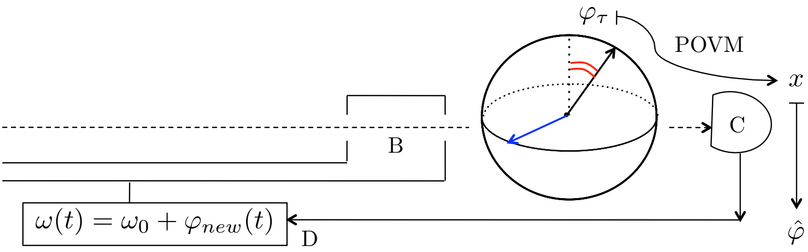

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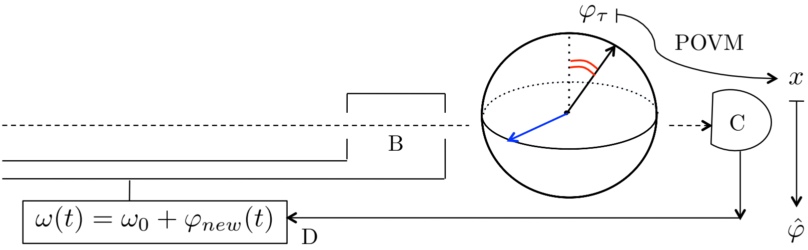

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A feedback uses $\hat{\varphi}$ to adjust the original frequency error φ.

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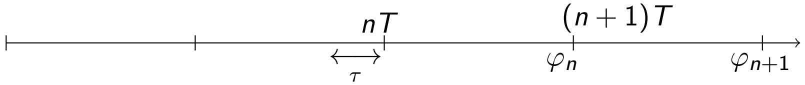

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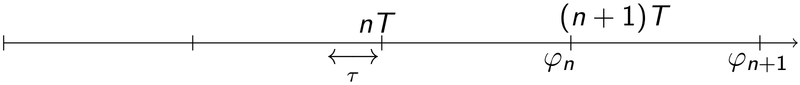

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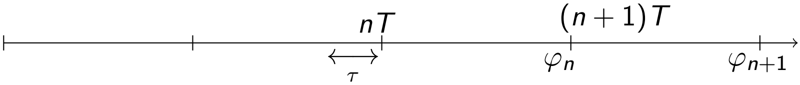

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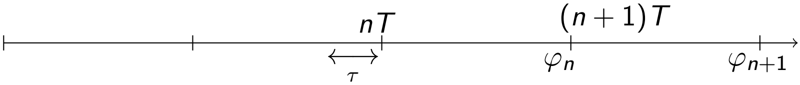

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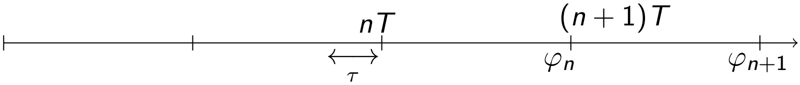

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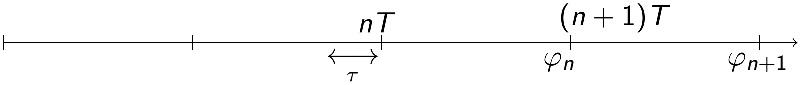

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- φ_n provides $\varphi(t)$, which gives the clock time;

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Definition (ζ-unbiased clock)

The clock is ζ-unbiased if the estimation procedure satisfies

$$\mathbb{E}[\varphi - \hat{\varphi}|\varphi] = \zeta\varphi, \quad |\zeta| < 1.$$

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Fisher information is inversely proportional to the width .

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where

$$g(\zeta, x) = \zeta^2 + \frac{1+\zeta-2\zeta^2}{3}x,$$

$$f(\zeta, x) = 1 + \zeta + \zeta^2 + (1+2\zeta)(1-\zeta)x + (1-\zeta)^2 x^2,$$

$$\frac{1}{F} = \mathbb{E}\left[\frac{1}{F(\tau\varphi_n)}\right].$$

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Quantitative analysis, the case $T = \tau$:

- The optimal interrogation time is determined by a balance of the dissipation and estimation precision. For fixed ζ:

$$4DT = (1-\zeta)^2 \frac{1}{FT^2}.$$

- For the optimal time, $\zeta \approx 0.35$ minimize the variance of the stationary state;

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- Cramer-Rao type inequality: Suppose $\mathbb{E}[\varphi\hat{\varphi}] = \zeta\mathbb{E}[\varphi^2]$ then

$$\mathbb{E}[(\varphi - \hat{\varphi})^2] \geq (1-\zeta)^2 \frac{1}{F} + \zeta^2 \mathbb{E}[\varphi^2];$$

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Outlooks:

- "Central limit theorem“;
- Beyond unbiased clock, unbiased stationary state;
- Entropy production;

Happy Birthday

Yosi!