The intriguing δ'

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with grateful remembrance of a common work which inspired various further explorations

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One is nevertheless tempted to choose a problem around which our paths crossed, especially if the subject proved inspirative and lead to various other investigations.

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$$\psi'(0+) = \psi'(0-) =: \psi'(0) \quad \text{and} \quad \psi(0+) - \psi(0-) = \beta\psi'(0)$$

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The name they choose was not particularly fortunate but it stuck. Recall that while while the δ interaction can be obtained as a limit of *scaled potentials*, the δ' is *not* the limit of scaled potentials of *zero mean* – cf. [Šeba'86, Zolotaryuk et al.'03].

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The intriguing features of the interaction came first to attention when Yosi proposed to look at the δ' version of the *Wannier-Stark model* combining singular periodic and linear potentials, i.e the system formally described by the Hamiltonian

$$H(\beta, F, a) = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{n\in Z} \beta\delta'_{na} - eFx$$

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(b) A heuristic argument: the tilted gaps are *classically forbidden regions*; if their widths grow they may prevent indefinite propagation.

(c) Make it rigorous by a *Simon-Spencer-type* argument. Inserting a sequence of Neumann conditions we get an operator with pure point spectrum; if the 'chops' are placed in the middle of the tilted gaps, one can check that the perturbation is *trace class*. $\square$

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(a1) U is nondecreasing, $\lim_{x\to\infty} U(x) = \infty$
(a2) U is C^2 smooth with $|U'(x)| \le c$ and $|U''(x)| \le \tilde{c}U(x)$ for some $c, \tilde{c} > 0$
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Under these conditions the standard solutions between neighboring δ's are

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}(x) = \begin{pmatrix} \cos \\ \sin \end{pmatrix} k_n(x-n)\,(1+\mathcal{O}(U(x)^{-1/2}))$$

Robustness of the effect, continued

In addition, we suppose that the coupling constants β_n are such that

(b) $|\beta_n| \geq \beta > 0$ for all n

(c) there is a monotonic sequence $\{n_\ell\} \subset \mathbb{Z}_+$ such that $\mathrm{Re}\, k_{n_\ell} = \pi(n_\ell + \epsilon_\ell)$ with $\epsilon_\ell \in (\frac{1}{4}, \frac{3}{4})$, and $\beta_{n_\ell}\beta_{n_{\ell+1}}^{-1}$ remains bounded as $n_\ell \to \infty$

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Using the same trace-class perturbation argument we can conclude:

Theorem (E'95)

Under the assumptions (a)–(c) the ac spectrum of

$$H(\{\beta_n\}, V, a) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{n \in Z_+} \beta_n \delta'_{na} + V(x)$$

is empty

How does the spectrum look like?

Character of the spectrum: using a KAM-type argument one can prove – cf. [Asch-Duclos-E'98] – that for all but a 'small set' of the parameters the spectrum of $H(\beta, F, a)$ is *pure point*, and extend this result to the result to a class of nonlinear background potentials.

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Spectrum as a set depends on parameter values, in particular, one is able to *conjecture* that the following relation holds,

$$\sigma_{\mathrm{ess}}(H(\beta, F, a)) = \left\{ \frac{4}{\beta a} + \left(\frac{m\pi}{a}\right)^2 - F\left(n + \frac{1}{2}\right) a \,:\, m, n \in \mathbb{Z} \right\}$$

This would imply a dichotomy:

- if $\gamma := \left(\frac{a}{\pi}\right)^2 Fa$ is rational, the spectrum is *nowhere dense*, and therefore automatically *pure point*.
- on the other hand, $\sigma((H(\beta, E, a)) = \sigma_{ess}((H(\beta, E, a)) = \mathbb{R}$ holds if γ is irrational.

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- a heuristic argument suggest that the spectrum might exhibit a transition from (singularly?) continuous at small F to a point one for larger field values
- such a behavior is observed in the random case with probability one [Delyon-Simon-Souillard'85], however, the deterministic problem remains open

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The first attempt we made in [Avron-E-Last'94] was *to replace points by 'onion' type graphs*,

Each 'onion' consists of N links of length L; we consider the limit $N \to \infty$ keeping the product $NL = \beta$ fixed.

‘Onion’ graph limits

- Assuming ‘Kirchhoff’ conditions at the graph vertices, one easily finds the reflection amplitude of a single ‘onion’ to be

$$r(kL; N) = \frac{-N^2 + 1}{N^2 + 2iN \cot(kL) + 1}$$

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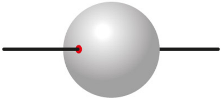
- hence the two halflines asymptotically decouple as as $k \to \infty$, even if the decoupling is Dirichlet instead of Neumann appropriate for the δ'
- for an 'onion' string we get similarly the *band-gap structure* of the δ' in the limit $k \to \infty$

Other geometric scatterers

As an example, consider a sphere with two leads attached



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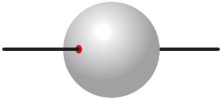

They are coupled through boundary conditions which involve boundary values on the halfline with the *generalized ones* on the sphere which are the coefficients in the expansion $\Phi(\vec{x}) = L_0(\Phi) \ln|\vec{x}| + L_1(\Phi_2) + \mathcal{O}(|\vec{x}|)$

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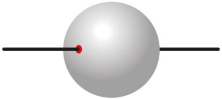

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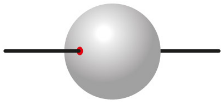

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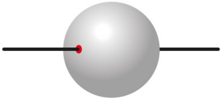

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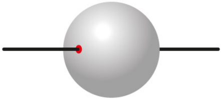

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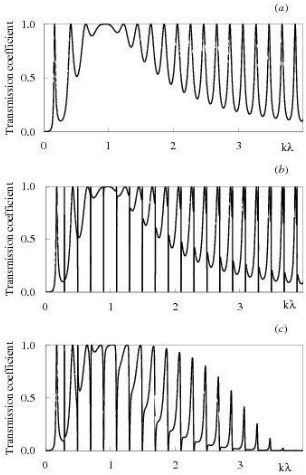
- the *background reflection dominates* at high energies, $k \to \infty$

Transmission through the sphere



(a) Junctions at opposed poles, (b) tilt 2°, (c) tilt 4°

(reproduced from [Brüning-Geyler-Margulis-Pyataev'02])

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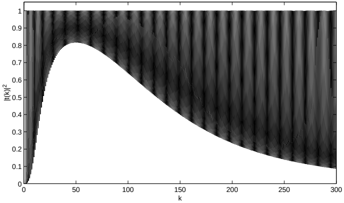
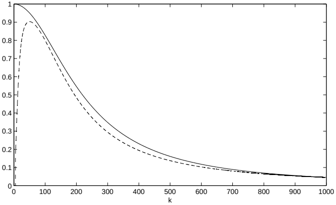
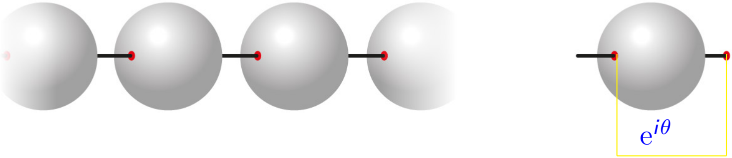
- The conjecture is supported to numerical results, for instance





Arrays of geometric scatterers

One can also consider 'necklaces' of such geometric scatterers:



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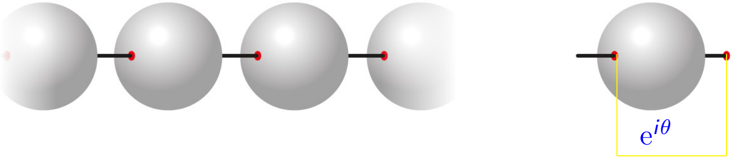

The band spectrum of such system can be found using standard Floquet analysis investigating the dependence of single cell eigenvalues on the *quasimomentum* θ

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Question: Is the *ac* spectrum again absent if we add an electric field parallel to the array?

Approximation by Schrödinger operators

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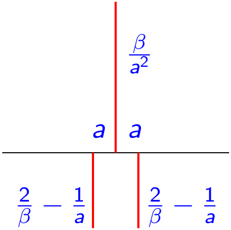
Then a scheme appeared in [Cheon-Shigehara'98] which gave formally the δ' conditions in the limit $a \to 0$, namely



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- convergence of transfer (thus also scattering) matrices was proven in [Albeverio-Nizhnik'00]

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- convergence of transfer (thus also scattering) matrices was proven in [Albeverio-Nizhnik'00]
- the norm-resolvent convergence, $\|(-\Delta_{\mathcal{A}_{a(\epsilon)}, Y_{a(\epsilon)}} + \kappa^2)^{-1} - (\Xi_{\beta,y} + \kappa^2)^{-1}\| \to 0$ as $a \to 0$ was proven in [E-Neidhardt-Zagrebnov'01]

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- One should note how subtle the convergence is: both resolvents are strongly divergent as $a \to 0$, but in the difference *the first four orders cancel* and the fifth gives a convergent result

Moreover, since δ interaction is a limit of squeezed potentials, one can approximate the δ' by *regular potentials*.

Approximation by Schrödinger operators, contd

Consider operator $H^a_{\epsilon,y} := -\Delta + W^a_{\epsilon,y}$ with the potential

$$W^a_{\epsilon,0}(x) = \frac{\beta}{\epsilon a(\epsilon)^2}\, V_0\left(\frac{x}{\epsilon}\right) + \left(\frac{2}{\beta} - \frac{1}{a(\epsilon)}\right)\left\{\frac{1}{\epsilon}\, V_{-1}\left(\frac{x + a(\epsilon)}{\epsilon}\right) + \frac{1}{\epsilon}\, V_1\left(\frac{x - a(\epsilon)}{\epsilon}\right)\right\}$$

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and let $\|V_j\|_{L^1} = 1$ and $\int_{-\infty}^{\infty} \mathrm{d}x\, |x|^{1/2}\, |V_0(x)| < \infty$ hold for $j = 0, \pm 1$.

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Moreover, assume that $a(\epsilon) \to 0$ as $\epsilon \to 0$ and $\lim_{\epsilon\to 0} \frac{\epsilon}{a(\epsilon)^{12}} = 0$.

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Theorem (E-Neidhardt-Zagrebnov'01)

Under the stated assumptions, one has

$$\lim_{\epsilon\to 0} \left\|(H^a_{\epsilon,y} + \kappa^2)^{-1} - (\Xi_{\beta,y} + \kappa^2)^{-1}\right\| = 0\,.$$

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Remark: The power 12 is for sure not optimal.

Passing to higher dimensions

The δ' is an essentially one-dimensional thing, however, on can obtain interesting models considering singular Schrödinger operators with a δ' interaction supported by a manifold of *codimension one*.

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Let us look a the radially periodic potentials in $\mathbb{R}^\nu$, $\nu \geq 2$. If they are regular, the spectrum mixes by [Hempel-Herbst-Hinz-Kalf'91] absolutely continuous and dense pure point components. The same is true by [E-Fraas'07] for concentric δ potentials.

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One can treat similarly other point interactions, in particular, the δ'. As long as the system has a radial symmetry, we can employ the partial-wave decomposition, $H_\beta := \bigoplus_l U^{-1} H_{\beta,l} U \otimes I_l$, where

$$H_{\beta,l} := -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r^2}\left[\frac{(\nu-1)(\nu-3)}{4} + l(l+\nu-2)\right]$$

defined on functions which are locally H^2 and satisfy the δ' conditions *with the same coupling constant* β at the radii r_n, $n = 1, 2, \dots$ with $r_{n+1} - r_n = d > 0$ (an extra condition at the origin needed if $\nu \leq 3$).

Comparison to the δ' KP model

The radial motion can be naturally compared to the one described by

$$h_\beta := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{n\in\mathbb{Z}} \beta\delta'_{x_n}$$

on $L^2(\mathbb{R})$ with the δ' interactions at the points $x_n = d(n+\frac{1}{2})$.

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In particular, one has $\sigma_{\mathrm{ess}}(H_{\beta,l}) = \sigma_{\mathrm{ess}}(h_\beta)$ which yields

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Remark: If $\nu = 2$ the operator H_β has infinitely many eigenvalues below $\inf \sigma_{\mathrm{ess}}(H_\beta)$ – they are analogous to the *'Welsh eigenvalues'* discussed in [Brown et al'98].

Character of the spectrum

Theorem (E-Fraas'08)

(a) For any gap (E_{2k-1}, E_{2k}) in the essential spectrum of h_β we have
(i) H_β has no continuous spectrum in (E_{2k-1}, E_{2k}),
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Remark: While the spectrum consists of interlaced intervals of *ac* and *dense p.p.* spectrum, in contrast to more regular potentials including δ, for δ' the *dense point component dominates at high energies*.

Strong coupling behavior

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For simplicity we consider operators in $L^2(\mathbb{R}^2)$ with the interaction support being a smooth closed curve Γ, being graph of a function $\Gamma : [0, L] \to \mathbb{R}^2$.

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The operator acts as Laplacian outside the interaction support,

$$(H_{\beta,\Gamma}\psi)(x) = -(\Delta\psi)(x)$$

for $x\in\mathbb{R}^2\setminus\Gamma$, with the domain $\mathcal{D}(H_{\beta,\Gamma}) = \{\psi\in H^2(\mathbb{R}^2\setminus\Gamma)\mid \partial_{n_\Gamma}\psi(x) = \partial_{-n_\Gamma}\psi(x) =: \psi'(x)|_\Gamma,\; \beta\psi'(x)|_\Gamma = \psi(x)|_{\partial_+\Gamma} - \psi(x)|_{\partial_-\Gamma}\}$, where n_Γ is the outer normal to Γ and $\psi(x)|_{\partial_\pm\Gamma}$ are the appropriate traces.

Strong coupling behavior, continued

Alternatively, the singular Schrödinger operator $H_{\beta,\Gamma}$ can be defined through its quadratic form. We introduce locally orthogonal coordinates (s, u) in the vicinity of Γ — s is the arc length of Γ and u the distance from the curve — and set

$$h_{\beta,\Gamma}[\psi] = \|\nabla\psi\|^2 + \beta^{-1} \int_\Gamma |\psi(s, 0_+) - \psi(s, 0_-)|^2 \, \mathrm{d}s$$

defined on functions $\psi \in C(\mathbb{R}^2) \cap H^1(\mathbb{R}^2 \setminus \Gamma)$ written, with an abuse of notation, also as $\psi(s, u)$.

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For this purpose we introduce a comparison operator

$$S := -\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{1}{4}\gamma(s)^2$$

on $L^2(0, L)$ with periodic boundary conditions, where $\gamma(s)$ denotes the *signed curvature* of Γ at the point s.

Strong coupling on a δ' loop

Theorem (E-Jex'13)

Let Γ be a C^4-smooth closed curve without self-intersections. Then $\sigma_{\mathrm{ess}}(H_{\beta,\Gamma}) = [0,\infty)$ and to any $n \in \mathbb{N}$ there is a $\beta_n > 0$ such that $\#\sigma_{\mathrm{disc}}(H_{\beta,\Gamma}) \geq n$ holds for $\beta \in (0,\beta_n)$. Denoting for such a β by $\lambda_j(\beta)$ the j-th eigenvalue of $H_{\beta,\Gamma}$, again counted with its multiplicity, we have the asymptotic expansion

$$\lambda_j(\beta) = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}\big(\beta \ln|\beta|\big)\,, \quad j = 1,\dots,n\,,$$

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valid as $\beta \to 0-$, where μ_j is the j-th eigenvalue of the comparison operator S introduced above. Moreover, for the counting function $\beta \mapsto \#\sigma_d(H_{\beta,\Gamma})$ we have

$$\#\sigma_{\rm disc}(H_{\beta,\Gamma}) = \frac{2L}{\pi|\beta|} + \mathcal{O}(-\ln|\beta|) \quad \text{as} \quad \beta \to 0-\,.$$

Strong coupling behavior, continued

Remark: Compare the above with the asymptotics for δ interaction,

$$\lambda_j(\beta) = -\frac{\alpha^2}{4} + \mu_j + \mathcal{O}\big(-\alpha^{-1} \ln|\alpha|\big)\,, \quad j = 1, \dots, n\,,$$

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We employ a bracketing argument: imposing Dirichlet and Neumann conditions at $\partial\Omega_a$, we get $H_N(\beta) \le H_\beta \le H_D(\beta)$. Furthermore, the 'outer' parts of the estimating operators are positive, hence for our purpose it is only the strip part which matters.

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$$h_{N/D,\beta}[f] = \|\nabla f\|^2 + \beta^{-1}\int_\Gamma |f(s,0_+) - f(s,0_-)|^2\,\mathrm{d}s$$

defined on $H^1(\Omega_a\setminus\Gamma)$ and $H^1_0(\Omega_a\setminus\Gamma)$, respectively.

Proof sketch

Next we ‘straighten’ Ω_a passing to the coordinates (s,u); in this way the estimating operators are equivalent to those associated with the forms

$$q_D[f] = \left\| \frac{1}{g}\partial_s f \right\|^2 + \|\partial_u f\|^2 + (f, Vf) + \beta^{-1}\int_0^L |f(s,0_+) - f(s,0_-)|^2\,\mathrm{d}s$$
$$+\frac{1}{2}\int_0^L \gamma(s)\big(|f(s,0_+)|^2 - |f(s,0_-)|^2\big)\,\mathrm{d}s$$

$$q_N[g] = q_D[g] - \int_0^L \frac{\gamma(s)}{2(1+a\gamma(s))}|f(s,a)|^2\,\mathrm{d}s + \int_0^L \frac{\gamma(s)}{2(1-a\gamma(s))}|f(s,-a)|^2\,\mathrm{d}s$$

on $H_0^1((0,L)\times((-a,0)\cup(0,a)))$ and $H^1((0,L)\times((-a,0)\cup(0,a)))$, respectively, with periodic boundary conditions in the variable s. The geometrically induced potential in these formulæ is given by $V = \frac{u\gamma''}{2g^3} - \frac{5(u\gamma')^2}{4g^4} - \frac{\gamma^2}{4g^2}$ with $g(s) := 1 + u\gamma(s)$.

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This is still not easy to handle, therefore we pass to slightly cruder estimates by the operators $Q_{a,\beta}^{\pm} = U_a^{\pm}\otimes I + \int_{[0,L)}^{\oplus} T_{a,\beta}^{\pm}(s)\,\mathrm{d}s$, where $U_a^{\pm}$ refers to a u-independent estimate of the first and the third terms.

Proof sketch, continued

The transverse part of the upper bound corresponds to the quadratic form

$$t^+_{a,\beta}(s)[f] := \|f'\|^2 - \frac{1}{\beta}\,|f(0_+) - f(0_-)|^2 + \frac{1}{2}\gamma(s)\big(|f(s,0_+)|^2 - |f(s,0_-)|^2\big)\,,$$

and has the following spectral properties:

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Lemma

The operators $T^{+}_{a,\beta}(s)$ has exactly one negative eigenvalue $t_+ = -\kappa_+^2$ provided $\frac{a}{\beta} > 2$ which is independent of s and such that

$$\kappa_+ = \frac{2}{\beta} - \frac{4}{\beta}\,\mathrm{e}^{-4a/\beta} + \mathcal{O}(\beta^{-1}\mathrm{e}^{-8a/\beta}) \quad \textit{holds as} \quad \beta \to 0\,.$$

Proof sketch, concluded

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Lemma

There is a positive C *independent of* a *and* j *such that*

$$|\mu_j^{\pm}(a) - \mu_j| \leq Caj^2$$

holds for $j \in \mathbb{N}$ *and* $0 < a < \frac{1}{2\gamma_+}$*, where* $\mu_j^{\pm}(a)$ *are the eigenvalues of* $U_a^{\pm}$*, respectively, with the multiplicity taken into account.*

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holds for $j \in \mathbb{N}$ and $0 < a < \frac{1}{2\gamma_+}$, where $\mu_j^{\pm}(a)$ are the eigenvalues of $U_a^{\pm}$, respectively, with the multiplicity taken into account.

Choosing finally $a(\beta) = \frac{3}{4}\beta \ln|\beta|$ and putting the estimates together, we get the first claim; the second one is demonstrated in a similar way. □

Geometrically induced bound states

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Consider a singular Schrödinger operator $H_{\beta,\Gamma}$ in $L^2(\mathbb{R}^2)$ with an attractive δ' interaction supported by an infinite curve Γ.

Theorem (Behrndt-E-Lotoreichik'13)

Suppose that Γ is piecewise C^1 smooth and obtained by a nontrivial local deformation of a straight line, then $\sigma_{\rm disc}(H_{\beta,\Gamma}) \neq \emptyset$ for any $\beta < 0$.

Geometrically induced bound states

Let us finally show that the δ' *need not be strong* to produce spectral effects related to the geometry of its support.

Consider a singular Schrödinger operator $H_{\beta,\Gamma}$ in $L^2(\mathbb{R}^2)$ with an attractive δ' interaction supported by an infinite curve Γ.

Theorem (Behrndt-E-Lotoreichik'13)

Suppose that Γ is piecewise C^1 smooth and obtained by a nontrivial local deformation of a straight line, then $\sigma_{\rm disc}(H_{\beta,\Gamma}) \neq \emptyset$ for any $\beta < 0$.

Proof idea: Choosing $\alpha = \frac{4}{\beta}$ we find by a comparison of the corresponding quadratic form and minimax principle that $\lambda_n(H^{\delta'}_{\beta,\Gamma}) \leq \lambda_n(H^{\delta}_{\alpha,\Gamma})$ holds for any $n \in \mathbb{N}$; combining this inequality with the existence result obtained in [E-Ichinose'01] for the δ interaction we get the sought claim. □

The talk was based, in particular, on

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[EJ13] P.E., M. Jex: Spectral asymptotics of a strong δ' interaction on a planar loop, `arXiv:1304.7696`

[BEL13] J. Behrndt, P.E., V. Lotoreichik: Schrödinger operators with δ and δ'-interactions on Lipschitz surfaces and chromatic numbers of associated partitions, `arXiv:1307.????`

It remains to say

Happy birthday, Yosi!