

Linear Representations of the Automorphism Group of a Free Group

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Abstract

Let F_n be the free group on $n \geq 2$ elements and $\text{Aut}(F_n)$ its group of automorphisms. In this paper we present a rich collection of linear representations of $\text{Aut}(F_n)$ arising through the action of finite index subgroups of it on relation modules of finite quotient groups of F_n . We show (under certain conditions) that the images of our representations are arithmetic groups.

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1 Introduction

Let F_n be the free group on $n \geq 2$ elements and $\text{Aut}(F_n)$ its group of automorphisms. The latter is a much studied group, but very little seems to be known about its (finite dimensional) complex representation theory (see for example [12], [6], [29], [30], [22]). In fact, as far as we know the only representations studied (at least when $n \geq 3$) are:

- The representation

$$\rho_1 : \text{Aut}(F_n) \rightarrow \text{Aut}(F_n/F'_n) \cong \text{GL}(n, \mathbb{Z}) \quad (1)$$

where F'_n is the commutator subgroup of F_n .

- The representations factoring through the homomorphisms

$$\rho_i : \text{Aut}(F_n) \rightarrow \text{Aut}(F_n/F_n^{(i+1)})$$

where $F_n^{(i+1)}$ ($i = 0, 1, \dots$) stands for the lower central series of F_n .

- Representations with finite image.

What is common to all of the above mentioned representations ρ is that if we denote by $\mathcal{H} := \overline{\rho(\text{Aut}(F_n))}$ the Zariski closure of $\rho(\text{Aut}(F_n))$, then the semi-simple part of the connected component \mathcal{H}° of \mathcal{H} is either trivial or $\text{SL}(n, \mathbb{C})$. Moreover, $\rho(\text{IA}(F_n))$ is always virtually solvable for these representations, where $\text{IA}(F_n)$ is the kernel of ρ_1 .

In this paper we will show that the representation theory of $\text{Aut}(F_n)$ is in fact much richer. Our main Theorem (Theorem 1.4 below) implies for example

Theorem 1.1 *Let $n \geq 2$, $k \geq 1$, $h_1 < \dots < h_k$, m_1, \dots, m_k be natural numbers. Let $\mathbb{Q}(\zeta_{m_i})$ be the field of m_i -th roots of unity and $\mathbb{Z}(\zeta_{m_i})$ its ring of integers. There is a subgroup $\Gamma \leq \text{Aut}(F_n)$ of finite index and a representation*

$$\rho : \Gamma \rightarrow \prod_{i=1}^k \text{SL}((n-1)h_i, \mathbb{Q}(\zeta_{m_i}))^{m_i}$$

such that $\rho(\Gamma)$ is commensurable with $\prod_{i=1}^k \text{SL}((n-1)h_i, \mathbb{Z}(\zeta_{m_i}))^{m_i}$.

Theorem 1.1 shows that $\text{Aut}(F_n)$ has a representation ρ such that the connected component of $\overline{\rho(\text{Aut}(F_n))}$ is isomorphic to

$$\prod_{i=1}^k \text{SL}((n-1)h_i, \mathbb{C})^{m_i}$$

as above. In fact the proof shows that this is even true for $\overline{\rho(\text{IA}(F_n))}$ and in particular the latter is very far from being virtually solvable (see Section 9.2 for more).

Specifying $m_1 = \dots = m_k = 1$ in Theorem 1.1, we deduce

Corollary 1.2 *Let $n \geq 2$, $k \geq 1$ be natural numbers. There is a subgroup $\Gamma \leq \text{Aut}(F_n)$ of finite index and a representation*

$$\rho : \Gamma \rightarrow \prod_{i=1}^k \text{SL}((n-1)i, \mathbb{Z})$$

such that $\rho(\Gamma)$ is of finite index in $\prod_{i=1}^k \text{SL}((n-1)i, \mathbb{Z})$.

Specialising Theorem 1.1 even further to $n = 3$, $k = 1$, $h_1 = m_1 = 1$ we get that $\text{Aut}(F_3)$ has a subgroup of finite index which can be mapped onto a subgroup of finite index in $\text{SL}(2, \mathbb{Z})$. This implies

Corollary 1.3 *The automorphism group $\text{Aut}(F_3)$ is large, that is it has a subgroup of finite index which can be mapped onto a free nonabelian group. In particular $\text{Aut}(F_3)$ does not have Kazhdan's property (T).*

The corollary is easy for $\text{Aut}(F_2)$, see the discussion in Section 7, but it is a well known open problem for larger n . Our solution for $n = 3$ does not indicate what should be the answer for $n > 3$.

We shall describe now how the representations to be considered here arise. Let G be a finite group and $\pi : F_n \rightarrow G$ a surjective homomorphism of the free group F_n onto G . Let R be the kernel of π . We define

$$\Gamma(R) := \{ \varphi \in \text{Aut}(F_n) \mid \varphi(R) = R \} \tag{2}$$

and

$$\Gamma(G, \pi) := \{ \varphi \in \Gamma(R) \mid \varphi \text{ induces the identity on } F_n/R \}. \quad (3)$$

These are subgroups of finite index in $\text{Aut}(F_n)$. We let

$$\bar{R} := R/R' \quad (4)$$

be the relation module of the presentation $\pi : F_n \rightarrow G$. The action of F_n on R by conjugation leads to an action of G on \bar{R} .

The structure of \bar{R} as a G -module is described by Gaschütz' theory (see Section 2.2 and the references therein). It says in particular that

$$\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R} \cong \mathbb{Q} \oplus \mathbb{Q}[G]^{n-1} \quad (5)$$

as a module over the rational group ring $\mathbb{Q}[G]$ (see also (18)). Our work can be described as an equivariant Gaschütz' theory where we try to describe the action of $\Gamma(G, \pi)$ on \bar{R} : Every $\varphi \in \Gamma(R)$ induces a linear automorphism $\bar{\varphi}$ of \bar{R} and (as an application of the theorem of Gaschütz) $\Gamma(G, \pi)$ consists exactly of those elements $\varphi \in \Gamma(R)$ for which $\bar{\varphi}$ is G -equivariant. Already now we can see from (5) why the number $n - 1$ plays such an important role in Theorem 1.1.

To be more precise: The relation module \bar{R} is a finitely generated free abelian group, let t be its \mathbb{Z} -rank. Then

$$\mathcal{G}_{G, \pi} := \text{Aut}_G(\mathbb{C} \otimes_{\mathbb{Z}} \bar{R}) \leq \text{GL}(t, \mathbb{C}) \quad (6)$$

is a \mathbb{Q} -defined algebraic subgroup of $\text{GL}(t, \mathbb{C})$, in fact $\mathcal{G}_{G, \pi}$ is the centraliser of the group G acting on $\mathbb{C} \otimes_{\mathbb{Z}} \bar{R}$ through matrices with rational entries. Let

$$\mathcal{G}_{G, \pi}^1 \leq \text{SL}(t, \mathbb{C}) \quad (7)$$

be the kernel of all \mathbb{Q} -defined homomorphisms from the complex algebraic group $\mathbb{C} \otimes \mathcal{G}_{G, \pi}$ to the multiplicative group. This is a \mathbb{Q} -defined subgroup of $\mathcal{G}_{G, \pi}$. We shall describe it in more detail in Section 2.2. We define

$$\mathcal{G}_{G, \pi}(\mathbb{Z}) := \{ \phi \in \mathcal{G}_{G, \pi} \mid \phi(\bar{R}) = \bar{R} \}. \quad (8)$$

This is an arithmetic subgroup of the \mathbb{Q} -defined algebraic group $\mathcal{G}_{G, \pi}$ which contains the arithmetic subgroup

$$\mathcal{G}_{G, \pi}^1(\mathbb{Z}) := \{ \phi \in \mathcal{G}_{G, \pi}^1 \mid \phi(\bar{R}) = \bar{R} \} \quad (9)$$

of $\mathcal{G}_{G, \pi}^1$. Following the definitions we obtain an integral linear representation

$$\rho_{G, \pi} : \Gamma(G, \pi) \rightarrow \mathcal{G}_{G, \pi}(\mathbb{Z}), \quad \rho_{G, \pi}(\varphi) = \bar{\varphi} \quad (\varphi \in \Gamma(G, \pi)). \quad (10)$$

Examples show (see Section 6) that in general $\rho_{G, \pi}$ is not onto, neither is its image of finite index in $\mathcal{G}_{R, \pi}(\mathbb{Z})$. However, our main result shows that, at least for redundant presentations, the image of $\rho_{G, \pi}$ captures the whole semi-simple part of $\mathcal{G}_{R, \pi}(\mathbb{Z})$. More precisely:

Theorem 1.4 *Assume n is a natural number with $n \geq 4$. Let $\pi : F_n \rightarrow G$ be a redundant presentation of the finite group G . Then $\rho_{G, \pi}(\Gamma(G, \pi)) \cap \mathcal{G}_{G, \pi}^1$ is of finite index in the arithmetic group $\mathcal{G}_{G, \pi}^1(\mathbb{Z})$.*

A presentation $\pi : F_n \rightarrow G$ is called *redundant* if there is a basis x_1, \dots, x_n of F_n such that $\pi(x_n) = 1$. This is equivalent to the kernel of π containing at least one primitive element (that is a member of a basis) of F_n .

In general we do not know whether $\rho_{G,\pi}(\Gamma(G, \pi)) \cap \mathcal{G}_{G,\pi}^1(\mathbb{Z})$ is of finite index in $\rho_{G,\pi}(\Gamma(G, \pi))$. Section 8 contains criteria under which this is going to happen.

Choosing the finite group in various ways we obtain a rich variety of linear representations of subgroups of finite index in $\text{Aut}(F_n)$ with arithmetic image groups. Let us start with some simple examples.

Example 1: Let $G = C_2$ be the group of order 2 and $\pi : F_n \rightarrow G$ be any surjective homomorphism. Since we generally have assumed $n \geq 2$, the representation is redundant. As an abelian group we have $\bar{R} \cong \mathbb{Z}^{2(n-1)+1}$. Also $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}$ can be decomposed into the +1-eigenspace of a generator of G and the corresponding -1-eigenspace. We obtain a decomposition $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R} = \mathbb{Q}^n \oplus \mathbb{Q}^{n-1}$ which has to be respected by $\rho_{G,\pi}(\Gamma(G, \pi))$. Thus we obtain a representation

$$\rho_{G,\pi} : \Gamma(G, \pi) \rightarrow \text{GL}(n, \mathbb{Q}) \times \text{GL}(n-1, \mathbb{Q})$$

which we show to have an image commensurable with $\text{SL}(n, \mathbb{Z}) \times \text{SL}(n-1, \mathbb{Z})$.

The first factor is the familiar one arising from the representation (1). But the second is somewhat less expected. It already gives rise to a representation which is suitable for proving Corollary 1.3.

Example 2: Let p be a prime number, $\zeta_p \in \mathbb{C}$ a non trivial p -th root of unity. Let $K := \mathbb{Q}(\zeta_p)$ be the p -th cyclotomic field and $\mathbb{Z}(\zeta_p)$ its ring of integers. Taking $G = C_p$ the cyclic group of order p we have $\bar{R} \cong \mathbb{Z}^{p(n-1)+1}$. Now $p(n-1) + 1 = (p-1)(n-1) + n$ and we get a representation

$$\rho_{G,\pi} : \Gamma(G, \pi) \rightarrow \text{GL}(n, \mathbb{Q}) \times \text{GL}(n-1, K)$$

which we show to have an image commensurable with $\text{SL}(n, \mathbb{Z}) \times \text{SL}(n-1, \mathbb{Z}(\zeta_p))$. This example is treated in Section 6 along with the case of a general cyclic group.

Let G be now an arbitrary finite group and $\pi : F_n \rightarrow G$ a redundant epimorphism. Decompose the group algebra $\mathbb{Q}[G]$ as

$$\mathbb{Q}[G] = \mathbb{Q} \oplus \prod_{i=2}^{\ell} M(h_i, D_i)$$

where the D_i are division algebras. Theorem 1.4 implies that there is a homomorphism ρ from a finite index subgroup Γ of $\text{Aut}(F_n)$ into a subgroup of $\prod_{i=2}^{\ell} \text{GL}((n-1)h_i, D_i)$ where the image contains an arithmetic subgroup of

$$\Lambda := \prod_{i=2}^{\ell} \text{SL}((n-1)h_i, \mathcal{R}_i)$$

where \mathcal{R}_i is an order in D_i . In general we do not know whether $\rho(\Gamma) \cap \Lambda$ is of finite index in $\rho(\Gamma)$. It will be especially interesting if it is not! In this case $\rho(\Gamma)/(\rho(\Gamma) \cap \Lambda)$ would be an infinite abelian group and this would imply that $\text{Aut}(F_n)$ does not have Kazhdan property (T). We have some results in the opposite direction showing, for example, that if G is metabelian then $\rho(\Gamma)$ is in fact commensurable to Λ .

In any event, whenever $\rho(\Gamma) \cap \Lambda$ is of finite index or not, it becomes of finite index after we divide by the center of $\prod_{i=2}^{\ell} \text{GL}((n-1)h_i, D_i)$. This shows that in all cases an arithmetic group

commensurable to Λ is a quotient of some finite index subgroup of $\text{Aut}(F_n)$. Thus Theorem 1.4 provides a very rich class of arithmetic subgroups which are quotients of (finite index subgroups) of $\text{Aut}(F_n)$. Theorem 1.1 is obtained from the main result by some special choices of the finite group G and by an application of Margulis super-rigidity (see Section 7).

The present paper opens up a large number of interesting problems. Among others it shows that the classical Torelli subgroup (i.e. $\text{IA}(F_n)$) is just a first example in a series of infinitely many. The kernel of $\rho_{G,\pi}$, for any (redundant) homomorphism $\pi : F_n \rightarrow G$ of F_n to a finite group G , can be considered as a natural generalisation of $\text{IA}(F_n)$ which corresponds to the case where G is the trivial group. Just as $\text{IA}(F_n)$, the kernel of $\rho_{G,\pi}$ is residually torsion-free nilpotent and, at least when π is redundant, the image of $\rho_{G,\pi}$ is an arithmetic group hence is finitely presented, which implies that $\ker(\rho_{G,\pi})$ is a finitely generated normal subgroup. Is $\ker(\rho_{G,\pi})$ a finitely generated group? In case of the classical Torelli group this is a result of Magnus (see [23]).

Let us now describe the method of the proof of the main result and the layout of the paper: In Section 2, we describe in detail the rational relation module $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}$ together with $\mathcal{G}_{G,\pi}$ which is the centraliser of G when acting on the relation module. In Section 3, we describe the homomorphism from $\Gamma(G, \pi)$ to $\mathcal{G}_{G,\pi}$, which is given by the action of $\Gamma(G, \pi)$ on $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}$. We compute the action of various elements of $\Gamma(G, \pi)$ on $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}$ in Section 4 using results on Fox calculus from Section 3. A careful choice of elements from $\Gamma(G, \pi)$ allows us to produce sufficiently many unipotent elements of $\mathcal{G}_{G,\pi}(\mathbb{Z})$ to appeal to results of Vaserstein [33] (see also Venkataramana [35] for generalisations) to deduce in Section 5 that these elements generate a finite index subgroup. Section 6 is devoted to a detailed study of the case when G is a cyclic group. In this case we give fairly complete results. Some of them are used in Section 7 for the proof of Theorem 1.1. In Section 8 we discuss the question whether the image of $\Gamma(G, \pi)$ is commensurable with $\mathcal{G}_{G,\pi}^1(\mathbb{Z})$ or an infinite extension of it. We solve it for cyclic groups and also for metabelian groups. Some remarks, computational results and open problems are given in Section 9.

In a subsequent paper [16] we apply similar methods to study linear representations of the mapping class group.

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2 Relation modules and representations of subgroups of finite index in $\text{Aut}(F_n)$

In this section we collect some results on the structure of the relation modules of finite groups. We also consider integrality properties of the linear representations of subgroups of finite index in $\text{Aut}(F_n)$ introduced in the introduction.

2.1 The rational group ring

Here G is a finite group and $\mathbb{Q}[G]$ is its rational group ring. Our basic reference for the structural results we need is the book of Curtis and Reiner [4] (see also [32]), in particular Sections IV and

XI therein.

We start to set up some notation. Let N be an irreducible left $\mathbb{Q}[G]$ -module. We write

$$\mathbf{I}_N(M)$$

for the N -isotypic component of the left $\mathbb{Q}[G]$ -module M . We also choose representatives N_1, \dots, N_ℓ for the isomorphism classes of irreducible left $\mathbb{Q}[G]$ -modules and abbreviate

$$\mathbf{I}_i(M) := \mathbf{I}_{N_i}(M) \quad (i = 1, \dots, \ell)$$

for a left $\mathbb{Q}[G]$ -module M . We take N_1 to be the one dimensional trivial module.

Viewing $\mathbb{Q}[G]$ as a left $\mathbb{Q}[G]$ -module we obtain a direct product decomposition

$$\mathbb{Q}[G] = \prod_{i=1}^{\ell} \mathbf{I}_i(\mathbb{Q}[G]). \quad (11)$$

In this case every $\mathbf{I}_i(\mathbb{Q}[G])$ is also a right submodule of $\mathbb{Q}[G]$. Furthermore every $\mathbf{I}_i(\mathbb{Q}[G])$ is an ideal of $\mathbb{Q}[G]$ having its own unity. The decomposition (11) is a decomposition of $\mathbb{Q}[G]$ as a direct product of \mathbb{Q} -algebras.

We let $P_i \in \mathbb{Q}[G]$ ($i = 1, \dots, \ell$) be the right projector of $\mathbb{Q}[G]$ onto $\mathbf{I}_i(\mathbb{Q}[G])$, that is we have

RP: $AP_i = A$ for all $A \in \mathbf{I}_i(\mathbb{Q}[G])$ and $AP_i = 0$ if A is in the direct product of the subrings $\mathbf{I}_j(\mathbb{Q}[G])$ with $\mathbf{I}_i(\mathbb{Q}[G])$ excluded.

Notice that we also have

LP: $P_iA = A$ for all $A \in \mathbf{I}_i(\mathbb{Q}[G])$ and $P_iA = 0$ if A is in the direct product of the subrings $\mathbf{I}_j(\mathbb{Q}[G])$ with $\mathbf{I}_i(\mathbb{Q}[G])$ excluded.

In fact we have $P_i \in \mathbf{I}_i(\mathbb{Q}[G])$ for $i = 1, \dots, \ell$ and P_i can be thought of as the unit element in $\mathbf{I}_i(\mathbb{Q}[G])$.

Given two left $\mathbb{Q}[G]$ -modules M_1, M_2 we define $\text{Hom}_G(M_1, M_2)$ to be the vector space of $\mathbb{Q}[G]$ -module homomorphisms from M_1 to M_2 . The \mathbb{Q} -algebra of $\mathbb{Q}[G]$ -module endomorphisms of M_1 is denoted by $\text{End}_G(M_1)$.

Let us write S^{op} for the *opposite ring* of an associative ring S . Mapping an element $A \in \mathbf{I}_i(\mathbb{Q}[G])$ to the multiplication from the right by A defines \mathbb{Q} -algebra isomorphisms

$$\mathbf{I}_i(\mathbb{Q}[G])^{\text{op}} \rightarrow \text{End}_G(\mathbf{I}_i(\mathbb{Q}[G])) \quad (i = 1, \dots, \ell). \quad (12)$$

The existence of the isomorphisms (12) implies that

$$\dim_{\mathbb{Q}}(\text{Hom}_G(\mathbf{I}_i(\mathbb{Q}[G])^f, \mathbf{I}_i(\mathbb{Q}[G]))) = f \dim_{\mathbb{Q}}(\mathbf{I}_i(\mathbb{Q}[G])) \quad (13)$$

for all $f \in \mathbb{N}$ and $i = 1, \dots, \ell$.

The algebras

$$D_i := \text{End}_G(N_i) \quad (i = 1, \dots, \ell)$$

of left $\mathbb{Q}[G]$ -module endomorphism of the irreducible modules N_i are finite dimensional division algebras with cyclotomic number fields as center. We set

$$h_i := \dim_{D_i}(N_i) \quad (i = 1, \dots, \ell).$$

Then the algebra $\mathbf{I}_i(\mathbb{Q}[G])$ is isomorphic to a $h_i \times h_i$ matrix algebra $M(h_i, D_i)$ over D_i , that is

$$\mathbf{I}_i(\mathbb{Q}[G]) \cong M(h_i, D_i) \quad (i = 1, \dots, \ell). \quad (14)$$

Using (12) we get isomorphisms of algebras

$$\text{End}_G(\mathbf{I}_i(\mathbb{Q}[G])) \cong M(h_i, D_i^{\text{op}}) \quad (i = 1, \dots, \ell). \quad (15)$$

We shall further discuss integral versions of the isomorphisms (11) and (14). Our reference here is [4] Section XI or [5] Section XX.

Let K be a number field with ring of integers \mathcal{O} and S a finite dimensional K -algebra. A subring \mathcal{R} of S is called an *order* if it contains a K -basis of S and if there is a subring \mathcal{O}_0 of finite index in \mathcal{O} such that \mathcal{R} is a finitely generated \mathcal{O}_0 -module (subrings are always assumed to contain the identity element of the bigger ring). Let h be a natural number and $S = M(h, D)$ where D is a finite dimensional division algebra over K with center K . We write $\text{End}_S(S^m)$ ($m \in \mathbb{N}$) for the algebra of S -left module endomorphisms of the S -left module S^m . We obtain an algebra isomorphism

$$\Theta : \text{End}_S(S^m) \rightarrow M(m, S)^{\text{op}} = M(m, S^{\text{op}}) = M(m, M(h, D^{\text{op}})). \quad (16)$$

Here we identify $M(m, S)^{\text{op}}$ in the usual way with $M(m, S^{\text{op}})$ and then with $M(m, M(h, D^{\text{op}}))$. We call a subgroup Λ of the additive group of S^m a *lattice* if Λ contains a K -basis of S^m and if there is a subring \mathcal{O}_0 of finite index in \mathcal{O} such that Λ is a finitely generated \mathcal{O}_0 -module. We write $\text{End}_S(S^m)_\Lambda$ for the ring of those endomorphisms stabilising Λ . The following is well known (see [4], [5]).

Lemma 2.1 *Let $\Lambda \leq S^m$ ($m \in \mathbb{N}$) be a lattice. There is a sublattice $\tilde{\Lambda} \leq \Lambda$ (of finite index in Λ) and an order \mathcal{R} in D such that*

$$\Theta(\text{End}_S(S^m)_{\tilde{\Lambda}}) \subset M(m, M(h, \mathcal{R}^{\text{op}})).$$

Let $\text{Aut}_S(S^m)_\Lambda$ be the group of invertible elements in $\text{End}_S(S^m)_\Lambda$. Note that $\text{Aut}_S(S^m)_{\tilde{\Lambda}}$ is commensurable with $\text{Aut}_S(S^m)_\Lambda$ in particular $\text{Aut}_S(S^m)_\Lambda \cap \text{Aut}_S(S^m)_{\tilde{\Lambda}}$ has finite index in $\text{Aut}_S(S^m)_\Lambda$ and we obtain a representation

$$\Theta : \text{Aut}_S(S^m)_\Lambda \cap \text{Aut}_S(S^m)_{\tilde{\Lambda}} \rightarrow \text{GL}(m, M(h, \mathcal{R}^{\text{op}})) = \text{GL}(mh, \mathcal{R}^{\text{op}}). \quad (17)$$

2.2 A result of Gaschütz

Next we need to report on a result of Gaschütz (see [13], [15] and [19]). For this, let G be a finite group and $\pi : F_n \rightarrow G$ a surjective homomorphism of a rank n free group F_n onto G . Let R be the kernel of π and $\bar{R} := R/R'$ the corresponding relation module with its action (from the left) of G . The result of Gaschütz says that there is a $\mathbb{Q}[G]$ -module isomorphism

$$\kappa : \mathbb{Q} \oplus \mathbb{Q}[G]^{n-1} \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}. \quad (18)$$

where \mathbb{Q} is to be the trivial one dimensional $\mathbb{Q}[G]$ -module. We fix a $\mathbb{Q}[G]$ -module isomorphism κ once for all and identify $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}$ with $\mathbb{Q} \oplus \mathbb{Q}[G]^{n-1}$. In particular we obtain $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}) = 1 + |G|(n-1)$, which is also clear from the formula of Reidemeister and Schreier. We deduce

$$\mathbf{I}_1(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}) = \mathbf{I}_1(\mathbb{Q}[G])^n = \mathbb{Q}^n, \quad (19)$$

$$\mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}) = \mathbf{I}_i(\mathbb{Q}[G])^{n-1} \quad (i = 2, \dots, \ell). \quad (20)$$

From (13) we get

Lemma 2.2 *Let G be a finite group and $\pi : F_n \rightarrow G$ a surjective homomorphism. Let $\bar{R} := R/R'$ be the corresponding relation module. Then we have:*

$$\dim_{\mathbb{Q}}(\text{Hom}_G(\mathbf{I}_1(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}), \mathbf{I}_1(\mathbb{Q}[G]))) = n$$

$$\dim_{\mathbb{Q}}(\text{Hom}_G(\mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}), \mathbf{I}_i(\mathbb{Q}[G]))) = (n-1)\dim_{\mathbb{Q}}(\mathbf{I}_i(\mathbb{Q}[G]))$$

The result of Gaschütz enables us further to decompose the algebraic group $\mathcal{G}_{G,\pi}$ and to describe the components. We define for $i = 1, \dots, \ell$:

$$\mathcal{G}_{G,\pi,i} := \left\{ \phi \in \mathcal{G}_{G,\pi} \mid \phi|_{\mathbb{C} \otimes \mathbf{I}_j(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R})} = \text{Id}_{\mathbb{C} \otimes \mathbf{I}_j(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R})} \text{ for } j \neq i \right\}. \quad (21)$$

Note that the groups $\mathcal{G}_{G,\pi,i}$ centralise each other and have pairwise trivial intersection. It is also easy to see that we have an internal product decomposition:

$$\mathcal{G}_{G,\pi} = \prod_{i=1}^{\ell} \mathcal{G}_{G,\pi,i}$$

of \mathbb{Q} -defined algebraic groups.

We write $\text{GL}(d, S)$ ($d \in \mathbb{N}$) for the group of invertible $d \times d$ -matrices with entries in an associative ring S . Following the discussion of the previous section we infer that the identification made using (18) defines isomorphisms

$$\Sigma_i : \mathcal{G}_{G,\pi,i}(\mathbb{Q}) \rightarrow \text{GL}(n-1, \mathbf{I}_i(\mathbb{Q}[G])^{\text{op}}) = \text{GL}((n-1)h_i, D_i^{\text{op}})$$

for $i = 2, \dots, \ell$ and also an isomorphism

$$\Sigma_1 : \mathcal{G}_{G,\pi,1}(\mathbb{Q}) \rightarrow \text{GL}(n, \mathbf{I}_1(\mathbb{Q}[G])^{\text{op}}) = \text{GL}(n, \mathbb{Q}).$$

Together they define an isomorphism

$$\Sigma : \mathcal{G}_{G,\pi}(\mathbb{Q}) \rightarrow \text{GL}(n, \mathbb{Q}) \times \prod_{i=2}^{\ell} \text{GL}((n-1)h_i, D_i^{\text{op}}).$$

Suppose that i is one of the numbers $1, \dots, \ell$ and let K_i be the center of $\mathbf{I}_i(\mathbb{Q}[G])^{\text{op}}$. As already mentioned, this is a cyclotomic number field. The group $\text{GL}((n-1)h_i, D_i^{\text{op}})$ is the group of K_i -rational points of a K_i -defined linear algebraic group \mathcal{H}_i . The above discussion shows that the

\mathbb{Q} -defined algebraic group $\mathcal{G}_{G,\pi,i}$ is equal to the base field restriction of \mathcal{H}_i from K_i to \mathbb{Q} . For this concept see [28]. We shall add a description of the groups $\mathcal{G}_{G,\pi}^1$ introduced in the introduction. Every element g of $\mathrm{GL}((n-1)h_i, D_i^{\mathrm{op}})$ ($i = 1, \dots, \ell$) acts by multiplication on $D_i^{(n-1)h_i}$. Taking a K_i -vector space identification $D_i = K_i^{d_i}$ we associate to g a square matrix of size $(n-1)h_i d_i$. The determinant of this matrix is called the reduced norm $\mathrm{Nrd}(g)$ of g . The group of elements of reduced norm 1 is conventionally denoted by $\mathrm{SL}((n-1)h_i, D_i^{\mathrm{op}})$. We have

$$\mathcal{G}_{G,\pi,i}^1(\mathbb{Q}) = \{g \in \mathcal{G}_{G,\pi,i}(\mathbb{Q}) \mid \mathrm{Nrd}(\Sigma_i(g)) = 1\}.$$

2.3 Representations of subgroups of finite index in $\mathrm{Aut}(F_n)$

In this section we describe a matrix version of the representation (10). To do this we use Lemma 2.1 which constructs an integral matrix representation of a subgroup of finite index in $\Gamma(G, \pi)$ which is compatible with the representations Σ_i ($i = 1, \dots, \ell$) and Σ from the previous section.

Let G be a finite group and $\pi : F_n \rightarrow G$ a surjective homomorphism of the free group F_n onto G . Let R be the kernel of π and \bar{R} the corresponding relation module. Let further $\Gamma(G, \pi)$ be the subgroup of $\mathrm{Aut}(F_n)$ defined in (3). In the sequel we shall often use the following fact without further notice.

Lemma 2.3 *An element $\varphi \in \Gamma(R)$ is in $\Gamma(G, \pi)$ if and only if the map $\bar{\varphi} : \bar{R} \rightarrow \bar{R}$ induced by φ is G -equivariant.*

Proof: If φ is in $\Gamma(G, \pi)$ then for every $w \in F_n$ we have a $r_w \in R$ with $\varphi(w) = wr_w$. Hence $\varphi(wrw^{-1}) = wr_w\varphi(r)r_w^{-1}w^{-1}$ holds for every $r \in R$. The latter is equal to $w\varphi(r)w^{-1}$ modulo R' which implies that $\bar{\varphi} : \bar{R} \rightarrow \bar{R}$ is G -equivariant.

If $\bar{\varphi} : \bar{R} \rightarrow \bar{R}$ is G -equivariant then for every $w \in F_n$ the congruence $\varphi(w)\varphi(r)\varphi(w)^{-1} \equiv w\varphi(r)w^{-1}$ holds modulo R' for every $r \in R$. This means that w and $\varphi(w)$ act the same on \bar{R} . The result of Gaschütz implies that the action of G on \bar{R} is faithful, that is no element except the identity of G acts as the identity. We conclude that $\varphi(w)w^{-1}$ is in R . \square

Keeping the notations set up in subsections 2.1, 2.2 we define

$$\Lambda_i := \bar{R} \cap \mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}) \quad (i = 1, \dots, \ell).$$

Each Λ_i is a $\mathbb{Z}[G]$ -submodule of $\bar{R} \cap \mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R})$ and $\prod_{i=1}^{\ell} \Lambda_i \leq \bar{R}$ is a finite index $\mathbb{Z}[G]$ -submodule. Notice that the Λ_i is a sublattice of $\mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}) = M(h_i, D_i)^{n-1}$ ($i = 2, \dots, \ell$) in the sense of the previous section, so is Λ_1 in $\mathbf{I}_1(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}) = \mathbb{Q}^n$. Applying Lemma 2.1 we choose orders $\mathcal{R}_i \subset D_i$ and finite index additive subgroups $\tilde{\Lambda}_i \leq \Lambda_i$ ($i = 1, \dots, \ell$). We define

$$\hat{\Gamma}(G, \pi) \leq \Gamma(G, \pi) \tag{22}$$

to be the simultaneous stabiliser of all the $\tilde{\Lambda}_i \leq \Lambda_i$ ($i = 1, \dots, \ell$). Notice further that $\hat{\Gamma}(G, \pi)$ is of finite index in $\Gamma(G, \pi)$. Using the various representations Θ from Lemma 2.1 we obtain representations

$$\sigma_{G,\pi,1} : \hat{\Gamma}(G, \pi) \rightarrow \mathrm{GL}(n, \mathbb{Z}), \quad \sigma_{G,\pi,i} : \hat{\Gamma}(G, \pi) \rightarrow \mathrm{GL}((n-1)h_i, \mathcal{R}_i^{\mathrm{op}})$$

($i = 2, \dots, \ell$), which come by projection from the analogously defined representation

$$\sigma_{G,\pi} : \hat{\Gamma}(G, \pi) \rightarrow \mathrm{GL}(n, \mathbb{Z}) \times \prod_{i=2}^{\ell} \mathrm{GL}(n-1, M(h_i, \mathcal{R}_i^{\mathrm{OP}})).$$

Notice that the diagram

$$\begin{array}{ccc} \hat{\Gamma}(G, \pi) & \xrightarrow{\quad} & \mathcal{G}_{G,\pi}(\mathbb{Q}) \\ \downarrow \sigma_{G,\pi} & & \swarrow \Sigma \\ \mathrm{GL}(n, \mathbb{Q}) \times \prod_{i=2}^{\ell} \mathrm{GL}(n-1, M(h_i, D_i^{\mathrm{OP}})) & & \end{array}$$

is commutative.

3 Derivations and the relation module

This section contains a brief discussion of the correspondences between Fox-derivatives, the relation module of a finite group and cohomology. We apply these results to obtain a certain canonical system of right module generators of $\mathrm{Hom}_G(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}, \mathbb{Q}[G])$ where G is a finite group and \bar{R} is the relation module coming from a presentation $\pi : F_n \rightarrow G$.

3.1 Fox calculus and cohomology

For $n \in \mathbb{N}$, let F_n be the free group with basis x_1, \dots, x_n . We write $\mathbb{Z}[F_n]$, $\mathbb{Q}[F_n]$ for the group rings of F_n over the integers or the rational numbers, respectively. Let

$$\frac{\partial}{\partial x_i} : F_n \rightarrow \mathbb{Z}[F_n] \quad (i = 1, \dots, n)$$

be the Fox-derivatives (see [3] for a more elaborate exposition). They are defined by the rules

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij}, \quad \frac{\partial w_1 w_2}{\partial x_i} = \frac{\partial w_1}{\partial x_i} + w_1 \frac{\partial w_2}{\partial x_i} \quad (23)$$

for $i, j = 1, \dots, n$ and $w_1, w_2 \in F_n$. As usual, δ_{ij} denoting the Kronecker symbol ($\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$). For later use we note some consequences of (23). For $i = 1, \dots, n$ and $e \in \mathbb{N}$ we have

$$\frac{\partial x_i^e}{\partial x_i} = \sum_{k=0}^{e-1} x_i^k, \quad \frac{\partial x_i^{-e}}{\partial x_i} = - \sum_{k=1}^e x_i^{-k}. \quad (24)$$

Let further $w := v_1 x_i^{e_1} v_2 x_i^{e_2} \dots v_k x_i^{e_k} v_{k+1}$ ($e_1, \dots, e_k \in \mathbb{Z}$) be an element of F_n with the subwords v_j ($j = 1, \dots, k+1$) not involving x_i , then

$$\frac{\partial w}{\partial x_i} = v_1 \frac{\partial x_i^{e_1}}{\partial x_i} + v_1 x_i^{e_1} v_2 \frac{\partial x_i^{e_2}}{\partial x_i} + \dots + v_1 x_i^{e_1} \cdot v_2 x_i^{e_2} \cdot \dots \cdot v_k \frac{\partial x_i^{e_k}}{\partial x_i}. \quad (25)$$

Let G be a finite group generated by its elements g_1, \dots, g_n ($n \in \mathbb{N}$) and let

$$\pi : F_n \rightarrow G \quad \pi(x_1) = g_1, \dots, \pi(x_n) = g_n$$

be the corresponding surjective group homomorphism. The homomorphism $\pi : F_n \rightarrow G$ extends by linearity to surjective ring homomorphisms:

$$\pi : \mathbb{Z}[F_n] \rightarrow \mathbb{Z}[G] \quad \pi : \mathbb{Q}[F_n] \rightarrow \mathbb{Q}[G].$$

Let $R \leq F_n$ be the kernel of π and set $\bar{R} := R/R'$ to be the commutator factor group of R . Let $\bar{\cdot} : R \rightarrow R/R'$ stand for the quotient homomorphism. The free group F_n acts by conjugation on R . Since its subgroup R acts trivially on \bar{R} we get an induced action of G on \bar{R} which we denote by $g \cdot u$ ($g \in G, u \in \bar{R}$). We generally write the composition in \bar{R} additively. The free abelian group \bar{R} then obtains the structure of a left $\mathbb{Z}[G]$ -module which extends to a left $\mathbb{Q}[G]$ -module structure on $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}$. We have

Lemma 3.1 *Let G be a group and $\pi : F_n \rightarrow G$ a homomorphism. Let R be the kernel of π . Then*

$$\pi \left(\frac{\partial r}{\partial x_i} \right) = 0$$

for all $r \in R'$ and $i = 1, \dots, n$.

Proof: Note first that (23) implies $\frac{\partial w^{-1}}{\partial x_i} = -w^{-1} \frac{\partial w}{\partial x_i}$ for all $w \in F_n$ and $i = 1, \dots, n$. It is enough to show the result for single commutators $r = r_1 r_2 r_1^{-1} r_2^{-1}$. A repeated application of (23) implies that

$$\frac{\partial r}{\partial x_i} = \frac{\partial r_1}{\partial x_i} + r_1 \frac{\partial r_2}{\partial x_i} - r_1 r_2 r_1^{-1} \frac{\partial r_1}{\partial x_i} - r_1 r_2 r_1^{-1} r_2^{-1} \frac{\partial r_2}{\partial x_i}$$

for all $i = 1, \dots, n$. As $\pi(r_1) = \pi(r_2) = 1$ the result follows. \square

The Fox-derivatives $\frac{\partial}{\partial x_i}$ now induce maps

$$\partial_i : \bar{R} \rightarrow \mathbb{Z}[G] \quad (i = 1, \dots, n)$$

defined as follows. For every $u \in \bar{R}$ choose $w_u \in R$ with $\bar{w}_u = u$ and define

$$\partial_i(u) := \pi \left(\frac{\partial w_u}{\partial x_i} \right) \quad (i = 1, \dots, n). \quad (26)$$

We have

Lemma 3.2 *The maps $\partial_1, \dots, \partial_n$ do not depend on the choices of the preimages w_u ($u \in \bar{R}$). They are $\mathbb{Z}[G]$ -module homomorphisms when $\mathbb{Z}[G]$ acts on itself from the left.*

Proof: The first statement follows from Lemma 3.1. For the second note that

$$\frac{\partial g r g^{-1}}{\partial x_i} = \frac{\partial g}{\partial x_i} + g \frac{\partial r}{\partial x_i} - g r g^{-1} \frac{\partial g}{\partial x_i}$$

for all $g, r \in F_n$ and all $i = 1, \dots, n$. When r is in R , $\pi(r) = 1$ and the projection of the right hand side into $\mathbb{Z}[G]$ is equal to $\pi(g)\pi(\frac{\partial r}{\partial x_i})$. \square

We also write $\partial_i : \mathbb{Q} \otimes_{\mathbb{Z}} \bar{R} \rightarrow \mathbb{Q}[G]$ for the induced $\mathbb{Q}[G]$ -module homomorphisms.

Let t be the \mathbb{Z} -rank of the free abelian group \bar{R} . Note that $t = 1 + |G|(n - 1)$ and also $t = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R})$. Let $B := (u_1, \dots, u_t)$ be a \mathbb{Q} -basis of $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}$. We consider here

$$J_B := \begin{pmatrix} \partial_1(u_1) & \dots & \partial_n(u_1) \\ \vdots & & \vdots \\ \partial_1(u_t) & \dots & \partial_n(u_t) \end{pmatrix}, \quad (27)$$

which is a $t \times n$ matrix with entries in $\mathbb{Q}[G]$.

Given a left $\mathbb{Q}[G]$ -module M and $s \in \mathbb{N}$ we write

$$M^s := \left\{ \left(\begin{array}{c} m_1 \\ \vdots \\ m_s \end{array} \right) \mid m_1, \dots, m_s \in M \right\}$$

for the corresponding $\mathbb{Q}[G]$ -module of column vectors. The matrix J_B induces a \mathbb{Q} -linear map $J_B : M^t \rightarrow M^n$. Notice for later use that given a $\mathbb{Q}[G]$ -submodule $N \leq M$ we have $J_B(N^t) \subseteq N^n$.

Given the left $\mathbb{Q}[G]$ -module M we write

$$\text{Der}(G, M) := \{ d : G \rightarrow M \mid d(gh) = gd(h) + d(g) \text{ for all } g, h \in G \}$$

for the vector space of derivations from G to M . Using the generators g_1, \dots, g_n of our finite group G we define for $d \in \text{Der}(G, M)$

$$L(d) := \begin{pmatrix} d(g_1) \\ \vdots \\ d(g_n) \end{pmatrix} \in M^n.$$

We have

Lemma 3.3 *The map $L : \text{Der}(G, M) \rightarrow M^n$ is injective and its image is equal to $\text{Ker}(J_B)$. Hence L defines a \mathbb{Q} -vector space isomorphism between $\text{Der}(G, M)$ and $\text{Ker}(J_B)$.*

Proof: Let d be a derivation in the kernel of L . We infer that $d(g_1) = \dots = d(g_n) = 0$. The derivation rule and the fact that the g_1, \dots, g_n generate G implies that d is identically zero. This shows that L is injective.

Let $D : F_n \rightarrow M$ be a derivation where F_n acts on M through the homomorphism $\pi : F_n \rightarrow G$. Since both sides are derivations and agree on the x_i 's we find

$$D(w) = \frac{\partial w}{\partial x_1} D(x_1) + \dots + \frac{\partial w}{\partial x_n} D(x_n) \quad (28)$$

for all $w \in F_n$.

Given a derivation $d : G \rightarrow M$, define an induced derivation $D : F_n \rightarrow M$ by setting $D(w) := d(\pi(w))$. Applying (28) we find for every $r \in R$

$$0 = d(1) = D(r) = \sum_{i=1}^n \frac{\partial r}{\partial x_i} D(x_i) = \sum_{i=1}^n \partial_i(r) d(g_i).$$

Applying this to $r = u_1, \dots, u_t$ shows that the image of L is contained in the kernel of J_B .

Let now

$$m := \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \in M^n$$

be an n -tuple of elements of M . It is well known that there is a derivation $D : F_n \rightarrow M$ with $D(x_1) = m_1, \dots, D(x_n) = m_n$. If m is an element of the kernel of J_B then (28) implies that $D(r) = 0$ for all $r \in R$. The derivation D then induces a derivation $d : G \rightarrow M$ with the property $d(g_1) = m_1, \dots, d(g_n) = m_n$. This shows that every element of the kernel of J_B is in the image of L . \square

Corollary 3.4 *Let M be a $\mathbb{Q}[G]$ -module then the kernel of the map $J_B : M^n \rightarrow M^t$ is zero if M is a trivial module and has \mathbb{Q} -vector space dimension equal to $\dim_{\mathbb{Q}}(M)$ if M does not have a trivial submodule.*

Proof: Given an element $m \in M$ we let $d_m : G \rightarrow M$ be the corresponding inner derivation $d_m(g) = (g - 1)m$. Let

$$\text{IDer}(G, M) := \{ d_m \mid m \in M \} \subseteq \text{Der}(G, M)$$

be the space of inner derivations. It is well known that the first cohomology group $H^1(G, M) = \text{Der}(G, M)/\text{IDer}(G, M)$ is zero (G is a finite group). Note also that the map $\mu : M \rightarrow \text{IDer}(G, M)$, defined by $\mu(m) := d_m$ ($m \in M$) is surjective.

If M is a trivial module then μ is identically zero, that implies $\text{IDer}(G, M) = \{0\}$ and hence also $\text{Der}(G, M) = \{0\}$. We find from Lemma 3.3 that $\dim_{\mathbb{Q}}(\ker(J_B)) = 0$.

If M does not contain the trivial module then μ is injective, that implies $\dim_{\mathbb{Q}}(\text{IDer}(G, M)) = \dim_{\mathbb{Q}}(M)$. We find from Lemma 3.3 that $\dim_{\mathbb{Q}}(\ker(J_B)) = \dim_{\mathbb{Q}}(\text{Der}(G, M)) = \dim_{\mathbb{Q}}(M)$. \square

The results of this section will be applied in the next section in case of the relation modules.

3.2 $\text{Hom}_G(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}, \mathbb{Q}[G])$ as right $\mathbb{Q}[G]$ -module

We keep here our notation from before, that is G is a finite group generated by its elements g_1, \dots, g_n ($n \in \mathbb{N}$) and $\pi : F_n \rightarrow G$ is the homomorphism defined by $\pi(x_1) = g_1, \dots, \pi(x_n) = g_n$. Furthermore R is the kernel of π and \bar{R} is the corresponding relation module. As the main result we shall show that the $\partial_1, \dots, \partial_n \in \text{Hom}_G(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}, \mathbb{Q}[G])$, which were constructed in the previous subsection generate $\text{Hom}_G(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}, \mathbb{Q}[G])$ if this space is considered as a $\mathbb{Q}[G]$ -module from the right in the following way.

Let M be a left $\mathbb{Q}[G]$ -module. For $f \in \text{Hom}_G(M, \mathbb{Q}[G])$ and $A \in \mathbb{Q}[G]$ we define $f^A \in \text{Hom}_G(M, \mathbb{Q}[G])$ by

$$f^A(x) := f(x)A \quad (x \in M). \quad (29)$$

Thus the \mathbb{Q} -vector space $\text{Hom}_G(M, \mathbb{Q}[G])$ becomes a right $\mathbb{Q}[G]$ -module.

We shall prove here:

Proposition 3.5 *The maps $\partial_1, \dots, \partial_n$ (defined in (26)) generate $\text{Hom}_G(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}, \mathbb{Q}[G])$ as right $\mathbb{Q}[G]$ -module.*

To prepare for the proof we decompose the matrix J_B into submatrices. To do this we choose the basis B adapted to the decomposition

$$\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R} = \prod_{i=1}^{\ell} \mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}).$$

That is we choose bases $B_i = (u_{i1}, \dots, u_{it_i})$ ($t_i := \dim_{\mathbb{Q}}(\mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}))$) for $i = 1, \dots, \ell$ and put them together to obtain a \mathbb{Q} -vector space basis

$$B = (u_1 = u_{11}, \dots, u_t = u_{\ell t_{\ell}})$$

of $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}$.

Consider the matrices

$$J_B^{P_i} := \begin{pmatrix} \partial_1(u_1)P_i & \dots & \partial_n(u_1)P_i \\ \vdots & & \vdots \\ \partial_1(u_t)P_i & \dots & \partial_n(u_t)P_i \end{pmatrix} = \begin{pmatrix} \partial_1^{P_i}(u_1) & \dots & \partial_n^{P_i}(u_1) \\ \vdots & & \vdots \\ \partial_1^{P_i}(u_t) & \dots & \partial_n^{P_i}(u_t) \end{pmatrix} \quad (30)$$

for $i = 1, \dots, \ell$. Notice that according to property **LP** of the right projector P_i (see section 2.1) the maps

$$J_B, J_B^{P_i} : \mathbf{I}_i(\mathbb{Q}[G])^n \rightarrow \mathbf{I}_i(\mathbb{Q}[G])^t$$

are the same. Furthermore we have $\partial_j(u) \in \mathbf{I}_i(\mathbb{Q}[G])$ ($j = 1, \dots, n$) for $u \in \mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R})$. This follows since each ∂_j is a $\mathbb{Q}[G]$ -module homomorphism which has to respect isotypic components. Thus property **RP** of the right projector P_i implies that all entries of $J_B^{P_i}$ are equal to zero except those in the submatrix

$$J_{B_i} := \begin{pmatrix} \partial_1(u_{i1})P_i & \dots & \partial_n(u_{i1})P_i \\ \vdots & & \vdots \\ \partial_1(u_{it_i})P_i & \dots & \partial_n(u_{it_i})P_i \end{pmatrix} \quad (31)$$

hence the kernels of the two maps $J_B : \mathbf{I}_i(\mathbb{Q}[G])^n \rightarrow \mathbf{I}_i(\mathbb{Q}[G])^t$ and $J_{B_i} : \mathbf{I}_i(\mathbb{Q}[G])^n \rightarrow \mathbf{I}_i(\mathbb{Q}[G])^{t_i}$ are the same. Note finally that we have

$$J_{B_i} = \begin{pmatrix} \partial_1^{P_i}(u_{i1}) & \dots & \partial_n^{P_i}(u_{i1}) \\ \vdots & & \vdots \\ \partial_1^{P_i}(u_{it_i}) & \dots & \partial_n^{P_i}(u_{it_i}) \end{pmatrix}$$

for all $i = 1, \dots, \ell$.

Proof of Proposition 3.5: We have the decomposition

$$\mathrm{Hom}_G(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}, \mathbb{Q}[G]) = \prod_{i=1}^{\ell} \mathrm{Hom}_G(\mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}), \mathbf{I}_i(\mathbb{Q}[G]))$$

of $\mathrm{Hom}_G(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}, \mathbb{Q}[G])$ as a right $\mathbb{Q}[G]$ -module. Since the $\partial_1^{P_i}, \dots, \partial_n^{P_i}$ are zero on $\mathbf{I}_j(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R})$ for $j \neq i$ it is enough to show that they generate $\mathrm{Hom}_G(\mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}), \mathbf{I}_i(\mathbb{Q}[G]))$ as a right $\mathbb{Q}[G]$ -module. Consider the \mathbb{Q} -linear map $\Phi_i : \mathbf{I}_i(\mathbb{Q}[G])^n \rightarrow \mathrm{Hom}_G(\mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}), \mathbf{I}_i(\mathbb{Q}[G]))$ given by

$$\Phi_i \left(\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \right) := \partial_1^{P_i m_1} + \dots + \partial_n^{P_i m_n}.$$

Notice that $\mathbf{I}_i(\mathbb{Q}[G]) \cdot \mathbb{Q}[G] = \mathbf{I}_i(\mathbb{Q}[G])$ and hence the image of Φ_i is the right $\mathbb{Q}[G]$ -submodule of $\mathrm{Hom}_G(\mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}), \mathbf{I}_i(\mathbb{Q}[G]))$ generated by the $\partial_1^{P_i}, \dots, \partial_n^{P_i}$. The key observation now is

$$\mathrm{kernel}(J_{B_i}) = \mathrm{kernel}(\Phi_i) \tag{32}$$

which is obvious by what is explained above. Notice that in fact $(m_1, \dots, m_n)^t$ is in the kernel of Φ_i if and only if $\partial_1^{P_i m_1}(u_{ik}) + \dots + \partial_n^{P_i m_n}(u_{ik}) = 0$ holds for all $k = 1, \dots, t_i$.

We can treat now the case $i = 1$ (the case of the trivial module). Here we have by (19) that $\dim_{\mathbb{Q}}(\mathbf{I}_1(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R})) = n$ and the kernel of Φ_1 is trivial (by Corollary 3.4). Since we have $\dim_{\mathbb{Q}}(\mathrm{Hom}_G(\mathbf{I}_1(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}), \mathbf{I}_1(\mathbb{Q}[G]))) = n$ (by Lemma 2.2) the map Φ_1 has to be surjective.

In the cases $i > 1$ we have $\dim_{\mathbb{Q}}(\mathrm{kernel}(\Phi_i)) = \dim_{\mathbb{Q}}(\mathbf{I}_i(\mathbb{Q}[G]))$ by (32) and Corollary 3.4. By Lemma 2.2 and (13) we have $\dim_{\mathbb{Q}}(\mathrm{Hom}_G(\mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}), \mathbf{I}_i(\mathbb{Q}[G]))) = (n-1)\dim_{\mathbb{Q}}(\mathbf{I}_i(\mathbb{Q}[G]))$ which implies that Φ_i has to be surjective. \square

As a corollary to Proposition 3.5 we note

Corollary 3.6 *The maps $\partial_1^{P_i}, \dots, \partial_n^{P_i}$ generate $\mathrm{Hom}_G(\mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}), \mathbf{I}_i(\mathbb{Q}[G]))$ as right $\mathbf{I}_i(\mathbb{Q}[G])$ -module for every $i = 1, \dots, \ell$.*

We shall give now a reformulation of the last corollary which will be needed later. Let $\mathrm{Id}_{i,k} \in \mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R})$ ($k = 1, \dots, n-1$) be the images under κ_i (see (20), (19)) of the unit elements of the components of $\mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R})^{n-1}$ respectively of $\mathbf{I}_1(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R})^n$ with $k = 1, \dots, n$. Using the identification (15) we think of $f \in \mathrm{Hom}_G(\mathbf{I}_i(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}), \mathbf{I}_i(\mathbb{Q}[G]))$ being identified with the collection of matrices $f(\mathrm{Id}_{i,1}) = A_{i,1}, \dots, f(\mathrm{Id}_{i,n-1}) = A_{i,n-1} \in M(h_i, D_i^{\mathrm{op}})$ or $f(\mathrm{Id}_{1,1}) = A_{1,1}, \dots, f(\mathrm{Id}_{1,n}) = A_{1,n} \in M(h_1, D_1^{\mathrm{op}}) = \mathbb{Q}$.

Corollary 3.7 *Let i be one of the numbers $2, \dots, \ell$. Given $A_1, \dots, A_{n-1} \in M(h_i, D_i^{\mathrm{op}})$ there are $B_1, \dots, B_n \in \mathbb{Q}[G]$ such that*

$$\partial_1^{P_i B_1} + \dots + \partial_n^{P_i B_n}(\mathrm{Id}_{i,k}) = A_k \quad (k = 1, \dots, n-1).$$

Given $A_1, \dots, A_n \in M(h_1, D_1^{\mathrm{op}})$ there are $B_1, \dots, B_n \in \mathbb{Q}[G]$ such that

$$\partial_1^{P_1 B_1} + \dots + \partial_n^{P_1 B_n}(\mathrm{Id}_{1,k}) = A_k \quad (k = 1, \dots, n).$$

Remark: We sketch here another (more conceptual) proof of Proposition 3.5. Features of the proof given above will play a role in the sequel.

Using $H^2(G, \mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}) = 0$ we obtain an injective homomorphism

$$\epsilon : F_n/R' \rightarrow (\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}) \rtimes G$$

from F_n/R' to the semidirect product on the right hand side. On $R/R' \leq F_n/R'$ the homomorphism ϵ induces the inclusion of R/R' into $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R} \leq (\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}) \rtimes G$. The second component of ϵ coincides with π . Associate to $f \in \text{Hom}_G(\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}, \mathbb{Q}[G])$ the homomorphism $\tau_f : (\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}) \rtimes G \rightarrow \mathbb{Q}[G] \rtimes G$ ($\tau_f((u, g)) = (f(u), g)$). Composition with ϵ creates a homomorphism

$$\lambda_f : F_n \rightarrow F_n/R' \rightarrow \mathbb{Q}[G] \rtimes G, \quad \lambda_f(w) = (d_f(w), \pi(w)), \quad (w \in F_n)$$

The map $d_f : F_n \rightarrow \mathbb{Q}[G]$ is a derivation. Hence we have for $w \in F_n$

$$d_f(w) = \frac{\partial w}{\partial x_1} d_f(x_1) + \dots + \frac{\partial w}{\partial x_n} d_f(x_n).$$

Restricting this formula to R/R' we have expressed f in the desired way.

4 Redundant presentations

In this section, we present a special decomposition of the relation module \bar{R} for redundant presentations of finite groups. We then describe the action of various Nielsen automorphisms of F_n on \bar{R} with respect to this decomposition.

Let F_n ($n \in \mathbb{N}, n \geq 2$) be the free group with basis x_1, \dots, x_{n-1}, y . Let G be a finite group generated by its elements g_1, \dots, g_{n-1} and let

$$\pi : F_n \rightarrow G \quad \pi(x_1) = g_1, \dots, \pi(x_{n-1}) = g_{n-1}, \pi(y) = 1 \quad (33)$$

be the surjective group homomorphism corresponding to these data. We keep the notation introduced above of objects related to the presentation π of G . That is R is the kernel of π and $\bar{R} = R/R'$ is the relation module, etc.. We shall construct certain Nielsen-like elements in $\hat{\Gamma}(G, \pi)$ (see 22) which will allow us to find many unipotent elements in the image of the representation

$$\sigma_{G, \pi} : \hat{\Gamma}(G, \pi) \rightarrow \text{GL}(n, \mathbb{Z}) \times \prod_{i=2}^{\ell} \text{GL}(n-1, M(h_i, \mathcal{R}_i^{\text{op}})) \quad (34)$$

defined in Section 2.3.

4.1 Decomposition of the relation module

With the notation fixed as indicated above we define

$$R_{\text{old}} := \{ x \in \langle x_1, \dots, x_{n-1} \rangle \mid \pi(x) = 1 \} \text{ and } \bar{R}_{\text{old}} := R_{\text{old}}/R'_{\text{old}}.$$

The injection $R_{\text{old}} \rightarrow R$ defines a $\mathbb{Z}[G]$ -module homomorphism $\bar{R}_{\text{old}} \rightarrow \bar{R}$. Since $y \in R$, we can consider the $\mathbb{Z}[G]$ -submodule of \bar{R} generated by $\bar{y} \in \bar{R}$.

Lemma 4.1 *The homomorphism $\bar{R}_{\text{old}} \rightarrow \bar{R}$ is injective and so is the homomorphism $\mathbb{Z}[G] \rightarrow \bar{R}$ defined by $A \mapsto A \cdot \bar{y}$. Identifying \bar{R}_{old} with its image in \bar{R} we obtain a direct sum decomposition*

$$\bar{R} = \bar{R}_{\text{old}} \oplus \mathbb{Z}[G] \cdot \bar{y}.$$

as a $\mathbb{Z}[G]$ -module.

Proof: Let us temporarily write \hat{R}_{old} for the image of R_{old} in \bar{R} . We first show that $\bar{R} = \hat{R}_{\text{old}} + \mathbb{Z}[G] \cdot \bar{y}$. Given any $w \in F_n$, by looking at the image of w modulo the normal closure of y in F_n , we see that w can be written in a unique way as $w = w_1 w_2$ with $w_1, w_2 \in F_n$ such that w_1 does not involve the letter y whereas w_2 is a product of conjugates of powers of y . If w is an element of R then w_1 has to be in R_{old} . This implies $\bar{R} = \hat{R}_{\text{old}} + \mathbb{Z}[G] \cdot \bar{y}$.

We infer that $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R} = \mathbb{Q} \otimes_{\mathbb{Z}} \hat{R}_{\text{old}} + \mathbb{Q}[G] \cdot \bar{y}$. We now count dimensions using the result of Gaschütz. As \bar{R} and \bar{R}_{old} are free over \mathbb{Z} , the lemma follows. \square

4.2 Nielsen Automorphisms

We study here the effect of certain Nielsen automorphisms on the relation module \bar{R} . We define elements $\alpha_i \in \text{Aut}(F_n)$ ($i = 1, \dots, n-1$) by

$$\alpha_i(x_i) := yx_i. \tag{35}$$

Our convention here is that values not given are identical to the argument, that is we assume $\alpha_i(x_j) := x_j$, for $j \neq i$ and $\alpha_i(y) = y$.

For $X \in \langle x_1, \dots, x_{n-1} \rangle$ we define elements $\beta_X \in \text{Aut}(F_n)$ by

$$\beta_X(y) := XyX^{-1} \tag{36}$$

and finally for $U \in R_{\text{old}}$ we define $\gamma_U \in \text{Aut}(F_n)$ by

$$\gamma_U(y) := Uy. \tag{37}$$

The $\alpha_i, \beta_X, \gamma_U$ extend to automorphisms of F_n . We have

Lemma 4.2 *The α_i ($i = 1, \dots, n-1$), β_X ($X \in \langle x_1, \dots, x_{n-1} \rangle$) and γ_U ($U \in R_{\text{old}}$) are in $\Gamma(G, \pi)$ and the following formulas hold for the corresponding automorphisms induced on \bar{R} :*

$$\bar{\alpha}_i(u) = u + \partial_i(u) \cdot \bar{y} \quad \text{for } u \in \bar{R}_{\text{old}} \text{ and} \quad \bar{\alpha}_i(\bar{y}) = \bar{y}. \tag{38}$$

$$\bar{\beta}_X(u) = u, \quad \text{for } u \in \bar{R}_{\text{old}} \text{ and} \quad \bar{\beta}_X(\bar{y}) = \pi(X) \cdot \bar{y}, \tag{39}$$

$$\bar{\gamma}_U(u) = u, \quad \text{for } u \in \bar{R}_{\text{old}} \text{ and} \quad \bar{\gamma}_U(\bar{y}) = \bar{U} + \bar{y}, \tag{40}$$

Proof: It is first of all clear from our definitions that all the $\alpha_i, \beta_X, \gamma_U$ are elements of $\Gamma(R) := \{\varphi \in \text{Aut}(F_n) \mid \varphi(R) = R\}$. Moreover all of them induce the identity on F_n/R and hence are even in $\Gamma(G, \pi)$. It follows (see Section 2.3) that they act as $\mathbb{Z}[G]$ -automorphisms on \bar{R} .

Next we prove formula (38). Fix $i \in \{1, \dots, n-1\}$ and define for $e \in \mathbb{N}$

$$W_e := y \cdot x_i y x_i^{-1} \cdot \dots \cdot x_i^{e-1} y x_i^{1-e}, \quad W_{-e} := x_i^{-1} y^{-1} x_i \cdot \dots \cdot x_i^{-e} y^{-1} x_i^e \in F_n$$

and add $W_0 := 1$. We then have $\alpha_i(x_i^e) = W_e \cdot x_i^e$ for all $e \in \mathbb{Z}$. For $u \in \bar{R}_{\text{old}}$ choose $w \in R_{\text{old}}$ with $\bar{w} = u$. We decompose $w \in \langle x_1, \dots, x_n \rangle$ as $w := v_1 x_i^{e_1} v_2 x_i^{e_2} \cdot \dots \cdot v_k x_i^{e_k} v_{k+1}$ ($e_1, \dots, e_k \in \mathbb{Z}$) such that the v_j do not involve x_i . Define further

$$V_1 := v_1, \quad V_2 := v_1 x_i^{e_1} v_2, \dots, \quad V_k := v_1 x_i^{e_1} v_2 \dots x_i^{e_k} v_k.$$

We then have

$$\alpha_i(w) = V_1 W_{e_1} V_1^{-1} \cdot V_2 W_{e_2} V_2^{-1} \cdot \dots \cdot V_k W_{e_k} V_k^{-1} \cdot w.$$

Switching to the additive notation and using formulas (24) we find

$$\bar{\alpha}_i(u) = u + \pi \left(v_1 \frac{\partial x_i^{e_1}}{\partial x_i} + v_1 x_i^{e_1} v_2 \frac{\partial x_i^{e_2}}{\partial x_i} + \dots + v_1 x_i^{e_1} v_2 x_i^{e_2} \cdot \dots \cdot v_k \frac{\partial x_i^{e_k}}{\partial x_i} \right) \cdot \bar{y}$$

and formula (38) follows from (25) and (26).

To prove formula (39) notice that β_X is the identity on \bar{R}_{old} , subsequently (39) is clear.

All statements made about the automorphisms γ_U ($U \in R_{\text{old}}$) are evident. Notice that the map $\bar{\gamma}_U$ depends only on the image \bar{U} of U in \bar{R}_{old} . \square

For later use we introduce now a notation for some elements in $\Gamma(G, \pi)$ obtained by composing the α_i and β_X . For every $g \in G$ we choose $X_g \in \langle x_1, \dots, x_{n-1} \rangle$ with $\pi(X_g) = g$ and define $\beta_g := \beta_{X_g}$. We infer from formulas (38) and (39)

$$\bar{\beta}_g \bar{\alpha}_i^m \bar{\beta}_{g^{-1}}(u + A \cdot \bar{y}) = u + (\partial_i(u) m g + A) \cdot \bar{y} \quad (u \in \bar{R}_{\text{old}}, A \in \mathbb{Z}[G])$$

for all $g \in G, m \in \mathbb{Z}$. Given an element $B = m_1 g_1 + \dots + m_k g_k \in \mathbb{Z}[G]$ ($m_1, \dots, m_k \in \mathbb{Z}, g_1, \dots, g_k \in G$) we define $\eta_{B,i} \in \text{Aut}(F_n)$ by

$$\eta_{B,i} := \beta_{g_1} \alpha_i^{m_1} \beta_{g_1^{-1}} \circ \dots \circ \beta_{g_k} \alpha_i^{m_k} \beta_{g_k^{-1}}. \quad (41)$$

for $i = 1, \dots, n-1$. By Lemma 4.2 it follows that $\eta_{B,i} \in \Gamma(G, \pi)$ for $i = 1, \dots, n$ and $B \in \mathbb{Z}[G]$. Using the notation of Section 3.2 we have

$$\bar{\eta}_{B,i}(u + A \cdot \bar{y}) = u + (\partial_i^B(u) + A) \cdot \bar{y} \quad (u \in \bar{R}_{\text{old}}, A \in \mathbb{Z}[G])$$

for $i = 1, \dots, n-1$ and $B \in \mathbb{Z}[G]$. This formula makes it clear that

$$\bar{\eta}_{m_1 B_1 + m_2 B_2, i} = \bar{\eta}_{B_1, i}^{m_1} \circ \bar{\eta}_{B_2, i}^{m_2}$$

holds for all $i = 1, \dots, n-1$ and $B_1, B_2 \in \mathbb{Z}[G]$ and $m_1, m_2 \in \mathbb{Z}$.

Using the representation (1) we set

$$\text{Aut}^+(F_n) := \{ \phi \in \text{Aut}(F) \mid \det(\rho_1(\phi)) = 1 \}.$$

The subgroup $\text{Aut}^+(F_n)$ is of index 2 in $\text{Aut}(F_n)$. For $n \geq 3$ it is a perfect group, that is $\text{Aut}^+(F_n)$ is equal to its own commutator subgroup. We note that all elements $\bar{\eta}_{B,i}$ are in $\text{Aut}^+(F_n)$.

5 Arithmeticity in case of a redundant presentation

In this section we prove Theorem 1.4: The decomposition of \bar{R} to its isotypic components shows that our main goal is to prove that the image of $\hat{\Gamma}(G, \pi)$ in (34) contains a finite index subgroup of each component

$$\mathrm{SL}(n-1, M(h_i, \mathcal{R}_i^{\mathrm{op}})) = \mathrm{SL}((n-1)h_i, \mathcal{R}_i^{\mathrm{op}}).$$

To prove this we will show that this image contains sufficiently many unipotent elements with nonzero entries in the bottom strip of h_i rows and also in the most right hand strip of h_i columns of this $\mathrm{SL}((n-1)h_i, \mathcal{R}_i^{\mathrm{op}})$. Then we can appeal to a theorem of Vaserstein which ensures that the elementary matrices with entries from a non-zero two-sided ideal of \mathcal{R}_i generate a finite index subgroup of the arithmetic group $\mathrm{SL}((n-1)h_i, \mathcal{R}_i^{\mathrm{op}})$.

5.1 Elementary matrices

This section sets up certain notations and facts about elementary matrices.

Let S be an associative ring with unity. We denote by $\mathrm{GL}(n, S)$ ($n \in \mathbb{N}$) the group of invertible $n \times n$ matrices with entries in S . We write I_n for the identity matrix. Given $1 \leq i, j \leq n$ with $i \neq j$ and $s \in S$ we define $E_{ij}(s) \in \mathrm{GL}(n, S)$ to be the corresponding elementary matrix, that is $E_{ij}(s)$ is equal to I_n except for the (i, j) position, where we put s . For $1 \leq i, j \leq n$ with $i \neq j$, $1 \leq k, l \leq n$ with $k \neq l$ and $s_1, s_2 \in S$ the Steinberg relations between the elementary matrices are

$$[E_{ij}(s_1), E_{kl}(s_2)] := \begin{cases} 1 & \text{for } j \neq k, \quad i \neq l, \\ E_{il}(s_1 s_2) & \text{for } j = k, \quad i \neq l, \\ E_{kj}(-s_2 s_1) & \text{for } j \neq k, \quad i = l, \end{cases} \quad (42)$$

where $[E_{ij}(s_1), E_{kl}(s_2)] = E_{ij}(s_1)E_{kl}(s_2)E_{ij}(s_1)^{-1}E_{kl}(s_2)^{-1}$. For every admissible pair i, j we have $E_{ij}(s_1)E_{ij}(s_2) = E_{ij}(s_1 + s_2)$ for all $s_1, s_2 \in S$.

Let $\mathfrak{a} \subset S$ be an ideal of S . Given natural numbers $m_1, m_2 \in \mathbb{N}$ we denote by $M(m_1, m_2; \mathfrak{a})$ the space of $m_1 \times m_2$ -matrices with entries in \mathfrak{a} . Suppose now that $m_1 + m_2 = n$. For $A \in M(m_1, m_2; \mathfrak{a})$ we define

$$H(A) := \begin{pmatrix} I_{m_2} & 0 \\ A & I_{m_1} \end{pmatrix}, \quad V(A) := \begin{pmatrix} I_{m_2} & A^t \\ 0 & I_{m_1} \end{pmatrix}$$

Here A^t stands for the transpose of the matrix A . Suppose that $A := (a_{ij})$ where $i = 1, \dots, m_1$, $j = 1, \dots, m_2$ with $a_{ij} \in \mathfrak{a}$, then we have

$$H(A) = \prod_{i=1}^{m_1} \prod_{j=1}^{m_2} E_{m_2+i, j}(a_{ij}), \quad V(A) = \prod_{i=1}^{m_1} \prod_{j=1}^{m_2} E_{j, m_2+i}(a_{ij}).$$

Given $m_1, m_2 \in \mathbb{N}$ with $m_1 + m_2 = n$ and an ideal $\mathfrak{a} \leq S$ we define

$$H(m_1, m_2; \mathfrak{a}) := \{ H(A) \mid A \in M(m_1, m_2; \mathfrak{a}) \},$$

$$V(m_1, m_2; \mathfrak{a}) := \{ V(A) \mid A \in M(m_1, m_2; \mathfrak{a}) \}.$$

Both $H(m_1, m_2; \mathfrak{a})$ and $V(m_1, m_2; \mathfrak{a})$ are abelian subgroups of $\mathrm{GL}(n, S)$.

We apply this notation in case of the orders $M(h_i, \mathcal{R}_i)$ of the simple factor rings $\mathbf{I}_i(\mathbb{Q}[G])$ of the rational group ring $\mathbb{Q}[G]$. So let K be a number field and D a finite dimensional division algebra over K with center K . Let \mathcal{R} be an order in D . It is well known that every two-sided ideal $\mathfrak{a} \leq \mathcal{R}$ is additively finitely generated and has finite index in \mathcal{R} if it is non-zero. We have

Proposition 5.1 *Let $n \in \mathbb{N}$ with $n \geq 3$ and let $m_1, m_2 \in \mathbb{N}$ satisfy $m_1 + m_2 = n$. Let \mathfrak{a} be a non-zero two-sided ideal of \mathcal{R} . Then $H(m_1, m_2; \mathfrak{a})$ and $V(m_1, m_2; \mathfrak{a})$ generate a subgroup of finite index in $\mathrm{SL}(n, \mathcal{R})$.*

Proof: Let U be the subgroup generated by $H(m_1, m_2; \mathfrak{a})$ and $V(m_1, m_2; \mathfrak{a})$. Using the Steinberg relations (42) we conclude that there is a non-zero two-sided ideal $\mathfrak{b} (= \mathfrak{a}^2)$ in \mathcal{R} such that every $E_{ij}(b)$ ($b \in \mathfrak{b}, i \neq j$) is in U . The main result of [33] implies that U has finite index in $\mathrm{SL}(n, \mathcal{R})$. \square

5.2 Constructing elements of $\mathcal{G}_{G, \pi}^1(\mathbb{Z})$

Let F_n ($n \in \mathbb{N}, n \geq 3$) be the free group with basis x_1, \dots, x_{n-1}, y , G a finite group with generators g_1, \dots, g_{n-1} and let

$$\pi : F_n \rightarrow G \quad \pi(x_1) = g_1, \dots, \pi(x_{n-1}) = g_{n-1}, \pi(y) = 1$$

be the redundant presentation resulting from the data. Notice the assumption $n \geq 3$, if $n = 2$ then G is cyclic. We include this case into the next section.

Given $i = 2, \dots, \ell$ and an ideal \mathfrak{a} of the order $\mathcal{R}_i^{\mathrm{op}}$, let us define

$$H_i(\mathfrak{a}) := H_i(h_i, (n-2)h_i; \mathfrak{a}) \leq \mathrm{GL}(n, \mathbb{Z}) \times \prod_{i=2}^{\ell} \mathrm{GL}(n-1, M(h_i, \mathcal{R}_i^{\mathrm{op}})) \quad (43)$$

to be $H(h_i, (n-2)h_i; \mathfrak{a}) \leq \mathrm{GL}((n-1)h_i, \mathcal{R}_i^{\mathrm{op}}) = \mathrm{GL}(n-1, M(h_i, \mathcal{R}_i^{\mathrm{op}}))$ but considered as a subgroup of the big direct product on the right hand side of (43) (with all components of the factors different from the i -th factor equal to the identity matrix). In case $i = 1$ we have $\mathcal{O}_1 = \mathbb{Z}$ and we define $H_1(\mathfrak{a}) = H_1(1, n-1; \mathfrak{a})$ with $n-1$ replaced by n . Let us also define

$$V_i(\mathfrak{a}) := V_i(h_i, (n-2)h_i; \mathfrak{a}) \leq \mathrm{GL}(n, \mathbb{Z}) \times \prod_{i=2}^{\ell} \mathrm{GL}(n-1, M(h_i, \mathcal{R}_i^{\mathrm{op}})) \quad (44)$$

in a similar way. We have

Lemma 5.2 *Let i be one of the $1, \dots, \ell$ and $g \in H_i(\mathcal{R}_i^{\mathrm{op}})$. There is an $e \in \mathbb{N}$ such that g^e is in $\sigma_{G, \pi}(\hat{\Gamma}(G, \pi))$ where $\sigma_{G, \pi}$ is as in (34).*

Proof: We prove the statement for $i \geq 2$, for $i = 1$ the proof is only different in notation.

Let $i \geq 2$ be fixed and consider $g \in H_i(\mathcal{R}_i^{\mathrm{op}})$. There are matrices $A_1, \dots, A_{n-2} \in M(h_i, \mathcal{R}_i^{\mathrm{op}})$ such that the i -th component of g_i of g is equal to

$$g_i = \begin{pmatrix} I_{h_i} & 0 & 0 & \dots & 0 & 0 \\ 0 & I_{h_i} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I_{h_i} & 0 \\ A_1 & A_2 & A_3 & \dots & A_{n-2} & I_{h_i} \end{pmatrix},$$

all other components being equal to the identity matrix in the corresponding component group. By Corollary 3.7 there are $B_1, \dots, B_{n-1} \in \mathbb{Q}[G]$ such that

$$\partial_1^{P_i B_1} + \dots + \partial_{n-1}^{P_i B_{n-1}} (\text{Id}_{i,k}) = A_k$$

for $k = 1, \dots, n-2$. Choose $e \in \mathbb{N}$ to have sufficiently many divisors so that $eP_i B_1, \dots, eP_i B_{n-2}$ are in $\mathbb{Z}[G]$. Put

$$\gamma := \eta_{eP_i B_{1,1}} \circ \dots \circ \eta_{eP_i B_{n-1,n-1}}$$

with the η defined in (41). Increase now (if necessary) e to ensure $\gamma \in \hat{\Gamma}(G, \pi)$ (see (22)). We have $\sigma_{G,\pi}(\gamma) = g^e$ and the lemma is proved. \square

Lemma 5.3 *Let i be one of the $1, \dots, \ell$ and $g \in V_i(\mathcal{R}_i^{\text{op}})$. There is an $e \in \mathbb{N}$ such that g^e is in $\sigma_{G,\pi}(\hat{\Gamma}(G, \pi))$.*

Proof: We prove the statement for $i \geq 2$, again for $i = 1$ the proof is only different in notation. Let $i \geq 2$ be fixed and consider $g \in V_i(\mathcal{R}_i^{\text{op}})$. There are matrices $A_1, \dots, A_{n-2} \in M(h_i, \mathcal{R}_i^{\text{op}})$ such that the i -th component of g_i of g is equal to

$$g_i = \begin{pmatrix} I_{h_i} & 0 & 0 & \dots & 0 & A_1 \\ 0 & I_{h_i} & 0 & \dots & 0 & A_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I_{h_i} & A_{n-2} \\ 0 & 0 & 0 & \dots & 0 & I_{h_i} \end{pmatrix},$$

The matrices A_1, \dots, A_{n-2} can be considered as elements of $M(h_i, \mathcal{R}_i)$, after all $M(h_i, \mathcal{R}_i^{\text{op}})$ and $M(h_i, \mathcal{R}_i)$ are the same sets. Using the convention of Section 2.3 we consider $u := (A_1, \dots, A_{n-2})$ as an element of $M(h_i, D_i)^{n-2} \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}_{\text{old}}$ (see (20)). Then there is $e \in \mathbb{N}$ such that $eu \in \bar{R}_{\text{old}}$. We choose $U \in R_{\text{old}}$ with $\bar{U} = eu$. We take $e \in \mathbb{N}$ to have sufficiently many divisors so that $\bar{\gamma}_U$ (see (37)) is in $\hat{\Gamma}(G, \pi)$. We have $\bar{\gamma}_U = g^e$ and the lemma is proved. \square

Since every $H_i(\mathcal{R}_i^{\text{op}})$ or $V_i(\mathcal{R}_i^{\text{op}})$ ($i = 1, \dots, \ell$) is a finitely generated abelian group the two previous lemmas immediately imply

Proposition 5.4 *There is $e \in \mathbb{N}$ such that*

$$\prod_{i=1}^{\ell} H_i(e\mathcal{R}_i) \leq \sigma_{G,\pi}(\hat{\Gamma}(G, \pi)) \quad \text{and} \quad \prod_{i=1}^{\ell} V_i(e\mathcal{R}_i) \leq \sigma_{G,\pi}(\hat{\Gamma}(G, \pi)). \quad (45)$$

Finally we can state

Theorem 5.5 *Let $n \in \mathbb{N}$ satisfy $n \geq 4$, then*

$$\sigma_{G,\pi}(\hat{\Gamma}(G, \pi)) \cap \text{SL}(n, \mathbb{Z}) \times \prod_{i=2}^{\ell} \text{SL}((n-1)h_i, \mathcal{R}_i^{\text{op}})$$

is of finite index in $\text{SL}(n, \mathbb{Z}) \times \prod_{i=2}^{\ell} \text{SL}((n-1)h_i, \mathcal{R}_i^{\text{op}})$.

Proof: Just use Proposition 5.4 together with Proposition 5.1. \square

6 Cyclic groups

In this section we present a detailed study of the case when G is a cyclic group. We restrict ourselves to a particular type of (redundant) presentation, but following the discussion in Section 9.1 this is in fact the general case. An important feature is that we are able to present a set of generators of $\Gamma(G, \pi)$, in the case $n \geq 3$. For a general group G we are not able to do this. As a consequence we can determine the precise image of $\Gamma(G, \pi)$ (not only up to commensurability) acting on its invariant pieces of the relation module. Some of the results here will be needed in Section 7, but we believe that this section is of independent interest. In fact, our method can be observed here in a very simple, almost elementary situation.

Let us call the attention of the reader to the fact that our results concerning cyclic groups, in fact, cover the general case of G being abelian. This follows since every irreducible representation of an abelian group factors through a cyclic quotient, see the discussion in Section 8.2.

Let the free group F_n ($n \geq 2$) be generated by x, y_1, \dots, y_{n-1} . Let us introduce the following elements of $\text{Aut}(F_n)$. Our convention here is again that values not given are identical to the argument.

- $\delta_i(x) := y_i x$ and $\epsilon_i(x) := x y_i$ for $i = 1, \dots, n-1$,
- $\varphi_i(y_i) := x y_i$ and $\psi_i(y_i) := y_i x$ for $i = 1, \dots, n-1$,
- $\lambda_{ij}(y_i) := y_j y_i$ and $\nu_{ij}(y_i) := y_i y_j$ for $i, j = 1, \dots, n-1$ with $i \neq j$.

A theorem of Nielsen (see [23]) asserts that these elements generate $\text{Aut}^+(F_n)$ where $\text{Aut}^+(F_n)$ is the kernel of the homomorphism $\rho_1 : \text{Aut}(F_n) \rightarrow \text{GL}(n, \mathbb{Z})$ followed by the determinant. If $\Gamma \leq \text{Aut}(F_n)$ is a subgroup, we define $\Gamma^+ := \Gamma \cap \text{Aut}^+(F_n)$. Let us further introduce

- $\kappa_{jk}(x) := x[y_j, y_k]$, $\kappa_{ijk}(y_i) := y_i[y_j, y_k]$ for $1 \leq i, j, k \leq n-1$ with $i \neq j, i \neq k$,
- $\tau_{ij}(y_i) := y_i[x, y_j]$ for $1 \leq i, j \leq n-1$ with $i \neq j$.

The set T_n consisting of the κ_{jk} , κ_{ijk} , τ_{ij} (with indices as above), the $\delta_i^{-1} \circ \epsilon_i$, $\varphi_i^{-1} \circ \psi_i$ for $i = 1, \dots, n-1$ and the $\lambda_{ij} \circ \nu_{ij}^{-1}$ for $1 \leq i, j \leq n-1$ with $i \neq j$ generates the group $\text{IA}(F_n)$ by another theorem of Nielsen (see [23]). Notice that T_n is contained in $\Gamma(G, \pi)$ for every presentation π of an abelian group G .

Let us also introduce the following notation concerning matrix groups. We write

$$\Gamma^1(n, m) \leq \text{SL}(n, \mathbb{Z})$$

for the subgroup of $\text{SL}(n, \mathbb{Z})$ consisting of those elements having a first row which is congruent to $(1, 0, \dots, 0)$ modulo $m \in \mathbb{N}$. We need

Lemma 6.1 *Let n and m be natural numbers with $n \geq 3$. The group $\Gamma^1(n, m)$ is generated by the elementary matrices*

$$E_{ij}(1) \quad (i, j = 1, \dots, n, i \neq j, i \neq 1), \quad E_{1j}(m) \quad (j = 2, \dots, n).$$

This lemma is an elementary exercise using the euclidean algorithm in \mathbb{Z} .

We first treat the cyclic group of order 2 which we call C_2 . We have singled out this case since it is completely elementary and also plays a special role in the proof of Corollary 1.3. Let g be a generator of C_2 . We choose the presentation

$$\pi : F_n \rightarrow C_2, \quad \pi(x) = g, \quad \pi(y_1) = \dots = \pi(y_{n-1}) = 1.$$

The corresponding relation module \bar{R} has the following elements as a \mathbb{Z} -basis.

$$\overline{x^2}, \overline{y_1}, \dots, \overline{y_{n-1}}, \overline{xy_1x^{-1}} = g\overline{y_1}, \dots, \overline{xy_{n-1}x^{-1}} = g\overline{y_{n-1}}.$$

The \mathbb{Q} -vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}$ decomposes as $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R} = V_1 \oplus V_{-1}$ where V_1, V_{-1} are the ± 1 eigenspaces of g respectively. Set $\bar{R}_1 := \bar{R} \cap V_1$ and $\bar{R}_{-1} := \bar{R} \cap V_{-1}$. Notice that $\bar{R}_1 + \bar{R}_{-1}$ is of finite index in \bar{R} . Introduce

$$v_i := \overline{y_i + xy_ix^{-1}}, \quad w_i := \overline{y_i - xy_ix^{-1}} \quad (i = 1, \dots, n-1). \quad (46)$$

Then $\overline{x^2}, v_1, \dots, v_{n-1}$ is a \mathbb{Z} -basis of \bar{R}_1 and w_1, \dots, w_{n-1} is a \mathbb{Z} -basis of \bar{R}_{-1} . Since $\Gamma(C_2, \pi)$ leaves \bar{R}_1 and \bar{R}_{-1} invariant we obtain, with the above \mathbb{Z} -bases being chosen, representations

$$\sigma_1 : \Gamma^+(C_2, \pi) \rightarrow \text{GL}(n, \mathbb{Z}), \quad \sigma_{-1} : \Gamma^+(C_2, \pi) \rightarrow \text{GL}(n-1, \mathbb{Z}). \quad (47)$$

With all these data given we have:

Proposition 6.2 *Let $n \geq 2$ be a natural number. The following hold.*

- (i) *The group $\Gamma^+(C_2, \pi)$ is generated by the automorphisms δ_i, ϵ_i ($i = 1, \dots, n-1$), the λ_{ij} and ν_{ij} ($i, j = 1, \dots, n-1, i \neq j$), the φ_i^2 ($i = 1, \dots, n-1$) and the elements of the set T_n introduced above.*
- (ii) *The image of σ_1 is equal to $\Gamma^1(n, 2)$.*
- (iii) *The index of $\Gamma^+(C_2, \pi)$ in $\text{Aut}^+(F_n)$ is $2^n - 1$.*
- (iv) *The representation σ_{-1} is surjective onto $\text{GL}(n-1, \mathbb{Z})$.*
- (v) *The group $\sigma_{-1}(\text{IA}(F_3)) \leq \text{GL}(2, \mathbb{Z})$ is generated by the matrices*

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

The proof of this proposition is elementary but repeated below in the more general case, so we skip it here. Items (ii), (iv) and (v) follow from formulas describing the action of the given generators of $\Gamma^+(C_2, \pi)$ on the bases in \bar{R}_1 and \bar{R}_{-1} . An important ingredient is Lemma 6.1. Note that $\Gamma^1(2, 2)$ is generated by the elementary matrices

$$E_{21}(1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad E_{12}(2) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

For $\Gamma^1(2, m)$ ($m \geq 3$) the analogous statement is not true. Proposition 6.2 already implies Corollary 1.3.

We turn now to the case of a general cyclic group $C_m = \langle g \rangle$ ($m \in \mathbb{N}$). We choose the presentation

$$\pi : F_n \rightarrow C_m, \quad \pi(x) = g, \quad \pi(y_1) = \dots = \pi(y_{n-1}) = 1.$$

The corresponding relation module \bar{R} has the following $1 + m(n - 1)$ elements as a \mathbb{Z} -basis

$$\overline{x^m}, \overline{x^k y_1 x^{-k}}, \dots, \overline{x^k y_{n-1} x^{-k}} \quad (k = 0, \dots, m - 1).$$

The rational group ring of C_m decomposes as

$$\mathbb{Q}[C_m] = \prod_{d|m} \mathbb{Q}(\zeta_d). \quad (48)$$

Here $\zeta_d \in \mathbb{C}$ is a primitive d -th root of unity. We consider each of the $\mathbb{Q}(\zeta_d)$ as a C_m -module letting g act by multiplication with ζ_d . The ring of integers $\mathbb{Z}(\zeta_d)$ is then a $\mathbb{Z}[C_m]$ -submodule of $\mathbb{Q}(\zeta_d)$. We mimic the decomposition (48) inside the relation module by introducing

$$v_i(d, m) := \sum_{k=0}^{m-1} \text{Tr}(\zeta_d^k) \overline{x^k y_i x^{-k}} \quad (49)$$

for $i = 1, \dots, n - 1$. Here $\text{Tr}(\zeta_d^k)$ is the trace of ζ_d^k taken from $\mathbb{Q}(\zeta_d)$ to \mathbb{Q} . Notice that the $v_i(d, m)$ are generalisations of the generators in (46). Let \bar{R}_d be the C_m -submodule of \bar{R} generated by the $v_i(d, m)$ for $i = 1, \dots, n - 1$ if $d > 1$ and define \bar{R}_1 to be the C_m -submodule of \bar{R} generated by the $v_i(1, m)$ for $i = 1, \dots, n - 1$ together with $\overline{x^m}$. By setting

$$\Lambda_d(v_i(d, m)) := \left(0, \dots, 0, \sum_{k=0}^{m-1} \text{Tr}(\zeta_d^k) \zeta_d^k, 0, \dots, 0 \right) \quad (i = 1, \dots, n - 1)$$

(with the non-zero entry in the i -th component) for $d > 1$ and extending C_m -linearly we obtain well defined additive homomorphisms

$$\Lambda_d : \bar{R}_d \rightarrow \mathbb{Z}(\zeta_d)^{n-1} \quad (d|m, d > 1).$$

The following are true

- the homomorphism Λ_d is C_m -equivariant and injective (onto an ideal in $\mathbb{Z}(\zeta_d)$),
- the \mathbb{Z} -rank of \bar{R}_d is $(n - 1)\varphi(d)$ for $d > 1$ and n for $d = 1$,
- $\bar{R}_d \cap \bar{R}_{d'} = \langle 0 \rangle$ for distinct divisors d, d' of m ,
- the (direct) sum of all \bar{R}_d ($d|m$) has finite index in \bar{R} ,
- each \bar{R}_d ($d|m$) is left invariant by $\Gamma(C_m, \pi)$.

As above we transport the action of $\Gamma(C_m, \pi)$ on \bar{R}_d to the image of λ_d and obtain our representations

$$\sigma_d : \Gamma(G, \pi) \rightarrow \text{GL}(n - 1, \mathbb{Q}(\zeta_d)) \quad (d|m, d > 1).$$

To control the image of σ_d we first show:

Proposition 6.3 *Let n and m be natural numbers with $n \geq 3$. The group $\Gamma^+(C_m, \pi)$ is generated by the automorphisms δ_i, ϵ_i ($i = 1, \dots, n - 1$), the λ_{ij} and ν_{ij} ($i, j = 1, \dots, n - 1, i \neq j$) and the φ_i^m , ($i = 1, \dots, n - 1$) and the elements of the set T_n introduced above.*

Proof: Since every element of $\Gamma(C_m, \pi)$ has to leave the normal closure of x^m in F_n invariant and has to fix x modulo R we see that $\rho_1(\Gamma^+(C_m, \pi))$ has to be contained in $\Gamma^1(n, m)$ (for ρ_1 see (1)). Examining the action of elements in the statement of the proposition on F_n/F'_n and using Lemma 6.1 we find that $\rho_1(\Gamma^+(C_m, \pi)) = \Gamma^1(n, m)$.

Now our set of generators contains enough elements to generate $\text{IA}(F_n)$ by the theorem of Nielsen and also enough elements to generate the image of $\Gamma^+(G, \pi)$ in $\text{SL}(n, \mathbb{Z})$ by Lemma 6.1. The proposition follows. \square

Finally we can state:

Proposition 6.4 *Let n and m be natural numbers with $n \geq 2$. The image of the representation σ_d ($d|m$, $d > 1$) is $\text{GL}^+(n-1, \mathbb{Z}(\zeta_d))$ where $\text{GL}^+(n-1, \mathbb{Z}(\zeta_d))$ is the subgroup consisting of those elements in $\text{GL}(n-1, \mathbb{Z}(\zeta_d))$ which have a power of ζ_d as determinant.*

Proof: Assume first that $n \geq 3$. Evaluating σ_d on the generators from Proposition 6.3 we see that $\sigma_d(\Gamma^+(C_m, \pi))$ is contained in $\text{GL}^+(n-1, \mathbb{Z}(\zeta_d))$. We now show that all elementary matrices $E_{ij}(\zeta_d^k)$ ($i, j = 1, \dots, n-1$, $k \in \mathbb{Z}$) are in the image of σ_d . This is done by evaluating σ_d on some of the generators from Proposition 6.3. We report only a special case: Take τ_{ij} for a pair (i, j) with $i \neq j$. We have $\tau_{ij}(y_i) = y_i[x, y_j]$. This leads to

$$\bar{\tau}_{ij}(v_i(d, m)) = v_i(d, m) + gv_j(d, m) - v_j(d, m)$$

which in turn shows that $E_{ij}(\zeta_d - 1)$ is in the image of σ_d . Since $E_{ij}(1)$ is also present (use λ_{ij}) we have shown that $E_{ij}(\zeta_d)$ is contained in the image of σ_d .

After more elementary considerations like that we can show that all the elementary matrices with entries in $\mathbb{Z}(\zeta_d)$ are in the image of σ_d . These matrices generate $\text{SL}(n-1, \mathbb{Z}(\zeta_d))$: For $n \geq 4$ this follows from results in [33], for $n = 3$ and $d \geq 5$ we apply the main result of [34], in case $n = 3$ and $d \leq 4$ the ring $\mathbb{Z}(\zeta_d)$ is euclidean, in case $n = 2$ nothing has to be proved.

As a last step we use the $\varphi_i^{-1} \circ \psi_i$ ($i = 1, \dots, n-1$) to show that the image of σ_d contains a matrix with determinant equal to ζ_d . This finishes the proof for $n \geq 3$.

Let us finally discuss the case $n = 2$ which seems trivial, but in fact is not. Note that $\text{GL}(1, \mathbb{Z}(\zeta_d))$ contains elements of infinite order whenever $d \geq 5$, $d \neq 6$ holds. Since we do not have generators for $\Gamma^+(2, m)$ ($m \geq 3$) at hand we cannot directly rule out the possibility that such an element lies in the image of σ_d . We argue as follows. Let $\varphi \in \Gamma^+(2, m)$ be such that for some divisor d of m the image $\sigma_d(\varphi)$ has a determinant of infinite order. We extend φ to an element of ψ of $\Gamma^+(3, m)$ by setting $\psi(x) := \varphi(x)$, $\psi(y_1) := \varphi(y_1)$, $\psi(y_2) := y_2$. Since the determinants of $\sigma_d(\varphi)$ and $\sigma_d(\psi)$ coincide we have finished also the case $n = 2$. \square

Proposition 6.4 allows us to conclude the following result concerning the abelianisations of the images of $\Gamma(C_m, \pi)$ under the representations σ_d ($d|m$, $d > 1$).

Corollary 6.5 *Let n and m be natural numbers with $n \geq 2$. Then $\sigma_d(\Gamma(C_m, \pi))$ ($d|m$, $d > 1$) has finite abelianisation.*

Proof: It is well known that $\text{GL}^+(n-1, \mathbb{Z}(\zeta_d))$ has finite abelianisation for all $d \in \mathbb{N}$: For $n = 2$ this fact is obvious, for $n = 3$ and $d = 2, 3, 4, 6$ a presentation of $\text{SL}(2, \mathbb{Z}(\zeta_d))$ (see [8]) can be used, for $d = 5$ or $d \geq 7$ this fact is contained in [31]. For other values of n , d the group $\text{GL}^+(n-1, \mathbb{Z}(\zeta_d))$ has

Kazhdan's property (T). Since $\sigma_d(\Gamma(C_m, \pi))$ contains $\sigma_d(\Gamma^+(C_m, \pi))$ as a subgroup of finite index the result follows. \square

We finally remark that

$$\sigma_d(\text{IA}(F_n)) \leq \text{GL}^+(n-1, \mathbb{Z}(\zeta_d)) \quad (d|m, d > 1)$$

is equal to the full congruence subgroup of $\text{GL}^+(n-1, \mathbb{Z}(\zeta_d))$ with respect to the ideal of $\mathbb{Z}(\zeta_d)$ generated by $\zeta_d - 1$. This can be shown by a more detailed analysis of the arguments involved in the proof of Proposition 6.4.

7 Completion of proofs

In this section we complete the proofs of all the results promised in the introduction.

Theorem 5.5 proves Theorem 1.4. Both theorems were formulated only for $n \geq 4$ but our method applies also to the case $n = 3$. The only obstacle for $n = 3$ is that congruence elementary matrices in $\text{SL}(2) = \text{SL}(n-1)$ do not always generate a finite index subgroup, for example the matrices

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

do not generate a subgroup of finite index in $\text{SL}(2, \mathbb{Z})$. By the result of Vaserstein [34] this is a real obstacle (in the case of commutative rings of endomorphisms) only for $\text{SL}(2, \mathbb{Z})$ and $\text{SL}(2)$ of a ring of integers in an imaginary quadratic number field. Thus our results still work for $n = 3$ in many cases. For example if the group algebra $\mathbb{Q}[G]$ of the finite group G has (except for the trivial module) no irreducible module N with $\dim_D(N) = 1$ ($D = \text{End}_G(N)$). If there are such modules N but if in each case D is a number field which is not \mathbb{Q} or imaginary quadratic we are also fine. The other cases remain open.

Proof of Theorem 1.1: $n = 2$: In this case we just consider the (surjective) representation

$$\rho_1 : \text{Aut}(F_2) \rightarrow \text{Aut}(F_2/F_2') \cong \text{GL}(2, \mathbb{Z}).$$

It is well known that $\text{GL}(2, \mathbb{Z})$ contains a free subgroup of any rank (≥ 2) of finite index. Hence any finitely generated group is the image of a subgroup of finite index in $\text{Aut}(F_2)$. Note that the groups appearing in Theorem 1.1 as image groups are finitely generated since they are arithmetic groups.

$n = 3$: Here we use the representation $\sigma_{-1} : \Gamma^+(C_2, \pi) \rightarrow \text{GL}(2, \mathbb{Z})$ from Section 6 to find for every $r \in \mathbb{N}$ with $r \geq 2$ a subgroup of finite index in $\text{Aut}(F_3)$ which can be mapped onto the free group of rank r . We then argue as in the case $n = 2$.

$n \geq 4$: We first consider the case $k = 1$. For $h, m \in \mathbb{N}$ we have to find a subgroup $\Gamma \leq \text{Aut}(F_n)$ of finite index and a representation

$$\rho : \Gamma \rightarrow \text{SL}((n-1)h, \mathbb{Q}(\zeta_m))^m$$

such that $\rho(\Gamma)$ is commensurable with $\text{SL}((n-1)h, \mathbb{Z}(\zeta_m))^m$. To do this we take the group

$$G := S_{h+1} \times C_m \times C_m.$$

As an input for Theorem 5.5 we need suitable $\mathbb{Q}[G]$ -modules. We let PM_{h+1} be the standard $h+1$ dimensional $\mathbb{Q}[S_{h+1}]$ -permutation module and let M_h be the kernel of the augmentation map from PM_{h+1} to \mathbb{Q} . The $\mathbb{Q}[S_{h+1}]$ -module M_h is irreducible and has \mathbb{Q} -dimension h . We put $N := \mathbb{Q}(\zeta_m) \otimes_{\mathbb{Q}} M_h$ and consider N as a \mathbb{Q} -vector space. Here, as before, $\zeta_m \in \mathbb{C}$ is a primitive m -th root of unity. Let $\epsilon : C_m \times C_m \rightarrow \langle \zeta_m \rangle$ be a surjective homomorphism. We turn the \mathbb{Q} -vector space $N = \mathbb{Q}(\zeta_m) \otimes_{\mathbb{Q}} M_h$ into a $\mathbb{Q}[G]$ -module N_ϵ by letting S_{h+1} act on M_h and letting $g \in C_m \times C_m$ act by multiplication by $\epsilon(g)$ on $\mathbb{Q}(\zeta_m)$. The following are clear for every surjective homomorphism $\epsilon : C_m \times C_m \rightarrow \langle \zeta_m \rangle$.

- $\dim_{\mathbb{Q}}(N_\epsilon) = \varphi(m)h$,
- $\text{End}_G(N_\epsilon) = \mathbb{Q}(\zeta_m)$,
- $\dim_{\mathbb{Q}(\zeta_m)}(N_\epsilon) = h$,
- if $\epsilon_1, \epsilon_2 : C_m \times C_m \rightarrow \langle \zeta_m \rangle$ are two surjective homomorphisms with distinct kernels then N_{ϵ_1} and N_{ϵ_2} are not isomorphic as $\mathbb{Q}[G]$ -modules.

Our group G can be generated by 3 elements: take $g_1, g_2, g_3 \in S_{h+1}$ such that g_3 and the commutator of g_1, g_2 generate S_{h+1} and consider the elements $(g_1, 1, 0), (g_2, 0, 1), (g_3, 0, 0)$ in G . Let $\pi : F_n \rightarrow G$ be a redundant presentation of G (remember the assumption is $n \geq 4$). Since the number of surjective homomorphisms $\epsilon : C_m \times C_m \rightarrow \langle \zeta_m \rangle$ with pairwise distinct kernels is bigger or equal to m we may apply Theorem 5.5 to obtain a subgroup $\Gamma \leq \text{Aut}(F_n)$ and a representation

$$\sigma : \Gamma \rightarrow \text{GL}((n-1)h, \mathbb{Q}(\zeta_m))^m$$

such that $\sigma(\Gamma) \cap \text{SL}((n-1)h, \mathbb{Z}(\zeta_m))^m$ is of finite index in $\text{SL}((n-1)h, \mathbb{Z}(\zeta_m))^m$. Here we do not know whether $\sigma(\Gamma) \cap \text{SL}((n-1)h, \mathbb{Z}(\zeta_m))^m$ is of finite index in $\sigma(\Gamma)$. However, there is a subgroup $\Delta \leq \text{GL}((n-1)h, \mathbb{Z}(\zeta_m))^m$ of finite index such that $\text{SL}((n-1)h, \mathbb{Z}(\zeta_m))^m \leq \Delta$ and such that $\Delta \leq Z \cdot \text{SL}((n-1)h, \mathbb{Z}(\zeta_m))^m$ with a subgroup $Z \leq \text{GL}((n-1)h, \mathbb{Z}(\zeta_m))^m$ which is central and satisfies $Z \cap \text{SL}((n-1)h, \mathbb{Z}(\zeta_m))^m = \langle 1 \rangle$. Let $\Gamma_0 \leq \Gamma$ be the inverse image of Δ under σ . The representation

$$\sigma_0 : \Gamma_0 \rightarrow \Delta/Z = \text{SL}((n-1)h, \mathbb{Z}(\zeta_m))^m$$

satisfies the requirements for $k=1$. Another way of looking at the last construction is to project a finite index subgroup of the image of $\sigma(\Gamma)$ into $\text{PGL}(n-1, \mathbb{Z}(\zeta_m))$ which contains $\text{PSL}(n-1, \mathbb{Z}(\zeta_m))$ as a finite index subgroup. We then intersect this subgroup in $\text{PGL}(n-1, \mathbb{Z}(\zeta_m))$ with $\text{PSL}(n-1, \mathbb{Z}(\zeta_m))$ and pull the resulting subgroup of $\text{PSL}(n-1, \mathbb{Z}(\zeta_m))$ back to obtain a finite index subgroup of Γ which is mapped into $\text{SL}(n-1, \mathbb{Z}(\zeta_m))$.

We turn now to the general case. Let $\Gamma_1, \dots, \Gamma_k \leq \text{Aut}(F_n)$ be the subgroups of finite index and

$$\sigma_i : \Gamma_i \rightarrow \text{SL}((n-1)h_i, \mathbb{Z}(\zeta_{m_i}))^{m_i} \quad (i = 1, \dots, k)$$

be the representations constructed above. Set $\Gamma := \Gamma_1 \cap \dots \cap \Gamma_k$ and consider the representation

$$\sigma : \Gamma \rightarrow \prod_{i=1}^k \text{SL}((n-1)h_i, \mathbb{Z}(\zeta_{m_i}))^{m_i}$$

which maps an element $\gamma \in \Gamma$ to the tuple $(\sigma_1(\gamma), \dots, \sigma_k(\gamma))$. The image of Γ projects onto an arithmetic subgroup in each of the factors of the direct product. We finish the proof by applying the following lemma.

Lemma 7.1 *Let $\mathcal{H}_1, \dots, \mathcal{H}_k$ be pairwise non-isogeneous \mathbb{Q} -defined simple linear algebraic groups which all have real rank greater or equal to 2. Let*

$$\Gamma \leq \prod_{i=1}^k \mathcal{H}_i^{m_i}(\mathbb{Q})$$

be a subgroup such that all projections into the factors $\mathcal{H}_i^{m_i}(\mathbb{Q})$ are arithmetic groups. Then Γ is an arithmetic subgroup of $\prod_{i=1}^k \mathcal{H}_i^{m_i}$, that is Γ is commensurable with $\prod_{i=1}^k \mathcal{H}_i^{m_i}(\mathbb{Z})$.

This lemma is proved using Margulis-superrigidity. See [20] item (iv) on pages 328-329 or [17] Section 2. \square

8 (SL) or not (SL)?

In this section we show some results concerning the question whether $\rho_{G,\pi}(\Gamma(G,\pi)) \cap \mathcal{G}_{G,\pi}^1$ is of finite index in $\rho_{G,\pi}(\Gamma(G,\pi))$ or not (for the notation see the introduction). The relevance of this question and the notation are explained in the introduction. We change here from the consideration of the rational relation module to its complex version. This brings certain technical advantages and still reflects the problem.

Let us set up some notation for this section. Let G be a finite group and $\pi : F_n \rightarrow G$ an epimorphism with kernel R . We define $\Gamma(G,\pi)$, $\mathcal{G}_{G,\pi}$ and $\mathcal{G}_{G,\pi}^1$ as in the introduction and consider the action of $\Gamma(G,\pi)$ now on the complex relation module $\mathbb{C} \otimes_{\mathbb{Z}} \bar{R}$. The result of Gaschütz now says that

$$\mathbb{C} \otimes_{\mathbb{Z}} \bar{R} = \mathbb{C} \oplus \mathbb{C}[G]^{n-1}$$

as $\mathbb{C}[G]$ -modules.

If Q is an irreducible $\mathbb{C}[G]$ -module and M is an arbitrary $\mathbb{C}[G]$ -module we keep the notation $\mathbf{I}_Q(M)$ for the Q -isotypic component inside M . We hope that this creates no confusion with the $\mathbf{I}_N(M)$ of Section 2.1 where N, M are $\mathbb{Q}[G]$ -modules.

Let Q be a non-trivial irreducible $\mathbb{C}[G]$ -module of complex dimension h_Q . Let $\mathcal{G}_{G,\pi,Q}$ be the stabiliser of $I_Q(\mathbb{C} \otimes_{\mathbb{Z}} \bar{R})$ inside $\mathcal{G}_{G,\pi}$ (see 21). By our usual procedure we obtain a representation

$$\rho_{G,\pi,Q} : \Gamma(G,\pi) \rightarrow \mathcal{G}_{G,\pi,Q} = \mathrm{GL}(n-1, M(h_Q, \mathbb{C})) = \mathrm{GL}((n-1)h_Q, \mathbb{C}). \quad (50)$$

We call the pair (G, Q) of *type (SL)* if $\rho_{G,\pi,Q}(\Gamma(G,\pi)) \cap \mathrm{SL}((n-1)h_Q, \mathbb{C})$ is of finite index in $\rho_{G,\pi,Q}(\Gamma(G,\pi))$ for every $n \in \mathbb{N}$ and every epimorphism $\pi : F_n \rightarrow G$. We say the finite group G is of *type (SL)* if for all non-trivial irreducible $\mathbb{C}[G]$ -module Q the pair (G, Q) is of type (SL).

Note that, since $\Gamma(G,\pi)$ is finitely generated, $\rho_{G,\pi,Q}(\Gamma(G,\pi)) \cap \mathrm{SL}((n-1)h_Q, \mathbb{C})$ is of finite index in $\rho_{G,\pi,Q}(\Gamma(G,\pi))$ if and only if $\rho_{G,\pi,Q}(\Gamma(G,\pi))$ contains no element with a determinant of infinite order.

Note that the above implies that once we find a finite group G which is not of type (SL), we have found a subgroup Γ of finite index in some $\mathrm{Aut}(F_n)$ which has an epimorphism onto \mathbb{Z} .

Let G be a finite group G and N_2, \dots, N_ℓ its non-trivial irreducible $\mathbb{Q}[G]$ -modules. Let $\pi : F_n \rightarrow G$ be a surjective homomorphism. The following are equivalent:

- For all $i = 2, \dots, \ell$ the intersection

$$\sigma_{G,\pi,i}(\Gamma(G, \pi)) \cap \mathrm{SL}((n-1)h_i, \mathcal{R}_i^{\mathrm{op}})$$

has finite index in $\sigma_{G,\pi,i}(\Gamma(G, \pi)) \leq \mathrm{GL}((n-1)h_i, \mathcal{R}_i^{\mathrm{op}})$.

- For all non-trivial irreducible $\mathbb{C}[G]$ -modules Q the intersection

$$\rho_{G,\pi,Q}(\Gamma(G, \pi)) \cap \mathrm{SL}((n-1)h_Q, \mathbb{C})$$

is of finite index in $\rho_{G,\pi,Q}(\Gamma(G, \pi))$.

This is seen by introducing the decomposition of the complexifications $\mathbb{C} \otimes_{\mathbb{Q}} N_i$ into irreducible $\mathbb{C}[G]$ -modules.

There are some non-trivial irreducible $\mathbb{C}[G]$ -modules Q which are obviously of type (SL):

Proposition 8.1 *Let G be a finite group and Q a non-trivial irreducible $\mathbb{C}[G]$ -module. If there is an (irreducible) $\mathbb{Q}[G]$ -module N with $Q = \mathbb{C} \otimes_{\mathbb{Q}} N$ then (G, Q) is of type (SL).*

Proof: We note that under the above assumptions the endomorphism algebra $D := \mathrm{End}_G(N)$ is a division algebra over the rational numbers with \mathbb{Q} as center. In this case $\mathrm{SL}(m, \mathcal{R})$ (i.e. the group of elements of reduced norm 1 in $M(m, \mathcal{R})$) is of finite index in $\mathrm{SL}(m, \mathcal{R})$ for any order $\mathcal{R} \leq D$ and any $m \in \mathbb{N}$. \square

In case of the symmetric groups S_m ($m \in \mathbb{N}$) it is known, that every $\mathbb{C}[S_m]$ -module satisfies the assumptions of Proposition 8.1, see [32], Section 13. This implies:

Corollary 8.2 *All the symmetric groups S_m are of type (SL).*

In case of the alternating group A_5 Proposition 8.1 applies to the irreducible $\mathbb{C}[A_5]$ -modules of dimensions 4 and 6 but not to the two 3-dimensional irreducible $\mathbb{C}[A_5]$ -modules. They fit together to form a 6-dimensional irreducible $\mathbb{Q}[A_5]$ -module N with $\mathrm{End}_G(N) = \mathbb{Q}(\sqrt{5})$.

8.1 Reduction Steps

After the definitions above we discuss some reduction steps which will allow us to recognise pairs (G, Q) of type (SL).

Lemma 8.3 *Let G be a finite group with normal subgroup H_0 and quotient $H = G/H_0$. Let Q be a non-trivial irreducible $\mathbb{C}[G]$ -module on which H_0 acts trivially. If (H, Q) is of type (SL) then (G, Q) also has this property.*

Proof: Let $\pi : F_n \rightarrow G$ ($n \geq 2$) be any epimorphism and let R_G be the kernel of π . Let $\pi_0 : F_n \rightarrow G \rightarrow H$ be the resulting presentation of H and R_H its kernel. Obviously R_G is a subgroup of finite index in R_H . It is also easily seen that $\Gamma(G, \pi)$ is a subgroup of finite index of $\Gamma(H, \pi_0)$. The inclusion of R_G into R_H gives rise to a surjective \mathbb{C} -linear map

$$\Theta : \mathbb{C} \otimes_{\mathbb{Z}} \bar{R}_G \rightarrow \mathbb{C} \otimes_{\mathbb{Z}} \bar{R}_H.$$

This linear map is equivariant for the induced actions of G and $\Gamma(G, \pi)$. By restriction we obtain a map

$$\Theta_Q : \mathbf{I}_Q(\mathbb{C} \otimes_{\mathbb{Z}} R_G) \rightarrow \mathbf{I}_Q(\mathbb{C} \otimes_{\mathbb{Z}} R_H) \quad (51)$$

which is surjective and H -equivariant. Gaschütz' result implies that Θ_Q is in fact an $(\Gamma(G, \pi)$ -equivariant) isomorphism.

We have to show that $\det(\rho_{G, \pi, Q}(\varphi))$ is of finite order for any $\varphi \in \Gamma(G, \pi)$. We compute this determinant on the right hand side of (51) where our assumption implies this property. \square

Lemma 8.4 *Let G be a finite group with normal subgroup H . Let Q be a non-trivial irreducible $\mathbb{C}[G]$ -module which is induced from H , that is, Q is of the form $Q = \text{Ind}_H^G(Q_0)$ where Q_0 is an irreducible $\mathbb{C}[H]$ -module. If (H, Q_0) is of type (SL) then (G, Q) also has this property.*

Proof: Let $\pi : F_n \rightarrow G$ ($n \geq 2$) be any epimorphism and let R be the kernel of π . The subgroup $\pi^{-1}(H) \leq F_n$ is a free group on $m = 1 + [G : H](n - 1)$ generators, we denote it by F_m . We then have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R & \longrightarrow & F_n & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & = & & & & & & \\ 1 & \longrightarrow & R & \longrightarrow & F_m & \xrightarrow{\pi_0} & H & \longrightarrow & 1 \end{array}$$

where π_0 is the restriction of π to F_m . Note that $R = \ker(\pi) = \ker(\pi_0)$.

An element $\varphi \in \Gamma(G, \pi)$ stabilises the subgroup $F_m \leq F_n$ and as an automorphism of F_m it lies in $\Gamma(H, \pi_0)$. On the vector space $\mathbb{C} \otimes_{\mathbb{Z}} \bar{R}$ the linear maps induced by φ as an element of $\Gamma(G, \pi)$ and by φ as an element of $\Gamma(H, \pi_0)$ coincide. We denote both by $\bar{\varphi}$.

The complex relation module $\mathbb{C} \times_{\mathbb{Z}} \bar{R}$ now has a structure as a $\mathbb{C}[G]$ -module and a structure of a $\mathbb{C}[H]$ -module. In fact the second is just the restriction of the first.

Let $Q = Q_0 \oplus Q_1 \oplus \dots \oplus Q_r$ be the decomposition of Q into irreducible $\mathbb{C}[H]$ -modules. We now prove that

$$\mathbf{I}_Q(\mathbb{C} \otimes_{\mathbb{Z}} \bar{R}) = \mathbf{I}_{Q_0}(\mathbb{C} \otimes_{\mathbb{Z}} \bar{R}) \oplus \mathbf{I}_{Q_1}(\mathbb{C} \otimes_{\mathbb{Z}} \bar{R}) \oplus \dots \oplus \mathbf{I}_{Q_r}(\mathbb{C} \otimes_{\mathbb{Z}} \bar{R}). \quad (52)$$

On the left hand side the isotypic component of the $\mathbb{C}[G]$ -module Q is taken whereas on the right hand side we see the the isotypic component of the corresponding $\mathbb{C}[H]$ -modules. Clearly the left hand side of (52) is contained in the right hand side. Note that, since H is normal in G the module Q is also induced from any of the Q_0, \dots, Q_r . The Frobenius reciprocity law ([32], chapter 7) implies that none of the Q_0, \dots, Q_r can occur as an irreducible constituent of the restriction of an irreducible $\mathbb{C}[G]$ -module different from Q .

We have to show that $\det(\rho_{G, \pi, Q}(\varphi))$ is of finite order for any $\varphi \in \Gamma(G, \pi)$. We compute this determinant on the right hand side of (52) where our assumption implies this property. \square

8.2 Abelian and metabelian groups

This section contains results which describe the image of $\rho_{G, \pi}$ for abelian and then for metabelian groups G up to commensurability. In Section 6 we have proved a very precise result in this direction

for cyclic groups G . This allowed us to conclude for $n \geq 2$ that the image of $\Gamma(G, \pi)$ (under certain linear representations) has finite abelianisation. We present here the following much stronger result for $n \geq 3$. This result is due to A. Rapinchuk (it appears in a somewhat hidden form in [30], page 150), we thank him for the permission to include it.

Proposition 8.5 *Let $n \geq 3$ be a natural number, G a finite abelian group G and let $\pi : F_n \rightarrow G$ be any surjective homomorphism. Then $\Gamma(G, \pi)$ has finite abelianisation.*

Proof: Since G is abelian we have $F'_n \leq R$ where R is, as always, the kernel of π . Consequently we have $\text{IA}(F_n) \leq \Gamma(G, \pi)$. Writing $\tilde{\Gamma}(G, \pi) := \rho_1(\Gamma(G, \pi)) \leq \text{GL}(n, \mathbb{Z})$ for the image of $\rho_1(\Gamma(G, \pi))$ in $\text{GL}(n, \mathbb{Z})$ we have the exact sequence of groups

$$1 \rightarrow \text{IA}(F_n) \rightarrow \Gamma(G, \pi) \rightarrow \tilde{\Gamma}(G, \pi) \rightarrow 1$$

Note that $\tilde{\Gamma}(G, \pi)$ is a subgroup of finite index in $\text{GL}(n, \mathbb{Z})$. Since $\Gamma(G, \pi)$ is a finitely generated group we need only show that

$$H^1(\Gamma(G, \pi), \mathbb{Q}) = \text{Hom}(\Gamma(G, \pi), \mathbb{Q}) = 0$$

We shall do that by applying the edge-term sequence of the Hochschild-Serre spectral sequence for the above short exact sequence of groups. We shall use that there is an exact sequence of \mathbb{Q} -vector spaces

$$H^1(\tilde{\Gamma}(G, \pi), \mathbb{Q}) \rightarrow H^1(\Gamma(G, \pi), \mathbb{Q}) \rightarrow H^1(\text{IA}(F_n), \mathbb{Q})^{\tilde{\Gamma}(G, \pi)} \quad (53)$$

First of all the left hand side of (53) is 0 since $n \geq 3$ and $\text{GL}(n, \mathbb{Z})$ has Kazhdan's property (T).

Next we evaluate the right hand side of (53). By a theorem of Bachmuth ([1] and also [11]) there is a $\text{Aut}(F_n)$ -equivariant isomorphism

$$H^1(\text{IA}(F_n), \mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(F_n/F'_n, F'_n/\gamma_2(F_n))$$

where $\gamma_2(F_n) := [[F_n, F_n], F_n]$ is the third term of the lower central series of F_n . Let $\text{GL}^{\pm}(n, \mathbb{C})$ be the subgroup of $\text{GL}(n, \mathbb{C})$ consisting of those elements having determinant ± 1 . Let \mathbb{C}^n be the standard $\text{GL}^{\pm}(n, \mathbb{C})$ -module of dimension n and $\Lambda^2(\mathbb{C}^n)$ be its outer square. Consider $\text{GL}^{\pm}(n, \mathbb{Z})$ as a subgroup of $\text{GL}^{\pm}(n, \mathbb{C})$. By identifying $(F'_n/\gamma_2(F_n))$ with $\Lambda^2(F_n/F'_n) = \Lambda^2(\mathbb{Z}^n)$ we obtain an $\text{GL}^{\pm}(n, \mathbb{Z})$ -equivariant isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}} \text{Hom}_{\mathbb{Z}}(F_n/F'_n, F'_n/\gamma_2(F_n)) \rightarrow \text{Hom}(\mathbb{C}^n, \Lambda^2(\mathbb{C}^n)).$$

It is well known (see [11], page 427) that the right hand side splits as a $\text{GL}^{\pm}(n, \mathbb{C})$ -module as

$$\text{Hom}(\mathbb{C}^n, \Lambda^2(\mathbb{C}^n)) = \mathbb{C}^n \oplus V_n$$

where V_n is a $\text{GL}^{\pm}(n, \mathbb{C})$ -module of dimension $\dim_{\mathbb{C}}(V_n) = n(n+1)(n-2)/2$ which is irreducible even as $\text{SL}(n, \mathbb{C})$ -module. Hence the space of $\text{SL}(n, \mathbb{C})$ -invariants in $\text{Hom}(\mathbb{C}^n, \Lambda^2(\mathbb{C}^n))$ is 0. Since the Zariski closure of $\tilde{\Gamma}(G, \pi)$ contains $\text{SL}(n, \mathbb{C})$ the group $\text{Hom}_{\mathbb{Z}}(F_n/F'_n, F'_n/\gamma_2(F_n))^{\tilde{\Gamma}(G, \pi)}$ is 0. This shows that the right hand side of (53) is 0. Together with the above the proposition is proved. \square

Proposition 8.5 is clearly not true for $n = 2$, since the left-most term in (53) contributes to the $H^1(\Gamma(G, \pi), \mathbb{Q})$. See also the tables in Section 9.4. As an immediate corollary from Proposition 8.5 we get:

Corollary 8.6 *All finite abelian groups are of type (SL).*

This corollary can also be deduced from Corollary 6.5 (in fact this corollary is needed in the case $n = 2$) and Lemma 8.3. But the statement incorporated in Proposition 8.5 (for $n \geq 3$) is in fact stronger.

A metabelian group is a group G with abelian commutator subgroup G' . We prove:

Proposition 8.7 *All metabelian finite groups G are of type (SL)*

Proof: We use induction on the order of G and the fact proved above that all finite abelian groups have property (SL).

Let Q be a non-trivial irreducible $\mathbb{C}[G]$ -module. We may assume by induction and Lemma 8.3 that G acts faithfully on Q .

Case 1: the commutator subgroup is not contained in the center $Z(G)$ of G . The restriction of the module Q to G' cannot be isotypic. Otherwise the abelian subgroup G' would act by scalar matrices on Q and G' would have to be contained in the center of G . Now Proposition 24 of [32] says that Q is either isotypic or induced. It follows that Q is induced from a proper subgroup H of G containing G' .

By our construction H is normal in G . Hence this case is finished by induction and application of Lemma 8.4.

Case 2: the commutator subgroup is contained in the center $Z(G)$ of G . In this case G is nilpotent of class 2. Unless G is abelian (the case treated above) we find $g \in G$ with $g \notin Z(G)$. The subgroup $A := \langle G', g \rangle$ generated by G' and G is abelian, normal in G and not contained in $Z(G)$. We then proceed as in case 1. \square

9 Concluding remarks and suggestions for further research

9.1 Dependence on the presentation

The results of the current paper can be considered as a first step toward a systematic study of the relation module of a finite group as a $\Gamma(G, \pi)$ -module, i.e. an equivariant Gaschütz's theory. Here we gave a quite satisfactory answer in the case of redundant presentations. It is not clear to what extent this represents the general case.

For a fixed finite group G let us look at all the possible epimorphisms from the free group F_n to G and $\mathbf{R}(n, G)$ the set of their kernels. The automorphism group $\text{Aut}(F_n)$ acts on this set. It is easy to see that the equivariant Gaschütz theory that we are trying to develop here depends only on the orbits of $\text{Aut}(F_n)$ on $\mathbf{R}(n, G)$ and not on the actual presentation. Various authors studied the transitivity properties of the action of $\text{Aut}(F_n)$ on $\mathbf{R}(n, G)$. It is not transitive in general (a first example was given by B. H. Neumann [25]) but it is in some interesting cases. For example, it is transitive if G is a cyclic group and $n \geq 2$. Thus our results in Section 6 give the full picture for all relation modules of cyclic groups. It is also transitive if $n > 2\log_2(|G|)$ or if G is solvable and $n > d(G)$, where $d(G)$ denotes the minimal number of generators of G (see [7]). An old conjecture of Wiegold predicts transitivity for finite simple groups if $n \geq 3$, in which case again $n > d(G)$

holds. For some partial results see [14], [9] and the references in [27]. It was even suggested in [27] that for every finite group G it is transitive if $n > d(G)$.

In [9] it is shown that if a finite group G has spread 2 (i.e. for every $g_1, g_2 \in G$ with $g_1 \neq 1 \neq g_2$ there is $h \in G$ such that $\langle g_1, h \rangle = \langle g_2, h \rangle = G$) then $\text{Aut}(F_n)$ acts transitively on the set of kernels of all redundant presentations. Guralnick and Shalev [18] proved that almost all finite simple groups have spread 2.

Theorem 1.4 is not true in general if the presentation is not redundant, at least when $n = 2$, see the A_5 -example in Section 9.4. The story for $n \geq 3$ may be different as $\text{Aut}(F_2)$ behaves in many ways differently from $\text{Aut}(F_n)$ for $n \geq 3$.

9.2 Representations of $\text{IA}(F_n)$

A well known theorem of Formanek and Procesi (see [12]) asserts that the automorphism group $\text{Aut}(F_n)$ has, for $n \geq 3$, no faithful linear representation over any field. Their proof suggests a stronger

Conjecture: For every linear representation ρ of $\text{Aut}(F_n)$ the image $\rho(\text{Inn}(F_n))$ is virtually solvable.

As far as we know, in all previously constructed cases of linear representations even $\rho(\text{IA}(F_n))$ had this property. It should be mentioned that in all the representations ρ studied in this paper, $\rho(\text{Inn}(F_n))$ is finite but $\rho(\text{IA}(F_n))$ is far from being virtually solvable and, in fact, our results show that $\rho(\text{IA}(F_n))$ can be Zariski dense in very large semi-simple groups. Our representations seem to be the first known with this property. We have:

Proposition 9.1 *Let $n \geq 3$, $k \geq 1$, $h_1 < \dots < h_k$, m_1, \dots, m_k be natural numbers. Let $\mathbb{Q}(\zeta_{m_i})$ be the field of m_i -th roots of unity and $\mathbb{Z}(\zeta_{m_i})$ its ring of integers. There is a subgroup $\Gamma \leq \text{IA}(F_n)$ of finite index and a representation*

$$\rho : \Gamma \rightarrow \prod_{i=1}^k \text{SL}((n-1)h_i, \mathbb{Q}(\zeta_{m_i}))^{m_i}$$

such that $\rho(\Gamma)$ is commensurable with $\prod_{i=1}^k \text{SL}((n-1)h_i, \mathbb{Z}(\zeta_{m_i}))^{m_i}$.

Proof: The proof of Theorem 1.1 shows that there is a subgroup of finite index $\Gamma_0 \leq \text{Aut}(F_n)$ and a representation

$$\rho_0 : \Gamma_0 \rightarrow \text{SL}(n, \mathbb{Z}) \times \prod_{i=1}^k \text{SL}((n-1)h_i, \mathbb{Q}(\zeta_{m_i}))^{m_i} \quad (54)$$

such that the image $\rho_0(\Gamma_0)$ is the internal direct product of its intersections with the factors in (54). These intersections in turn have finite index in the corresponding factor. Using the fact that a normal subgroup of a subgroup of finite index in $\text{SL}(n, \mathbb{Z})$ ($n \geq 3$) is either finite or has finite index we conclude the proof of the proposition. \square

Another easy consequence of Theorem 1.1 concerns representations of the the outer automorphism group $\text{Out}(F_n)$. We have:

Proposition 9.2 *Let $n \geq 2$, $k \geq 1$, $h_1 < \dots < h_k$, m_1, \dots, m_k be natural numbers. Let $\mathbb{Q}(\zeta_{m_i})$ be the field of m_i -th roots of unity and $\mathbb{Z}(\zeta_{m_i})$ its ring of integers. There is a subgroup $\Gamma \leq \text{Out}(F_n)$ of finite index and a representation*

$$\rho : \Gamma \rightarrow \prod_{i=1}^k \text{SL}((n-1)h_i, \mathbb{Q}(\zeta_{m_i}))^{m_i}$$

such that $\rho(\Gamma)$ is commensurable with $\prod_{i=1}^k \text{SL}((n-1)h_i, \mathbb{Z}(\zeta_{m_i}))^{m_i}$.

Here we only have to note that the image of the inner automorphisms under any of the representations from Theorem 1.1 is finite. This finite subgroup of the image can be avoided by going to a subgroup of finite index in Γ .

As a consequence of Propositions 9.1, 9.2 we get:

Corollary 9.3 *The groups $\text{IA}(F_3)$ and $\text{Out}(F_3)$ are large.*

Our result from Theorem 1.1 can conveniently be summarised in the language of the pro-algebraic completion $\mathcal{A}(\text{Aut}(F_n))$ of $\text{Aut}(F_n)$. For a definition and discussion of properties of the pro-algebraic completion (also called the Hochschild-Mostow group) of a group see [21] and [2]. We have

Proposition 9.4 *Let $n \geq 2$ be a natural number and $\mathcal{S}(\text{Aut}(F_n))$ be the semisimple part of the connected component of the identity of the pro-algebraic completion $\mathcal{A}(\text{Aut}(F_n))$. Then for every $h \in \mathbb{N}$, the group $\text{SL}((n-1)h, \mathbb{C})$ appears infinitely many times as a factor in $\mathcal{S}(\text{Aut}(F_n))$. That is, there is a surjective homomorphism*

$$\mathcal{S}(\text{Aut}(F_n)) \rightarrow \prod_{h=1}^{\infty} \prod_{i=1}^{\infty} \text{SL}((n-1)h, \mathbb{C}).$$

Propositions 9.1, 9.2 imply similar results for the groups $\text{IA}(F_n)$, $\text{Out}(F_n)$ ($n \geq 3$).

9.3 Representations of $\Gamma(G, \pi)$ into affine groups

In this section we give another construction of a linear representation of subgroups of finite index in $\text{Aut}(F_n)$. This representation takes values in affine groups (which have a non-trivial unipotent radical). The construction originates from the proof of Gaschütz' result as presented in [19]. Besides giving new types of image groups, it is of importance for the computer algorithms which we have used to create the computational results described in the next subsection.

Let G be a finite group and $\pi : F_n \rightarrow G$ a surjective homomorphism of the free group F_n onto G . Let R be the kernel of π and \bar{R} the corresponding relation module. Let further $\Gamma(G, \pi)$ be the subgroup of $\text{Aut}(F_n)$ defined in (3). Given a group H and a commutative ring S we write $S[H]$ for the corresponding group ring and $\mathcal{I}(S[H])$ for its augmentation ideal. If $H_0 \leq H$ is a subgroup we define $\mathcal{I}(S[H], H_0)$ to be the two-sided ideal of $S[H]$ generated by the $h - 1$ for $h \in H_0$.

Gaschütz' result (18) is proved (in [19]) by considering the exact sequence

$$0 \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \bar{R} \rightarrow \mathcal{I}(\mathbb{Q}[F_n]) / (\mathcal{I}(\mathbb{Q}[F_n], R) \cdot \mathcal{I}(\mathbb{Q}[F_n])) \rightarrow \mathcal{I}(\mathbb{Q}[G]) \rightarrow 0 \quad (55)$$

of G -equivariant homomorphisms. The right hand map is induced by $\pi : F_n \rightarrow G$ while the left hand map comes from the map from R to $\mathcal{I}(\mathbb{Q}[F_n])$ which sends $r \in R$ to $r - 1$. The free group F_n acts on the middle term in (55) by multiplication from the left. This leads to an action of the finite group G on this term.

We now let $\Gamma(G, \pi)$ act on the left hand term as before, on the middle term by its action on F_n and trivially on $\mathbb{Q}[G]$. It is straightforward to see that the sequence (55) is then $\Gamma(G, \pi)$ -equivariant. We obtain a representation

$$\eta_{G, \pi} : \Gamma(G, \pi) \rightarrow \text{Hom}_G(\mathcal{I}(\mathbb{Q}[G]), \mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}) \rtimes \mathcal{G}_{G, \pi}(\mathbb{Q}). \quad (56)$$

The semi-direct product in (56) is formed with respect to the action of $\mathcal{G}_{G, \pi}(\mathbb{Q})$ on $\mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}$.

Our methods show:

Theorem 9.5 *Assume n is a natural number with $n \geq 4$. Let $\pi : F_n \rightarrow G$ be a redundant presentation of the finite group G . Then*

$$\eta_{G, \pi}(\Gamma(G, \pi)) \cap \text{Hom}_G(\mathcal{I}(\mathbb{Q}[G]), \mathbb{Q} \otimes_{\mathbb{Z}} \bar{R}) \rtimes \mathcal{G}_{G, \pi}^1(\mathbb{Q})$$

is of finite index in the arithmetic group $\text{Hom}_G(\mathcal{I}(\mathbb{Z}[G]), \bar{R}) \rtimes \mathcal{G}_{G, \pi}^1(\mathbb{Z})$.

Theorem 9.5 can then be used to prove

Theorem 9.6 *Let $n \geq 2$, h, m be natural numbers. Let $\mathbb{Q}(\zeta_m)$ be the field of m -th roots of unity and $\mathbb{Z}(\zeta_m)$ its ring of integers. There is a subgroup $\Gamma \leq \text{Aut}(F_n)$ of finite index and a representation*

$$\eta : \Gamma \rightarrow \left(\mathbb{Q}(\zeta_m)^{(n-1)h} \rtimes \text{SL}((n-1)h, \mathbb{Q}(\zeta_m)) \right)^m$$

such that $\eta(\Gamma)$ is commensurable with $(\mathbb{Z}(\zeta_m)^{(n-1)h} \rtimes \text{SL}((n-1)h, \mathbb{Z}(\zeta_m)))^m$.

9.4 Computational results

An important feature of our methods is that subgroups of quite high indices in $\text{Aut}(F_n)$ may be computationally located. We describe here some of the results of our computer calculations. The computations were done using the computer algebra system MAGMA.

We shall describe our approach now in detail for $n = 2$. Let $F_2 := \langle x, y \rangle$ be the free group. The automorphism group $\text{Aut}^+(F_2)$ is generated by α, β which are given by

$$\alpha(x) = y^{-1}, \quad \alpha(y) = x \quad \beta(x) = x^{-1}y^{-1}, \quad \beta(y) = x.$$

The group $\text{Aut}^+(F_2)$ is finitely presented, a presentation is (see [24])

$$\text{Aut}^+(F_2) = \langle \alpha, \beta \mid \alpha^4 = \beta^3 = \alpha^2 \beta^2 \alpha^2 \beta \alpha \beta \alpha^2 \beta^2 \alpha = 1 \rangle. \quad (57)$$

Let $\pi : F_2 \rightarrow G$ be a presentation of a finite group. If $\varphi \in \text{Aut}^+(F_2)$ is given by its values on x, y it is easy to check whether φ is in $\Gamma(G, \pi)$ or not. Using a random process we generate a list of words $\varphi_1, \dots, \varphi_k \in \Gamma(G, \pi)$ in the automorphisms α, β . We then wait until

$$\Delta := \langle \varphi_1, \dots, \varphi_k \rangle$$

has finite index in $\text{Aut}^+(F_2)$. This is checked using a Todd-Coxeter algorithm using the presentation (57). By this approach we first of all find the following table. The first column contains the finite group G , which we always think of being given as a permutation group. The third and fourth column contain the images x, y under the surjective homomorphism $\pi : F_2 \rightarrow G$. The fifth column gives the index of $\Delta \leq \Gamma(G, \pi)$ in $\text{Aut}^+(F_2)$. We give the abelianised group Δ^{ab} in the last column.

G	$ G $	$\pi(x)$	$\pi(y)$	$[\text{Aut}^+(F_2) : \Delta]$	Δ^{ab}
C_2	2	(1,2)	(1)	3	$\mathbb{Z}^2 \times C_2 \times C_4$
C_3	3	(1,2,3)	(1)	8	$\mathbb{Z} \times C_3^2$
C_4	4	(1,2,3,4)	(1)	12	$\mathbb{Z}^2 \times C_4$
$C_2 \times C_2$	4	(1,2)	(3,4)	6	$\mathbb{Z}^2 \times C_2^3$
C_5	5	(1,2,3,4,5)	(1)	24	$\mathbb{Z}^3 \times C_5$
C_6	6	(1,2,3,4,5,6)	(1)	24	$\mathbb{Z}^3 \times C_6$
S_3	6	(1,2,3)	(1,2)	18	$\mathbb{Z}^2 \times C_2$
C_7	7	(1,2,3,4,5,6,7)	(1)	48	$\mathbb{Z}^5 \times C_7$
D_4	8	(1,2,3,4)	(1,4)(2,3)	24	$\mathbb{Z}^3 \times C_2$
Q_8	8	(1,7,2,8)(3,6,4,5)	(1,4,2,3)(5,7,6,8)	24	$\mathbb{Z}^2 \times C_4$
D_5	10	(1,2,3,4,5)	(1,5)(2,4)	30	$\mathbb{Z}^2 \times C_2$
A_4	12	(1,2,3)	(1,2)(3,4)	96	\mathbb{Z}^3
$S_3 \times C_2$	12	(1,3)(4,5)	(1,2)	36	$\mathbb{Z}^3 \times C_2$
$\text{Sm}(12, 1)$	12	σ_1	σ_2	72	$\mathbb{Z}^3 \times C_2$
A_5	60	(1,2,3,4,5)	(1,2,3)	1080	\mathbb{Z}^{17}

The notation for the finite groups is: C_n is the cyclic group of order n , D_n is the dihedral group of order $2n$, Q_8 is the quaternion group of order 8 and $\text{Sm}(12, 1)$ is the non-abelian group of order 12 not isomorphic to A_4 or $S_3 \times C_2$. The two permutations σ_1, σ_2 are:

$$\sigma_1 := (1, 8, 4, 11)(2, 9, 5, 12)(3, 7, 6, 10), \quad \sigma_2 := (1, 9, 4, 12)(2, 7, 5, 10)(3, 8, 6, 11).$$

Proceeding with our computer calculations, we then evaluated the $\varphi_1, \dots, \varphi_k$ (which now generate a subgroup of finite index in $\Gamma(G, \pi)$) on the middle term of the sequence (55). Locating the isotypic component of an irreducible $\mathbb{Q}[G]$ -module N inside all three terms of (55) we were able to compute $\rho_{G, \pi, N}(\varphi_i)$ ($i = 1, \dots, k$) as matrices. Computing determinants we could decide whether $\rho_{G, \pi}(\Gamma(G, \pi)) \cap \mathcal{G}_{G, \pi}^1(\mathbb{Z})$ is of finite index in $\rho_{G, \pi}(\Gamma(G, \pi))$ or not.

We have considered at least one presentation $\pi : F_2 \rightarrow G$ for every group G with $|G| \leq 60$ which can be generated by 2 elements and have found that $\rho_{G, \pi}(\Gamma(G, \pi)) \cap \mathcal{G}_{G, \pi}^1(\mathbb{Z})$ is of finite index in $\rho_{G, \pi}(\Gamma(G, \pi))$. Of course we have run experiments on many more groups (like $\text{PSL}(2, 7)$) but always found a similar result.

Let us report some more on the particularly interesting case $G = A_5$. Let N be the irreducible $\mathbb{Q}[G]$ -module of dimension 6. We have $\text{End}_G(N) = \mathbb{Q}(\sqrt{5})$. Its ring of integers is $\mathcal{O} = \mathbb{Z}((1+\sqrt{5})/2)$. Identifying the isotypic component of N in $\mathbb{Q}[G]$ with $M(3, \mathbb{Q}(\sqrt{5}))$ we obtain a representation

$$\rho : \Gamma(G, \pi) \rightarrow \text{GL}(1, M(3, \mathbb{Q}(\sqrt{5}))) = \text{GL}(3, \mathbb{Q}(\sqrt{5})).$$

Running the programm described above we have found

Proposition 9.7 *For every presentation $\pi : F_2 \rightarrow G = A_5$ there is a subgroup $\Delta \leq \Gamma(G, \pi)$ of finite index such that $\rho_{A_5, \pi}(\Delta)$ is contained up to change of bases in the subgroup of $\text{SL}(3, \mathcal{O})$ consisting of elements which have $(1, 0, 0)$ as a first row.*

Proposition 9.7 shows that Theorem 1.4 is not true for $n = 2$ and non-redundant presentations. It is not clear what Proposition 9.7 suggests toward the general case. It may suggest that Theorem 1.4 is not true for non-redundant presentations but it may also be that $n = 2$ is exceptional.

We have developed similar programmes also for the cases $n = 3$ and $n = 4$. They lead to the following tables. The notation is the same as in the above table for $n = 2$. The presentations used for $n = 4$ take the value 1 on the fourth generator of the free group.

Subgroups in $\text{Aut}(F_3)$

G	$ G $	$\pi(x)$	$\pi(y)$	$\pi(z)$	$[\text{Aut}(F_3) : \Delta]$	Δ^{ab}
C_2	2	(1,2)	(1)	(1)	7	C_2^3
C_3	3	(1,2,3)	(1)	(1)	26	C_6
S_3	6	(1,2,3)	(1,2)	(1)	168	C_2^3
D_4	8	(1,3)	(1,4,3,2)	(1)	336	C_2^8
Q_8	8	(1,7,2,8)(3,6,4,5)	(1,4,2,3)(5,7,6,8)	(1)	336	C_2^7
D_5	10	(1,2,3,4,5)	(1,5)(2,4)	(1)	840	C_2^5
A_4	12	(1,2,3)	(1,2)(3,4)	(1)	1560	C_6
$S_3 \times C_2$	12	(1,3)(4,5)	(1,2)	(1)	1008	C_2^9
$\text{Sm}(12, 1)$	12	σ_1	σ_2	(1)	1344	$C_2^2 \times C_4$
A_5	60	(1,2,3,4,5)	(1,2,3)	(1)	200160	?

Subgroups in $\text{Aut}(F_4)$

G	$ G $	$[\text{Aut}(F_4) : \Delta]$	Δ^{ab}
C_2	2	15	C_2^2
C_3	3	80	C_6
S_3	6	80	C_6
D_4	8	3360	C_2^4
Q_8	8	840	C_2^6
D_5	10	930	C_2^3
A_4	12	1680	$C_3 \times C_6$
$S_3 \times C_2$	12	2730	C_2^4
$\text{Sm}(12, 1)$	12	3120	$C_2^2 \times C_4$
A_5	60	213098	?

We have run experiments on many finite groups G and presentations $\pi : F_n \rightarrow G$, we have always found that $\rho_{G, \pi}(\Gamma(G, \pi)) \cap \mathcal{G}_{G, \pi}^1(\mathbb{Z})$ is of finite index in $\rho_{G, \pi}(\Gamma(G, \pi))$.

References

- [1] S. Bachmuth, *Induced automorphisms of free groups and free metabelian groups*. Trans. Amer. Math. Soc. **122**, (1966), 1–17

- [2] H. Bass, A. Lubotzky, A.R. Magid, S. Mozes, *The proalgebraic completion of rigid groups*. Geometriae Dedicata **95**, (2002), 19–58
- [3] J. Birman, *Braids, links and mapping class groups.*, Ann. of Math. Stud. **82**, Princeton Univ. Press, Princeton (1974)
- [4] C.W. Curtis, I. Reiner, *Representation Theory of finite Groups and associative Algebras*. Interscience Publishers (1962)
- [5] M. Deuring, *Algebren*. Ergebnisse **41**, Springer Verlag, New York, Berlin, Heidelberg (1968)
- [6] D. Djoković, V. Platonov, *Low dimensional representations of $\text{Aut}(F_2)$* . Manuscripta Math. **89**, (1996), 475–509
- [7] M. Dunwoody, *Nielsen Transformations*. In: Computational Problems in Abstract Algebra (1970), Pergamon, Oxford, 45–46
- [8] J. Elstrodt, F. Grunewald, J. Mennicke, *Groups acting on Hyperbolic Space: Harmonic Analysis and Number Theory*. Springer Monographs in Math., (1998), 524 pp
- [9] M. Evans, *Presentations of groups involving more generators than are necessary*. Proc London Math. Soc. **67**, (1993), 106–126
- [10] M. Evans, *T-systems of certain finite simple groups*. Math. Proc. Cambridge Philos. Soc. **113**, (1993), 9–22
- [11] E. Formanek, *Characterizing a free group in its automorphism group*. J. Algebra, **133**, (1990), 424–432
- [12] E. Formanek, C. Procesi, *The automorphism group of a free group is not linear*. J. Algebra, **149**, (1992), 494–499
- [13] W. Gaschütz, *Über modulare Darstellungen endlicher Gruppen, die von freien Gruppen induziert werden*. Math. Z. **60**, (1954), 274–286
- [14] R. Gilman, *Finite quotients of the automorphism group of a free group*. Canad. J. Math. **29**, (1977), 541–551
- [15] K. Gruenberg, *Relation modules of finite groups*. CBMS **25**, Amer. Math. Soc. (1976)
- [16] F. Grunewald, A. Lubotzky, *Linear representations of the mapping class group*. In preparation
- [17] F. Grunewald, V. Platonov, *Rigidity results for groups with radical, cohomology of finite groups and arithmeticity problems*. Duke Math. J. **100**, (1999), 321–358
- [18] R. Guralnick, A. Shalev, *On the spread of finite simple groups*. Combinatorica **23**, (2003), 73–87
- [19] M. Jarden, J. Ritter, *Normal Automorphisms of absolute Galois Groups of \wp -adic Fields*. Duke Math. J. **47**, (1980), 47–56
- [20] A. Lubotzky, *Torsion in profinite completions of torsion-free groups*. Quart. J. Math. Oxford Series **44**, (1993), 327–332

- [21] A. Lubotzky, A.R. Magid, *Varieties of representations of finitely generated groups*. Mem. Amer. Math. Soc. **58**, (1985)
- [22] A. Lubotzky, I. Pak, *The product replacement algorithm and Kazhdan's property (T)*. J. Amer. Math. Soc. **14**, (2001), 347–363
- [23] W. Magnus, A. Karass, D. Solitar, *Combinatorial Group Theory*. Interscience Publishers (1966)
- [24] B.H. Neumann, *Die Automorphismengruppe der freien Gruppen*. Math. Ann. **107**, (1933), 367–386
- [25] B.H. Neumann, *On a question of Gaschütz*. Archiv der Math. **7**, (1956), 87–90
- [26] J. Nielsen, *Die Isomorphismengruppe der freien Gruppen*. Math. Ann. **91**, (1924), 169–208
- [27] I. Pak, *What do we know about the product replacement algorithm?* Groups and Computation III (Columbus, Ohio, 1999), Ohio State Univ. Math. Res. Inst. Publ. **8**, 301–347, de Gruyter, Berlin, (2001)
- [28] V. Platonov, A. Rapinchuk, *Algebraic Groups and Number Theory*. Pure and Applied Mathematics **139**, Academic Press, (1994)
- [29] A. Potapchik, A. Rapinchuk, *Low-dimensional linear representations of $\text{Aut}(F_n)$, $n \geq 3$* . Transactions of the Amer. Math. Soc. **352**, (1999), 1437–1451
- [30] A. Rapinchuk, *n -dimensional linear representations of $\text{Aut}(F_n)$, and more*. J. of Algebra **197**, (1997), 146–152
- [31] J.-P., Serre, *Le problème des groupes des congruence pour SL_2* . Ann. of Math., **92**, (1970), 489–527
- [32] J.-P., Serre, *Linear Representations of Finite Groups*. Graduate Text in Math. **42**, Springer Verlag, New York, Berlin, Heidelberg (1993)
- [33] L. Vaserstein, *The structure of classical arithmetic groups of rank greater than one*. Math. USSR Sbornik (English translation), **20**, (1973), 465–492
- [34] L. Vaserstein, *The group SL_2 over Dedekind rings of arithmetic type*. Mat. Sb. **89**, (1972), 313–322
- [35] T.N. Venkataramana, *On systems of generators of arithmetic subgroups of higher rank groups*. Pacific Journal of Math., **166**, **1**, (1994), 193–212