The Mathematics of Boris Weisfeiler

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A Tribute to Boris Weisfeiler

In 1985 the mathematician Boris Weisfeiler disappeared while hiking alone in a remote area of Chile. At the time, he was a professor of mathematics at Pennsylvania State University and was widely recognized for his work in algebraic groups. What happened to him remains a mystery, and to this day it is not known whether he is still alive.

Born in the Soviet Union, Weisfeiler received his Ph.D. in 1970 from the Leningrad branch of the Steklov Institute, where his adviser was E. B. Vinberg. Weisfeiler emigrated to the United States in 1975 and worked with Armand Borel at the Institute for Advanced Study. The next year he joined the faculty at Pennsylvania State University. In 1981 he became an American citizen.

Weisfeiler's disappearance has been the subject of several newspaper articles (see, for example, "Chilean Mystery: Clues to Vanished American", by Larry Rohter, *New York Times*, May 19, 2002; and "Tracing a Mystery of the Missing in Chile", by Pascale Bonnefoy, *Washington Post*, January 18, 2003). Further information about media coverage, as well as the present status of the investigation into his disappearance, may be found at `http://weisfeiler.com/boris`.

On the occasion of the publication in Chile of a book about Weisfeiler's disappearance, the *Notices* decided to present a brief tribute to his life and work. What follows is a short summary of his mathematical work and a review of the book. This is not an obituary, as hope remains that Weisfeiler is still alive. Nevertheless, it seems appropriate to commemorate this lost member of the mathematical community, whose absence is keenly felt.

—*Allyn Jackson*

Boris Weisfeiler's mathematical activity extended over more than twenty years of research in the USSR and the USA, during which time he published three dozen research papers. All this ended abruptly in early 1985 with his disappearance—right after he had published his most influential papers.

Weisfeiler's area was algebraic groups in all their aspects and directions. During his years of activity the theory of algebraic groups over algebraically closed fields had already been well understood and Weisfeiler was part of the trend of studying the more difficult questions concerning the case when the field is not algebraically closed and the groups do not split, or even worse—are nonisotropic. His specialty was the positive characteristic case. In these cases the connection between the algebraic group and its Lie algebra is more subtle and a great deal of effort was needed to study questions whose solutions in characteristic zero were pretty standard.

A popular subject of that period was the study of "abstract homomorphisms" between algebraic groups; i.e., assume k and k' are fields and G and G' are algebraic groups defined over k and k', respectively, and let $\varphi : G(k) \to G'(k')$ be homomorphism of groups. The standard "wishful" result in this theory is the claim that, under suitable conditions, such a homomorphism is algebraic; i.e. there is a monomorphism of fields $k \to k'$ and after identifying k as a subfield of k', the homomorphism φ is a homomorphism in the category of algebraic groups, i.e., given by polynomial maps. Various methods have developed to tackle such problems. Usually, one associates some geometry with the algebraic groups in the spirit of Klein's Erlangen program, as projective geometry is associated with

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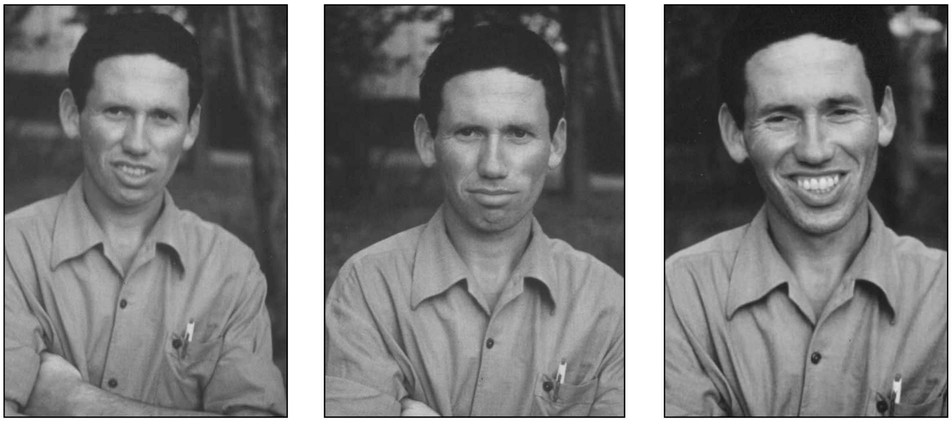

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$GL_n(k)$, and hence the map φ induces a map between the associated geometries which turns out to be a useful way to study the original φ. The works of Borel, Tits, O'Meara, and others settled most questions for isotropic groups, but the nonisotropic cases were always of special difficulty. This was a direction in which Weisfeiler made some important contributions, but unfortunately they are too technical to be elaborated upon here.

Weisfeiler is mainly remembered and quoted for the two contributions he made in the last year (1984) just before his disappearance. The first one is truly spectacular, although it took some time to realize that: In [W1] (following a partial result in [MVW]) Weisfeiler proved a strong approximation theorem for general linear groups. To get a feeling of what it says, let us take a very special case: Let $SL_n(\mathbb{Z})$ be the group of $n \times n$ integral matrices of determinant one, and let $\Gamma \leq SL_n(\mathbb{Z})$ be a Zariski-dense subgroup. Then, Weisfeiler's theorem says (in fact, this case is already covered by [MVW]) that Γ is almost dense in $SL_n(\mathbb{Z})$ in the congruence topology of $SL_n(\mathbb{Z})$; i.e., its closure there is of finite index. The congruence topology of $SL_n(\mathbb{Z})$ is the one for which the groups

$$\{\Gamma(m) = Ker(SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/m\mathbb{Z}) | m \in \mathbb{Z}\}$$

serve as a system of neighborhoods of the identity.

On the surface, this is a technical theorem comparing two topologies of $SL_n(\mathbb{Z})$, but, in fact, this result has many extremely useful consequences. Being Zariski dense is a very weak condition, but having the quality of being congruence dense gives a lot of corollaries on the finite quotients of Γ. It says, for example, that for almost every prime number p, the finite simple group $PSL_n(p)$ is a quotient of Γ. For a general, finitely generated linear group Γ, it implies, for example, that either Γ is almost solvable or it has a finite index subgroup with infinitely many different finite simple quotients. It is not surprising that Weisfeiler's strong approximation theorem has since played an important role in the study of infinite linear groups, in asymptotic group theory in general and subgroup growth in particular. (The reader is referred to [LS] and especially to Window 9, "Strong Approximation for Linear Groups", which describes some of the applications). Interestingly enough Weisfeiler used the (at that time brand new) classification of finite simple groups (CFSG) for his proof. He was probably the first to see how CFSG could be used for infinite linear groups. (By now, there are proofs which do not require the CFSG; see [LS] and the references therein.)

In [W2], Weisfeiler made another remarkable use of the CFSG for linear groups. He announced a sharp bound on the index of the abelian normal subgroup of finite linear subgroups of GL_n. The existence of such a bound was proved by Jordan, with extensions and better bounds proven by many including Brauer and Feit. Weisfeiler's result is still, twenty years later, the best known result. Unfortunately, a detailed proof has never appeared.

Boris Weisfeiler's mathematical work was suddenly and tragically cut—but his name is still remembered by all those who knew him or who work in related areas.

References

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[W1] Boris Weisfeiler, Strong approximation for Zariski-dense subgroups of semisimple algebraic groups, *Ann. of Math.* (2) **120** (1984), no. 2, 271–315.

[W2] ____ , Post-classification version of Jordan's theorem on finite linear groups, *Proc. Nat. Acad. Sci. USA* **81** (1984), no. 16, Phys. Sci., 5278–5279.