

STABILITY AND INVARIANT RANDOM SUBGROUPS

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Abstract

Consider $\text{Sym}(n)$ endowed with the normalized Hamming metric d_n . A finitely generated group Γ is P -stable if every almost homomorphism $\rho_{n_k} : \Gamma \rightarrow \text{Sym}(n_k)$ (i.e., for every $g, h \in \Gamma$, $\lim_{k \rightarrow \infty} d_{n_k}(\rho_{n_k}(gh), \rho_{n_k}(g)\rho_{n_k}(h)) = 0$) is close to an actual homomorphism $\varphi_{n_k} : \Gamma \rightarrow \text{Sym}(n_k)$. Glebsky and Rivera observed that finite groups are P -stable, while Arzhantseva and Păunescu showed the same for abelian groups and raised many questions, especially about the P -stability of amenable groups. We develop P -stability in general and, in particular, for amenable groups. Our main tool is the theory of invariant random subgroups, which enables us to give a characterization of P -stability among amenable groups and to deduce the stability and instability of various families of amenable groups.

1. Introduction

Let $(G_n, d_n)_{n=1}^\infty$ be a sequence of groups G_n equipped with bi-invariant metrics d_n , and let Γ be a finitely presented¹ group generated by a finite set $S = \{s_1, \dots, s_m\}$ subject to the relations $E = \{w_1, \dots, w_r\} \subset \mathbb{F}$, where $\mathbb{F}$ is the free group on S . In recent years, there has been some interest in the *stability* of Γ with respect to $\mathcal{G} = (G_n, d_n)_{n=1}^\infty$ (see [4], [10], [15] and the references therein). Specifically, we have the following definition.

Definition 1.1

The group Γ is *stable* with respect to $\mathcal{G}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\bar{g} = (g_1, \dots, g_m) \in G_n^m$ satisfies $\sum_{i=1}^r d_n(w_i(\bar{g}), \text{id}_{G_n}) < \delta$, then $\exists \bar{g}' = (g'_1, \dots, g'_m) \in G_n^m$ with $\sum_{i=1}^m d_n(g_i, g'_i) < \epsilon$ and $w_i(\bar{g}') = \text{id}$ for every $i = 1, \dots, m$ (i.e., $\bar{g}$ and $\bar{g}'$ are “ ϵ -close” and $\bar{g}'$ is a “solution” for $w_1 = 1, \dots, w_r = 1$).

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¹The main results of this paper will be formulated and proved for general finitely generated groups, but for the simplicity of the exposition in this [Introduction](#), we will assume that Γ is finitely presented.

In other words, every “almost homomorphism” from Γ to G_n is close to an actual homomorphism. It is not difficult to show (see [4]) that the stability of Γ with respect to $(G_n, d_n)_{n=1}^\infty$ depends only on the group Γ , rather than the chosen presentation—so the notion is well defined.

The roots of this definition lie in some classical questions, asked by Halmos, Turing, Ulam, and others, about whether “almost solutions” are always just small deformations of precise solutions. The most popular question of this sort, with origins in mathematical physics, refers to the case where the G_n ’s are some groups of matrices and the question asks whether “almost commuting matrices” are “near” commuting matrices (which is the same as the stability of $\Gamma = \mathbb{Z} \times \mathbb{Z}$ defined above). The answer in this case depends very much on the metrics d_n ; for example, if $G_n = \mathrm{U}(n)$ is the unitary group of degree n , then $\mathbb{Z}^2$ is stable with respect to the Hilbert–Schmidt norm but not with respect to the operator norm (see [14], [28]; see also the [Introduction](#) of [4] for a short survey of this problem).

In recent years (starting in [15] and in a more systematic way in [4]), an interest has grown in a discrete version, that is, in the case where $G_n = \mathrm{Sym}(n)$ is the symmetric group on $[n] = \{1, \dots, n\}$, where d_n is the normalized Hamming distance $d_n(\sigma, \tau) = \frac{1}{n} \cdot |\{x \in [n] \mid \sigma(x) \neq \tau(x)\}|$. We will refer to this as *permutation stability* (or *P-stability* for short). One of the motivations to study this comes from the “local testability” of systems of equations in permutation groups (see [4], [7], [15]). Another motivation comes from the hope to find a nonsofic group; as was observed in [15], groups which are P-stable and not residually finite are *not* sofic. So it is desirable to have criteria for a group to be P-stable (see [10] for a similar strategy, which led to the construction of non-Frobenius-approximated groups).

But, as of now, there are very few methods and results for proving the P-stability of groups. Clearly, free groups are P-stable, Glebsky and Rivera in [15] showed that finite groups are P-stable, and Arzhantseva and Păunescu in [4] showed the same for abelian groups. Now, a free product of P-stable groups is P-stable, but this is not known, in general, for direct products!

In this paper, we develop P-stability and non-P-stability criteria. Some of these are for general groups, but they are especially effective for amenable groups. Here is a sample of some conclusions of our work (see Corollaries 8.2, 8.4, and 8.7).

THEOREM 1.2

- (i) *Every polycyclic-by-finite group is P-stable.*
- (ii) *For every $n \in \mathbb{Z}$, the Baumslag–Solitar group $\mathrm{BS}(1, n) = \langle x, y \mid xyx^{-1} = y^n \rangle$ is P-stable.*
- (iii) *There exists a finitely presented solvable subgroup of $\mathrm{GL}_4(\mathbb{Q})$ which is not P-stable.*

Part (i) of the theorem above answers a question raised in [18] (the very special case of the group $BS(1, -1)$ was previously asked about in [4]; see the paragraph after [4, Example 7.3]). Part (ii) completes the classification of the P-stability of the Baumslag–Solitar groups $BS(m, n)$ which was started in [4, Example 7.3], except for the case $|m| = |n| \geq 2$. Part (iii) shows that there is a finitely presented amenable residually finite group which is not P-stable, answering a question posed in [4] (see, in [4], the paragraph before Theorem 7.2 and also Theorem 7.2(iii)).

The main novel method in the current paper is the use of the theory of *invariant random subgroups* (IRS). This theory, which formally goes back to the seminal work of Stuck and Zimmer in [27], got new life in recent years starting with the work of Abert, Glasner, and Virag in [2]: an IRS is defined as a Γ -invariant probability measure on the compact space $\text{Sub}(\Gamma)$ of all (closed) subgroups of Γ . Let $\text{IRS}(\Gamma)$ be the space of IRS of Γ . Fixing a surjective map $\pi : \mathbb{F} \twoheadrightarrow \Gamma$ from the free group on S onto Γ , we can think of $\text{IRS}(\Gamma)$ as a subspace of $\text{IRS}(\mathbb{F})$.

If Γ is a discrete group, then $\mu \in \text{IRS}(\Gamma)$ will be called a *finite-index IRS* if it is atomic and all of its atoms are finite-index subgroups of Γ . The IRS which are a limit of the finite-index ones are called *cosofic* (see [13, Definition 15]). We then prove the following (see Theorem 7.10).

THEOREM 1.3

Let $\pi : \mathbb{F} \twoheadrightarrow \Gamma$ be as before, and let $\text{IRS}(\Gamma) \subset \text{IRS}(\mathbb{F})$. Then the following hold.

- (i) If Γ is P-stable and $\mu \in \text{IRS}(\Gamma)$ is cosofic in $\mathbb{F}$, then μ is cosofic in Γ .
- (ii) If Γ is amenable, then Γ is P-stable if and only if every $\mu \in \text{IRS}(\Gamma)$ is cosofic (in Γ).

Theorem 1.3(ii) gives an “if and only if” criterion for the P-stability of amenable groups. A crucial ingredient in the proof of this criterion is a result of Newman and Sohler from [23] and [24] which gives the testability of properties of hyperfinite families of graphs (see [7] for more in this direction). Actually, in the sequel, it will be more convenient for us to use Elek’s treatment in [11] of the aforementioned theorem. The amenability assumption in Theorem 1.3(ii) turns out to be essential. Indeed, by [6], the groups $SL_n(\mathbb{Z})$ are not P-stable for $n \geq 3$; but, as a corollary of the Stuck–Zimmer theorem (see [27, Theorem 2.1]), all of their IRSs are supported on finite-index subgroups (and the trivial subgroup $\{1\}$ and the center), and, in particular, they are cosofic.

Let us sketch the argument for the $\Leftarrow$ direction of Theorem 1.3(ii). We think of the hypothesis that every IRS of Γ is cosofic as a “density condition.” Let $(X_n)_{n=1}^\infty$ be a sequence of finite sets with almost actions of Γ , where Γ is amenable. Then, as n tends to infinity, X_n converges to a *probability measure preserving (pmp.)* action of Γ ,

and hence, one obtains an IRS of Γ . Now, by assuming the density condition, this IRS also arises as a limit of finite Γ -actions. A little argument ensures that these actions can happen on the same sets X_n . Both the sequence of actions and the sequence of almost actions are hyperfinite, since the group Γ is amenable. Hence, by the Elek–Newman–Sohler result (on almost isomorphisms of hyperfinite graphs with almost the same local statistics), the almost actions are almost conjugate to the actions if n is large enough, which is the end of the proof. The role of the density condition in the argument above is to ensure that there are enough actions to model any possible IRS that could come up. For the $\Rightarrow$ direction, any IRS (if Γ is amenable) actually arises as a limit of almost actions of Γ on finite sets, so that the condition of density of finite-index IRS is also necessary.

In general, it is not easy to check the criterion of Theorem 1.3(ii), but if Γ has only countably many subgroups (see [9] for a characterization of solvable groups with this property), then every $\mu \in \text{IRS}(\Gamma)$ is atomic and hence supported only on almost-normal subgroups, that is, subgroups H for which $[\Gamma : N_\Gamma(H)] < \infty$. This enables us to prove the following (see Proposition 8.1).

THEOREM 1.4

If $\text{Sub}(\Gamma)$ is countable and every almost-normal subgroup of Γ is profinitely closed in Γ , then every $\mu \in \text{IRS}(\Gamma)$ is cosofic in Γ , and if Γ is also amenable, then Γ is P-stable.

The first two points of Theorem 1.2 are deduced from Theorem 1.4. We also show that if there exists a *finitely generated* almost-normal subgroup of Γ which is *not* profinitely closed, then Γ is not P-stable (assuming that Γ is amenable, but also under a milder condition related to soficity), and this is used to prove part (iii) of Theorem 1.2.

The present article is organized as follows. In Section 3, we give the definitions of P-stable equations and groups and explain the relation between the two notions. In Sections 4 and 5, we review the needed facts regarding IRS and the profinite topology, respectively. In Section 6, we review the theories of hyperfinite actions and graphs and adapt the Newman–Sohler theorem to our needs. In Section 7, we prove Theorem 1.3. Finally, in Section 8, we use Theorem 1.3 to prove Theorems 1.2 and 1.4.

Let us end by saying that, while our results give far-reaching extensions of the groups for which P-stability or non-P-stability is known, we are still far from having the complete picture even for amenable groups (or even for solvable groups). We still cannot answer the question whether, for given P-stable groups Γ_1 and Γ_2 , $\Gamma_1 \times \Gamma_2$ is also P-stable. Is *locally extended residually finite (LERF)* a sufficient condition? More specifically, is the Grigorchuk group P-stable? (See Question 8.6 and the discussion

surrounding it.) Our work gives further motivation to understand and classify the IRS of various finitely generated groups.

2. Notation and conventions

Throughout the paper, we fix the following. Let Γ be a finitely generated group. Present Γ as a quotient of a finitely generated free group $\mathbb{F}$ with quotient map $\pi: \mathbb{F} \rightarrow \Gamma$. Fix a finite basis $S = \{s_1, \dots, s_m\}$ for $\mathbb{F}$. Note that every result we prove for Γ applies to $\mathbb{F}$ as well as a special case by viewing $\mathbb{F}$ as a quotient of itself with π being the identity map.

Recall that a Γ -set is a set X endowed with an action of Γ , that is, a homomorphism $\rho: \Gamma \rightarrow \text{Sym}(X)$ called the *structure homomorphism* of the action. When ρ is understood from the context, we write $g \cdot x$ for $\rho(g)(x)$, where $g \in \Gamma$ and $x \in X$. We also write $\Gamma \curvearrowright X$ when we want to refer to an action of Γ on a set X , but suppress the structure homomorphism ρ . For a subgroup H of Γ , we endow the coset space Γ/H , by default, with the action given by $g \cdot (g_1 H) = (gg_1)H$.

For a subset A of Γ : Write $A^{-1} = \{a^{-1}\}_{a \in A}$ and $A^{\pm 1} = A \cup A^{-1}$. Write $\langle A \rangle$ for the subgroup generated by A and $\langle\langle A \rangle\rangle$ for the normal closure of A in Γ (i.e., the smallest normal subgroup of Γ which contains A or, equivalently, the subgroup consisting of products of Γ -conjugates of elements of $A^{\pm 1}$).

For a Γ -set X , define a metric d_X on X where $d_X(x, y)$ is the length, with respect to $S^{\pm 1}$, of the shortest word $w \in \mathbb{F}$ for which $w \cdot x = y$ (or ∞ if no such word exists, but we shall always work within connected components anyway). For an element $g \in \Gamma$ and a subset $A \subset \Gamma$, write $g \cdot A = \{g \cdot a \mid a \in A\}$. We use the notation $\coprod$ for disjoint unions and write $X^{\sqcup k}$ for the disjoint union of k copies of X .

For a metric space X , for an integer $r \geq 0$ and a point $x \in X$, write $B_X(x, r) = \{y \in X \mid d_X(x, y) \leq r\}$. For an integer $r \geq 0$ and a subset $A \subset X$, write $B_X(A, r) = \bigcup_{a \in A} B_X(a, r)$. In the case in which $X = \Gamma$, write $B_\Gamma(r)$ for $B_\Gamma(1_\Gamma, r)$.

For a logical formula φ , we write $\mathbf{1}_\varphi$ to mean 1 if φ holds in the given context and 0 otherwise. For a subgroup H of Γ , write H^Γ for the set of subgroups of Γ which are conjugate to H , that is, $H^\Gamma = \{H^g \mid g \in \Gamma\}$. Then, $|H^\Gamma| = [\Gamma : N_\Gamma(H)]$, and we say that H is *almost normal* in Γ if $|H^\Gamma| < \infty$. For an element x in a measurable space X , we write δ_x for the Dirac measure at x . For $n \in \mathbb{N}$, denote $[n] = \{1, \dots, n\}$.

3. Definitions

3.1. P -stable equations

We refer to the elements of the basis S of $\mathbb{F}$ as *letters* and to the elements of $\mathbb{F}$ as *words*. For a word $w \in \mathbb{F}$, an integer $n \geq 1$, and a tuple of permutations $(\sigma_1, \dots, \sigma_m) \in \text{Sym}(n)^m$, we write $w(\sigma_1, \dots, \sigma_m)$ for the element of $\text{Sym}(n)$ result-

ing from the substitution $s_1 \mapsto \sigma_1, \dots, s_m \mapsto \sigma_m$ applied to the word w . That is, if $w = s_{i_1}^{\epsilon_1} \cdots s_{i_l}^{\epsilon_l}$ for some integers $l \geq 0$, $i_1, \dots, i_l \in [m]$, and $\epsilon_1, \dots, \epsilon_l \in \{+1, -1\}$, then $w(\sigma_1, \dots, \sigma_m) = \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_l}^{\epsilon_l} \in \text{Sym}(n)$.

Definition 3.1

For $n \in \mathbb{N}$, the *normalized Hamming distance* d_n on $\text{Sym}(n)$ is defined by $d_n(\sigma_1, \sigma_2) = \frac{1}{n} |\{x \in [n] \mid \sigma_1(x) \neq \sigma_2(x)\}|$, where $\sigma_1, \sigma_2 \in \text{Sym}(n)$.

Note that d_n is a bi-invariant metric on $\text{Sym}(n)$.

Definition 3.2

Let $n \in \mathbb{N}$, let $E \subset \mathbb{F}$, and let $(\sigma_1, \dots, \sigma_m) \in \text{Sym}(n)^m$. Then the following hold.

- (i) The tuple $(\sigma_1, \dots, \sigma_m)$ is a *solution* for the system of equations $\{w = 1\}_{w \in E}$ if $w(\sigma_1, \dots, \sigma_m) = 1$ for each $w \in E$.
- (ii) Assume that E is a finite set. For $\delta > 0$, the tuple $(\sigma_1, \dots, \sigma_m)$ is a δ -*solution* for the system of equations $\{w = 1\}_{w \in E}$ if

$$\sum_{w \in E} d_n(w(\sigma_1, \dots, \sigma_m), 1) \leq \delta.$$

Definition 3.3

For $n \in \mathbb{N}$ and $\bar{\sigma} = (\sigma_1, \dots, \sigma_m), \bar{\tau} = (\tau_1, \dots, \tau_m) \in \text{Sym}(n)^m$, define $d_n(\bar{\sigma}, \bar{\tau}) = \sum_{i=1}^m d_n(\sigma_i, \tau_i)$. For $\epsilon > 0$, if $d_n(\bar{\sigma}, \bar{\tau}) \leq \epsilon$, then we say that $\bar{\sigma}$ and $\bar{\tau}$ are ϵ -close.

Definition 3.4

For $E \subset \mathbb{F}$, we say that the system of equations $\{w = 1\}_{w \in E}$ is *stable in permutations* (or *P-stable* for short) if for every $\epsilon > 0$ there are $\delta > 0$ and a finite subset $E_0 \subset E$ such that, for every $n \in \mathbb{N}$ and δ -solution $(\sigma_1, \dots, \sigma_m) \in \text{Sym}(n)^m$ for $\{w = 1\}_{w \in E_0}$, there is a solution $(\tau_1, \dots, \tau_m) \in \text{Sym}(n)^m$ for $\{w = 1\}_{w \in E}$ such that $(\sigma_1, \dots, \sigma_m)$ and $(\tau_1, \dots, \tau_m)$ are ϵ -close.

Remark 3.5

The notion of a “stable system,” introduced in [4], is a special case of Definition 3.4 for a *finite* $E \subset \mathbb{F}$. Indeed, Definition 3.2 of [4] says that a finite $E \subset \mathbb{F}$ is a *stable system* if, for every $\epsilon > 0$, there is $\delta > 0$ such that every δ -solution for $\{w = 1\}_{w \in E}$ is ϵ -close to a solution for $\{w = 1\}_{w \in E}$. This is indeed equivalent to our Definition 3.4 in light of Remark 3.6 below.

Remark 3.6

For $E_1 \subset E_2 \subset \mathbb{F}$, every solution for $\{w = 1\}_{w \in E_2}$ is a solution for $\{w = 1\}_{w \in E_1}$. Moreover, if E_2 is finite, then for $\delta > 0$, every δ -solution for $\{w = 1\}_{w \in E_2}$ is a

δ -solution for $\{w = 1\}_{w \in E_1}$. If $w_1, w_2 \in \mathbb{F}$, then every simultaneous solution for $\{w_1 = 1\}$ and $\{w_2 = 1\}$ is a solution for $\{w_1 \cdot w_2 = 1\}$. If $t, w \in \mathbb{F}$, then every solution for $\{w = 1\}$ is a solution for $\{t \cdot w \cdot t^{-1} = 1\}$. By the above, if $E \subset \mathbb{F}$ and $n \in \mathbb{N}$, then a tuple $(\sigma_1, \dots, \sigma_m) \in \text{Sym}(n)^m$ is a solution for $\{w = 1\}_{w \in E}$ if and only if it is a solution for $\{w = 1\}_{w \in \langle\langle E \rangle\rangle}$.

LEMMA 3.7

Let $E \subset \mathbb{F}$. Take $\tilde{\delta} > 0$, and take a finite subset $\tilde{E}_0 \subset \langle\langle E \rangle\rangle$. Then, there are $\delta > 0$ and a finite subset $E_0 \subset E$ such that every δ -solution for $\{w = 1\}_{w \in E_0}$ is a $\tilde{\delta}$ -solution for $\{w = 1\}_{w \in \tilde{E}_0}$.

Proof

For every $w \in \tilde{E}_0$, write $w = \prod_{i=1}^{l_w} t_{w,i} \cdot q_{w,i}^{\epsilon_{w,i}} \cdot t_{w,i}^{-1}$, where $l_w \geq 0$, $\{q_{w,i}\}_{i=1}^{l_w} \subset E$, $\{\epsilon_{w,i}\}_{i=1}^{l_w} \subset \{1, -1\}$, and $\{t_{w,i}\}_{i=1}^{l_w} \subset \mathbb{F}$. Let $E_0 = \{q_{w,i} \mid w \in \tilde{E}_0, 1 \leq i \leq l_w\}$, and let $C = \sum_{w \in \tilde{E}_0} l_w$. Define $\delta = \frac{1}{C} \cdot \tilde{\delta}$. Take $n \in \mathbb{N}$, and take a δ -solution $(\sigma_1, \dots, \sigma_m) \in \text{Sym}(n)^m$ for $\{w = 1\}_{w \in E_0}$. For every $x \in [n]$ and $w \in \mathbb{F}$, write $w \cdot x$ for $w(\sigma_1, \dots, \sigma_m)(x)$. For every $w \in E_0$, write $F_w = \{x \in [n] \mid w \cdot x \neq x\}$. Then, $\sum_{w \in E_0} |F_w| \leq \delta n$. A fortiori, $|F_w| \leq \delta n$ for each $w \in E_0$.

Let $w \in \tilde{E}_0$. Define $P_w = \bigcup_{i=1}^{l_w} t_{w,i} \cdot F_{q_{w,i}}$. Then $|P_w| \leq \sum_{i=1}^{l_w} |F_{q_{w,i}}| \leq l_w \cdot \delta n$. For $x \in [n]$ and $1 \leq i \leq l_w$, if $t_{w,i}^{-1} \cdot x \notin F_{q_{w,i}}$, then

$$(t_{w,i} \cdot q_{w,i} \cdot t_{w,i}^{-1}) \cdot x = t_{w,i} \cdot q_{w,i} \cdot (t_{w,i}^{-1} \cdot x) = t_{w,i} \cdot (t_{w,i}^{-1} \cdot x) = x,$$

and so if $x \notin P_w$, then

$$w \cdot x = \left(\prod_{i=1}^{l_w} t_{w,i} \cdot q_{w,i} \cdot t_{w,i}^{-1} \right) \cdot x = x.$$

Therefore, $d_n(w(\sigma_1, \dots, \sigma_m), 1) \leq \frac{1}{n} \cdot |P_w| \leq \frac{1}{n} \cdot l_w \cdot \delta n = l_w \cdot \delta$. Finally, $\sum_{w \in \tilde{E}_0} d_n(w(\sigma_1, \dots, \sigma_m), 1) \leq C \cdot \delta = \tilde{\delta}$. In other words, $(\sigma_1, \dots, \sigma_m)$ is a $\tilde{\delta}$ -solution for $\{w = 1\}_{w \in \tilde{E}_0}$. $\square$

LEMMA 3.8

Let $E \subset \mathbb{F}$. Then, $\{w = 1\}_{w \in E}$ is P -stable if and only if $\{w = 1\}_{w \in \langle\langle E \rangle\rangle}$ is P -stable.

Proof

Assume that $\{w = 1\}_{w \in E}$ is P -stable. Let $\epsilon > 0$. Then, there are $\delta > 0$ and a finite subset $E_0 \subset E$ (and so $E_0 \subset \langle\langle E \rangle\rangle$) such that every δ -solution for $\{w = 1\}_{w \in E_0}$ is ϵ -close to a solution for $\{w = 1\}_{w \in E}$. The latter is a solution for $\{w = 1\}_{w \in \langle\langle E \rangle\rangle}$ as well by Remark 3.6, and so $\{w = 1\}_{w \in \langle\langle E \rangle\rangle}$ is P -stable. The reverse implication follows similarly using Lemma 3.7. $\square$

3.2. P -stable groups

Each Γ -set X is naturally an $\mathbb{F}$ -set. Conversely, for an $\mathbb{F}$ -set X , if the structure homomorphism $\rho: \mathbb{F} \rightarrow \text{Sym}(X)$ factors through Γ by $\pi: \mathbb{F} \twoheadrightarrow \Gamma$, then X is naturally a Γ -set. This condition is equivalent to the following: for every $w \in \mathbb{F}$ and $x \in X$, if $\pi(w) = 1_\Gamma$, then $w \cdot x = x$.

Definition 3.9

For $\delta > 0$ and a finite subset $E_0 \subset \text{Ker } \pi$, a finite $\mathbb{F}$ -set X is a (δ, E_0) -almost- Γ -set if $\sum_{w \in E_0} \Pr_{x \in X}(w \cdot x \neq x) \leq \delta$ (where X is endowed with the uniform distribution).

Definition 3.10

Let X and Y be finite $\mathbb{F}$ -sets of the same cardinality. For a bijection $f: X \rightarrow Y$, define

$$\|f\|_{\text{gen}} = \frac{1}{|S|} \cdot \sum_{s \in S} \Pr_{x \in X}(f(s \cdot x) \neq s \cdot f(x)).$$

Finally,

$$d_{\text{gen}}(X, Y) = \min\{\|f\|_{\text{gen}} \mid f: X \rightarrow Y \text{ is a bijection}\}.$$

We refer to d_{gen} as the *generator metric*.

(Definition 3.10 will be generalized by Definition 6.2.)

For $n \in \mathbb{N}$ and a tuple $\bar{\sigma} = (\sigma_1, \dots, \sigma_m) \in \text{Sym}(n)^m$, write $\mathbb{F}(\bar{\sigma})$ for the $\mathbb{F}$ -set whose point set is $[n]$, with the action given by $s_i \cdot x = \sigma_i(x)$ for each $1 \leq i \leq m$. Note that, for $\bar{\sigma}, \bar{\tau} \in \text{Sym}(n)^m$, $\|\text{id}\|_{\text{gen}} = d_n(\bar{\sigma}, \bar{\tau})$, where $\text{id}: \mathbb{F}(\bar{\sigma}) \rightarrow \mathbb{F}(\bar{\tau})$ is the identity map $[n] \rightarrow [n]$.

In Definition 3.4, we generalized (see Remark 3.5) the notion of a P -stable system of equations from finite systems (as studied in [15] and [4]) to possibly infinite systems. Analogously, we now generalize the notion of P -stable groups, studied in the aforementioned papers for finitely presented groups and coinciding with Definition 1.1 in the [Introduction](#), to finitely generated groups.

Definition 3.11

The group Γ is *stable in permutations* (or P -stable for short) if for every $\epsilon > 0$ there are $\delta > 0$ and a finite subset $E_0 \subset \text{Ker } \pi$ such that, for every finite $\mathbb{F}$ -set X , if X is a (δ, E_0) -almost- Γ -set, then there is a Γ -set Y such that $|X| = |Y|$ and $d_{\text{gen}}(X, Y) \leq \epsilon$.

LEMMA 3.12

The group Γ is P -stable if and only if the system of equations $\{w = 1\}_{w \in \text{Ker } \pi}$ is P -stable.

Proof

Assume that Γ is P-stable. Then, for $\epsilon > 0$, there are $\delta > 0$ and $E_0 \subset \text{Ker } \pi$ satisfying the condition in Definition 3.11. Let $\bar{\sigma} \in \text{Sym}(n)^m$ be a δ -solution for $\{w = 1\}_{w \in E_0}$. Let $X = \mathbb{F}(\bar{\sigma})$. Then, X is a (δ, E_0) -almost- Γ -set. Therefore, there are a Γ -set Y and a bijection $f: X \rightarrow Y$ satisfying $\|f\|_{\text{gen}} \leq \epsilon$. Define a tuple $\bar{\tau} = (\tau_1, \dots, \tau_m) \in \text{Sym}(n)^m$ by $\tau_i(x) = f^{-1}(s_i \cdot f(x))$. Then, $\bar{\tau}$ is a solution for E , and it is ϵ -close to $\bar{\sigma}$.

In the other direction, assume that $\{w = 1\}_{w \in \text{Ker } \pi}$ is P-stable. Then, for every $\epsilon > 0$, there are $\delta > 0$ and $E_0 \subset \text{Ker } \pi$ satisfying the condition in Definition 3.4. Let X be a (δ, E_0) -almost- Γ -set. Denote $|X| = n$, take an arbitrary bijection $f: [n] \rightarrow X$, and define a tuple $\bar{\sigma} = (\sigma_1, \dots, \sigma_m) \in \text{Sym}(n)^m$ by $\sigma_i(x) = f^{-1}(s_i \cdot f(x))$. Then, $\bar{\sigma}$ is a δ -solution for E_0 . Therefore, there is a solution $\bar{\tau} \in \text{Sym}(n)^m$ for $\text{Ker } \pi$ which is ϵ -close to $\bar{\sigma}$. Let $Y = \mathbb{F}(\bar{\tau})$. Consider f as a function from Y to X . Then, $\|f\|_{\text{gen}} \leq \epsilon$, and so $d_{\text{gen}}(X, Y) \leq \epsilon$. $\square$

Remark 3.13

Definition 3.11 introduces the notion of a P-stable group Γ using a given presentation of Γ as a quotient of a finitely generated free group. Nevertheless, the definition depends only on Γ as an abstract group. Indeed, consider two finitely generated free groups $\mathbb{F}_S$ and $\mathbb{F}_T$ with bases S and T , respectively. Denote the generator metrics on finite $\mathbb{F}_S$ -sets and on finite $\mathbb{F}_T$ -sets by d_{gen}^S and d_{gen}^T , respectively. Present Γ in two ways: $\pi_S: \mathbb{F}_S \twoheadrightarrow \Gamma$ and $\pi_T: \mathbb{F}_T \twoheadrightarrow \Gamma$. For every $t \in T$, let $v_t \in \mathbb{F}_S$ be a word for which $\pi_S(v_t) = \pi_T(t)$. Define a homomorphism $\alpha: \mathbb{F}_T \rightarrow \mathbb{F}_S$ by extending the law $\alpha(t) = v_t$. Then, every $\mathbb{F}_S$ -set is naturally an $\mathbb{F}_T$ -set. There is a constant $C > 0$ such that, for every pair X and Y of finite $\mathbb{F}_S$ -sets of the same cardinality, $d_{\text{gen}}^T(X, Y) \leq C \cdot d_{\text{gen}}^S(X, Y)$. Moreover, for $E_0 \subset \text{Ker } \pi_T$, $\delta > 0$, and an $\mathbb{F}_S$ -set X , if X is a $(\delta, \alpha(E_0))$ -almost- Γ -set, then as an $\mathbb{F}_T$ -set it is a (δ, E_0) -almost- Γ -set. Running the same arguments with S and T reversed, we see that Γ is P-stable with respect to π_S if and only if it is P-stable with respect to π_T . More concisely, we have shown that the metrics d_{gen}^S and d_{gen}^T are bi-Lipschitz equivalent and that the notions of almost- Γ -sets with respect to $\mathbb{F}_S$ and to $\mathbb{F}_T$ are essentially equivalent.

4. Invariant random subgroups

We recall the notion of an IRS (see [2], [12], [13]). Write 2^Γ for the set of functions $f: \Gamma \rightarrow \{0, 1\}$ and identify 2^Γ with the power set of Γ by associating each function $f: \Gamma \rightarrow \{0, 1\}$ with the set $\{g \in \Gamma \mid f(g) = 1\}$. Denote the set of subgroups of Γ by $\text{Sub}(\Gamma)$. Endow $\text{Sub}(\Gamma)$ with the Chabauty topology, which, for discrete groups, is just the subspace topology induced from the product topology on 2^Γ and the inclusion $\text{Sub}(\Gamma) \subset 2^\Gamma$. A sequence $(U_n)_{n=1}^\infty$ in 2^Γ converges if and only if, for every $g \in \Gamma$, either $g \in U_n$ for all large enough n or $g \notin U_n$ for all large enough n . In this case,

$\limsup U_n = \liminf U_n = U$, where U is the limit of the sequence. This also shows that $\text{Sub}(\Gamma)$ is a closed subspace of 2^Γ , and so it is compact. The group Γ acts on $\text{Sub}(\Gamma)$ continuously by conjugation. Write $\text{Sub}_{\text{f.g.}}(\Gamma)$, $\text{Sub}_{\text{f.i.}}(\Gamma)$, and $\text{Sub}_{\text{a.n.}}(\Gamma)$ for the subspaces of $\text{Sub}(\Gamma)$ of finitely generated subgroups, finite-index subgroups, and almost-normal subgroups, respectively.

For an element $w \in \Gamma$, define $C_w = \{H \leq \Gamma \mid w \in H\}$. For an integer $r \geq 0$ and a subset $W \subset \Gamma$, let $C_{r,W} = \{H \leq \Gamma \mid H \cap B_\Gamma(r) = W \cap B_\Gamma(r)\}$. Note that such sets C_w and $C_{r,W}$ are clopen in $\text{Sub}(\Gamma)$, and so their characteristic functions are continuous. For a given subgroup $K \leq \Gamma$, the subspace $\{H \leq \Gamma \mid K \leq H\}$ of $\text{Sub}(\Gamma)$ is closed since it equals $\bigcap_{k \in K} C_k$.

We exhibit a metric generating the topology of $\text{Sub}(\Gamma)$. Fix an enumeration $(g_i)_{i=1}^\infty$ of the elements of Γ . The metric on 2^Γ defined by $d_{\text{prod}}(U_1, U_2) = \sum_{i=1}^\infty 2^{-i} \cdot \mathbf{1}_{U_1 \cap \{g_i\} = U_2 \cap \{g_i\}}$ induces the product topology on 2^Γ , and so its restriction to $\text{Sub}(\Gamma)$ induces the Chabauty topology. Note that, for every $\epsilon > 0$, there is an integer $r \geq 1$ such that, for all $H_1, H_2 \leq \Gamma$, if $H_1 \cap B_\Gamma(r) = H_2 \cap B_\Gamma(r)$, then $d_{\text{prod}}(H_1, H_2) < \epsilon$.

Consider the space $\text{Prob}(\text{Sub}(\Gamma))$ of Borel regular probability measures on $\text{Sub}(\Gamma)$. We shall refer to elements of $\text{Prob}(\text{Sub}(\Gamma))$ as *random subgroups*. The group Γ acts on $\text{Prob}(\text{Sub}(\Gamma))$ by conjugation, that is, $(g \cdot \mu)(A) = \mu(g^{-1}Ag)$. We write $\text{IRS}(\Gamma)$ for the subspace of $\text{Prob}(\text{Sub}(\Gamma))$ of conjugation-invariant random subgroups, namely, $\text{IRS}(\Gamma) = \text{Prob}(\text{Sub}(\Gamma))^\Gamma$. We shall refer to elements of $\text{IRS}(\Gamma)$ as *IRS*. Endow $\text{IRS}(\Gamma)$ with the weak-* topology. A sequence $(\mu_n)_{n=1}^\infty$ in $\text{IRS}(\Gamma)$ converges in the weak-* topology to $\mu \in \text{IRS}(\Gamma)$ if and only if $\int f d\mu_n \rightarrow \int f d\mu$ for every continuous function $f: \text{Sub}(\Gamma) \rightarrow \mathbb{R}$. It follows from the Riesz–Markov and Banach–Alaoglu theorems that $\text{IRS}(\Gamma)$ is a compact space. Moreover, under the weak-* topologies, $\text{IRS}(\Gamma)$ is metrizable by the Lévy–Prokhorov metric. We shall only use the metrizability of $\text{IRS}(\Gamma)$ to identify the closure of a given subset A of $\text{IRS}(\Gamma)$ with the set of limits of convergent sequences (rather than nets) in A .

The space $\text{Sub}(\Gamma)$ enjoys a useful sequence $\mathcal{P}_n(\Gamma)$ of partitions into finitely many clopen sets. For $n \in \mathbb{N}$, define the partition

$$\mathcal{P}_n = \mathcal{P}_n(\Gamma) = \{C_{n,W} \mid W \subset B_\Gamma(n), C_{n,W} \neq \emptyset\}.$$

For a continuous function $f: \text{Sub}(\Gamma) \rightarrow \mathbb{R}$, define a sequence of continuous functions $f_n: \text{Sub}(\Gamma) \rightarrow \mathbb{R}$ by $f_n = \sum_{A \in \mathcal{P}_n} f(K_A) \cdot \mathbf{1}_A$, where K_A is an arbitrary element of A for each $A \in \mathcal{P}_n$. Then, since f is uniformly continuous, $\|f_n - f\|_\infty \rightarrow 0$. This shows that, for $\mu \in \text{IRS}(\Gamma)$ and a sequence $(\mu_n)_{n=1}^\infty$ in $\text{IRS}(\Gamma)$, $\mu_n \rightarrow \mu$ in the weak-* topology if and only if, for every integer $r \geq 1$ and $W \subset B_\Gamma(r)$, $\mu_n(C_{r,W}) \xrightarrow{n \rightarrow \infty} \mu(C_{r,W})$.

Let $\mu \in \text{IRS}(\Gamma)$ be an atomic IRS. Then, all atoms of μ must be almost-normal subgroups of Γ . Fix $n \in \mathbb{N}$. Take pairwise nonconjugate subgroups $H_1, \dots, H_k$ of Γ

such that $M > 1 - \frac{1}{n}$, where $M = \sum_{i=1}^k m_i$ and $m_i = \mu(H_i^\Gamma)$. Write $A = \bigcup_{i=1}^k H_i^\Gamma$. Let $\mu_n \in \text{IRS}(\Gamma)$ be the atomic IRS assigning measure $\frac{m_i}{M \cdot |H_i^\Gamma|}$ to each conjugate of H_i for every $1 \leq i \leq k$. Take a continuous function $f: \text{Sub}(\Gamma) \rightarrow \mathbb{R}$. Then,

$$\begin{aligned} \left| \int f d\mu_n - \int f d\mu \right| &\leq \left| \int_A f d\mu_n - \int_A f d\mu \right| \\ &\quad + \left| \int_{\text{Sub}(\Gamma) \setminus A} f d\mu_n - \int_{\text{Sub}(\Gamma) \setminus A} f d\mu \right| \\ &\leq \sum_{i=1}^k \frac{1}{|H_i^\Gamma|} \cdot \sum_{K \in H_i^\Gamma} \left(\frac{m_i}{M} - m_i \right) \cdot f(K) \\ &\quad + \left| 0 - \int_{\text{Sub}(\Gamma) \setminus A} f d\mu \right| \\ &\leq \left(\frac{1}{M} - 1 \right) \cdot \left(\sum_{i=1}^k m_i \right) \cdot \|f\|_\infty + \frac{1}{n} \cdot \|f\|_\infty \\ &= \left((1 - M) + \frac{1}{n} \right) \cdot \|f\|_\infty \\ &\leq \frac{2}{n} \cdot \|f\|_\infty \end{aligned}$$

and so $\mu_n \rightarrow \mu$. We have thus shown that every atomic IRS $\mu \in \text{IRS}(\Gamma)$ is the limit of a sequence $(\mu_n)_{n=1}^\infty$ of finitely supported atomic IRS with $\text{supp}(\mu_n) \subset \text{supp}(\mu)$.

Recall that a standard Borel space is a measurable space which is isomorphic, as a measurable space, to a compact metric space with its Borel σ -algebra.

Definition 4.1

A *probability space* is a standard Borel space endowed with a Borel regular probability measure. A Γ -*probability space* X is a probability space endowed with a Borel action $\Gamma \curvearrowright X$. If the action is pmp, we say for short that X is a *pmp Γ -space*.

Let (X, ν) be a Γ -probability space. Then, the stabilizer map $f: X \rightarrow \text{Sub}(\Gamma)$ defined by $f(x) = \text{Stab}_\Gamma(x)$ is a Borel map, and so we may define the pushforward measure $\mu = f_*\nu \in \text{Prob}(\text{Sub}(\Gamma))$. By definition, $\mu(A) = \nu(f^{-1}(A))$ for every Borel set $A \subset \text{Sub}(\Gamma)$. If X is a pmp Γ -space, then $\mu \in \text{IRS}(\Gamma)$, and we refer to μ as the *IRS associated with X* . For a sequence $(X_n)_{n=1}^\infty$ of pmp Γ -spaces with associated sequence of IRS $(\mu_n)_{n=1}^\infty$, if $\mu_n \rightarrow \mu$, then we say that μ is the *limiting IRS* of $(X_n)_{n=1}^\infty$.

We will consider both $\text{IRS}(\Gamma)$ and $\text{IRS}(\mathbb{F})$. The discussion above applies to $\text{IRS}(\mathbb{F})$ as a special case. We identify $\text{IRS}(\Gamma)$ with the subspace of $\text{IRS}(\mathbb{F})$ of measures supported on subgroups which contain $\text{Ker } \pi$. With this identification, $\text{IRS}(\Gamma)$ is a closed subspace of $\text{IRS}(\mathbb{F})$.

Definition 4.2

A random subgroup $\mu \in \text{Prob}(\text{Sub}(\Gamma))$ is a *finite-index random subgroup* if it is atomic and all of its atoms are finite-index subgroups of Γ . Write $\text{IRS}_{\text{f.i.}}(\Gamma)$ for the subspace of $\text{IRS}(\Gamma)$ consisting of the finite-index IRS.

Definition 4.3

An IRS $\mu \in \text{IRS}(\Gamma)$ is cosofic if it is the weak-* limit of a sequence $(\mu_n)_{n=1}^\infty$ in $\text{IRS}(\Gamma)$ of finite-index IRS.

By the discussion above regarding approximation of atomic IRS by finitely supported atomic IRS, and since $\text{IRS}(\Gamma)$ is metrizable, an IRS $\mu \in \text{IRS}(\Gamma)$ is cosofic if and only if it is the limit of a sequence of *finitely supported* finite-index IRS.

LEMMA 4.4

Let $\mu \in \text{IRS}(\Gamma)$ be a cosofic IRS. Then, there is a sequence $(X_n)_{n=1}^\infty$ of finite Γ -sets whose associated sequence of IRS $(\mu_n)_{n=1}^\infty$ converges to μ .

Proof

Let $(\mu_n)_{n=1}^\infty$ be a sequence in $\text{IRS}(\Gamma)$ of finitely supported finite-index IRS converging to μ . Fix $n \in \mathbb{N}$. Take pairwise nonconjugate subgroups $H_1, \dots, H_k$ such that μ_n is supported on $\bigcup_{i=1}^k H_i^\Gamma$. Take positive integers $l_1, \dots, l_k$ satisfying $|\frac{l_i}{S} - m_i| < \frac{1}{kn}$, where $S = \sum_{i=1}^k l_i$ and $m_i = \mu(H_i^\Gamma)$. Let $X_n = \bigsqcup_{i=1}^k (\Gamma/H_i)^{\bigsqcup l_i}$, and write $\nu_n \in \text{IRS}(\Gamma)$ for the IRS associated with X_n . Take a continuous function $f: \text{Sub}(\Gamma) \rightarrow \mathbb{R}$. Then,

$$\begin{aligned} \left| \int f d\nu_n - \int f d\mu_n \right| &= \left| \sum_{i=1}^k \left(\frac{l_i}{S} - m_i \right) \cdot \left(\frac{1}{|H_i^\Gamma|} \cdot \sum_{K \in H_i^\Gamma} f(K) \right) \right| \\ &\leq \sum_{i=1}^k \frac{1}{kn} \cdot \|f\|_\infty \\ &= \frac{1}{n} \cdot \|f\|_\infty \end{aligned}$$

and so $\nu_n \rightarrow \mu$. □

LEMMA 4.5

Let $\mu \in \text{IRS}(\Gamma)$ be a cosofic IRS. Then, $\text{supp}(\mu) \subset \overline{\text{Sub}_{\text{f.i.}}(\Gamma)}$.

Proof

Let $H \in \text{supp}(\mu)$. Take a sequence $(\mu_n)_{n=1}^\infty$ of finite-index IRS converging to μ . Let $r \in \mathbb{N}$. Since $\mu(C_{r,H}) > 0$, there is $n \in \mathbb{N}$ for which $\mu_n(C_{r,H}) > 0$. Therefore, there is a finite-index subgroup H_r of Γ satisfying $H_r \in C_{r,H}$. Then $H_r \rightarrow H$, and so $H \in \overline{\text{Sub}_{\text{f.i.}}(\Gamma)}$. $\square$

Given an IRS $\mu \in \text{IRS}(\Gamma)$, we say that μ is *cosofic in $\mathbb{F}$* if it is cosofic as an element of $\text{IRS}(\mathbb{F})$ under the natural inclusion of $\text{IRS}(\Gamma)$ in $\text{IRS}(\mathbb{F})$, that is, if it is the limit of a sequence $(\mu_n)_{n=1}^\infty$ in $\text{IRS}(\mathbb{F})$ of finite-index IRS. Therefore, for emphasis, we sometimes say “cosofic in Γ ” instead of “cosofic.”

5. Remarks on the profinite topology on an abstract group

Recall that the profinite topology on the group Γ is the topology, making Γ a topological group, for which the finite-index subgroups form a basis of neighborhoods of 1_Γ . The closure of a subgroup H of Γ under the profinite topology of Γ equals the intersection of the finite-index subgroups of Γ containing H . We refer to this closure as the *profinite closure* $\overline{H}$ of H in Γ , and if $\overline{H} = H$, we say that H is *profinately closed* in Γ . If H is normal in Γ , then $\overline{H}$ equals the intersection of the *normal* finite-index subgroups of Γ which contain H . Write $\text{Sub}_{\text{p.c.}}(\Gamma)$ for the subspace of $\text{Sub}(\Gamma)$ of profinitely closed subgroups of Γ . Note that the trivial subgroup $\{1\}$ of Γ is profinitely closed if and only if Γ is residually finite.

LEMMA 5.1

We have $\overline{\text{Sub}_{\text{f.i.}}(\Gamma)} \cap \text{Sub}_{\text{f.g.}}(\Gamma) \subset \text{Sub}_{\text{p.c.}}(\Gamma) \subset \overline{\text{Sub}_{\text{f.i.}}(\Gamma)}$.

Proof

For the right inclusion, take $H \in \text{Sub}_{\text{p.c.}}(\Gamma)$. Then, there is a sequence $(H_n)_{n=1}^\infty$ of finite-index subgroups of Γ such that $H = \bigcap_{n=1}^\infty H_n$. Hence, $H_n \rightarrow H$. For the left inclusion, take $H \in \overline{\text{Sub}_{\text{f.i.}}(\Gamma)} \cap \text{Sub}_{\text{f.g.}}(\Gamma)$. Take a sequence H_n of finite-index subgroups of H converging to H . Fix a finite generating set T for H . There is $n_0 \geq 1$ such that, for $n \geq n_0$, $T \subset H_n$; hence, $H \subset H_n$. Therefore, $H = \bigcap_{n=n_0}^\infty H_n$, and so H is profinitely closed. $\square$

The group Γ is *LERF* if every finitely generated subgroup of Γ is profinitely closed. Equivalently, Γ is LERF if every subgroup of Γ is a limit in $\text{Sub}(\Gamma)$ of finite-index subgroups.

6. Benjamini–Schramm convergence, hyperfiniteness, and applications

Consider the compact space $[0, 1]^{\mathbb{N}}$ and the metric d_{prod} on $[0, 1]^{\mathbb{N}}$ defined by $d_{\text{prod}}((a_k)_{k=1}^{\infty}, (b_k)_{k=1}^{\infty}) = \sum_{k=1}^{\infty} 2^{-k} \cdot |a_k - b_k|$ and generating the product topology of $[0, 1]^{\mathbb{N}}$. Fix an enumeration $((\rho_i, W_i))_{i=1}^{\infty}$ of all pairs $(\rho, W) \in \mathbb{Z}_{\geq 0} \times 2^{\mathbb{F}}$ satisfying $W_i \subset B_{\mathbb{F}}(\rho_i)$, namely, W_i is a subset of the ball of radius ρ_i in $\mathbb{F}$. For an $\mathbb{F}$ -probability space X , define $p_i(X) = \Pr_{x \in X}(\text{Stab}_{\mathbb{F}}(x) \cap B_{\mathbb{F}}(\rho_i) = W_i)$ and $\mathcal{L}(X) = (p_i(X))_{i=1}^{\infty} \in [0, 1]^{\mathbb{N}}$. Note that if $\mu \in \text{IRS}(\mathbb{F})$ is the IRS associated with the action $\mathbb{F} \curvearrowright X$, then $p_i(X) = \mu(C_{\rho_i, W_i})$. For a pair X, Y of $\mathbb{F}$ -probability spaces, define $d_{\text{stat}}(X, Y) = d_{\text{prod}}(\mathcal{L}(X), \mathcal{L}(Y))$. So, $\{p_i(X)\}_{i=1}^{\infty}$ gives the “local statistics” of the stabilizers of the action of $\mathbb{F}$ on X . Note that d_{stat} defines a pseudometric on the space of (equivalence classes of) $\mathbb{F}$ -probability space, which becomes an actual metric when restricted to finite $\mathbb{F}$ -sets. Convergence under the d_{stat} metric is called *Benjamini–Schramm convergence*. (More precisely, it is a directed, edge-labeled version of Benjamini–Schramm convergence.)

LEMMA 6.1

Let $(X_n)_{n=1}^{\infty}$ and $(Y_n)_{n=1}^{\infty}$ be pmp $\mathbb{F}$ -spaces. Write μ_n and ν_n for the associated IRS of X_n and Y_n , respectively. Then the following hold.

- (i) If $\mu_n \rightarrow \mu$ for some $\mu \in \text{IRS}(\mathbb{F})$ and $d_{\text{stat}}(X_n, Y_n) \rightarrow 0$, then $\nu_n \rightarrow \mu$.
- (ii) If $\mu_n \rightarrow \lambda$ and $\nu_n \rightarrow \lambda$ for some $\lambda \in \text{IRS}(\mathbb{F})$, then $d_{\text{stat}}(X_n, Y_n) \rightarrow 0$.

Proof

- (i) Take $r \geq 1$ and $W \subset B_{\mathbb{F}}(r)$. Under the hypothesis of (i), $\mu_n(C_{r,W}) \xrightarrow{n \rightarrow \infty} \mu(C_{r,W})$ and $|\mu_n(C_{r,W}) - \nu_n(C_{r,W})| \xrightarrow{n \rightarrow \infty} 0$. Hence, $\nu_n(C_{r,W}) \xrightarrow{n \rightarrow \infty} \mu(C_{r,W})$, and so $\nu_n \rightarrow \mu$.
- (ii) Take $r \geq 1$ and $W \subset B_{\mathbb{F}}(r)$. Under the hypothesis of (ii), $\mu_n(C_{r,W}) \xrightarrow{n \rightarrow \infty} \lambda(C_{r,W})$ and $\nu_n(C_{r,W}) \xrightarrow{n \rightarrow \infty} \lambda(C_{r,W})$. Hence, $|\mu_n(C_{r,W}) - \nu_n(C_{r,W})| \xrightarrow{n \rightarrow \infty} 0$, and so $d_{\text{stat}}(X_n, Y_n) \rightarrow 0$. $\square$

We now generalize Definition 3.10 and we relate d_{gen} and d_{stat} .

Definition 6.2

Let (X, μ) and (Y, ν) be pmp $\mathbb{F}$ -spaces. For a measured-space isomorphism $f: X \rightarrow Y$, define

$$\|f\|_{\text{gen}} = \frac{1}{|S|} \cdot \sum_{s \in S} \mu(\{x \in X \mid f(s \cdot x) \neq s \cdot f(x)\}).$$

Finally, let

$$d_{\text{gen}}(X, Y) = \inf\{\|f\|_{\text{gen}} \mid f: X \rightarrow Y \text{ is a measured-space isomorphism}\}.$$

PROPOSITION 6.3

Let $\{(X_n, \nu_n)\}_{n=1}^\infty$ and $\{(Y_n, \lambda_n)\}_{n=1}^\infty$ be sequences of pmp $\mathbb{F}$ -spaces satisfying $d_{\text{gen}}(X_n, Y_n) \rightarrow 0$. Then, $d_{\text{stat}}(X_n, Y_n) \rightarrow 0$.

Proof

Take a sequence $\{f_n\}_{n=1}^\infty$ of measured-space isomorphisms $f_n: X_n \rightarrow Y_n$ such that $\|f_n\|_{\text{gen}} \rightarrow 0$. For $n \in \mathbb{N}$, let

$$P_n = \bigcup_{s \in S} \{x \in X_n \mid f_n(s \cdot x) \neq s \cdot f_n(x)\},$$

and for $r \in \mathbb{N}$, let $Q_n^r = X_n \setminus B_{X_n}(P_n, r)$ (in the notation of Section 2). Then, $\{Q_n^r\}_{n,r \in \mathbb{N}}$ are Borel sets, and for each $r \in \mathbb{N}$,

$$\nu_n(B_{X_n}(P_n, r)) \leq (2 \cdot |S|)^{r+1} \cdot \nu_n(P_n) \xrightarrow{n \rightarrow \infty} 0;$$

hence, $\nu_n(Q_n^r) \xrightarrow{n \rightarrow \infty} 1$. Furthermore, for each $x \in Q_n^r$, $\text{Stab}_{\mathbb{F}}(x) \cap B_{\mathbb{F}}(r) = \text{Stab}_{\mathbb{F}}(f_n(x)) \cap B_{\mathbb{F}}(r)$. Hence, for each $i \in \mathbb{N}$, there is $r \in \mathbb{N}$ such that $|p_i(X_n) - p_i(Y_n)| \leq 1 - \nu_n(Q_n^r)$ for all $n \in \mathbb{N}$, and so $|p_i(X_n) - p_i(Y_n)| \xrightarrow{n \rightarrow \infty} 0$. Thus, $d_{\text{stat}}(X_n, Y_n) \rightarrow 0$. $\square$

In Proposition 6.8 below, we will give a partial converse to Proposition 6.3 in the context of actions of an amenable group on finite sets.

Let X be a standard Borel space. Let E be a *Borel equivalence relation* on X ; that is, $E \subset X \times X$ is a Borel set which is an equivalence relation. We write $x \sim_E y$ if $(x, y) \in E$. Then, E is *finite* (resp., *countable*) if all of its equivalence classes are finite (resp., countable). A countable equivalence relation E is *hyperfinite* if it can be written as an *ascending* union of finite Borel equivalence relations. If μ is a probability measure on X , then E is *hyperfinite μ -a.e.* if there is a μ -conull Borel subset $X_0 \subset X$, respecting E , such that the restriction of E to X_0 is hyperfinite. A Borel action $\Gamma \curvearrowright X$ gives rise to a Borel equivalence relation on X which we denote by E_X^Γ . If (X, μ) is a pmp Γ -space, then the action $\Gamma \curvearrowright (X, \mu)$ is called *hyperfinite* if the equivalence relation E_X^Γ is hyperfinite μ -a.e. A well-known theorem of Ornstein–Weiss (see [8, Theorem 10]) says that every action of an amenable group is hyperfinite. (For a thorough treatment of Borel equivalence relations, see [17].)

Definition 6.4

Let $\mathcal{X}$ be a family of finite graphs. Then, $\mathcal{X}$ is *hyperfinite* if for every $\epsilon > 0$ there is $K \in \mathbb{N}$ such that, for each graph $X \in \mathcal{X}$, there is a set $Z \subset V(X)$, $|Z| < \epsilon \cdot |V(X)|$, such that, after removing from X all edges incident to Z , each component of the resulting graph is of size at most K .

PROPOSITION 6.5

Assume that Γ is amenable. Then, the sequence $\{X_n\}_{n=1}^\infty$ of all finite Schreier graphs of Γ is hyperfinite.

Proof

For $n \in \mathbb{N}$ and $Z \subset V(X_n)$, let $c_{n,Z}$ be the size of the largest component of the graph resulting from removing all edges incident to Z from X_n . For $\epsilon > 0$, let $c_{n,\epsilon} = \min\{c_{n,Z} \mid Z \subset V(X_n), |Z| < \epsilon \cdot |V(X_n)|\}$. Assume, for the sake of contradiction, that $\{X_n\}_{n=1}^\infty$ is not a hyperfinite family. Then, there are $\epsilon > 0$ and an increasing sequence $\{n_k\}_{k=1}^\infty$ such that $c_{n_k,\epsilon} \xrightarrow{k \rightarrow \infty} \infty$. Write $\mu_n \in \text{IRS}(\Gamma)$ for the IRS associated with X_n . Since $\text{IRS}(\Gamma)$ is compact, we may further assume that $\mu_n \xrightarrow{n \rightarrow \infty} \mu$ for some $\mu \in \text{IRS}(\Gamma)$. By [2, Proposition 13], there is a pmp Γ -space X whose associated IRS is μ . By [25], the action $\Gamma \curvearrowright X$ is hyperfinite since Γ is amenable. Then, by [26, Theorem 1.1] (see also [11, Theorem 1]), $\{X_{n_k}\}_{k=1}^\infty$ is a hyperfinite family, a contradiction. $\square$

Recall that a bijection $f : X \rightarrow Y$ between measured spaces (X, μ) and (Y, ν) is a *measured-space isomorphism* if f and f^{-1} are both Borel maps and, for each Borel set $A \subset X$, $\mu(A) = \nu(f(A))$. It is well known that a Borel bijection between standard Borel spaces is automatically a measured-space isomorphism (see [16, Corollary 15.2]). A measured-space isomorphism from a measured space to itself is called a *measured-space automorphism*.

A well-known question of Aldous–Lyons from [3] is whether every $\mu \in \text{IRS}(\mathbb{F})$ is cosofic. The following proposition shows that if μ arises from an amenable quotient of $\mathbb{F}$, then it is cosofic. Note that the same conclusion can be drawn if μ arises from a quotient which is isomorphic to $\text{SL}_3(\mathbb{Z})$, since this group is residually finite and all of its IRS are supported on finite-index subgroups and the trivial subgroup. The same properties hold for a wide range of other lattices in simple Lie groups of higher rank (see [13] and the references within).

PROPOSITION 6.6

Assume that Γ is amenable and $\mu \in \text{IRS}(\Gamma)$. Then, μ is cosofic in $\mathbb{F}$.

Proof

By [2, Proposition 13], there is a pmp Γ -space (X, ν) for which μ is the associated IRS. Since Γ is amenable, the action $\Gamma \curvearrowright X$ is hyperfinite by [25]. From now on, regard X as an $\mathbb{F}$ -probability space, and for each $s \in S$ (recall that S is our fixed basis for $\mathbb{F}$), let $f_s : X \rightarrow X$ be the measured-space automorphism defined by $f_s(x) = s \cdot x$.

Let $n \geq 1$. Since the equivalence relation $E_X^{\mathbb{F}}$ is hyperfinite μ -a.e., there is a Borel set $Z \subset X$, $\nu(Z) < \frac{1}{n}$, such that all orbits of the restriction of $E_X^{\mathbb{F}}$ to $X \setminus Z$ are finite (see [20, Section 21.1.2]). Let $E = E_X^{\mathbb{F}}|_{X \setminus Z} \cup \{(x, x) \mid x \in Z\}$. For each $s \in S$, Lemma 6.7 below gives us a measured-space automorphism $h_s: X \rightarrow X$ which respects the equivalence relation E and agrees with f_s on $X \setminus (Z \cup (f_s)^{-1}(Z))$. Let X_n be the pmp $\mathbb{F}$ -space which, as a probability space, equals X , endowed with the pmp action of $\mathbb{F}$ given by $s \cdot x = h_s(x)$ for each $s \in S$. Then, $d_{\text{gen}}(X_n, X) \rightarrow 0$, and so, by Proposition 6.3, $d_{\text{stat}}(X_n, X) \rightarrow 0$. Write $\mu_n \in \text{IRS}(\mathbb{F})$ for the IRS associated with X_n . Then μ_n is a finite-index IRS since each h_s respects the finite Borel equivalence relation E . By Lemma 6.1(i) applied to the sequence $(X_n)_{n=1}^{\infty}$ against the constant sequence $(X)_{n=1}^{\infty}$, $\mu_n \rightarrow \mu$, and so, μ is cosofic in $\mathbb{F}$. $\square$

LEMMA 6.7

Let X be a probability space, let $f: X \rightarrow X$ be a measured-space automorphism, and let $E \subset X \times X$ be a finite Borel equivalence relation on X . Write $X_{E,f} = \{x \in X \mid f(x) \sim_E x\}$. Then, there is a measured-space automorphism $h: X \rightarrow X$ such that $h(x) \sim_E x$ for every $x \in X$ and h agrees with f on $X_{E,f}$.

Proof

The idea behind the construction of the map h is as follows: Since the equivalence relation E is finite, the space X can be decomposed as a disjoint union of *finite f -cycles* and *finite maximal f -chains*, that is, sets of the form $x, f(x), f^{(2)}(x), \dots, f^{(m)}(x)$ for $x \in X$ and $m \geq 0$ such that $f^{(i)}(x) \sim_E f^{(i+1)}(x)$ for each $0 \leq i < m$ and such that either $f^{(m+1)}(x) = x$ and $f^{(m)}(x) \sim_E x$ (these are the f -cycles) or $f^{-1}(x) \sim_E x$ and $f^{(m)}(x) \sim_E f^{(m+1)}(x)$ (these are the maximal f -chains). For each maximal f -chain, as above, we define $h(f^{(i)}(x)) = f^{(i+1)}(x)$ for each $0 \leq i < m$ and $h(f^{(m)}(x)) = x$. On the f -cycles, we make h identical to f . The resulting function h is a bijection. We now formalize this construction in a way that enables us to see that the resulting map h is a Borel measure-preserving automorphism of X .

For each $n \geq 0$, let $A_n = \{x \in X \setminus X_{E,f} \mid (f^{-1})^{(n)}(x) \sim_E x\}$. For each $n \geq 0$, set $X_n = \bigcap_{i=0}^n A_i \setminus A_{n+1}$. So, a point $x \in X \setminus X_{E,f}$ satisfies $x \in X_n$ if and only if $(f^{-1})^{(k)}(x) \sim_E x$ for every $1 \leq k \leq n$, but $(f^{-1})^{(n+1)}(x) \not\sim_E x$. The sets X_n are disjoint by construction. Furthermore, since each equivalence class of E is finite, every $x \in X \setminus X_{E,f}$ belongs to X_n for some $n \in \mathbb{N}$. Therefore, $\mathcal{C} = \{X_{E,f}\} \cup \{X_n\}_{n=0}^{\infty}$ forms a partition of X into countably many Borel sets. We define $h: X \rightarrow X$: for $x \in X_{E,f}$, set $h(x) = f(x)$, and for $x \in X_n$, set $h(x) = (f^{-1})^{(n)}(x)$. Then, h is a bijection. By the definition of h and since f is a Borel automorphism, h maps every Borel subset of each set in the partition $\mathcal{C}$ to a Borel subset of X . Thus, h maps every Borel subset of X to a Borel set. This shows that h^{-1} is a Borel bijection,

and so, since X is a standard Borel space, h is a Borel bijection as well. Similarly, h preserves the measure on X because it does so when restricted to each set in the partition $\mathcal{C}$. $\square$

The following result, which is essential for our needs, gives a converse to Proposition 6.3 in the case in which the $\mathbb{F}$ -sets are finite and the actions in one of the sequences factor through an amenable quotient.

PROPOSITION 6.8

Assume that Γ is amenable. Let $(X_n)_{n=1}^\infty$ be a sequence of finite $\mathbb{F}$ -sets, and let $(Y_n)_{n=1}^\infty$ be a sequence of finite Γ -sets, satisfying $|X_n| = |Y_n|$ and $d_{\text{stat}}(X_n, Y_n) \rightarrow 0$. Then, $d_{\text{gen}}(X_n, Y_n) \rightarrow 0$.

Proof

The statement of this proposition is an adaptation of a theorem of Newman and Sohler (see [23] and [24, Theorem 3.1]) from the context of finite undirected graphs to the context of group actions on finite sets. We begin by describing the Newman–Sohler theorem (see [11, Theorem 5] for this formulation and a different proof). First, we need some definitions. Fix $q \in \mathbb{N}$. Let $\mathbf{P}_q$ be the collection of finite undirected graphs for which the degree of each vertex is at most q . We begin by defining the notion of the statistical distance between finite undirected graphs. For $H \in \mathbf{P}_q$ and a vertex h_0 of H , we say that (H, h_0) is a pointed graph of radius r if each vertex $h \in H$ is at distance at most r from h_0 . Write $\mathbf{P}_{q,r}$ for the set of pointed graphs (H, h_0) of radius r with $H \in \mathbf{P}_q$. Enumerate the disjoint union $\bigsqcup_{r \in \mathbb{N}} \mathbf{P}_{q,r}$ by $\{H_i\}_{i=1}^\infty$. For $r \in \mathbb{N}$, $H_i \in \mathbf{P}_{q,r}$, and $G \in \mathbf{P}_q$, write $p_i(G)$ for the probability, under a uniformly random choice of a vertex v of G , that the ball of radius r , centered at v , is pointed-isomorphic to H_i . Write $\mathcal{L}(G) = (p_i(G))_{i=1}^\infty \in [0, 1]^\mathbb{N}$. For $G_1, G_2 \in \mathbf{P}_q$, the statistical distance between G_1 and G_2 is defined as $d_{\text{stat}}(G_1, G_2) = \sum_{i=1}^\infty 2^{-i} \cdot |p_i(G_1) - p_i(G_2)|$. We now define another notion of distance between graphs in $\mathbf{P}_q$. (The generator metric d_{gen} is its analogue in the context of group actions.) For $G_1, G_2 \in \mathbf{P}_q$, $n := |V(G_1)| = |V(G_2)|$, and a bijection $f : V(G_1) \rightarrow V(G_2)$, let $\mathcal{Q}_f$ be the set of pairs (v_1, v_2) of vertices of G_1 such that (v_1, v_2) is an edge of G_1 , but $(f(v_1), f(v_2))$ is not an edge of G_2 , or vice versa. Let $\|f\| = \frac{1}{n} \cdot |\mathcal{Q}_f|$. Define $d(G_1, G_2)$ as the minimum of $\|f\|$, running over all bijections f between the vertex sets.

The Newman–Sohler theorem says that if $\mathbf{G} \subset \mathbf{P}_q$ is a hyperfinite family, then for every $\epsilon > 0$, there is $f(\epsilon) > 0$ such that, for every $G_1 \in \mathbf{P}_q$ and $G_2 \in \mathbf{G}$, if $|V(G_1)| = |V(G_2)|$ and $d_{\text{stat}}(G_1, G_2) < f(\epsilon)$, then $d(G_1, G_2) < \epsilon$. In other words, if $(G_n^{(1)})_{n=1}^\infty$ is a sequence in $\mathbf{P}_q$ and $(G_n^{(2)})_{n=1}^\infty$ is a sequence in the hyperfinite family $\mathbf{G}$, satisfying $|G_n^{(1)}| = |G_n^{(2)}|$, then $d_{\text{stat}}(G_n^{(1)}, G_n^{(2)}) \rightarrow 0$ implies $d(G_n^{(1)}, G_n^{(2)}) \rightarrow 0$.

To adapt the Newman–Sohler theorem to the context of group actions, we use a standard encoding of actions of $\mathbb{F}$ by undirected graphs. (The details of this type of encoding are described, for example, in the proof of [11, Theorem 9].) There are $q \in \mathbb{N}$ and a mapping U from the set of actions of $\mathbb{F}$ on finite sets to the set $\mathbf{P}_q$ with the following properties: (1) if $(X_n)_{n=1}^\infty$ and $(Y_n)_{n=1}^\infty$ are sequences of finite $\mathbb{F}$ -sets, then $d_{\text{stat}}(X_n, Y_n) \rightarrow 0$ if and only if $d_{\text{stat}}(U(X_n), U(Y_n)) \rightarrow 0$, and (2) if, in addition, $|X_n| = |Y_n|$, then $d_{\text{gen}}(X_n, Y_n) \rightarrow 0$ if and only if $d(U(X_n), U(Y_n)) \rightarrow 0$. The proposition follows at once from the Newman–Sohler theorem and the above properties of the encoding function U . $\square$

Remark 6.9

The assumption that Γ is amenable in Proposition 6.8 is essential. Indeed, for $d \geq 2$, take $\Gamma = \mathbb{F}_d$, the free group on d generators. Then, there are sequences $(\Lambda_n)_{n=1}^\infty$ and $(\Delta_n)_{n=1}^\infty$ of finite quotients of Γ , $|\Lambda_n| = 2 \cdot |\Delta_n|$, giving rise to $2d$ -regular Cayley graphs $X_n = \text{Cay}(\Lambda_n)$ and $Y_n = \text{Cay}(\Delta_n)$, such that $(X_n)_{n=1}^\infty$ is a family of expander graphs and such that the girths of both X_n and Y_n approach infinity as $n \rightarrow \infty$ (see [21, Theorem 7.3.12] for examples of families of expander graphs with large girth). Then, $d_{\text{stat}}(X_n, Y_n) \rightarrow 0$ since, for every radius $r \geq 1$, any ball of radius r in X_n and in Y_n is a tree for large enough n . But, since $(Y_n)_{n=1}^\infty$ is a highly nonexpanding family, $d_{\text{gen}}(X_n, Y_n)$ does not approach 0 as $n \rightarrow \infty$.

7. The main theorem

Definition 7.1

A sequence $(X_n)_{n=1}^\infty$ of finite $\mathbb{F}$ -sets is *convergent* if it has a limiting IRS; that is, the sequence $(\mu_n)_{n=1}^\infty$ of IRS associated with $(X_n)_{n=1}^\infty$ converges in $\text{IRS}(\mathbb{F})$.

Definition 7.2

A sequence $(X_n)_{n=1}^\infty$ of finite $\mathbb{F}$ -sets is a *stability challenge* for Γ if $\Pr_{x \in X_n}(w \cdot x = x) \rightarrow 1$ for each $w \in \text{Ker } \pi$.

Every sofic approximation $(X_n)_{n=1}^\infty$ for Γ is a stability challenge. Indeed, a sofic approximation can be defined as a stability challenge with the following additional requirement: for every $w \in \mathbb{F}$, if $w \notin \text{Ker } \pi$, then $\Pr_{x \in X_n}(w \cdot x \neq x) \rightarrow 1$. Note that $\text{IRS}(\mathbb{F})$ is compact, and so every stability challenge for Γ has a convergent subsequence.

Definition 7.3

Let $(X_n)_{n=1}^\infty$ be a stability challenge for Γ . Then, a sequence $(Y_n)_{n=1}^\infty$ of finite Γ -sets, satisfying $|X_n| = |Y_n|$, is

- (i) a solution for $(X_n)_{n=1}^\infty$ if $d_{\text{gen}}(X_n, Y_n) \xrightarrow{n \rightarrow \infty} 0$,
- (ii) a statistical solution for $(X_n)_{n=1}^\infty$ if $d_{\text{stat}}(X_n, Y_n) \xrightarrow{n \rightarrow \infty} 0$.

By Proposition 6.3, if $(X_n)_{n=1}^\infty$ is a stability challenge for Γ , then any solution for $(X_n)_{n=1}^\infty$ is a statistical solution. By Proposition 6.8, the converse holds as well if Γ is amenable. Note also that Γ is P-stable (Definition 3.11) if and only if every stability challenge for Γ has a solution. In fact, it suffices to consider *convergent* stability challenges.

LEMMA 7.4

The group Γ is P-stable if and only if every convergent stability challenge for Γ has a solution.

Proof

We only need to prove the “if” direction. Assume that Γ is not P-stable. We would like to show that Γ has a convergent stability challenge which does not have a solution. Take $\epsilon > 0$ and a sequence $(X_n)_{n=1}^\infty$ such that X_n is a (δ_n, E_n) -almost- Γ -set for $\delta_n = \frac{1}{n}$ and $E_n = \text{Ker } \pi \cap B_{\mathbb{F}}(n)$, but there is no Γ -set Y_n for which $d_{\text{gen}}(X_n, Y_n) < \epsilon$. Then, every subsequence of $(X_n)_{n=1}^\infty$ is a stability challenge for Γ which has no solution. Since $\text{IRS}(\mathbb{F})$ is compact, $(X_n)_{n=1}^\infty$ has a subsequence which is a convergent stability challenge for Γ with no solution. $\square$

LEMMA 7.5

Let $(X_n)_{n=1}^\infty$ be a convergent sequence of finite $\mathbb{F}$ -sets, and write $\mu \in \text{IRS}(\mathbb{F})$ for its limiting IRS. Then, $(X_n)_{n=1}^\infty$ is a stability challenge for Γ if and only if $\mu \in \text{IRS}(\Gamma)$.

Proof

Let $(\mu_n)_{n=1}^\infty$ be the sequence of IRS associated with $(X_n)_{n=1}^\infty$. By definition, $\mu_n \rightarrow \mu$. Then, $(\mu_n)_{n=1}^\infty$ is a stability challenge for Γ if and only if $\mu_n(C_w) \rightarrow 1$ for each $w \in \text{Ker } \pi$. If $\mu \in \text{IRS}(\Gamma)$, then $\mu_n(C_w) \rightarrow \mu(C_w) = 1$ for every $w \in \text{Ker } \pi$, which says that $(\mu_n)_{n=1}^\infty$ is a stability challenge for Γ . In the other direction, if $(\mu_n)_{n=1}^\infty$ is a stability challenge for Γ , then for every $w \in \text{Ker } \pi$, $\mu_n(C_w) \rightarrow 1$, but $\mu_n(C_w) \rightarrow \mu(C_w)$, forcing $\mu(C_w) = 1$. Therefore,

$$\mu(\{H \leq \mathbb{F} \mid \text{Ker } \pi \leq H\}) = \mu\left(\bigcap_{w \in \text{Ker } \pi} C_w\right) = 1,$$

and so $\mu \in \text{IRS}(\Gamma)$. $\square$

The proof of the following lemma is an adaptation of the argument in the proof of [4, Proposition 6.1].

LEMMA 7.6

Let $(X_n)_{n=1}^\infty$ be a sequence of finite Γ -sets satisfying $|X_n| \rightarrow \infty$, with associated sequence of IRS $(\mu_n)_{n=1}^\infty$, satisfying $\mu_n \rightarrow \mu$ for some $\mu \in \text{IRS}(\Gamma)$. Let $(m_k)_{k=1}^\infty$ be a sequence of positive integers satisfying $m_k \rightarrow \infty$. Then, there is a sequence $(Y_k)_{k=1}^\infty$ of Γ -sets, satisfying $|Y_k| = m_k$, with associated IRS sequence $(\nu_k)_{k=1}^\infty$, such that $\nu_k \rightarrow \mu$ as well.

Proof

For an integer $r \geq 0$, write Z_r for the Γ -set on r points on which Γ acts trivially. Take an increasing sequence $(i_n)_{n=1}^\infty$ of positive integers such that, for every $n \in \mathbb{N}$ and $i_n \leq k < i_{n+1}$, $\frac{|X_n|}{m_k} < \frac{1}{n}$. For each $k \geq i_1$, take the unique $n \in \mathbb{N}$ for which $i_n \leq k < i_{n+1}$, write $m_k = q_k \cdot |X_n| + r_k$ for integers $q_k \geq n$ and $0 \leq r_k < |X_n|$, and let $Y_k = (X_n)^{\sqcup q_k} \sqcup Z_{r_k}$. So, $|Y_k| = m_k$. For $1 \leq k < i_1$, define $Y_k = Z_{m_k}$. Let $(\nu_k)_{k=1}^\infty$ be the sequence of IRS associated with $(Y_k)_{k=1}^\infty$. We would like to show that $\nu_k \rightarrow \mu$. Take a continuous function $f : \text{Sub}(\Gamma) \rightarrow \mathbb{R}$. Let $n \in \mathbb{N}$, and let $i_n \leq k < i_{n+1}$. Then,

$$\begin{aligned} \left| \int f d\nu_k - \int f d\mu_n \right| &= \left| \frac{1}{m_k} \cdot \left(q_k \cdot |X_n| \cdot \int f d\mu_n + r_k \cdot f(\Gamma) \right) - \int f d\mu_n \right| \\ &= \frac{1}{m_k} \cdot \left| (q_k \cdot |X_n| - m_k) \cdot \int f d\mu_n + r_k \cdot f(\Gamma) \right| \\ &= \frac{r_k}{m_k} \cdot \left| - \int f d\mu_n + f(\Gamma) \right| \\ &< \frac{2}{n} \cdot \|f\|_\infty. \end{aligned}$$

Therefore, $\int f d\nu_k \rightarrow \int f d\mu$, and so $\nu_k \rightarrow \mu$. $\square$

Definition 7.7

Let $(X_n)_{n=1}^\infty$ be a convergent stability challenge for Γ whose limiting IRS is $\mu \in \text{IRS}(\Gamma)$. Then, $(X_n)_{n=1}^\infty$ is *cosofic* if μ is cosofic.

LEMMA 7.8

Let $(X_n)_{n=1}^\infty$ be a convergent stability challenge for Γ . Then, $(X_n)_{n=1}^\infty$ is cosofic if and only if $(X_n)_{n=1}^\infty$ has a statistical solution.

Proof

Write $\mu \in \text{IRS}(\Gamma)$ for the limiting IRS of $(X_n)_{n=1}^\infty$. Assume that $(X_n)_{n=1}^\infty$ is cosofic.

By Lemma 4.4, there is a sequence $(Y_n)_{n=1}^\infty$ of finite Γ -sets whose associated sequence of IRS converges to μ . By Lemma 7.6, we may assume that $|X_n| = |Y_n|$ for all $n \in \mathbb{N}$. Then, by Lemma 6.1(ii), $(Y_n)_{n=1}^\infty$ is a statistical solution for $(X_n)_{n=1}^\infty$.

In the other direction, assume that $(X_n)_{n=1}^\infty$ has a statistical solution $(Y_n)_{n=1}^\infty$. Write $(v_n)_{n=1}^\infty$ for the sequence of IRS associated with $(Y_n)_{n=1}^\infty$. Then, $(v_n)_{n=1}^\infty$ is a sequence of finite-index IRS in $\text{IRS}(\Gamma)$, which, by Lemma 6.1(i), converges to μ , and so $(X_n)_{n=1}^\infty$ is cosofic. $\square$

LEMMA 7.9

Let $\mu \in \text{IRS}(\Gamma)$, and assume that μ is cosofic in $\mathbb{F}$. Then, there is a convergent stability challenge for Γ whose limiting IRS is μ .

Proof

By Lemma 4.4 applied to $\mathbb{F}$ (rather than Γ), there is a sequence $(X_n)_{n=1}^\infty$ of finite $\mathbb{F}$ -sets whose associated sequence of IRS converges to μ . But $\mu \in \text{IRS}(\Gamma)$, and so, by Lemma 7.5, $(X_n)_{n=1}^\infty$ is a stability challenge for Γ . $\square$

The following proves Theorem 1.3 of the Introduction.

THEOREM 7.10

- (i) *Assume that Γ is P-stable, and assume that $\mu \in \text{IRS}(\Gamma)$ is cosofic in $\mathbb{F}$. Then, μ is cosofic in Γ .*
- (ii) *Assume that Γ is amenable. Then, Γ is P-stable if and only if every $\mu \in \text{IRS}(\Gamma)$ is cosofic in Γ .*

Proof

(i) By Lemma 7.9, there is a convergent stability challenge $(X_n)_{n=1}^\infty$ for Γ whose limiting IRS is μ . Then, $(X_n)_{n=1}^\infty$ has a solution; a fortiori, it has a statistical solution. Thus, by Lemma 7.8, $(X_n)_{n=1}^\infty$ is cosofic, that is, μ is cosofic in Γ .

(ii) Assume that Γ is P-stable. Let $\mu \in \text{IRS}(\Gamma)$. By Proposition 6.6, μ is cosofic in $\mathbb{F}$. Hence, by (i), μ is cosofic in Γ .

In the other direction, assume that every $\mu \in \text{IRS}(\Gamma)$ is cosofic in Γ . Let $(X_n)_{n=1}^\infty$ be a convergent stability challenge for Γ . Then, $(X_n)_{n=1}^\infty$ is cosofic, and so by Lemma 7.8, it has a statistical solution $(Y_n)_{n=1}^\infty$. By Proposition 6.8, $(Y_n)_{n=1}^\infty$ is, in fact, a solution for $(X_n)_{n=1}^\infty$, and so Γ is P-stable by Lemma 7.4. $\square$

8. Applications of the main theorems

In this section, we give several applications of the results of Section 7 and, in particular, prove Theorem 1.2. The next proposition proves Theorem 1.4.

PROPOSITION 8.1

Assume that $\text{Sub}(\Gamma)$ is countable, and assume that every almost-normal subgroup of Γ is profinitely closed. Then, every $\mu \in \text{IRS}(\Gamma)$ is cosofic. If, further, Γ is amenable, then Γ is P -stable.

Proof

The latter statement follows from the former by Theorem 7.10(ii). We turn to proving the former. Since $\text{Sub}(\Gamma)$ is countable, every IRS in $\text{IRS}(\Gamma)$ is atomic, and so all of its atoms are almost-normal subgroups. Let $\mu \in \text{IRS}(\Gamma)$. Take a sequence $(\mu_n)_{n=1}^\infty$ in $\text{IRS}(\Gamma)$ of finitely supported atomic IRS, converging to μ . Since $\text{IRS}(\Gamma)$ is metrizable, it suffices to prove that each μ_n is cosofic. Let H be an almost-normal subgroup of Γ , and let $\nu \in \text{IRS}(\Gamma)$ be the atomic IRS assigning measure $\frac{1}{|H\Gamma|}$ to each conjugate of H . It suffices to prove that ν is cosofic. Take representatives $g_1, \dots, g_k$ for the left cosets of $N_\Gamma(H)$ in Γ . Since H is profinitely closed in Γ and $[\Gamma : N_\Gamma(H)] < \infty$, H is profinitely closed and normal in $N_\Gamma(H)$. Therefore, there is a sequence $(H_n)_{n=1}^\infty$ of finite-index normal subgroups of $N_\Gamma(H)$ such that $H = \bigcap_{n=1}^\infty H_n$. For $1 \leq i \leq k$, $g_i H_n \xrightarrow{n \rightarrow \infty} g_i H$, and so $\delta_{g_i H_n} \xrightarrow{n \rightarrow \infty} \delta_{g_i H}$. Hence, writing $\nu_n = \frac{1}{k} \sum_{i=1}^k \delta_{g_i H_n}$, we have $\nu_n \xrightarrow{n \rightarrow \infty} \nu$, that is, ν is a limit of finite-index random subgroups. It remains to show that each random subgroup ν_n is an IRS. Take $g \in \Gamma$. Let $\sigma \in \text{Sym}(k)$ be the permutation for which $gg_i N_\Gamma(H) = g_{\sigma(i)} N_\Gamma(H)$. Since H_n is normal in $N_\Gamma(H)$, $gg_i H_n = g_{\sigma(i)} H_n$ for each $1 \leq i \leq k$. So, $g \cdot \nu_n = \frac{1}{k} \sum_{i=1}^k \delta_{gg_i H_n} = \nu_n$, that is, ν_n is an IRS. Hence, ν is cosofic. $\square$

The following corollary provides a proof for part (i) of Theorem 1.2.

COROLLARY 8.2

Virtually polycyclic groups are P -stable.

Proof

Assume that Γ is a virtually polycyclic group. Then, every subgroup of Γ is finitely generated, and so $\text{Sub}(\Gamma)$ is countable. Furthermore, Γ is LERF (see [22]) and amenable. Hence, all of the conditions of Proposition 8.1 are met. $\square$

Remark 8.3

Nevertheless, not every solvable group is P -stable, even if it is residually finite (see Corollary 8.7).

The following corollary provides a proof for part (ii) of Theorem 1.2.

COROLLARY 8.4

For every $n \in \mathbb{Z}$, the Baumslag–Solitar group $BS(1, n)$ is P -stable.

Proof

Let $\Gamma = BS(1, n)$. Note that $\Gamma \cong \mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z}$, where $1 \in \mathbb{Z}$ acts on $\mathbb{Z}[\frac{1}{n}]$ by multiplication by n . We use Proposition 8.1 to show that Γ is P -stable. First, Γ is amenable since it is solvable. The group Γ is an example of a constructible solvable group. Every constructible solvable group is residually finite, and the class of constructible solvable groups is closed under taking quotients and finite-index subgroups (see [5] or [19, Section 11.2]). Therefore, every almost-normal subgroup of Γ is profinitely closed. It remains to show that $\text{Sub}(\Gamma)$ is countable. In general, for a countable group G and $N \triangleleft G$, if G/N is Noetherian (i.e., every subgroup is finitely generated) and $\text{Sub}(N)$ is countable, then $\text{Sub}(G)$ is countable. In our case, by taking $N = \mathbb{Z}[\frac{1}{n}]$, Γ/N is infinite cyclic, and $\text{Sub}(N)$ is countable. To see that $N = \mathbb{Z}[\frac{1}{n}]$ indeed has only countably many subgroups, we argue as follows. Let H be a subgroup of $\mathbb{Z}[\frac{1}{n}]$. Then, H is determined by the sequence $(H_i)_{i=0}^\infty$, where $H_i = H \cap \frac{1}{n^i} \mathbb{Z}$. The latter is determined by the sequence $(l_i)_{i=0}^\infty$, where $l_i = [\frac{1}{n^i} \mathbb{Z} : H \cap \frac{1}{n^i} \mathbb{Z}]$. By assuming $H \neq \{0\}$, each l_i is a positive integer (i.e., $l_i \neq \infty$). For each $i \geq 0$, $l_i = l_{i+1} / \gcd(n, l_{i+1})$. This shows that $(\frac{l_{i+1}}{l_i})_{i=0}^\infty$ is a nondecreasing sequence of natural numbers which divide n . In particular, this sequence of quotients stabilizes at some divisor q of n . Let n_0 be the minimal natural number for which $l_{n_0+1} = l_{n_0} \cdot q$. Then, $l_i = l_{n_0} \cdot q^{i-n_0}$ for each $i \geq n_0$. The sequence $(l_i)_{i=0}^\infty$ is determined by n_0 , l_{n_0} , and q . Subsequently, $\mathbb{Z}[\frac{1}{n}]$ has only countably many subgroups. $\square$

PROPOSITION 8.5

Assume that Γ is P -stable. Let H be an almost-normal subgroup of Γ such that the IRS $\mu \in \text{IRS}(\Gamma)$, assigning probability $\frac{1}{|H^\Gamma|}$ to each conjugate of H , is cosofic in $\mathbb{F}$. Then, H is a limit in $\text{Sub}(\Gamma)$ of finite-index subgroups. If, in addition, H is finitely generated, then H is profinitely closed.

Proof

The latter statement follows from the former by Lemma 5.1. We turn to proving the former. By Theorem 7.10(i), μ is cosofic in Γ . Since $H \in \text{supp}(\mu)$, Lemma 4.5 implies that $H \in \overline{\text{Sub}_{\text{f.i.}}(\Gamma)}$. $\square$

Specializing to $H = \{1\}$ in Proposition 8.5, we see that a sofic P -stable group must be residually finite, as proved by Glebsky and Rivera in [15, Theorem 2] and by Arzhantseva and Păunescu in [4, Theorem 7.2(ii)]. By Propositions 8.5 and 6.6, if Γ is amenable and P -stable, then every almost-normal subgroup of Γ is a limit in

$\text{Sub}(\Gamma)$ of finite-index subgroups. If the converse is true as well (under the amenability assumption), it would give a positive answer to the following question.

Question 8.6

Is every amenable LERF group P-stable?

A related question was asked by Arzhantseva and Păunescu (see [4, Conjecture 1.2]): (*) Among finitely presented amenable groups, is P-stability equivalent to the following: every normal subgroup of Γ is profinitely closed? In fact, Conjecture 1.2 in [4] was stated differently, without assuming amenability, but (*) is an equivalent formulation under the amenability assumption (see [4, Theorem 7.2(iii)]).

Arzhantseva and Păunescu asked whether every finitely presented amenable residually finite group is P-stable (see the paragraph before Theorem 7.2 of [4] and Theorem 7.2(iii) of the same paper). We recall the construction of Abels's groups and show that they provide a negative answer. Fix a prime p . Abels's group (for the prime p) is

$$A_p = \left\{ \left(\begin{pmatrix} 1 & * & * & * \\ & p^m & * & * \\ & & p^n & * \\ & & & 1 \end{pmatrix} \middle| m, n \in \mathbb{Z} \right\} \leq \text{GL}_4\left(\mathbb{Z}\left[\frac{1}{p}\right]\right).$$

COROLLARY 8.7

Abels's group A_p is finitely presented, amenable, and residually finite, but not P-stable.

Proof

The group A_p is amenable since it is solvable. It is residually finite since it is finitely generated and linear. In fact, in [1], Abels showed that A_p is finitely presented. By Proposition 8.5, to show that A_p is not P-stable it suffices to exhibit a finitely generated almost-normal subgroup H of A_p which is not profinitely closed. Note that the center of A_p is

$$Z(A_p) = \left\{ \left(\begin{pmatrix} 1 & 0 & 0 & x \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \middle| x \in \mathbb{Z}\left[\frac{1}{p}\right] \right\} \cong \mathbb{Z}\left[\frac{1}{p}\right].$$

Let

$$H = \left\{ \left(\begin{pmatrix} 1 & 0 & 0 & n \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\} \cong \mathbb{Z}.$$

Since H is cyclic and central, we are left with showing that H is not profinitely closed in A_p . In general, if a group G is endowed with its profinite topology, then the subspace topology on a subgroup $L \leq G$ is coarser (or equal) to the profinite topology of L . Therefore, it suffices to prove that H is not closed in the profinite topology of $Z(A_p)$. Consider the inclusion $\mathbb{Z} \subset \mathbb{Z}[\frac{1}{p}]$. It suffices to show that the only finite quotient of $L = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ is the trivial group. Note that, for every $x \in L$, there is $n \in \mathbb{N}$ such that $p^n x = 0$, that is, L is a p -group, so a finite quotient of L must be a finite p -group, and if this finite quotient is nontrivial, then L has a quotient which is cyclic of order p . At the same time, L is p -divisible (i.e., every element is a p th multiple), and hence so is every quotient of L . But the cyclic group of order p is not p -divisible. $\square$

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