REPRESENTATION VARIETIES OF FUCHSIAN GROUPS

DEDICATED TO THE MEMORY OF LEON EHRENPREIS

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Abstract. We estimate the dimension of varieties of the form $\mathrm{Hom}(\Gamma, G)$ where Γ is a Fuchsian group and G is a simple real algebraic group, answering along the way a question of I. Dolgachev.

0. Introduction

Let G be an almost simple real algebraic group, i.e., a non-abelian linear algebraic group over $\mathbb{R}$ with no proper normal $\mathbb{R}$-subgroups of positive dimension. Let Γ be a finitely generated group. The set of representations $\mathrm{Hom}(\Gamma, G(\mathbb{R}))$ coincides with the set of real points of the representation variety $X_{\Gamma,G} := \mathrm{Hom}(\Gamma, G)$. (We note here, that by a *variety*, we mean an affine scheme of finite type over $\mathbb{R}$; in particular, we do not assume that it is irreducible or reduced.)

Let $X^{\mathrm{epi}}_{\Gamma,G}$ denote the Zariski-closure in $X_{\Gamma,G}$ of the set of Zariski-dense homomorphisms $\Gamma \to G(\mathbb{R})$. In this paper, we estimate the dimension of $X^{\mathrm{epi}}_{\Gamma,G}$ when Γ is a cocompact Fuchsian group. Our main results assert that in most cases, this dimension is roughly $(1 - \chi(\Gamma)) \dim G$, where $\chi(\Gamma)$ is the Euler characteristic of Γ.

To formulate our results more precisely, we need some notation and definitions. A cocompact oriented Fuchsian group Γ (and all Fuchsian groups in this paper will be assumed to be cocompact and oriented without further mention) always admits a presentation of the following kind: Consider non-negative integers m and g and integers $d_1, \ldots, d_m$ greater than or equal to 2, such that

$$2 - 2g - \sum_{i=1}^{m}(1 - d_i^{-1}) \tag{0.1}$$

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is negative. For some choice of m, g, and d_i, Γ has a presentation

$$\Gamma := \langle x_1,\dots,x_m,y_1,\dots,y_g,z_1,\dots,z_g \mid x_1^{d_1},\dots,x_m^{d_m}, \\ x_1\cdots x_m[y_1,z_1]\cdots[y_g,z_g]\rangle, \tag{0.2}$$

and its Euler characteristic $\chi(\Gamma)$ is given by (0.1). If $g=0$ in the presentation (0.2), we sometimes denote Γ by $\Gamma_{d_1,\dots,d_m}$. For $m=3$, Γ is called a *triangle group*, and its isomorphism class does not depend on the order of the subscripts. Note that the parameter g and the multiset $\{d_1,\dots,d_m\}$ are determined by the isomorphism class of Γ. Every non-trivial element of Γ of finite order is conjugate to a power of one of the x_i, which is an element of order exactly d_i.

Definition 0.1. *Let H be an almost simple algebraic group. We say that a Fuchsian group Γ is H-*dense *if and only if there exists a homomorphism $\phi\colon \Gamma\to H(\mathbb{R})$ such that $\phi(\Gamma)$ is Zariski-dense in H and ϕ is injective on all finite cyclic subgroups of Γ (equivalently, $\phi(x_i)$ has order d_i for all i).*

We can now state our main theorems.

Theorem 0.2. *For every Fuchsian group Γ and every integer $n\ge 2$*

$$\dim X^{\mathrm{epi}}_{\Gamma,\mathrm{SU}(n)} = (1-\chi(\Gamma))\dim \mathrm{SU}(n) + O(1),$$

where the implicit constants depend only on Γ.

In particular, this answers a question of Igor Dolgachev, proving the existence in sufficiently high degree, of uncountably many absolutely irreducible, pairwise non-conjugate, representations.

Theorem 0.3. *For every Fuchsian group Γ and every split simple real algebraic group G,*

$$\dim X^{\mathrm{epi}}_{\Gamma,G} = (1-\chi(\Gamma))\dim G + O(\operatorname{rank} G),$$

where the implicit constants depend only on Γ.

Theorem 0.4. *For every $\mathrm{SO}(3)$-dense Fuchsian group Γ and every compact simple real algebraic group G,*

$$\dim X^{\mathrm{epi}}_{\Gamma,G} = (1-\chi(\Gamma))\dim G + O(\operatorname{rank} G),$$

where the implicit constants depend only on Γ.

Let us mention here that all but finitely many Fuchsian groups are $\mathrm{SO}(3)$-dense (see Proposition 5.1 for the complete list of exceptions).

The proof of the theorems is based on deformation theory. It is a well-known result of Weil [We] that the Zariski tangent space to $X_{\Gamma,G}$ at any point $\rho\in X_{\Gamma,G}(\mathbb{R})$ is equal to the space of 1-cocycles $Z^1(\Gamma,\mathrm{Ad}\circ\rho)$, where $\mathrm{Ad}\circ\rho$ is the representation of Γ on the Lie algebra $\mathfrak{g}$ of G determined by ρ. (For brevity, we often denote $\mathrm{Ad}\circ\rho$ by $\mathfrak{g}$, where the action of Γ is understood.) In general, the dimension of the tangent space to $X_{\Gamma,G}$ at ρ can be strictly larger than the dimension of a component of $X_{\Gamma,G}$ containing ρ, thanks to obstructions in $H^2(\Gamma,\mathrm{Ad}\circ\rho)$. Weil showed that if the coadjoint representation $(\mathrm{Ad}\circ\rho)^*$ has no Γ-invariant vectors, then ρ is a non-singular point of $X_{\Gamma,G}$, i.e., it lies on a unique component of $X_{\Gamma,G}$ whose dimension is given by $\dim Z^1(\Gamma,\mathrm{Ad}\circ\rho)$, the dimension of the Zariski-tangent space to $X_{\Gamma,G}$ at ρ. Computing this dimension is easy; the difficulty is to find ρ for which the obstruction space vanishes. A basic technique is to find a subgroup H of G for which the homomorphisms $\Gamma\to H$ are better understood and to choose ρ to factor through H. In this paper, we make particular use of the homomorphisms from $H=\mathsf{A}_n$ to $G=\mathrm{SO}(n-1)$ and of the principal homomorphisms from $H=\mathrm{PGL}(2)$ and $H=\mathrm{SO}(3)$ to various groups G—see §3 and §4 respectively.

It is interesting to compare our results (Theorems 0.2–0.4) to the results of Liebeck and Shalev [LS]. They also estimate $\dim X_{\Gamma,G}$ (and implicitly $\dim X^{\mathrm{epi}}_{\Gamma,G}$), but their methods work only for genus $g\geq 2$, while the most difficult (and interesting) case is $g=0$. A striking point is that they deduce their information about $X_{\Gamma,G}$ from deep results on the finite quotients of Γ, while we work directly with $X^{\mathrm{epi}}_{\Gamma,G}$ and can deduce that various families of finite groups of Lie type can be realized as quotients of Γ (see [LLM]).

The paper is organised as follows. In §1, we give a uniform proof of the upper bound in Theorems 0.2, 0.3 and 0.4. This requires estimating the dimensions of suitable cohomology groups and boils down to finding lower bounds on dimensions of centralizers.

To prove the lower bounds of these three theorems, we present in each case a representation of Γ which is "good" in the sense that it is a non-singular point of the representation variety to which it belongs. We then compute the dimension of the tangent space at the good point. In §2, we explain how one can go from a good representation of Γ into a smaller group H to a good representation into a larger group G. The initial step of this kind of induction is via a representation of Γ into an alternating group, $\mathrm{SO}(3)$, or $\mathrm{PGL}_2(\mathbb{R})$. We discuss the alternating group strategy in §3, where we prove Theorem 0.2 and begin the proof of Theorem 0.3. In §4, we discuss the principal homomorphism strategy, treating the remaining cases of Theorem 0.3, proving Theorem 0.4, and proving the existence of dense homomorphisms from SO(3)-dense Fuchsian groups to exceptional compact Lie groups (Proposition 4.3).

Proposition 5.2 in §5 shows that there are only six Fuchsian groups which are not SO(3)-dense. We do not have a good strategy for finding dense homomorphisms from these groups to compact simple Lie groups, since the methods of §3 are not effective. Y. William Yu found explicit surjective homomorphisms, described in the Appendix, from these groups to small alternating groups, which may serve as base cases for inductively constructing dense homomorphisms $\Gamma \to G(\mathbb{R})$ for these groups. We are grateful to him for his help.

All *Fuchsian groups* in this paper are assumed to be cocompact and oriented. A *variety* is an affine scheme of finite type over $\mathbb{R}$. Its *dimension* is understood to mean its Krull dimension. *Points* are $\mathbb{R}$-points, and *non-singular* points should be understood scheme-theoretically; i.e., a point x is non-singular if and only if it lies in only one irreducible component X, and the dimension of X equals the dimension of the Zariski-tangent space at x. An *algebraic group* will mean a linear algebraic group over $\mathbb{R}$. Unless otherwise stated, all topological notions will be understood in the sense of the Zariski-topology. In particular, a *closed subgroup* is taken to be Zariski-closed. Note, however, that an algebraic group G is *compact* if $G(\mathbb{R})$ is so in the real topology.

This paper is dedicated to the memory of Leon Ehrenpreis who was a leading figure in Fuchsian groups and was an inspiration in several other directions—not only mathematically.

1. Upper Bounds

We recall some results from [We]. For every finitely generated group Γ, the Zariski tangent space to $\rho \in X_{\Gamma,G}(\mathbb{R})$ is equal to $Z^1(\Gamma, \mathrm{Ad} \circ \rho)$ where $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is the adjoint representation of G on its Lie algebra. We will often write this more briefly as $Z^1(\Gamma, \mathfrak{g})$. Note that $\dim Z^1(\Gamma, \mathfrak{g})$ is always at least as great as the dimension of any component of $X_{\Gamma,G}$ in which ρ lies. Moreover, if Γ is a Fuchsian group and the coadjoint representation $\mathfrak{g}^* = (\mathrm{Ad} \circ \rho)^*$ has no Γ-invariant vectors, then ρ is a non-singular point of $X_{\Gamma,G}$.

If V denotes any finite dimensional real vector space V on which Γ acts, then

(1.1)

$$\dim Z^1(\Gamma, V) := (2g-1)\dim V + \dim(V^*)^\Gamma + \sum_{j=1}^{m}(\dim V - \dim V^{\langle x_j\rangle}).$$

$$= (1-\chi(\Gamma))\dim V + \dim(V^*)^\Gamma + \sum_{j=1}^{m}\Big(\frac{\dim V}{d_j} - \dim V^{\langle x_j\rangle}\Big).$$

The following proposition essentially gives the upper bounds in Theorems 0.2, 0.3 and 0.4, since for every irreducible component C of $X_{\Gamma,G}^{\mathrm{epi}}$ there exists a representation $\rho\colon \Gamma \to G(\mathbb{R})$ with Zariski-dense image in $C(\mathbb{R})$; $\dim Z^1(\Gamma, \mathfrak{g})$ is at least as great as the dimension of any irreducible component of $X_{\Gamma,G}$ to which ρ belongs and therefore at least as great as $\dim C$.

Proposition 1.1. *For every Fuchsian group Γ, every reductive $\mathbb{R}$-algebraic group G with a Lie algebra $\mathfrak{g}$ and every representation ρ : $\Gamma \to G(\mathbb{R})$ with Zariski dense image, we have:*

$$\dim Z^1(\Gamma, \mathfrak{g}) \leq (1-\chi(\Gamma))\dim G + (2g+m+\operatorname{rank} G) + \frac{3}{2}m \operatorname{rank} G,$$

where g and m are as in (0.2).

Proof. Weil's formula (1.1) yields

(1.2)

$$\dim Z^1(\Gamma, \mathfrak{g}) = (1-\chi(\Gamma))\dim G + \dim(\mathfrak{g}^*)^\Gamma + \sum_{j=1}^{m}\left(\frac{\dim G}{d_j} - \dim \mathfrak{g}^{\langle x_j\rangle}\right).$$

Note that if $\mathfrak{g}$ is the real Lie algebra of G then $\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$ is the complex Lie algebra of G. By abuse of notation we will also denote it by $\mathfrak{g}$. Of course they have the same dimensions over $\mathbb{R}$ and $\mathbb{C}$, respectively.

Lemma 1.2. *Under the above assumptions:*

$$\dim(\mathfrak{g}^*)^\Gamma \leq 2g + m + \operatorname{rank} G.$$

Proof of Lemma 1.2. The dimension of the Γ-invariants on $\mathfrak{g}^*$, $\dim(\mathfrak{g}^*)^\Gamma$, is equal to the dimension of the Γ-coinvariants on $\mathfrak{g}$. As Γ is Zariski dense in G, this is equal to the dimension of the coinvariants of G acting on $\mathfrak{g}$ via Ad. Letting G^0 act first, we deduce that the space of G-coinvariants is a quotient space of $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. More precisely, it is equal to the coinvariants of $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ acted upon by the finite group G/G^0. As $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is a characteristic zero vector space, the dimension of the coinvariants is the same as that of the G/G^0-invariant subspace. Now, the space of linear maps $\mathrm{Hom}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}],\mathbb{R})$ corresponds to the homomorphisms from G^0 to $\mathbb{R}$ and the G/G^0-invariants are those which can be extended to G. So, altogether $\dim(\mathfrak{g}^*)^\Gamma$ is bounded by $\dim \mathrm{Hom}(G,\mathbb{R})$. Now

$$\dim \mathrm{Hom}(G,\mathbb{R}) = \dim G^{\mathrm{ab}},$$

where $G^{\mathrm{ab}} = G/[G,G]$, and

$$G^{\mathrm{ab}} = U \times T \times A,$$

where U is a unipotent group, T a torus, and A a finite group. So $\dim G^{\mathrm{ab}} = \dim U + \dim T$. As Γ is Zariski dense in G, its image is Zariski dense in U and hence

$$\dim U \leq d(\Gamma) \leq 2g + m,$$

where $d(\Gamma)$ denotes the number of generators of Γ. Now, T, being a quotient of G, satisfies $\dim T \leq \mathrm{rank}\, G$. Altogether,

$$\dim(\mathfrak{g}^*)^\Gamma \leq 2g + m + \mathrm{rank}\, G,$$

as claimed. This completes the proof of Lemma 1.2.

Lemma 1.3. *If G is a complex reductive group and α an automorphism of G of order k, then*

$$\dim \mathrm{Fix}_G(\alpha) \geq \frac{\dim G}{k} - \frac{3}{2}\mathrm{rank}\, G,$$

where $\mathrm{Fix}_G(\alpha)$ denotes the subgroup of the fixed points of α.

Let us say that an automorphism α of G of order k is a *pure outer automorphism* of G if α^l is not inner for any l satisfying $1 \leq l < k$.

For inner or pure automorphisms we have a slightly stronger result:

Lemma 1.4. *Let α be either an inner or a pure outer automorphism of G of order k. Then*

$$\dim \mathrm{Fix}_G(\alpha) \geq \frac{\dim G}{k} - \mathrm{rank}\, G. \tag{1.3}$$

Proof of Lemma 1.4. Without loss of generality, we can assume G is connected. Let $\mathfrak{g}$ be the Lie algebra of G. Then α acts also on $\mathfrak{g}$, and $\dim \mathrm{Fix}_G(\alpha) = \dim \mathfrak{g}^\alpha$, so we can work at the level of Lie algebras. As α respects the decomposition of $\mathfrak{g}$ into $[\mathfrak{g},\mathfrak{g}] \oplus \mathfrak{z}$ where $\mathfrak{z}$ is the Lie algebra of the central torus. As $\mathrm{rank}\,\mathfrak{g} = \mathrm{rank}[\mathfrak{g},\mathfrak{g}] + \dim \mathfrak{z}$, we can restrict α to $[\mathfrak{g},\mathfrak{g}]$ and assume $\mathfrak{g}$ is semisimple.

Moreover we can write $\mathfrak{g}$ as a direct sum $\mathfrak{g} = \bigoplus_{i=1}^s \mathfrak{g}_i$ where each $\mathfrak{g}_i$ is itself a direct sum of isomorphic simple Lie algebras such that for each i, α acts transitively on the simple components. As both sides of the inequality are additive on a direct sum of α-invariant subalgebras, we can assume $\mathfrak{g}$ is a sum of t isomorphic simple algebras, $t|k$, and α acts transitively on the summands. If α is inner, then $t = 1$. If α is pure outer, it is equivalent to an action of the form

$$\alpha(x_1, \ldots, x_t) = (\beta(x_t), x_1, \ldots, x_{t-1}),$$

where β is a pure outer automorphism of a simple factor $\mathfrak{h}$, of order k/t. Thus,

$$\dim \mathfrak{g}^\alpha = \dim\{(x, x, \ldots, x) \mid x \in \mathfrak{h}^\beta\} = \dim \mathfrak{h}^\beta.$$

Thus, for the outer case, it suffices to prove the result when $t = 1$. If $k = 1$, the result is trivial. The possibilities for $(\mathfrak{g}, \mathfrak{h})$ are well-known (see, e.g., [He, Chapter X, Table 1]). For $k = 2$, they are $(\mathfrak{sl}(2n), \mathfrak{sp}(2n))$, $(\mathfrak{sl}(2n+1), \mathfrak{so}(2n+1))$, $(\mathfrak{so}(2n), \mathfrak{so}(2n-1))$, and $(\mathfrak{e}_6, \mathfrak{f}_4)$, and for $k = 3$, there is the unique case $(\mathfrak{so}(8), \mathfrak{g}_2)$.

Now assume α is inner. By an exhaustive computer search, one checks the cases of exceptional Lie algebras $\mathfrak{g}$. (It suffices to check for $k \leq (h + 1)/2$, when h is the Coxeter number of $\mathfrak{g}$.)

It remains to prove the lemma for simple algebras of type A, B, C, and D. Here, it is convenient to work at the level of groups. Let x be the element of $G(\mathbb{C})$ of order k so that α corresponds to conjugation by x and $\mathrm{Fix}_G(\alpha) = Z_G(x)$.

We start with type A, setting $G = \mathrm{SL}_n$. Let a_j denote the multiplicity of $e^{2\pi i k/j}$ as an eigenvalue of x. By the Cauchy-Schwartz inequality,

$$\dim Z_G(x) + 1 = \sum_{j=0}^{k-1} a_j^2 \geq \frac{\left(\sum_{j=0}^{k-1} a_j\right)^2}{k} = n^2/k > \frac{\dim G}{k}, \tag{1.4}$$

so (1.3) holds. Note that in this case, a stronger inequality holds, namely

$$\dim \mathrm{Fix}_G(\alpha) \geq \frac{\dim G}{k} - 1.$$

This will be needed for the upper bound in Theorem 0.2.

Next we consider types B and D, assuming that G is an orthogonal group. Let $x \in \mathrm{SO}_n(\mathbb{C})$ have order k. The multiplicity of the eigenvalue $e^{2\pi i j/k}$ in the natural n-dimensional representation V of G will be denoted a_j. For $1 \leq j < k/2$, $a_{k-j} = a_j$. For k odd, we define $a_{k/2} = 0$. Identifying the Lie algebra $\mathfrak{g}$ with $\bigwedge^2 V$, we see that

$$\dim Z_G(x) = \frac{a_0^2 - a_0}{2} + \frac{a_{k/2}^2 - a_{k/2}}{2} + \sum_{1 \leq j < k/2} a_j^2. \tag{1.5}$$

For k even,

$$2\dim Z_G(x)+\frac{1}{2}=(a_0-\frac{1}{2})^2+(a_{k/2}-\frac{1}{2})^2+\sum_{j=1}^{k/2-1}(a_j^2+a_{k-j}^2)$$
$$\geq\frac{(n-1)^2}{k}.$$

As

$$\dim G-\frac{(n-1)^2}{2}=\frac{n-1}{2},$$

(1.3) holds.
For k odd,

$$2\dim Z_G(x)+\frac{1}{4}=(a_0-\frac{1}{2})^2+\sum_{j=1}^{(k-1)/2}a_j^2+a_{k-j}^2$$
$$\geq\frac{(n-1/2)^2}{k}.$$

and again (1.3) holds.

Finally we consider type C, assuming that $G=\mathrm{Sp}_n$ and $x\in G(\mathbb{C})$ of order k. We define a_j as before, using the natural n-dimensional representation V of G. Identifying the Lie algebra $\mathfrak{g}$ with $\mathrm{Sym}^2 V$, we see that

$$2\dim Z_G(x)+\frac{1}{2}=(a_0+\frac{1}{2})^2+(a_{k/2}+\frac{1}{2})^2+\sum_{j=1}^{k/2-1}(a_j^2+a_j^2)$$
$$\geq\frac{(n+1)^2}{k}.$$

for k even and

$$2\dim Z_G(x)+\frac{1}{4}=(a_0+\frac{1}{2})^2+\sum_{j=1}^{(k-1)/2}(a_j^2+a_j^2)$$
$$\geq\frac{(n+1/2)^2}{k}.$$

for k odd. In both cases, (1.3) holds. This finishes the proof of Lemma 1.4.

Proof of Lemma 1.3. To prove the statement, we still need to handle the case where α is neither an inner nor a pure outer automorphism. This means that for some l dividing k, with $1<l<k$, α^l is inner while

α is not. Let $H = Z_G(\alpha^l) = \mathrm{Fix}_G(\alpha^l)$. As α^l is an inner automorphism of order k/l, Lemma 1.4 implies that

$$\dim H \geq \frac{\dim G}{k/l} - \operatorname{rank} G.$$

Now α acts on the reductive group H as a pure outer automorphism of order at most l. Thus, again by Lemma 1.4

$$\begin{aligned}
\dim \mathrm{Fix}_G(\alpha) &= \dim \mathrm{Fix}_H(\alpha) \\
&\geq \frac{\dim H}{l} - \operatorname{rank} H \\
&\geq \frac{1}{l}\left(\frac{\dim G}{k/l} - \operatorname{rank} G\right) - \operatorname{rank} G \\
&\geq \frac{\dim G}{k} - \left(1 + \frac{1}{l}\right) \operatorname{rank} G.
\end{aligned}$$

As $l > 1$, we get

$$\dim \mathrm{Fix}_G(\alpha) \geq \frac{\dim G}{k} - \frac{3}{2} \operatorname{rank} G$$

completing the proof of Lemma 1.3.

We are now ready to put Lemmas 1.2 and 1.3 into (1.1). Note that $\dim \mathfrak{g}^{\langle x_j \rangle}$ there is equal to $\dim \mathrm{Fix}_G(x_j)$ and we have:

$$\dim Z^1(\Gamma, \mathfrak{g}) \leq (1 - \chi(\Gamma)) \dim G + (2g + m + \operatorname{rank} G) + \frac{3}{2} m \operatorname{rank} G.$$

$\square$

In summary, we have proved the upper bounds for Theorems 0.2, 0.3, and 0.4. For Theorems 0.3 and 0.4, the bounds follow immediately from Proposition 1.1, while the bound for Theorem 0.2 requires the better estimate proved in (1.4).

2. A Density Criterion

The results in this section are valid for general finitely generated groups Γ. The main result is Theorem 2.4, which gives a criterion for an irreducible component C of $X_{\Gamma,G}$ to be contained in $X^{\mathrm{epi}}_{\Gamma,G}$, i.e. to have the property that there exists a Zariski-dense subset of $C(\mathbb{R})$ consisting of representations ρ such that $\rho(\Gamma)$ is Zariski-dense in G. We begin with the technical results needed in the proof of Theorem 2.4.

Proposition 2.1. *Let G be a linear algebraic group over $\mathbb{R}$, and $H \subset G$ a closed subgroup such that $G(\mathbb{R})/H(\mathbb{R})$ is compact. Let C denote an irreducible component of $X_{\Gamma,H}$. The condition on $\rho \in X_{\Gamma,G}(\mathbb{R})$ that*

ρ is not contained in any $G(\mathbb{R})$-conjugate of $C(\mathbb{R})$ is open in the real topology.

Proof. The conjugation map $H \times X_{\Gamma,H} \to X_{\Gamma,H}$ restricts to a map

$$H^\circ \times C \to X_{\Gamma,H}.$$

As H° and C are irreducible, the image of this morphism lies in an irreducible component of $X_{\Gamma,H}$, which must therefore be C.

The proposition can be restated as follows: the condition on ρ that ρ *is* contained in some $G(\mathbb{R})$-conjugate of $C(\mathbb{R})$ is closed in the real topology. To prove this, consider a sequence $\rho_i \in X_{\Gamma,G}(\mathbb{R})$ converging to ρ. Suppose that for each ρ_i there exists $g_i \in G(\mathbb{R})$ such that $\rho_i \in g_i C(\mathbb{R}) g_i^{-1}$. Let $\bar{g}_i$ denote the image of g_i in $G(\mathbb{R})/H^\circ(\mathbb{R})$. As this set is compact, there exists a subsequence which converges to some $\bar{g} \in G(\mathbb{R})/H^\circ(\mathbb{R})$. Passing to this subsequence, we may assume that $\bar{g}_1, \bar{g}_2, \ldots$ converges to $\bar{g}$. If $g \in G(\mathbb{R})$ represents the coset $\bar{g}$, we claim that $\rho \in gC(\mathbb{R})g^{-1}$. The claim implies the proposition

By the implicit function theorem, there exists a continuous section $s\colon G(\mathbb{R})/H^\circ(\mathbb{R}) \to G(\mathbb{R})$ in a neighborhood of $\bar{g}$, and we may normalize so that $s(\bar{g}) = g$. For i sufficiently large, $s(\bar{g}_i)$ is defined, and $g_i = s(\bar{g}_i)h_i$ for some $h_i \in H^\circ(\mathbb{R})$. As conjugation by elements of $H^\circ(\mathbb{R})$ preserves C, we may assume without loss of generality that $g_i = s(\bar{g}_i)$ for all i sufficiently large. As $\lim_{i\to\infty} g_i = g$ and $C(\mathbb{R})$ is closed in the real topology in $X_{\Gamma,G}(\mathbb{R})$,

$$g^{-1}\rho g = \lim_{i\to\infty} g_i^{-1}\rho_i g_i \in C(\mathbb{R}).$$

□

The following proposition is surely well-known, but for lack of a precise reference, we give a proof.

Proposition 2.2. *Let G be an almost simple real algebraic group. There exists a finite set $\{H_1, \ldots, H_k\}$ of proper closed subgroups of G such that every proper closed subgroup is contained in some group of the form gH_ig^{-1}, where $g \in G(\mathbb{R})$.*

Proof. The theorem is proved for $G(\mathbb{R})$ compact in [La, 1.3], so we may assume henceforth that G is not compact.

First we prove that every proper closed subgroup K is contained in a maximal closed subgroup of positive dimension. If $\dim K > 0$, then for every infinite ascending chain $K_1 = K \subsetneq K_2 \subsetneq \cdots \subset G$ of closed subgroups of dimension $\dim K$, there exists a proper subgroup L of G which contains every K_i and for which $\dim L > \dim K$. Indeed, we can take $L := N_G(K^\circ)$, which contains all K_i, since $K_i^\circ = K^\circ$. It cannot equal G since G is almost simple, and if $\dim K = \dim L$, then $L^\circ = K^\circ$, and there are only finitely many groups between K and L. Thus every proper subgroup of G of positive dimension is either contained in a maximal subgroup of G of the same dimension or in a proper subgroup of higher dimension. It follows that each such proper subgroup is contained in a maximal subgroup. For finite subgroups K, we can embed K in a maximal compact subgroup of G, which lies in a conjugacy class of proper closed subgroups of positive dimension since G itself is not compact.

We claim that every maximal closed subgroup H of positive dimension is either parabolic or the normalizer of a connected semisimple subgroup or the normalizer of a maximal torus. Indeed, H is contained in the normalizer of its unipotent radical U. If U is non-trivial, this normalizer is contained in a parabolic P [Hu, 30.3, Cor. A], so $H = P$. If U is trivial, H is reductive and is contained in the normalizer of the derived group of its identity component H°. If this is non-trivial, H is the normalizer of a semisimple subgroup. If not, H° is a torus T. Then H is contained in the normalizer of the derived group of $Z_G(T)^\circ$, which is again the normalizer of a semisimple subgroup unless $Z_G(T)^\circ$ is a torus. In this case, it is a maximal torus, and H is the normalizer of this torus. Since a real semisimple group has finitely many conjugacy classes of parabolics and maximal tori, we need only consider the normalizers of semisimple subgroups. There are finitely many conjugacy classes of these by a theorem of Richardson [Ri]. □

The proof of Proposition 2.2 gives some additional information, which we employ in the following lemma:

Lemma 2.3. *If H is a maximal proper subgroup of a split almost simple algebraic group G over $\mathbb{R}$, then either H is parabolic or* $\dim H \leq \frac{9}{10}\dim G$.

Proof. For exceptional groups, all proper subgroups have dimension $\leq \frac{9}{10}\dim G$. Indeed, this is true for exceptional groups G over finite fields as a consequence of the Landazuri-Seitz estimates for the minimal degree of a non-trivial complex representation of $G(\mathbb{F}_q)$ [LZ], and the same result follows in characteristic zero by a specialization argument. We therefore consider only the case that G is of type A, B, C, or D. Also, we can ignore isogenies and assume that G is either SL_n, a split orthogonal group, or a split symplectic group. Let V be the natural representation of G. If $\dim V = n$, then $\dim G$ is n^2-1, $n(n-1)/2$, or $n(n+1)/2$, depending on whether G is linear, orthogonal, or symplectic.

It suffices to consider the case that H is the normalizer of a semisimple subgroup $K \subset G$. The action of H must preserve the decomposition of V into K-irreducible factors. Therefore, H lies in a parabolic subgroup unless all factors have equal dimension. If all factors have equal dimension and there are at least three factors, then $\dim H \leq n^2/3$, so the theorem holds in such cases. If H° respects a decomposition $V = W_1 \oplus W_2$ where $\dim W_i = n/2$, then either G is linear and $\dim H < (1/2)\dim G + 1$, G is orthogonal and $\dim H \leq (n/2)^2$, or G is symplectic and $\dim H \leq (n/2)(n/2+1)$. If $V \otimes \mathbb{C}$ is reducible, it decomposes into two factors of degree $n/2$, and the same estimates apply.

We have therefore reduced to the case that K is semisimple and $V \otimes \mathbb{C}$ is irreducible, so we may and do extend scalars to $\mathbb{C}$ for the remainder of the proof. If K is not almost simple, then any element of G which normalizes K must respect a non-trivial tensor decomposition and therefore H respects such a decomposition. This implies

$$\dim H \leq m^2 + (n/m)^2 - 1 \leq 3 + n^2/4.$$

We may therefore assume that K is almost simple and V is associated to a dominant weight of K. It is easy to deduce from the Weyl dimension formula that every non-trivial irreducible representation of a simple Lie algebra L of rank r, other than the natural representation and its dual, has dimension at least $(r^2+r)/2$, we need only consider the case that V is a natural representation. As $H \subsetneq G$, we need only consider the inclusions $\mathrm{SO}(n) \subset \mathrm{SL}_n$ and $\mathrm{Sp}(n) \subset \mathrm{SL}_n$. In all cases, we have $\dim H \leq \frac{2}{3}\dim G$.

□

Theorem 2.4. *Let Γ be a finitely generated group, G an almost simple real algebraic group, and $\rho_0 \in \mathrm{Hom}(\Gamma, G(\mathbb{R}))$ a non-singular $\mathbb{R}$-point of $X_{\Gamma,G}$. For every closed subgroup H of G such that $\rho_0(\Gamma) \subset H(\mathbb{R})$, let t_H denote the dimension of the Zariski tangent space of $X_{\Gamma,H}$ at ρ_0 (i.e., $t_H = \dim Z^1(\Gamma, \mathfrak{h})$, where $\mathfrak{h}$ is the Lie algebra of $H(\mathbb{R})$ with the adjoint action of Γ.) We assume*

(1) *If H is any maximal closed subgroup such that $\rho_0(\Gamma) \subset H(\mathbb{R})$, then*

$$t_G - \dim G > t_H - \dim H,$$

(2) *If H is any maximal closed subgroup such that $G(\mathbb{R})/H(\mathbb{R})$ is not compact, then*

$$t_G - \dim G > \dim X_{\Gamma,H} - \dim H.$$

Then $X^{\mathrm{epi}}_{\Gamma,G}$ contains the irreducible component of $X_{\Gamma,G}$ to which ρ_0 belongs.

Proof. Let C denote the irreducible component of $X_{\Gamma,G}$ containing ρ_0, which is unique since ρ_0 is a non-singular point of $X_{\Gamma,G}$. Again, since ρ_0 is a non-singular point, there is an open neighborhood U of ρ_0 in $C(\mathbb{R})$ which is diffeomorphic to $\mathbb{R}^n$, where $n := \dim C = t_G$.

Let $\{H_1, \dots, H_k\}$ represent the conjugacy classes of maximal proper closed subgroups of G given by Lemma 2.2. Let $C_{i,j}$ denote the irreducible components of X_{Γ,H_i}. For each component we consider the conjugation morphism $\chi_{i,j} \colon G \times C_{i,j} \to X_{\Gamma,G}$. We claim that the fibers of this morphism have dimension at least $\dim H_i$. Indeed, the action of H_i° on $G \times C_{i,j}$ given by

$$h.(g, \rho_0) = (gh^{-1}, h\rho_0 h^{-1})$$

is free, and $\chi_{i,j}$ is constant on the orbits of the action. Thus, the closure of the image of $\chi_{i,j}$ has dimension at most $\dim C_{i,j} + \dim G - \dim H_i$. Condition (2) guarantees that if $G(\mathbb{R})/H_i(\mathbb{R})$ is not compact, then a non-empty Zariski-open subset of C lies outside the image of $\chi_{i,j}$ for all j. Condition (1) guarantees the same thing if $G(\mathbb{R})/H_i(\mathbb{R})$ is compact, and some conjugate of ρ_0 lies in $C_{i,j}(\mathbb{R})$. Note that $\dim C_{i,j} \le t_{H_i}$ if $\rho_0 \in C_{i,j}(\mathbb{R})$.

Finally, we consider components $C_{i,j}$ for which $G(\mathbb{R})/H_i(\mathbb{R})$ is compact, but no conjugate of ρ_0 lies in $C_{i,j}(\mathbb{R})$. By Proposition 2.1, the $G(\mathbb{R})$-orbit of each such $C_{i,j}(\mathbb{R})$ meets $C(\mathbb{R})$ in a set which is closed in the real topology. Since ρ_0 belongs to none of these sets, there is a neighborhood U of ρ_0 consisting of homomorphisms ρ such that no conjugate of ρ lies in any such $C_{i,j}$. The intersection of U with any non-empty Zariski-open subset of $C(\mathbb{R})$ is therefore Zariski-dense in C, and for every ρ in this set, $\rho(\Gamma)$ is Zariski-dense in $G(\mathbb{R})$. It follows that $X^{\mathrm{epi}}_{\Gamma,G}$ contains C. □

Note that if G is compact, condition (2) is vacuous.

Corollary 2.5. *If G is a compact almost simple algebraic group over $\mathbb{R}$, H is a connected maximal proper closed subgroup of G with finite center, and $\rho_0 \colon \Gamma \to H(\mathbb{R})$ has dense image, then $t_G - \dim G > t_H - \dim H$ implies $X^{\mathrm{epi}}_{\Gamma,G}$ contains the irreducible component of $X_{\Gamma,G}$ to which ρ_0 belongs.*

Proof. To apply the theorem, we need only prove that ρ_0 is a non-singular point of $X_{\Gamma,G}$. As H is maximal, the product $Z_G(H)H$ must equal H, which means $Z_G(H) = Z(H)$ is finite. Thus, $\mathfrak{g}^\Gamma = \mathfrak{g}^H = \{0\}$, and since $\mathfrak{g}$ is a self-dual $G(\mathbb{R})$-representation, this implies $(\mathfrak{g}^*)^\Gamma = \{0\}$, which implies that ρ_0 is a non-singular point of $X_{\Gamma,G}$. $\square$

3. The Alternating Group Method

In this section Γ is any (cocompact, oriented) Fuchsian group. We first consider $G = \mathrm{SO}(n)$.

Proposition 3.1. *For Γ a Fuchsian group and $G = \mathrm{SO}(n)$, we have*

$$\dim X^{\mathrm{epi}}_{\Gamma,\mathrm{SO}(n)} = (1-\chi(\Gamma))\dim \mathrm{SO}(n) + O(n)$$

where the implicit constant depends only on Γ.

Proof. Proposition 1.1 gives the upper bound, so it suffices to prove

$$\dim X^{\mathrm{epi}}_{\Gamma,\mathrm{SO}(n)} \geq (1-\chi(\Gamma))\dim \mathrm{SO}(n) + O(n).$$

Let $d_1,\ldots,d_m$ be defined as in (0.2). For large n, denote C_i, for $i = 1,\ldots,m$, the conjugacy class in the alternating group A_{n+1} which consists of even permutations of $\{1,2,\ldots,n+1\}$ with only d_i-cycles and 1-cycles and with as many d_i-cycles as possible. Thus, any element of C_i has at most $2d_i-1$ fixed points. Theorem 1.9 of [LS] ensures that for large enough n, there exist epimorphisms ρ_0 from Γ onto A_{n+1}, sending x_i to an element of C_i for $i = 1,\ldots,m$ and x_i as in (0.2).

Now $\mathsf{A}_{n+1} \subset \mathrm{SO}(n)$ and moreover the action of A_{n+1} on the Lie algebra $\mathfrak{so}(n)$ of $\mathrm{SO}(n)$ is the restriction to A_{n+1} of the irreducible S_{n+1} representation associated to the partition $(n-1)+1+1$ ([FH, Ex. 4.6]). If $n \geq 5$, this partition is not self-dual, so the restriction to A_{n+1} is irreducible. By (1.2),

$$\dim Z^1(\Gamma, \mathrm{Ad}\circ\rho_0) = (1-\chi(\Gamma))\dim \mathfrak{so}(n) + \sum_{i=1}^m \left(\frac{\dim \mathfrak{so}(n)}{d_i} - \dim \mathfrak{so}(n)^{\langle x_i\rangle}\right).$$

Now $\dim \mathfrak{so}(n)^{\langle x_i\rangle}$ is equal to the multiplicity of the eigenvalue 1 of $x = \rho_0(x_i)$ acting via Ad on $\mathfrak{so}(n)$. Note that the multiplicity of every d_ith root of unity as an eigenvalue for our element $x = \rho_0(x_i)$, when acting on the natural n-dimensional representation, is of the form $\frac{n}{d_i} + O(1)$, where the implied constant depends only on d_i. Thus using the same arguments as in the proof of Lemma 1.4 (see (1.5)), we can deduce that

$$\left|\frac{\dim \mathfrak{so}(n)}{d_i} - \dim \mathfrak{so}(n)^{\langle x_i\rangle}\right| = O(n),$$

where again the constant depends only on d_i.

As $\mathfrak{so}(n)^*$ has no A_{n+1}-invariants, $X_{\Gamma,\mathrm{SO}(n)}$ is non-singular at ρ_0. By Theorem 2.4, as long n is large enough that

$$\begin{aligned} t_{\mathrm{SO}(n)} &= \dim Z^1(\Gamma, \mathrm{Ad} \circ \rho_0) \\ &> \dim \mathrm{SO}(n) - \dim \mathsf{A}_{n+1} + t_{\mathsf{A}_{n+1}} \\ &= \dim \mathrm{SO}(n), \end{aligned}$$

$X^{\mathrm{epi}}_{\Gamma,\mathrm{SO}(n)}$ contains the component of $X_{\Gamma,\mathrm{SO}(n)}$ to which ρ_0 belongs, and this has dimension $t_{\mathrm{SO}(n)} = (1 - \chi(\Gamma)) \dim \mathrm{SO}(n) + O(n)$. $\square$

We remark that in this case, there is a more elementary alternative argument. The condition on $X_{\Gamma,\mathrm{SO}(n)}$ of irreducibility on $\mathfrak{so}(n)$ is open. It is impossible that all representations in a neighborhood of ρ_0 have finite image and those with infinite image should have Zariski dense image (since the Lie algebra of the connected component of the Zariski closure is $\rho(\Gamma)$-invariant).

We can now prove Theorem 0.2.

Proof. The upper bound has already been proved in §1. It therefore suffices to prove

$$\dim X^{\mathrm{epi}}_{\Gamma,\mathrm{SU}(n)} \geq (1 - \chi(\Gamma)) \dim \mathrm{SU}(n) + O(1).$$

Throughout the argument, we may always assume that n is sufficiently large,

We begin by defining ρ_0 as in the proof of Proposition 3.1. Let C denote the irreducible component of $X_{\Gamma,\mathrm{SO}(n)}$ to which ρ_0 belongs. We may choose $\rho_0' \in C(\mathbb{R})$ such that $\rho_0'(\Gamma)$ is Zariski-dense in $\mathrm{SO}(n)$. As there are finitely many conjugacy classes of order d_i in $\mathrm{SO}(n)$, the conjugacy class of $\rho(x_i)$ does not vary as ρ ranges over the irreducible variety C, so $\rho_0(x_i)$ is conjugate to $\rho_0'(x_i)$ in $\mathrm{SO}(n)$.

We have no further use for ρ_0 and now redefine ρ_0 to be the composition of ρ_0' with the inclusion $\mathrm{SO}(n) \hookrightarrow \mathrm{SU}(n)$. The eigenvalues of $\rho_0(x_i)$ are d_ith roots of unity, and each appears with multiplicity $n/d_i + O(1)$, where the implicit constant may depend on d_i but does not depend on n. The representation $\mathrm{SO}(n) \to \mathrm{SU}(n)$ is irreducible, so $(\mathfrak{su}(n))^{\mathrm{SO}(n)} = \{0\}$. As $\mathfrak{su}(n)$ is a self-dual representation of $\mathrm{SU}(n)$, it is a self-dual representation of $\mathrm{SO}(n)$, so as $\rho_0(\Gamma)$ is dense in $\mathrm{SO}(n)$,

$$(\mathfrak{su}(n)^*)^{\Gamma} = (\mathfrak{su}(n)^*)^{\mathrm{SO}(n)} = \{0\}.$$

It follows that $X_{\Gamma,\mathrm{SU}(n)}$ is non-singular at ρ_0. Since each eigenvalue of $\rho_0(x_i)$ has multiplicity $n/d_i + O(1)$,

$$t_{\mathrm{SU}(n)} = \dim Z^1(\Gamma, \mathrm{Ad} \circ \rho_0) = (1 - \chi(\Gamma)) \dim \mathrm{SU}(n) + O(1).$$

We claim that $\mathrm{SO}(n)$ is contained in a unique maximal closed subgroup of $\mathrm{SU}(n)$. Indeed, if G is any intermediate group, the Lie algebra $\mathfrak{g}$ of G must be an $\mathrm{SO}(n)$-subrepresentation of $\mathfrak{su}(n)$ which contains $\mathfrak{so}(n)$. Since $\mathfrak{su}(n)/\mathfrak{so}(n)$ is an irreducible $\mathrm{SO}(n)$-representation (namely, the symmetric square of the natural representation of $\mathrm{SO}(n)$), it follows that $\mathfrak{g} = \mathfrak{su}(n)$ or $\mathfrak{g} = \mathfrak{so}(n)$. In the former case, $G = \mathrm{SU}(n)$. In the latter case, G is contained in $N_G(\mathrm{SO}(n))$. This is therefore the unique maximal proper closed subgroup of $\mathrm{SU}(n)$ containing $\mathrm{SO}(n)$, or (equivalently) $\rho_0(\Gamma)$. The theorem now follows from Theorem 2.4 together with the upper bound estimate Proposition 1.1 applied to $N_G(\mathrm{SO}(n))$. □

We can also deduce Theorem 0.3 for G of type A and D from Proposition 3.1.

Proof. If $G_1 \to G_2$ is an isogeny, the morphism $X_{\Gamma,G_1} \to X_{\Gamma,G_2}$ is quasi-finite, and so

$$\dim X_{\Gamma,G_2} \geq \dim X_{\Gamma,G_1}.$$

Likewise, the composition of a homomorphism with dense image with an isogeny still has dense image, so

$$\dim X^{\mathrm{epi}}_{\Gamma,G_2} \geq \dim X^{\mathrm{epi}}_{\Gamma,G_1}.$$

In particular, to prove our dimension estimate for an adjoint group, it suffices to prove it for any covering group. We begin by proving it for $G = \mathrm{SL}_n$, which also gives it for PGL_n.

Let ρ_0 now denote a homomorphism $\Gamma \to \mathrm{SO}(n) \subset \mathrm{SL}_n(\mathbb{R})$ with dense image and such that every eigenvalue of $\rho_0(x_i)$ has multiplicity $n/d_i + O(1)$. Such a homomorphism exists by the proof of Proposition 3.1. It is well-known that $\mathrm{SO}(n)$ is a maximal closed subgroup of SL_n, and $\mathfrak{g}^{\mathrm{SO}(n)} = \{0\}$. Thus ρ_0 is a non-singular point of $X_{\Gamma,G}(\mathbb{R})$. Let C denote the unique irreducible component to which it belongs. In applying Theorem 2.4, we do not need to consider parabolic subgroups at all since $\rho_0(\Gamma)$ is not contained in any and $G(\mathbb{R})/H(\mathbb{R})$ is compact when H is parabolic. All other maximal subgroups are reductive, and we may therefore apply Proposition 1.1 to get an upper bound

$$\dim X_{\Gamma,H} \leq (1 - \chi(\Gamma)) \dim H + 2g + m + (3m/2 + 1)n$$

By Lemma 2.3, $\dim H < \frac{9}{10}(n^2 - 1)$, so for n sufficiently large,

$$\dim X_{\Gamma,H} - \dim H < \dim X_{\Gamma,G} - \dim G.$$

Thus condition (2) of Theorem 2.4 holds, and so the component C of $X_{\Gamma,G}$ to which ρ_0 belongs lies in $X^{\mathrm{epi}}_{\Gamma,G}$. It is therefore a non-singular point of C, and it follows that

$$\dim X_{\Gamma,G}^{\mathrm{epi}} \geq \dim C = \dim Z^1(\Gamma, \mathfrak{g}) = (1 - \chi(\Gamma)) \dim \mathrm{SL}_n + O(n).$$

The argument for type D is very similar. Here we work with $G = \mathrm{SO}(n,n)$, which is a double cover of the split adjoint group of type D_n over $\mathbb{R}$. Our starting point is a homomorphism $\rho_0 \colon \Gamma \to \mathrm{SO}(n) \times \mathrm{SO}(n)$ with dense image and such that the eigenvalues of

$$\rho(x_i) \in \mathrm{SO}(n) \times \mathrm{SO}(n) \subset \mathrm{SO}(n,n) \subset \mathrm{GL}_{2n}(\mathbb{C})$$

have multiplicity $(2n)/d_i + O(1)$. Such a ρ_0 is given by a pair (σ, τ) of dense homomorphisms $\Gamma \to \mathrm{SO}(n)$ satisfying a balanced eigenvalue multiplicity condition and the additional condition that σ and τ do not lie in the same orbit under the action of $\mathrm{Aut}(\mathrm{SO}(n))$ on $X_{\Gamma,\mathrm{SO}(n)}$. This additional condition causes no harm, since $\dim \mathrm{Aut}\,\mathrm{SO}(n) = \dim \mathrm{SO}(n)$, while the components of $\dim X_{\Gamma,\mathrm{SO}(n)}^{\mathrm{epi}}$ constructed above (which satisfy the balanced eigenvalue condition) have dimension greater than $\dim \mathrm{SO}(n)$ for large n. Given a pair (σ, τ) as above, the closure H of $\rho_0(\Gamma)$ is a subgroup of $\mathrm{SO}(n) \times \mathrm{SO}(n)$ which maps onto each factor but which does not lie in the graph of an isomorphism between the two factors. By Goursat's lemma, $H = \mathrm{SO}(n) \times \mathrm{SO}(n)$. From here, one passes from H to $G = \mathrm{SO}(n,n)$ just as in the case of groups of type A. □

4. Principal Homomorphisms

It is a well-known theorem of de Siebenthal [dS] and Dynkin [D1] that for every (adjoint) simple algebraic group G over $\mathbb{C}$ there exists a conjugacy class of *principal* homomorphisms $\mathrm{SL}_2 \to G$ such that the image of any non-trivial unipotent element of $\mathrm{SL}_2(\mathbb{C})$ is a regular unipotent element of $G(\mathbb{C})$. The restriction of the adjoint representation of G to SL_2 via the principal homomorphism is a direct sum of V_{2e_i}, where $e_1, \ldots, e_r$ is the sequence of exponents of G, and V_m denotes the mth symmetric power of the 2-dimensional irreducible representation of SL_2, which is of dimension $m+1$ [Ko]. In particular,

$$\dim G = \sum_{i=1}^{r} (2e_i + 1),$$

where r denotes $\mathrm{rank}\, G$. As each V_{2e_i} factors through PGL_2, the same is true for the homomorphism $\mathrm{SL}_2 \to \mathrm{Ad}(G)$. More generally, if G is defined and split over any field K of characteristic zero, the principal homomorphism can be defined over K.

The following proposition is due to Dynkin:

Proposition 4.1. *Let G be an adjoint simple algebraic group over $\mathbb{C}$ of type A_1, A_2, B_n $(n \geq 4)$, C_n $(n \geq 2)$, E_7, E_8, F_4, or G_2. Let H denote the image of a principal homomorphism of G. Let K be a closed subgroup of G whose image in the adjoint representation of G is conjugate to that of H. Then K is a maximal subgroup of G.*

Proof. As K is conjugate to H in $\mathrm{GL}(\mathfrak{g})$, in particular the number of irreducible factors of $\mathfrak{g}$ restricted to H and to K are the same. By [Ko], this already implies that H and K are conjugate in G. The fact that H is maximal is due to Dynkin. The classical and exceptional cases are treated in [D3] and [D2] respectively. □

As SL_2 is simply connected, the principal homomorphism $\mathrm{SL}_2 \to G$ lifts to a homomorphism $\mathrm{SL}_2 \to H$ if H is a split semisimple group which is simple modulo its center. Again, this is true for split groups over any field of characteristic zero. We also call such homomorphisms principal.

If G is an adjoint simple group over $\mathbb{R}$ with $G(\mathbb{R})$ compact and $\phi\colon \mathrm{PGL}_{2,\mathbb{C}} \to G_{\mathbb{C}}$ is a principal homomorphism over $\mathbb{C}$, ϕ maps the maximal compact subgroup $\mathrm{SO}(3) \subset \mathrm{PGL}_2(\mathbb{C})$ into a maximal compact subgroup of $G(\mathbb{C})$. Thus ϕ can be chosen to map $\mathrm{SU}(2)$ to $G(\mathbb{R})$, and such a homomorphism will again be called principal. Likewise, if H is almost simple and $H(\mathbb{R})$ is compact, a principal homomorphism $\phi\colon \mathrm{SL}_{2,\mathbb{C}} \to H_{\mathbb{C}}$ can be chosen so that $\phi(\mathrm{SU}(2)) \subset H(\mathbb{R})$.

Proposition 4.2. *Let G be an adjoint compact simple real algebraic group of type A_1, A_2, B_n $(n \geq 4)$, C_n $(n \geq 2)$, E_7, E_8, F_4, or G_2, and let Γ be an $\mathrm{SO}(3)$-dense Fuchsian group. Let $\rho_0\colon \Gamma \to G$ denote the composition of the map $\Gamma \to \mathrm{SO}(3)$ and the principal homomorphism $\phi\colon \mathrm{SO}(3) \to G$. If*

$$-\chi(\Gamma)\dim G + \sum_{j=1}^m \frac{\dim G}{d_j} - \sum_{j=1}^m \sum_{i=1}^r (1 + 2\lfloor e_i/d_j \rfloor) > -\chi(\Gamma)\dim \mathrm{SO}(3) + \sum_{j=1}^m \frac{\dim \mathrm{SO}(3)}{d_j} - m,$$

then

$$\dim X^{\mathrm{epi}}_{\Gamma,G} \geq (1-\chi(\Gamma))\dim G + \sum_{j=1}^m \frac{\dim G}{d_j} - \sum_{j=1}^m \sum_{i=1}^r (1+2\lfloor e_i/d_j \rfloor). \tag{4.1}$$

Proof. Let x_j denote the jth generator of finite order in the presentation (0.2). If $\phi(x_j)$ lifts to an element of $\mathrm{SU}(2)$ whose eigenvalues are $\zeta^{\pm 1}$, where ζ is a primitive $2d_j$-root of unity, the eigenvalues of the image of x_j in $\mathrm{Aut}(\mathfrak{g})$ are

$$\zeta^{-2e_1}, \zeta^{2-2e_1}, \zeta^{4-2e_1}, \ldots, 1, \ldots, \zeta^{2e_1}, \zeta^{-2e_2}, \ldots, \zeta^{2e_2}, \ldots, \zeta^{-2e_r}, \ldots, \zeta^{2e_r}.$$

The multiplicity of 1 as eigenvalue is therefore $\sum_{i=1}^r (1+2\lfloor e_i/d_j \rfloor)$. By (1.2), the left hand side of (4.1) is $\dim Z^1(\Gamma, \mathfrak{g})$. By Corollary 2.5, we need only check that

$$t_G - \dim G = -\chi(\Gamma) \dim G + \sum_{j=1}^m \frac{\dim G}{d_j} - \sum_{j=1}^m \sum_{i=1}^r (1 + 2\lfloor e_i/d_j \rfloor).$$

is greater than

$$t_{\mathrm{SO}(3)} - \dim \mathrm{SO}(3) = -\chi(\Gamma) \dim \mathrm{SO}(3) + \sum_{j=1}^m \frac{\dim \mathrm{SO}(3)}{d_j} - \sum_{j=1}^m 1,$$

which is true by hypothesis. □

We can now prove Theorem 0.4.

Proof. Recall that if $G_1 \to G_2$ is an isogeny, we can prove the theorem for G_1 and immediately deduce it for G_2. Theorem 0.2 and Proposition 3.1 therefore cover groups of type A, B, and D. This leaves only the symplectic case, where Proposition 4.2 applies. Note that

$$\begin{aligned}
\sum_{j=1}^m \frac{\dim G}{d_j} &- \sum_{j=1}^m \sum_{i=1}^r (1 + 2\lfloor e_i/d_j \rfloor) \\
&= \sum_{j=1}^m \sum_{i=1}^r \frac{1+2e_i}{d_j} - \sum_{j=1}^m \sum_{i=1}^r (1 + 2\lfloor e_i/d_j \rfloor) \\
&= \sum_{i=1}^r \sum_{j=1}^m \Bigl(\frac{1+2e_i}{d_j} - 1 + 2\lfloor e_i/d_j \rfloor\Bigr).
\end{aligned}$$

As

$$-1 < 2x + 1/d_j - 1 - 2\lfloor x \rfloor < 1,$$

the error term is at most mr in absolute value. □

The following proposition illustrates the fact that the methods of this section are not only useful in the large rank limit. We make essential use of the technique illustrated below in [LLM].

Proposition 4.3. *Every* $\mathrm{SO}(3)$*-dense Fuchsian group is also* $F_4(\mathbb{R})$*-dense,* $E_7(\mathbb{R})$*-dense, and* $E_8(\mathbb{R})$*-dense, where* F_4*,* E_7*, and* E_8 *denote the compact simple exceptional real algebraic groups of absolute rank* 4*,* 7*, and* 8 *respectively.*

Proof. Let G be one of F_4, E_7, and E_8. Let E denote the set of exponents of G, other than 1, which is the only exponent of SO(3). We map Γ to $G(\mathbb{R})$ via the principal homomorphism $\mathrm{SO}(3) \to G$ and apply Corollary 2.5. To show that there exists a homomorphism from Γ to $G(\mathbb{R})$ with dense image, we need only check that

$$t_G - \dim G > t_{\mathrm{SO}(3)} - \dim \mathrm{SO}(3).$$

The proof of Theorem 2.4 proceeds by deforming the composed homomorphism $\Gamma \to \mathrm{SO}(3) \to G(\mathbb{R})$, and under continuous deformation, the order of the image of a torsion element remains constant. We therefore obtain more, namely that Γ is $G(\mathbb{R})$-dense.

By replacing t_G and $t_{\mathrm{SO}(3)}$ by the middle expression in (1.1) for $V = \mathfrak{g}$ and $V = \mathfrak{so}(3)$ respectively, the desired inequality can be rewritten

$$(2g-2+m)(\dim G - \dim \mathrm{SO}(3)) - \sum_{j=1}^{m} \sum_{e\in E} (1 + 2\lfloor e/d_j \rfloor) > 0. \tag{4.2}$$

The summand is non-increasing with each d_j. In particular,

$$\sum_{j=1}^{m} \sum_{e\in E} (1 + 2\lfloor e/d_j \rfloor) \le \sum_{j=1}^{m} \sum_{e\in E} (1 + 2\lfloor e/2 \rfloor) < \sum_{j=1}^{m} \sum_{e\in E} (1 + 2e)$$
$$= \dim G - \dim \mathrm{SO}(3).$$

Therefore, if $g \ge 1$, the expression (4.2) is positive. For $g = 0$, $(d_1, \ldots, d_m)$ is dominated by $(2, 2, \ldots, 2)$ for $m \ge 5$, $(2, 2, 2, 3)$ for $m = 4$, and $(2, 3, 7)$, $(2, 4, 5)$, or $(3, 3, 4)$ for $m = 3$.

The following table presents the value of

$$\sum_{i=1}^{r} \Big((1 + 2\lfloor d_i/n \rfloor) - \frac{2d_i + 1}{n} \Big)$$

for each root system of exceptional type and for each $n \le 7$.

n	A_1	E_6	E_7	E_8	F_4	G_2
2	$-1/2$	-1	$-7/2$	-4	-2	-1
3	0	-2	$-4/3$	$-8/3$	$-4/3$	$-2/3$
4	$1/4$	$1/2$	$-1/4$	-2	-1	$1/2$
5	$2/5$	$2/5$	$2/5$	$-8/5$	$8/5$	$6/5$
6	$1/2$	-1	$-7/6$	$-4/3$	$-2/3$	$-1/3$
7	$4/7$	$6/7$	0	$4/7$	$4/7$	0

By (1.2), the relevant values of $t_G - \dim G$ are given in the following table:

d_i vector	A_1	E_6	E_7	E_8	F_4	G_2
$(2,2,2,3)$	2	18	34	56	16	6
$(2,3,7)$	0	4	8	12	4	2
$(2,4,5)$	0	4	10	20	4	0
$(3,3,4)$	0	10	14	28	8	2

For $(\underbrace{2,\ldots,2}_{m})$, $m \geq 5$, the values of $t_G - \dim G$ for A_1, E_6, E_7, E_8, F_4, G_2 are $2m-6$, $40m-136$, $70m-266$, $128m-496$, $28m-104$, $8m-28$ respectively. In all cases except $(2,4,5)$ for G_2, the desired inequality holds. $\square$

We conclude by proving Theorem 0.3 in the remaining cases, i.e., for adjoint groups G of type B or C.

Proof. We begin with a Zariski-dense homomorphism $\rho_0\colon \Gamma \to \mathrm{PGL}_2(\mathbb{R})$. Such a homomorphism always exists since Γ is Fuchsian. We now embed PGL_2 via the principal homomorphism in a split adjoint group G of type B_n or C_n. Assuming $n \geq 4$, the image is a maximal subgroup, and we can apply Theorem 2.4 as in the A and D cases. $\square$

5. SO(3)-dense Groups

In this section we show that almost all Fuchsian groups are SO(3)-dense and classify the exceptions.

Lemma 5.1. *Let $d \geq 2$ be an integer.*

(1) *If $d \neq 6$, there exists an integer a relatively prime to d such that*
$$\frac{1}{4} \leq \frac{a}{d} \leq \frac{1}{2},$$
with equality only if $d \in \{2,4\}$.

(2) *If $d \notin \{4,6,10\}$, then a can be chosen such that*
$$\frac{1}{3} \leq \frac{a}{d} \leq \frac{1}{2},$$
with equality only if $d \in \{2,3\}$.

(3) *If $d \notin \{2,3,18\}$, there exists a such that*
$$\frac{1}{12} < \frac{a}{d} < \frac{4}{15},$$
with equality only if $d = 12$.

Proof. For (1) and (2), let

$$a = \begin{cases} \frac{d-1}{2} & \text{if } d \equiv 1 \pmod 2, \\ \frac{d-4}{2} & \text{if } d \equiv 2 \pmod 4, \\ \frac{d-2}{2} & \text{if } d \equiv 0 \pmod 4. \end{cases}$$

As long as $d > 12$, these fractions satisfy the desired inequalities, and for $d \leq 12$, this can be checked by hand.

For (3), let $a = \frac{d-b}{6}$, where b depends on $d \pmod{36}$ and is given as follows:

b	$d \pmod 4$	$d \pmod 9$
-12	2	3
-6	0	6
-4	2	$2, 5, 8$
-3	$1, 3$	3
-2	0	$1, 4, 7$
-1	$1, 3$	$2, 5, 8$
1	$1, 3$	$1, 4, 7$
2	0	$2, 5, 8$
3	$1, 3$	$0, 6$
4	2	$1, 4, 7$
6	0	$0, 3$
12	2	$0, 6$

As long as $d > 24$, these fractions satisfy the desired inequalities, and the cases $d \leq 24$ can be checked by hand. □

Proposition 5.2. *A cocompact oriented Fuchsian group is* SO(3)*-dense if and only if it does not belong to the set*

$$\{\Gamma_{2,4,6}, \Gamma_{2,6,6}, \Gamma_{3,4,4}, \Gamma_{3,6,6}, \Gamma_{2,6,10}, \Gamma_{4,6,12}\}. \tag{5.1}$$

Proof. We recall that every proper closed subgroup of SO(3) is contained in a subgroup of SO(3) isomorphic to O(2), A_5, or S_4. The set of homomorphisms $\mathrm{O}(2) \to \mathrm{SO}(3)$, $A_5 \to \mathrm{SO}(3)$, and $S_4 \to \mathrm{SO}(3)$ have dimension 2, 3, and 3 respectively. Furthermore, $\dim X_{\Gamma,\mathrm{O}(2)} \leq 2g+m$, while $\dim X_{\Gamma,S_4} = \dim X_{\Gamma,A_5} = 0$.

Every non-trivial conjugacy class in SO(3) has dimension 2. As the commutator map $\mathrm{SO}(3) \times \mathrm{SO}(3) \to \mathrm{SO}(3)$ is surjective and every fiber has dimension at least 3, if $g \geq 1$, we have $\dim X_{\Gamma,\mathrm{SO}(3)} \geq 3 + 3(2g - 2) + 2m$. For $g \geq 2$ or $g = 1$ and $m \geq 2$, the dimension of $\dim X_{\Gamma,\mathrm{SO}(3)}$ exceeds the dimension of the space of all homomorphisms whose image lies in a proper closed subgroup, so there exists a homomorphism with dense image with $\rho(x_i)$ of order d_i for all i. If $g = m = 1$, and $\rho(\Gamma) \subset$

$\mathrm{O}(2)$, then the commutator $\rho([y_1, z_1])$ lies in $\mathrm{SO}(2)$, so $\rho(x_1) \in \mathrm{SO}(2)$. The set of elements of order d_1 in $\mathrm{SO}(2)$ is finite, so $\dim X_{\Gamma,\mathrm{O}(2)} \leq 2$, and the set of elements of $X_{\Gamma,\mathrm{SO}(3)}$ which can be conjugated into a fixed $\mathrm{O}(2)$ has dimension ≤ 4; again there exists ρ with dense image and with $\rho(x_i)$ of order d_i for all i.

This leaves the case $g = 0$, $m \geq 3$. By (0.1), $\sum 1/d_i < m - 2$. We claim that unless we are in one of the cases of (5.1), there exist elements $\bar{x}_1, \ldots, \bar{x}_m \in \mathrm{SO}(3)$ of orders $d_1, \ldots, d_m$ respectively such that $\bar{x}_1 \cdots \bar{x}_m = e$ and the elements $\bar{x}_i$ generate a dense subgroup of $\mathrm{SO}(3)$. For $m = 3$, the order of terms in the sequence d_1, d_2, d_3 does not matter since $\bar{x}_1\bar{x}_2\bar{x}_3 = e$ implies $\bar{x}_2\bar{x}_3\bar{x}_1 = e$ and $\bar{x}_3^{-1}\bar{x}_2^{-1}\bar{x}_1^{-1} = e$. Without loss of generality we may therefore assume that $d_1 \leq d_2 \leq d_3$ when $m = 3$. If the base case $m = 3$ holds whenever d_3 is sufficiently large, the higher m cases follow by induction, since one can replace the $m+1$-tuple $(d_1, \ldots, d_{m+1})$ by the m-tuple $(d_1, \ldots, d_{m-1}, d)$ and the triple (d_m, d_{m+1}, d), where d is sufficiently large.

If $\alpha_1, \alpha_2, \alpha_3 \in (0, \pi]$ satisfy the triangle inequality, by a standard continuity argument, there exists a non-degenerate spherical triangle whose sides have angles α_i. If α_1, α_2, and α_3 are of order d_1, d_2, and d_3 respectively, then there exists a homomorphism from the triangle group Γ_{d_1,d_2,d_3} to $\mathrm{SO}(3)$ such that the generators x_i map to elements of order d_i, and these elements do not commute. We claim that except in the cases $(2,4,6)$, $(2,6,6)$, $(3,6,6)$, $(2,6,10)$, and $(4,6,12)$, there always exist positive integers $a_i \leq d_i/2$ such that a_i is relatively prime to d_i and a_i/d_i satisfy the triangle inequality. We can therefore set $\alpha_i = 2a_i\pi/d_i$.

Every non-decreasing triple from the interval $[1/4, 1/2]$ except for $1/4, 1/4, 1/2$ satisfies the triangle inequality. As (d_1, d_2, d_3) cannot be $(2,4,4)$, Lemma 5.1 (1) implies the claim unless at least one of d_1, d_2, d_3 equals 6. We therefore assume that at least one of the d_i is 6. As $1/6$ and any two elements of $[1/3, 1/2]$ other than $1/3$ and $1/2$ satisfy the triangle inequality and as $(d_1, d_2, d_3) \neq (2,3,6)$, Lemma 5.1 (2) implies the claim except if one of the d_i is 4, one of the d_i is 10, or two of the d_i are 6. By Lemma 5.1 (3), the remaining a_i/d_i can then be chosen to lie in $(1/12, 4/15)$ unless this $d_i \in \{2, 3, 12, 18\}$. If a_i/d_i is in this interval, the triangle inequality follows. Examination of the remaining 12 cases reveal five exceptions: $(2,4,6)$, $(2,6,6)$, $(2,6,10)$, $(3,6,6)$, and $(4,6,12)$.

Assuming that we are in none of these cases, there exist non-commuting elements $\bar{x}_i$ in $\mathrm{SO}(3)$ of order d_1, d_2, and d_3, such that $\bar{x}_1\bar{x}_2\bar{x}_3 = e$. They cannot all lie in a common $\mathrm{SO}(2)$. In fact, they cannot all lie in a common $\mathrm{O}(2)$, since any element in the non-trivial coset of $\mathrm{O}(2)$ has order

2, $d_3 \geq d_2 > 2$, and if three elements multiply to the identity, it is impossible that exactly two lie in SO(2). If Γ maps to S_4 or A_5, then $\{d_1, d_2, d_3\}$ is contained in $\{2,3,4\}$ or $\{2,3,5\}$ respectively. The possibilities for (d_1, d_2, d_3) are therefore $(2,5,5)$, $(3,3,5)$, $(3,5,5)$, $(5,5,5)$, $(3,4,4)$, $(3,3,4)$, and $(4,4,4)$. The realization of $\Gamma_{a,b,b}$ as an index-2 subgroup of $\Gamma_{2,2a,b}$ implies the proposition for $\Gamma_{2,5,5}$, $\Gamma_{3,3,5}$, $\Gamma_{3,5,5}$, $\Gamma_{5,5,5}$, $\Gamma_{3,3,4}$, and $\Gamma_{4,4,4}$. The only remaining case is $\Gamma_{3,4,4}$.

Lastly, we show that none of the groups in (5.1) are SO(3)-dense. Suppose there exist elements x_1, x_2, x_3 of orders d_1, d_2, d_3 respectively such that $x_1x_2x_3$ equals the identity and $\langle x_1, x_2, x_3 \rangle$ is dense in SO(3). These elements can be regarded as rotations through angles $2\pi a_1$, $2\pi a_2$, $2\pi a_3$ respectively, where the a_i can be taken in $[0, 1/2)$, and no two axes of rotation coincide. Choosing a point P on the great circle of vectors perpendicular to the axis of rotation of x_1, the three points $P, x_2^{-1}(P), x_1(P) = x_3^{-1}x_2^{-1}(P)$ satisfy the strict spherical triangle inequality, so $a_1 < a_2 + a_3$. Likewise $a_2 < a_3 + a_1$ and $a_3 < a_1 + a_2$. However, one easily verifies in each of the cases (5.1) that one cannot find rational numbers $a_1, a_2, a_3 \in (0, 1/2]$ with denominators d_1, d_2, d_3 respectively such that a_1, a_2, a_3 satisfy the strict triangle inequality. $\square$

6. Appendix by Y. William Yu

The following triples of permutations, which evidently multiply to 1, have been checked by machine to generate the full alternating groups in which they lie:

- $\Gamma_{2,4,6} \to \mathsf{A}_{14}$:

$$\begin{aligned} x_1 &= (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12) \\ x_2 &= (1\ 10\ 9\ 8)(2\ 14\ 13\ 3)(4\ 5)(6\ 7\ 12\ 11) \\ x_3 &= (1\ 3\ 5\ 11\ 7\ 9)(2\ 8\ 6\ 4\ 13\ 14) \end{aligned}$$

- $\Gamma_{2,6,6} \to \mathsf{A}_{14}$:

$$\begin{aligned} x_1 &= (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12) \\ x_2 &= (1\ 14\ 8\ 7\ 4\ 2)(3\ 5\ 13\ 11\ 9\ 6) \\ x_3 &= (1\ 4\ 6\ 3\ 7\ 14)(5\ 9\ 10\ 11\ 12\ 13) \end{aligned}$$

- $\Gamma_{3,6,6} \to \mathsf{A}_{12}$:

$$\begin{aligned} x_1 &= (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12) \\ x_2 &= (1\ 12\ 11\ 6\ 2\ 3)(4\ 10\ 8\ 9\ 5\ 7) \\ x_3 &= (1\ 2\ 3\ 6\ 9\ 10)(4\ 11)(5\ 7\ 8) \end{aligned}$$

- $\Gamma_{3,4,4} \to \mathsf{A}_{14}$:

$$x_1 = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)$$
$$x_2 = (1\ 14\ 11\ 12)(2\ 3\ 4\ 5)(7\ 10\ 13\ 9)(6\ 8)$$
$$x_3 = (1\ 2\ 12\ 14)(3\ 5)(4\ 8\ 9\ 6)(7\ 13\ 10\ 11)$$

- $\Gamma_{2,6,10} \to \mathsf{A}_{12}$:

$$x_1 = (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12)$$
$$x_2 = (1\ 8\ 6\ 7\ 5\ 3)(4\ 10\ 11)(9\ 12)$$
$$x_3 = (1\ 2\ 3\ 11\ 9\ 4\ 5\ 8\ 6\ 7)(10\ 12)$$

- $\Gamma_{4,6,12} \to \mathsf{A}_{12}$:

$$x_1 = (1\ 4\ 3\ 2)(5\ 8\ 7\ 6)(9\ 10)(11\ 12)$$
$$x_2 = (1\ 2\ 5\ 9\ 10\ 3)(4\ 7\ 11\ 8\ 6\ 12)$$
$$x_3 = (2\ 10\ 5\ 8)(3\ 12\ 7\ 11\ 6\ 4)$$

In each case, one can use (1.1) to compute that

$$\dim Z^1(\Gamma, \mathfrak{so}(n)) - \dim \mathrm{SO}(n) > 0.$$

The reasoning of Proposition 3.1 therefore applies to give a homomorphism $\Gamma \to \mathrm{SO}(n)$ either for $n = 11$ or for $n = 13$, with dense image.

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