FREE QUOTIENTS AND THE FIRST BETTI NUMBER OF SOME HYPERBOLIC MANIFOLDS

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Abstract. In this note we present a very simple method of proving that some hyperbolic manifolds M have finite sheeted covers with positive first Betti number. The method applies to the standard arithmetic subgroups of $SO(n,1)$ (a case which was proved previously by Millson [Mi]), to the non-arithmetic lattices in $SO(n,1)$ constructed by Gromov and Piatetski-Shapiro [GPS] and to groups generated by reflections. In all these cases we actually show that $\Gamma = \pi_1(M)$ has a finite index subgroup which is mapped onto a nonabelian free group.

1. Introduction

Let M^n be a finite-volume hyperbolic manifold. A well known conjecture of Thurston (cf. [Bo, p. 88] and [T, p. 380]) asserts that M^n has a finite sheeted covering N with a nonzero first Betti number, $\beta_1(N) > 0$. Various authors have applied several different techniques to prove this conjecture for many cases where $\pi_1(M^n)$ is an *arithmetic* lattice in $PO(n,1)$. The conjecture is known to be true for all arithmetic lattices in $PO(n,1)$, provided $n \neq 3$ and 7 (see [Lu3], [Ra] and the references therein). While $\pi_1(M^n)$ is always a lattice in $PO(n,1)$, it is not necessarily an arithmetic one. In this note we present a very simple method which proves the conjecture for the known examples of nonarithmetic lattices in $PO(n,1)$, at least when $n \neq 3$. When it can be applied, the method actually gives more; it shows that $\Gamma = \pi_1(M^n)$ is mapped onto a virtually-(nonabelian)-free group. This implies that Γ has a finite index subgroup Γ_0 which is mapped onto a nonabelian free group. In particular, M^n has finite sheeted covers with arbitrary large β_1. Our main result says:

Theorem 3.5. *Let M be an oriented n-dimensional finite-volume hyperbolic manifold. Assume M has a codimension one totally geodesic submanifold F. Then $\Gamma = \pi_1(M)$ has a virtually (nonabelian) free quotient. In*

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particular, M has finite sheeted covers with arbitrarily large first Betti numbers.

Our method works for the "standard" arithmetic lattices in $PO(n,1)$ (see Section 3.1 below; they include all the nonuniform arithmetic lattices for every n and all arithmetic lattices if n is even). But, it does not seem to work for the other arithmetic hyperbolic manifolds which *lack* totally geodesic hypersurfaces.

If Γ is mapped onto a virtually-free group then its subgroup growth is the same as that of a free group, i.e., super exponential. Thus, the results of this paper partially confirm a conjecture in [Lu2] which asserts that the subgroup growth of every lattice in $PO(n,1)$ is super exponential.

When Γ is arithmetic and mapped onto a virtually-free group, it has a negative solution to the congruence subgroup problem. Moreover, its congruence kernel contains a free pro-finite group of countable rank (see [Lu1]). This holds, therefore, for the standard arithmetic lattices in $PO(n,1)$.

The paper is organized as follows: In Section 2, we establish two simple group theoretic lemmas which play the key role in our method. In Section 3, we apply it first to the standard arithmetic lattices; here we simplify the proof of Millson [Mi] for this case. His proof uses in an essential way a symmetry between the two parts of the manifold (an idea which is also reproduced in [H1]). Our proof does not use this symmetry. We can therefore adapt it to the nonarithmetic lattices constructed by Gromov and Piatetski-Shapiro ([GPS]) where there is no such symmetry. This plus a common generalization is shown in Section 3. In Section 4, we bring in few more applications to Haken 3-manifolds.

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2. Some simple group theoretic lemmas

Let's start with a definition:

Definition 2.1. (i) A group Λ will be called *virtually-free* if it contains a subgroup of finite index which is isomorphic to a *nonabelian* free group.
(ii) A group Γ has a *virtually positive first Betti number* (Γ has $v\beta_1 > 0$, for short) if Γ has a finite index subgroup Λ which is mapped onto $\mathbb{Z}$ – the infinite cyclic group.
(iii) A group Γ has a *virtually-free quotient* (Γ has vFQ, for short) if Γ is mapped onto a virtually free group. Note that vFQ implies $v\beta_1 > 0$;

moreover, it implies that for every n, Γ has finite index subgroups which are mapped onto $\mathbb{Z}^n$, i.e., Γ has finite index subgroups with arbitrary large first Betti numbers.

Lemma 2.2. *Let $\Gamma = A_1 \underset{C}{*} A_2$ be a free product with amalgam. Assume Γ has a finite quotient $\pi\colon \Gamma \to S$, such that $\bar{C} = \pi(C) \underset{\neq}{<} \pi(A_i) = \bar{A}_i, i = 1, 2$. Then:*

(i) *Γ has $v\beta_1 > 0$.*
(ii) *If $([\bar{A}_1 \colon \bar{C}] - 1)([\bar{A}_2 \colon \bar{C}] - 1) > 1$ then Γ has vFQ.*
(iii) *$K = Ker(\pi)$ has a quotient which is a free group of rank $r = [S\colon \bar{C}] - 1 - \sum\limits_{i=1}^{2} ([S\colon \bar{A}_i] - 1)$.*

Proof. By the universal property of a free product with amalgam, Γ is mapped onto $\Lambda = \bar{A}_1 \underset{\bar{C}}{*} \bar{A}_2$. This is a free product with amalgam of finite groups, hence it is virtually free (cf. [Se2, Prop. 11, p. 120]). Moreover, a simple calculation using [Se2, Exercise 3, p. 123] shows that $H = Ker(\Lambda \to S)$ is a free group of rank r as in (iii). This proves (iii). Now, (i) follows, since by our assumptions $[S\colon C] \geq 2[S\colon A_i]$ for $i = 1, 2$ and hence $r \geq 1$. The assumption of (ii) implies further that $r \geq 2$ and (ii) is also proved.

It is interesting to observe that while the assumptions of the lemma are fairly simple, they are crucial. Not every residually finite free product with amalgam has a virtually free quotient:

Example 2.3. Let p be a prime and $\Gamma = SL_2(\mathbb{Z}[\frac{1}{p}])$. It is well known that $\Gamma = A_1 \underset{C}{*} A_2$ where $A_1 = SL_2(\mathbb{Z}), A_2 = \{\begin{pmatrix} a & pb \\ p^{-1}c & d \end{pmatrix} | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})\} \simeq SL_2(\mathbb{Z})$ and $C = A_1 \cap A_2 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | c \equiv 0 (\text{mod } p)\}$ (see [Se2, Corollary 2, p. 80]). It is also known that every normal subgroup of Γ is either central or of finite index ([Ma, IV 4.9]). In particular, Γ has no infinite virtually free quotient. One can see that the assumptions of Lemma 2.2 do not hold here: Indeed, Γ is known to have the congruence subgroup property([Se1]), so its finite quotients are all obtained via congruence subgroups, $\Gamma/\Gamma(m)$ where $\Gamma(m) = \{\gamma \in \Gamma | \gamma \equiv 1 (\text{mod } m)\}$ and m is prime to p. For these quotients, $\Gamma/\Gamma(m) \simeq SL_2(\mathbb{Z}/m\mathbb{Z})$ and the images of A_1, A_2 and C are all onto and in particular are all the same!

An analogue to Lemma 2.2 holds for HNN- constructions. Let A be a group with a subgroup B and a monomorphism $\varphi : B \to A$. The HNN-construction $\Gamma = A_{*_B}$ is the group with presentation $\Gamma = \langle A, t | t^{-1}bt = \varphi(b)$, for $b \in B \rangle$.

Lemma 2.4. *Let $\Gamma = A_{*_B}$ be an HNN-construction as above. Assume Γ has a finite quotient $\pi: \Gamma \to S$ with $\bar{B} = \pi(B) \underset{\neq}{<} \pi(A) = \bar{A}$. Then:*

(i) *Γ has vFQ.*
(ii) *$H = Ker(\pi)$ is mapped onto a free group of rank $r = [S:\bar{B}] - [S:\bar{A}] + 1$.*

Proof. By the universal property of the HNN-construction, Γ is mapped onto $\Lambda = \bar{A}_{*_{\bar{B}}}$. This is a fundamental group of a finite graph of finite groups and hence virtually free ([Se2, Prop. 11, p. 120]). Again [Se2, Exercise 3, p. 123] gives that $H = Ker(\Lambda \to S)$ is a free group of rank r.

It is convenient to reformulate the above lemmas in terms of the pro-finite topology of Γ. Recall the following definitions:

Definition 2.5. Let Γ be a group.
(i) The *pro-finite topology* of Γ is the topology for which the finite index subgroups of Γ serve as a fundamental system of neighborhoods of the identity.
(ii) If H is a subgroup of Γ then H^* denotes the closure of H with respect to the pro-finite topology of Γ. It is equal to the intersection of all the finite index subgroups of Γ containing H.

In this language, Lemmas 2.2 and 2.4 give:

Lemma 2.6. (i) *Let $\Gamma = A_1 \underset{C}{*} A_2$ be a free product with amalgam. If $C^* \underset{\neq}{<} A_1^*, A_2^*$ then Γ has $v\beta_1 > 0$. If, in addition, $([A_1^*: C^*] - 1)([A_2^*: C^*] - 1) > 1$, then Γ has vFQ.*
(ii) *Let $\Gamma = A_{*_B}$ be a HNN-construction. If $B^* \underset{\neq}{<} A^*$ then Γ has vFQ.*

3. Virtually free quotients of hyperbolic lattices

In this section we apply the group theoretic results of the previous section to prove nonvanishing of the first Betti number for some hyperbolic groups.

3.1 The standard arithmetic lattices

We start with reproving the result of Millson [Mi] concerning the standard arithmetic lattices in $SO(n,1)$.

Let us first recall the construction (see [Mi] for more details).

Let K be a totally real number field of degree m over the rational numbers, $\mathfrak{O}$ its ring of integers and $\sigma_1, \dots, \sigma_m$ the embeddings of K into $\mathbb{R}$. Let $f(X_1, X_2, \dots, X_{n+1}) = a_1 X_1^2 + \ldots + a_n X_n^2 - a_{n+1} X_{n+1}^2$ be a diagonal quadratic form with $a_i \in \mathfrak{O}$. Assume f^{σ_1} has signature $(n,1)$ and f^{σ_i} are positive definite for $i = 2, 3, \dots, m$. The subgroup Γ of $GL_{n+1}(\mathfrak{O})$ preserving f is a lattice in $\prod_{i=1}^{m} O_{\mathbb{R}}(f^{\sigma_i})$ and so is its projection to $O_{\mathbb{R}}(f^{\sigma_1}) \simeq O(n,1)$. For an ideal $\mathcal{P}$ of $\mathfrak{O}$, the congruence subgroup $\Gamma(\mathcal{P}) = \{A \in \Gamma | A \equiv I (\mathrm{mod}\ \mathcal{P})\}$ is a normal finite index subgroup of Γ and for almost all $\mathcal{P}, \Gamma(\mathcal{P})$ is torsion free. Fix such a $\mathcal{P}$ whose norm is different than 2 and let $\Gamma_n = \Gamma(\mathcal{P})$. Γ_n is in $SO_0(n,1)$ — the connected component of $SO(n,1)$ (see [Mi, p. 239]). As explained there the reflection τ through the plane $X_1 = 0$ normalizes Γ_n; its centralizer in Γ_n will be denoted Γ_{n-1}. Γ_{n-1} is the congruence subgroup mod $\mathcal{P}$ of the subgroup of $SO(n-1,1)$ preserving the form f_0 which is equal to f restricted to the hyperplane $X_1 = 0$. It is a lattice in $SO_0(n-1,1)$. Moreover, $Y_{n-1} = \Gamma_{n-1} \setminus SO_0(n-1,1)/SO(n-1)$ is an embedded totally geodesic hyperplane in the manifold $Y_n = \Gamma_n \setminus SO_0(n,1)/SO(n)$ (see [Mi, p. 240] or [GPS, 2.8. A]). Y_{n-1} might separate Y_n or not. (In fact, most of the work in [Mi] is to ensure that, if $\mathcal{P}$ is deep enough, then Y_{n-1} does not separate which gives the desired nonzero first Betti number). We can work with both cases: In the first case $\Gamma_n = A \underset{\Gamma_{n-1}}{*} B$ where A and B are the fundamental groups of the two "halves" $\mathfrak{a}$ and $\mathfrak{b}$, respectively, of Y_n. In the second case $\Gamma_n = A_{*\Gamma_{n-1}}$, an HNN-construction of A over Γ_{n-1}. In either case Γ_{n-1} is a proper subgroup of infinite index in A (and in B in the first case). Actually, A (and B) is Zariski dense in $SO(n,1)$ ([GPS, 1.7]).

We can now prove:

Theorem 3.1. *Γ_n has vFQ.*

Proof. As before either $\Gamma_n = A \underset{\Gamma_{n-1}}{*} B$ or $\Gamma = A_{*\Gamma_{n-1}}$. We only have to check that we can apply Lemma 2.2 and 2.4 respectively.

We have that $\Gamma_n \subseteq GL_{n+1}(\mathfrak{O})$ with $\Gamma_{n-1} \subseteq GL_n(\mathfrak{O})$ and $\Gamma_{n-1} = \Gamma_n \cap GL_n(\mathfrak{O})$. Thus, Γ_{n-1} is a proper closed subgroup of Γ_n with its congruence topology and Γ_{n-1} is of infinite index in A and B. It follows that for some ideal $\mathfrak{q}$ of $\mathfrak{O}$, $[\pi(A) : \pi(\Gamma_{n-1})] \geq 3$ (and $[\pi(B) : \pi(\Gamma_{n-1})] \geq 3$ in the first case) where π is the natural projection $\pi : GL_{n+1}(\mathfrak{O}) \to GL_{n+1}(\mathfrak{O}/\mathfrak{q})$. We thus have all the assumptions of Lemmas 2.2 and 2.4.

3.2 The hybrid manifolds

In [GPS], Gromov and Piatetski-Shapiro used a method of "interbreeding" two arithmetic groups Γ_n and Γ'_n in $G = SO(n,1)$ to get a new lattice Γ in G. Under suitable assumptions (essentially when Γ_n and Γ'_n come from different $\mathbb{Q}$-forms of G) the resulting lattice Γ is a nonarithmetic lattice of G.

The arithmetic lattices which are suitable for this interbreeding process are the standard ones – those we discussed in Subsection 3.1. We will also use the notations of Subsection 3.1. Let f and f' be two quadratic forms defined over the same number field K and take $\mathcal{P}$ to be the same ideal. We assume that the restrictions f_0 and f'_0 of f and f', respectively, to the hyperplanes

$X_1 = 0$ are equal to each other. Let Y_n and Y'_n be two manifolds as in (3.1). By our assumptions the hypersurfaces Y_{n-1} and Y'_{n-1} are isometric and the groups Γ_{n-1} and Γ'_{n-1} are isomorphic and actually can be identified with each other.

The hypersurface Y_{n-1} (resp. Y'_{n-1}) can either separate Y_n (resp. Y'_n) or not. Assume either both separate or both do not.

In the separating case the "hybrid manifold" is obtained by gluing one "half" of Y_n with a "half" of Y'_n along $Y_{n-1} \simeq Y'_{n-1}$. As explained in [GPS], the resulting manifold V is an oriented hyperbolic manifold whose fundamental group Γ is a lattice in $SO(n,1)$. (It can be either cocompact or not). In group theoretical terms one sees that $\Gamma = A \underset{\Gamma_{n-1}}{*} A'$ where A (resp. A') is the fundamental group of the half taken from Y_n (resp. Y'_n).

We can now argue in a way similar to the proof of Theorem 3.1: A and A' are subgroups of $GL_{n+1}(\mathfrak{O})$ which contain Γ_{n-1} as a proper subgroup of infinite index. Moreover, Γ_{n-1} is closed in the congruence topology and so, for most congruence quotients of $GL_{n+1}(\mathfrak{O})$, the image of Γ_{n-1} is different (and of index > 2) from the images of A and A'. By Lemma 2.2, Γ is mapped onto a virtually free group.

Let us consider now the nonseparating case: In this case Y_n (resp. Y'_n) is cut along Y_{n-1} (resp. Y'_{n-1}) and the resulting manifold V_n (resp. V'_n) has two boundary components, each isometric to Y_{n-1} (resp. Y'_{n-1}). As $Y_{n-1} \simeq Y'_{n-1}$ we can glue V_n to V'_n to get a complete hyperbolic manifold V (see [GPS]) whose fundamental group we denote by Γ. Here is the group theoretic description of Γ: as explained in (3.1), $\Gamma_n = \pi_1(Y_n) = A_{*_{\Gamma_{n-1}}} = \langle A, t\rangle$ and $\Gamma'_n = \pi_1(Y'_n) = A'_{*_{\Gamma'_{n-1}}} = \langle A', t'\rangle$. By gluing one boundary component of Y_n with one boundary component of Y'_n we get a manifold $U = Y_n \cup Y'_n$ with $\pi_1(U) = A \underset{\Gamma_{n-1}=\Gamma'_{n-1}}{*} A'$. Then we close U by gluing its two boundary components one to the other to get the resulting manifold V with

$$\pi_1(V) = \langle \pi_1(U), t'' \mid t''^{-1}(t^{-1}\Gamma_{n-1}t)t'' = t'^{-1}\Gamma_{n-1}t'\rangle.$$

Note that $\pi_1(U)$ is still a subgroup of $GL_{n+1}(\mathfrak{O})$. This is not clearly the case for $\pi_1(V)$. Still, $t'' \in \pi_1(V)$ which conjugates $t^{-1}\Gamma_{n-1}t$ $(\leq A \leq GL_n(\mathfrak{O}))$ to $t^{-1}\Gamma_{n-1}t'$ $(\leq A' \leq GL_n(\mathfrak{O}))$ preserves the congruence structure. Thus if we take all this mod $\mathfrak{q}$ for some ideal $\mathfrak{q}$ prime to $\mathcal{P}$, we get a map from $\pi_1(V)$ onto the HNN-construction of $\pi_1(U)(\text{mod } \mathfrak{q})$ over the image of $t^{-1}\Gamma_{n-1}t^{-1}$ (mod q). The latter is a proper subgroup of the first (for most q). Thus Γ is mapped onto a virtually free group (by [Se2, Prop. 11 p. 120 and Exercise 3 p. 123]).

To summarize we have:

Theorem 3.2. *Let M be a hyperbolic manifold obtained as the hybrid manifold of two arithmetic manifolds as in* [GPS]. *Then $\pi_1(M)$ is mapped onto a virtually-(nonabelian) free group.*

Corollary 3.3. *M has finite sheeted covers with arbitrary large first Betti number.*

Remarks 3.4. (a) In case $v_n = Y_n \setminus v'_n = Y_{n-1}$ and $Y'_n \setminus Y'_{n-1}$ are connected (the nonseparating case) it is clear that the resulting hybrid manifold has a nonzero first Betti number. It is less clear (but proved by our method) that by passing to finite covers this number can be made arbitrary large. For arithmetic lattices there is a general argument which ensures that once $\beta_1 \neq 0$, then β_1 is arbitrary large for some congruence subgroups (see [Bo, 2.8 and 4.2]). For nonarithmetic lattices we do not know any such general principle (and it is clearly not true for a general manifold).

(b) One can easily generalize the construction of hybrid manifolds: it is possible to cut few manifolds, each one in few disjoint hypersurfaces and then to glue them together in various different ways à la [GPS] (see also [VS, p. 228–231]). Theorem 3.3 would hold for such constructions as well.

3.3 Manifolds with totally geodesic hypersurfaces

A common generalization of Theorems 3.1 and 3.2 is:

Theorem 3.5. *Let M be an oriented n-dimensional finite-volume hyperbolic manifold. Assume M has a codimension one totally geodesic submanifold F. Then $\Gamma = \pi_1(M)$ has vFQ and, in particular, M has finite sheeted covers with arbitrarily large first Betti number.*

Proof. Γ is a lattice in $PO(n,1)$ and after conjugation we can assume that $\pi_1(F)$ is mapped into $PO(n-1,1)$. In fact it is a lattice there. The hypersurface F may cut M into two part $\mathfrak{a}$ and $\mathfrak{b}$ or be a nonseparating one. In the first case $\Gamma = A \underset{\pi_1(F)}{*} B$ where A (resp. B) is the fundamental group of $\mathfrak{a}$ (resp. $\mathfrak{b}$). In the second case $\Gamma = A_{*_{\pi_1(F)}}$ where this time A is the fundamental group of the open manifold obtained by cutting M along F. In either case $\pi_1(F)$ is of infinite index in A (and B). By the Borel density theorem $\pi_1(F)$ is Zariski dense in $PO(n-1,1)$ and as $PO(n-1,1)$ is a maximal algebraic subgroup of $PO(n,1)$, we deduce that A (and B) is Zariski dense in $PO(n,1)$. This implies that with respect to the pro-finite topology of $\Gamma, \pi_1(F)^*$ is of infinite index in A^* (and in B^*). Indeed, if $\pi_1(F)^*$ would be of finite index in A^*, then by [MS, Proposition 3] the Zariski closure of $\pi_1(F)$ would be of finite index in the Zariski closure of A. But $SO(n-1,1)$ is of infinite index in $SO(n,1)$. We can now apply Lemma 2.6 to deduce that Γ has vFQ.

3.4 Groups generated by reflections

An easy corollary of Theorem 3.5 is:

Corollary 3.6. *Let Γ be a torsion-free lattice in $PO(n,1)$. Assume there exists a reflection $\tau \in PO(n,1)$ normalizing Γ. Then Γ has vFQ.*

Proof. The reflection τ acts on the manifold $\Gamma \setminus \mathbb{H}^n$ and its fixed point set is a codimension one totally geodesic submanifold. Theorem 3.5 now applies for $\Gamma = \pi_1(\Gamma \setminus \mathbb{H}^n)$.

This in particular says that if Γ is generated by reflections then a finite index subgroup of Γ has vFQ. For $n \geq 4$, all known examples of nonarithmetic groups are either "hybrid manifolds" (3.2) or are commensurable to ones generated by reflections (cf. [VS, pp. 227-228]). So, all of them satisfy Thurston's conjecture, i.e., have finite covers with positive Betti number. For $n = 3$, there is another kind of nonarithmetic lattices: those obtained by (closing) knots' complements à la Thurston. Most of them are nonarithmetic and for those Thurston's conjecture is open. Note however that all torsion-free nonuniform lattices in $PO(3,1)$ have positive Betti number, so Thurston's conjecture is open only for the cocompact ones. We do not know, however, whether an arbitrary nonuniform lattice in $PO(3,1)$ has a finite index subgroup which is mapped onto a non-abelian free group. This is the case for the arithmetic ones as we now show.

3.5 The Bianchi groups

The group $PO(3,1)$ is locally isomorphic to $SL_2(\mathbb{C})$ and every arithmetic nonuniform lattice in it is commensurable (up to conjugacy) to $SL_2(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers in $Q(\sqrt{-d})$ for some $0 < d \in \mathbb{Z}$. The fact that $SL_2(\mathcal{O})$ has a finite index subgroup with a nonabelian free quotient was proved by Grunewald and Schwermer [GS] using number theoretical results. It also follows from our Theorem 3.1 since $SL_2(\mathcal{O})$ are commensurable to the standard arithmetic lattices discussed there. We however show here that the proof is "constructive" allowing us to present explicit subgroups of finite index with free quotients of explicitly calculated ranks. (In fact Theorems 3.1 and 3.2 are special cases of 3.5, we have proved them separately as the direct proof is effective).

Theorem 3.7. *Let $\mathcal{O} = \mathcal{O}_d$ be the ring of integers in $Q(\sqrt{-d}), 0 < d \in \mathbb{Z}$ and Γ_d a finite index torsion free subgroup of $SL_2(\mathcal{O})$. Then Γ_d has vFQ. Moreover if $p \in \mathbb{Z}$ is a prime which does not split in $\mathcal{O}$, then if p is large enough, $\Gamma_d(p) = \ker(SL_2(\mathcal{O}) \to SL_2(\mathcal{O}/p\mathcal{O}))$ is mapped onto a free group of rank $p^3 + p - 1$.*

Proof. Let $\Delta_d = \Gamma_d \cap SL_2(\mathbb{R}) = \Gamma_d \cap SL_2(\mathbb{Z})$. Then $\Delta_d \setminus SL_2(\mathbb{R})/SO(2)$ is a totally geodesic hypersurface in $\Gamma_d \setminus SL_2(\mathbb{C})/SU(2)$. Thus either $\Gamma_d =$

$A \underset{\Delta_d}{*} B$ for some subgroups A and B of Γ_d containing Δ_d as a subgroup of infinite index (see (3.1)) or $\Gamma_d = A_{*_{\Delta_d}}$, an HNN-construction for some $\Delta_d < A < \Gamma_d$, where $(A : \Delta_d)$ is infinite. As Δ_d is closed in the congruence topology of Γ_d, Lemma 2.6 implies that Γ_d has vFQ.

Moreover, let π be the projection of Γ_d to $SL_2(\mathcal{O}/p\mathcal{O}) = D_p$ (which is onto for almost all p). If p is large enough $\pi(A)$ (and $\pi(B)$) is equal to D_p while $\pi(\Delta_p)$ is equal to $SL_2(\mathbb{Z}/p\mathbb{Z}) \subsetneqq D_p$. Indeed, for large enough p, $\pi(A)$ contains $\pi(\Delta_p)$ properly and $SL_2(p)$ is a maximal subgroup of $D_p = SL_2(p^2)$. We can apply Lemma 2.2 (iii) and Lemma 2.4 (ii) to $\ker(\pi) = \Gamma_d(p)$, to deduce that $\Gamma_d(p)$ is mapped onto a free group of rank $r = \frac{|SL_2(p^2)|}{|SL_2(p)|} - 1 = p^3 + p - 1$.

4. Haken 3-manifolds

Throughout this section, M is a compact orientable irreducible 3-manifold with an infinite fundamental group. M is called Haken if it contains a 2-sided incompressible surface. A well known conjecture of Waldhausen asserts that M is always virtually Haken, i.e., has a finite cover which is Haken. This is a weaker conjecture than the one mentioned in the introduction asserting that M has a finite cover with positive β_1, since $\beta_1(M) > 0$ implies M is Haken.

In this section we give few results asserting that under suitable conditions, if M is Haken it has a finite cover with positive Betti number.

4.1 Haken nonhyperbolic 3-manifolds

Proposition 4.1. *If M is Haken and nonhyperbolic then $\pi_1(M)$ is either virtually-solvable or has vFQ. In any case it has a finite cover with a positive first Betti number.*

Proof. The Jaco-Shalen-Johanson decomposition Theorem gives a splitting of M by incompressible tori into pieces which are either Seifert fibered, solv or, by the work of Thurston, have hyperbolic structure (see [H2] and [T]). If this set of tori is non empty then $\pi_1(M)$ is decomposed as a free product with amalgam or HNN- construction where the edge group is abelian (isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ – the fundamental group of a torus). The edge group cannot be dense in the vertex groups (with the pro finite topology) as the vertex groups are non abelian and $\pi_1(M)$ is residually finite ([H]). Thus Lemma 2.6 implies that Γ has vFQ. If this set of tori is empty then, as M is not hyperbolic, M is Seifert fibered or a solv manifold. In case it is solv, its fundamental group is solvable. In case it is Seifert fibered, it is either nil or Euclidean in which case its fundamental group is virtually nilpotent, otherwise $\pi_1(M)$ is mapped onto a nonabelian free group.

The last proposition is a slight improvement of the main result of [Ko] (see also [LN, Corollary on page 212]).

Anyway, the conjecture Haken $\Rightarrow v\beta_1 > 0$ has to be proven only for hyperbolic manifolds.

4.2 Haken manifold where $\pi_1(M)$ is LERF

A group Γ is called LERF (or: *subgroup separable*) if for every finitely generated subgroup H and every $g \in \Gamma \backslash H$, there exists a homomorphism φ from Γ to a finite group such that $\varphi(g) \notin \varphi(H)$. It is widely believed that fundamental groups of hyperbolic (3-)manifolds are LERF. It is therefore of interest to observe:

Proposition 4.2. *If M is Haken and $\pi_1(M)$ is LERF but not virtually solvable, then $\Gamma = \pi_1(M)$ has vFQ.*

Proof. LERF means that for every finitely generated subgroup H of Γ, $H^* = H$. Thus, if M is Haken $\Gamma = A \underset{\pi_1(F)}{*} B$ or $\Gamma = A_{*_{\pi_1(F)}}$ where $\pi_1(F)$ is a proper subgroup of A (and B). By LERF-ness, $\pi_1(F)^* \underset{\neq}{<} A^*$ (and $\pi_1(F)^* \underset{\neq}{<} B^*$) and hence by Lemma 2.6, Γ has vFQ. (It is impossible that $(A^* \colon \pi_1(F)^*) = (B^* \colon \pi_1(F)^*) = 2$ as in such a case $\pi_1(F)$ would be normal in Γ which is impossible).

4.3 Some remarks on Haken $\Rightarrow v\beta_1 > 0$

Assume M is a closed hyperbolic Haken 3-manifold. So there exists a cocompact torsion free lattice Γ is $PSL_2(\mathbb{C})$ s.t. $M = \Gamma \backslash \mathbb{H}^3$. M has an incompressible surface F. If F does not separate M, then $\beta_1(M) > 0$. So we assume F separates M into two parts $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Denote $A_i = \pi_i(\mathfrak{A}_i), i = 1, 2$ and $C = \pi_1(F)$. Then C is a subgroup (of infinite index) in $A_i (i = 1, 2)$ and $\Gamma = A_i \underset{C}{*} A_2$. To prove that Γ has $v\beta_1 > 0$, it suffices to show that *in the pro-finite topology of* Γ, the closure of C is a proper subgroup of the closure of $A_i (i = 1, 2)$, (see Lemma 2.6 (i)). An elegant argument of Long and Niblo [LN] shows that C is closed in the pro-finite topology of A_i. Indeed, let $\mathcal{B}_i$ be the *double* of $\mathfrak{A}_i$ along its boundary F. Then $\mathcal{B}_i$ is a hyperbolic manifold. Hence $\Gamma_i = A_i \underset{C}{*} A_i$ is a residually finite group. There is an involution τ_i of Γ_i sending one copy of A_i to the other and fixing C. Moreover, C is exactly the set of fixed points of τ_i. It is easy to see that such a set is closed in the pro-finite topology of Γ_i and hence in that of A_i (for i = 1 and 2).

This argument plus our methods gives:

Proposition 4.3. *If $\mathfrak{A}$ is an irreducible, orientable 3-manifold with an incompressible boundary F, and M is a double of $\mathfrak{A}$ along F. Then $\pi_1(M)$ has vFQ.*

But, by no means the above argument shows that in our original M, C is closed in the pro-finite topology of Γ. Look for example at Example 2.3 above. There, C is closed (in fact of finite index) in A_i for $i = 1, 2$, but, C is dense in $\Gamma = A_1 \underset{C}{*} A_2$. The topology induces from Γ to A_i may be weaker than the pro finite topology of A_i! An interesting test case is the following: Assume M is a double of a manifold $\mathfrak{A}$ with a boundary F such that the identification of the two halves is done after twisting the F by an automorphism φ. Algebraically, this means $\Gamma = A \underset{C=\tilde{\varphi}(C)}{*} A$ where $\tilde{\varphi}$ is the automorphism of $C = \pi_1(F)$ induced by φ. Even in this case we do not know how to prove that $v\beta_1 > 0$.

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