Finite representations in the unitary dual and Ramanujan groups

Alexander Lubotzky and Yehuda Shalom

ABSTRACT. We define and study two "opposite" representation theoretic group properties pertaining to the place of the finite representations in the unitary dual. The first describes an abstract group theoretic counterpart to the (asymptotically optimal) Ramanujan graph property. The second, in which the finite representations are dense in the Fell topology, is shown to be shared by some lattices in $SL_2(\mathbb{C})$, thereby answering, in particular, two questions raised by Bekka and Louvet. This, together with various other examples of arithmetic groups satisfying one property or the other, suggests yet another feature of the well known "rank one" versus "higher rank" dichotomy.

1. Introduction

Let Γ be a residually finite discrete group, and $\tilde{\Gamma}$ be its unitary dual, i.e., the set of equivalence classes of **all** unitary Γ-representations, equipped with the Fell topology. The set of finite Γ-representations, namely, those factoring through a finite quotient of Γ, is the simplest collection of Γ-representations, and will be denoted by $\tilde{\Gamma}_F$. The main purpose of this paper is to study the "location" of $\tilde{\Gamma}_F$ in $\tilde{\Gamma}$, as reflected in $F(\Gamma)$ – the closure of $\tilde{\Gamma}_F$ in $\tilde{\Gamma}$ in the Fell topology, i.e., the set of all unitary Γ-representations which are weakly contained (denoted $\prec$) in the direct sum of the representations in $\tilde{\Gamma}_F$.

One may heuristically think of two "opposite sides" of the spectrum $\tilde{\Gamma}$: the first being the regular representation $l^2(\Gamma)$, and the second being the trivial representation 1_Γ. When Γ is amenable, the picture blurs, and the former becomes all of $\tilde{\Gamma}$. But as soon as Γ is non-amenable the two sides are separated, and the situation becomes interesting. In order to make this phenomenon more precise, one is naturally led to discuss norms of the so called "averaging" (or "Hecke") operators, i.e. norms of elements in the group algebra $\mathbb{C}\Gamma$, which act in any Γ-representation. Their intrinsic relation to the Fell topology becomes transparent by Eymard's result [Ey] stating that $\pi \prec \sigma$ iff for any such operator T one has $||T||_\pi \leq ||T||_\sigma$. Thus, any $T \in \mathbb{C}\Gamma$ and any value of α, may be used to isolate a "corner" in $\tilde{\Gamma}$, defined by the (closed) set of Γ-representations π with $||T||_\pi \leq \alpha$.

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We can now make our goal more concrete by distinguishing between two extremal types of behavior: the first is that $F(\Gamma) = \tilde{\Gamma}$, i.e., the finite representations are dense in the unitary dual, in which case we shall say that Γ has **Property FD**. The second is that $F(\Gamma)$ lies in the same "corner(s)" as does $l^2(\Gamma)$. Notice however, that we cannot require this to hold for all "corners", nor can we simply ask that $F(\Gamma)$, or parts of it, be contained in (the closure of) $l^2(\Gamma)$, as by Eymard's aforementioned result the two amount to the same, and for non-amenable Γ **no** finite dimensional representation π is weakly contained in $l^2(\Gamma)$. Rather, it appears natural to restrict to positive elements only, i.e, to elements $T \in \mathbb{C}\Gamma$ with non-negative real coefficients. For such operators, on the one hand $||T||$ always achieves its maximal value in 1_Γ, and on the other hand, over a large class of representations, it always achieves its minimal value on $l^2(\Gamma)$ (cf. [Sh2] Lemma 2.3 and the Remark thereafter), so our heuristic "opposite sides" picture becomes sharp and quantitative.

Thus, a first natural attempt to define the opposite type of behavior to property FD would be the property that $||T||_{\pi\in\tilde{\Gamma}_F} \leq ||T||_{l^2(\Gamma)}$ for **all positive** T, of course, with the restriction that π has no non-zero invariant vectors. Unfortunately, as appealing as it may seem, we are currently very far away from constructing any example of a (residually finite, non-amenable) group Γ satisfying this property, and it is not clear whether such an example exists at all. Instead, we shall require that the above inequality holds for at least **one** fixed positive (non-trivial) operator, and in fact, often also restrict to π's coming from some prescribed (non-trivial) family of finite quotients. This leads to the following notion, which as is well known by now, already involves some deep mathematics:

Definition. Let Γ be a finitely generated residually finite group, $T \in \mathbb{C}\Gamma$ a positive operator whose support generates Γ, and $\mathcal{L} = \{\Gamma_i\}_{i\in I}$ a family of finite index normal subgroups which is closed under finite intersections, and has trivial intersection. We say that the group Γ is $(T, \mathcal{L})-$**Ramanujan** if for every unitary Γ-representation π without invariant vectors and factoring through a quotient Γ/Γ_i, one has: $\quad ||T||_\pi \leq ||T||_{l^2(\Gamma)} \quad (*)$

We shall say for brevity that Γ is **Ramanujan** if there exists some T and $\mathcal{L}$ for which the above is satisfied.

The outstanding examples of Ramanujan groups are free groups, as follows from the construction of Ramanujan graphs by Lubotzky-Phillips-Sarnak [LPS]. Thus, from the point of view of combinatorics it may be natural to consider the more special operators T arising as an average over a finite (symmetric) generating subset. However, in representation theoretic perspective the above definition is both more natural, and has one additional important advantage: Recall that in the construction of [LPS], one should exclude from $(*)$ a certain 1-dimensional (±1-valued) character π_0, which gives rise to the maximal possible value in the left hand side of $(*)$. However, the proof in [LPS] can be modified to show that in fact any positive operator $T \in \mathbb{C}\Gamma$ which is spherically symmetric in the associated Cayley graph of the free group Γ, satisfies $(*)$ for any $\pi \neq \pi_0$ (see Cor. 4.2 below). Thus, as any element of the sphere of radius 1 (resp. 2) in Γ acts by the scalar -1 (resp. 1) in π_0, one can easily define a positive linear combination of the two spheres which will be annihilated in π_0, so that now $(*)$ is satisfied for π_0 as well. While in the present case this seems like a mere technicality, in other examples of

Ramanujan groups constructed below there are (finitely many) more exceptional 1-dimensional representations, which can then be handled similarly (although we shall not elaborate here further on this issue). Finally, to put matters in perspective, we remark that for any family $\mathcal{L}$ as in the Definition, the set of representations π appearing in the left hand side of $(*)$ **contains weakly** $l^2(\Gamma)$ (see Lemma 2.1 below), hence for **any** operator $T \in \mathbb{C}\Gamma$ the supermum over the representations π in the left hand side of $(*)$ is always $\geq$ the right hand side of $(*)$. This is what makes the Ramanujuan property so highly non-trivial.

In this paper we shall construct various groups Γ with, and without, one of the above two properties, all coming from the distinguished class of $(S-)$arithmetic groups. Property FD, discussed in Sections 2 and 3, deserves some additional comments. In [Ch] Choi proved that for any free group Γ, the set of **Finite Dimensional** Γ-representations is dense in the Fell topology of $\tilde{\Gamma}$. Modulo natural modifications, it remained the only group known to satisfy this property, which is obviously weaker (at least a priori) than property FD. In [BL] Bekka and Louvet asked whether that property is satisfied also by surface groups and the Picard group $SL_2(\mathbb{Z}[i])$. We shall answer affirmatively their questions in Section 2, by showing directly that these groups (along with some other arithmetic groups $\Gamma < SL_2(\mathbb{C})$) have the stronger property FD. The proof involves a mixture of algebraic, geometric and representation theoretic tools. The Ramanujan property is discussed in Sections 4 and 5. We shall see that a product of non-amenable groups is **never** Ramanujan, and observe that the deep number theoretic tools related to the so-called Ramanujan conjecture (including Drinfeld's and Lafforgue's work) can easily be translated into this property for the relevant groups. Combined with the previous result, this provides an example of two groups Γ, Λ generated by subsets $S = S_\Gamma, S_\Lambda$ resp., such that the two associated Cayley graphs are **isometric**; still, with respect to $T = T_S$ one is Ramanujan and the other is not (relative to **any** family $\mathcal{L}$).

2. Groups with property FD

As mentioned in the introduction, when Γ is amenable $l^2(\Gamma)$ is dense in $\tilde{\Gamma}$ and this property in fact characterizes amenability. Thus for Γ which is amenable and residually finite, we have $F(\Gamma) = \tilde{\Gamma}$, since for **any** residually finite group Γ, it is easy to see that $F(\Gamma)$ contains $l^2(\Gamma)$ (see a direct argument at the end of the proof of Lemma 2.4 below). In fact, the latter can be generalized:

Lemma 2.1. *Assume that the subgroup $H < \Gamma$ is of infinite index, and is closed in the profinite topology defined by $\mathcal{L}$. Then $l^2(\Gamma/H) \prec \oplus_{\Gamma_i \in \mathcal{L}} l_0^2(\Gamma/\Gamma_i)$. In particular, for every $T \in \mathbb{C}\Gamma$ one has: $||T||_{l^2(\Gamma/H)} \leq sup\, ||T||_{l_0^2(\Gamma/\Gamma_i)}$.*

The point of this observation of course is that in the right hand side we have l_0^2- the zero mean functions, rather than l^2. The Lemma also generalizes Alon-Boppana's well known result on the liminf of λ_1 of regular graphs. For a proof see [Sh1] (Part 1 of Theorem 4.1 and Theorem 2.4).

We now establish the first non-trivial class of groups with property FD:

Theorem 2.2. *If $\Gamma = F_r$ is a free group on r generators, then $F(\Gamma) = \tilde{\Gamma}$.*

Because of the "local nature" of the Fell topology, this automatically extends also to the case $r = \aleph_0$. The heart of the proof lies in the following result, for which we thank Tim Steger:

Proposition 2.3. *Assume that* Γ *as above acts continuously on a compact metric space* X*, preserving a probability measure* μ*. Then the induced unitary* Γ*-representation on* $L^2(X,\mu)$ *belongs to* $F(\Gamma)$*.*

Proof of the Proposition: The idea of the proof is to show that there are "arbitrarily small perturbations" of any given action, which factor through a finite quotient. First however, observe that we may assume μ has no atoms since for any $\alpha > 0$ there can be only finitely many atoms of weight α (μ has total mass 1), hence they form Γ-invariant subset on which the Γ-action factors through a finite quotient. By removing all the atoms of μ we are reduced to the non-atomic case. Let $S \subseteq \Gamma$ be a free generating subset. Abusing notation, we shall denote both the Γ-action on X and on $L^2(X)$ by ρ, i.e., for $f \in L^2(X)$ we have: $[\rho(g)f](x) = f(\rho(g^{-1})x)$. Our strategy now goes as follows: Because the subspace of continuous functions on X is dense in $L^2(X)$, it is enough, given any finite number of continuous functions $f_1, \cdots, f_n$ on X, and any $\epsilon > 0$, to construct a **new** (μ-)measure preserving Γ-action ρ' on X, which factors through a **finite quotient** of Γ, such that for every $g \in S \cup S^{-1}$ (as well as $g = e$) one has for the L^2 norm: $||\rho(g)f_i - \rho'(g)f_i|| < \epsilon$ for all $1 \leq i \leq n$.

Let then ϵ and $f_1, \cdots, f_n$ be given. Choose $\delta > 0$ so that $d(x,y) < 2\delta$ implies $|f_i(x) - f_i(y)| < \epsilon$ for all i and $x, y \in X$. Now, again using continuity, choose some $0 < \theta < \delta$ such that $d(x,y) < \theta$ implies $d(\rho(g)x, \rho(g)y) < \delta$ for $g \in S \cup S^{-1}$. Next observe that X can be divided into a disjoint union of subsets $B_1, \cdots, B_k$ which have equal measure (of size $1/k$) and diameter less than θ. We shall leave the straightforward, yet rather tedious verification of this fact to the reader, hinting only that it is easier to find first such a subdivision where the measure of every B_i is rational (and then subdivide arbitrarily all B_i's to get pieces of equal measure), which of course uses the fact that μ is non-atomic (it seems most convenient here to homemorphically embed X in $[0,1]^N$ and work with the latter).

We can now define the new Γ-action ρ' on X. By freeness, it is enough to define it for any $g \in S$ and we fix such an element. Consider the two disjoint subdivisions of X: $X = \cup B_i = \cup C_i$, where $C_i = g(B_i)$. Let us call a couple B_i and C_j "matched" if $\mu(B_i \cap C_j) > 0$. Because $\mu(B_i) = \mu(C_j) = 1/k$ for all i, j, it is easy to see that for every family of m B_i's, the family of C_j's matched to at least one of its members has least m elements (and vice versa), that is, Hall's marriage theorem applies. We can therefore find a permutation $\sigma_g \in S_k$ with the property that $g(B_i) \cap B_{\sigma_g(i)}$ is non empty for all i. Notice that defining $\sigma_{g^{-1}} = (\sigma_g)^{-1}$, the latter holds also when g is replaced by g^{-1}.

By freeness the permutation action of S on the set of B_i's extends to a permutation action of Γ, factoring through a finite quotient F ($\subseteq S_k$). In order to define a measure preserving Γ-action on X (and not only on the collection of subsets B_i), which factors through an action of F and induces the previous permutation action on the B_i's, we choose as a "model space" any non-atomic standard Lebesgue space Y on which S_k acts by permuting a disjoint subdivision of it to k subsets Y_i. Identifying measure preservingly each B_i and Y_i, induces an F- (hence also $\Gamma-$)action on X, which in turn induces the previously defined permutations $\{\sigma_g\}$ of the sets B_i. This defines the new Γ-action ρ' on X.

Finally, given any $1 \leq i \leq n$ we show that $||\rho(g)f_i - \rho'(g)f_i|| < \epsilon$ for any $g \in S \cup S^{-1}$. In fact, we show that for all $x \in X$: $|\rho(g)f_i(x) - \rho'(g)f_i(x)| < \epsilon$. Let $1 \leq j \leq k$ be such that $x \in B_j$. Then $\rho(g^{-1})x \in \rho(g^{-1})B_j$, and $\rho'(g^{-1}x) \in B_{\sigma_{g^{-1}}(j)}$.

By the construction of σ the two B_i's on the right hand sides intersect, and by the choice of θ and δ they both have diameter $< \delta$. By the triangle inequality it follows that $d(\rho(g^{-1})x, \rho'(g^{-1})x) < 2\delta$, which by the choice of δ implies $|\rho(g)f_i(x) - \rho'(g)f_i(x)| = |f_i(\rho(g^{-1})x) - f_i(\rho'(g^{-1})x)| < \epsilon$. As this holds for every $x \in X$ and μ is a probability measure, this establishes the Proposition.

Proof of Theorem 2.2: Let π be a unitary Γ-representation. As is well known, it is a general fact that there exists a probability measure preserving Γ-action on a standard Lebesgue space (X, μ) such that π is a subrepresentation of the unitary Γ-representation on $L^2(X, \mu)$ (cf. [Zi] 5.2.13). It is also a general fact that every such action admits a topological model, i.e., measure theoretically it can always be realized on a compact metric Γ-space on which Γ acts continuously (cf. [Zi] 2.1.19). Our result now follows immediately from the previous Proposition.

It is not difficult to see that property FD passes to subgroups (see Lemma 3.2 below). Our next purpose is to provide situations where the converse phenomenon takes place.

Lemma 2.4. *Let Γ be a discrete group and $\Lambda < \Gamma$ be a subgroup. Assume that Λ has property FD and that every finite index subgroup of Λ is an intersection of finite index subgroups of Γ. Then for every unitary Γ-representation π, $F(\Gamma)$ contains the representation $\pi \otimes l^2(\Gamma/\Lambda)$.*

Proof. Consider the restriction $\pi|_\Lambda$, which by assumption belongs to $F(\Lambda)$. Thus, for a sequence of finite index subgroups $M_i < \Lambda$ we have a weak containment $\pi|_\Lambda \prec \oplus l^2(\Lambda/M_i)$. Inducing both sides to Γ (and using "continuity of induction in the Fell topology") yields: $\pi \otimes l^2(\Gamma/\Lambda) \prec \oplus l^2(\Gamma/M_i)$. Therefore it only remains to be shown that for every finite index subgroup $M < \Lambda$, $l^2(\Gamma/M)$ is in $F(\Gamma)$, which is quite standard (compare with the less obvious Lemma 2.1 above). Indeed, by assumption one can write $M = \cap N_i$ for a decreasing sequence of finite index subgroups $N_i < \Gamma$, which means that for every finite subset $S \subseteq \Gamma$ one can find i large enough so that for any $\gamma \in S$: $\langle \gamma 1_e, 1_e \rangle_{l^2(\Gamma/M)} = \langle \gamma 1_e, 1_e \rangle_{l^2(\Gamma/N_i)}$, where 1_e denotes the Dirac function at the identity coset of the corresponding space. Therefore every matrix coefficient associated with functions which are finite linear combination of Γ-translations of 1_e (which form a dense subspace of $l^2(\Gamma/M)$), can be approximated by matrix coefficients in $\oplus l^2(\Gamma/N_i)$, as required.

□

Corollary 2.5. *Let Γ be a discrete group and $\Lambda < \Gamma$ be a normal subgroup such that Γ/Λ is amenable. Assume either that Λ has property FD and all its finite index subgroups are closed in the profinite topology of Γ; or, in case Λ is not finitely generated, that every finitely generated subgroup of Λ satisfies the same properties. Then Γ has property FD.*

Proof. Fix a unitary Γ-representation π; we show that $\pi \in F(\Gamma)$. In the first case this follows immediately from Lemma 2.4, since by assumption we have $1 \prec l^2(\Gamma/\Lambda)$, hence: $\pi \prec \pi \otimes l^2(\Gamma/\Lambda) \in F(\Gamma)$. As for the second, denoting by $\Lambda_i < \Lambda$ an increasing exhausting sequence of finitely generated subgroups, we have by the Lemma for any i: $\pi \otimes l^2(\Gamma/\Lambda_i) \in F(\Gamma)$, hence $\pi \otimes (\oplus l^2(\Gamma/\Lambda_i)) \cong \oplus(\pi \otimes l^2(\Gamma/\Lambda_i)) \in F(\Gamma)$. However since $1_\Lambda \prec \oplus l^2(\Lambda/\Lambda_i)$, inducing to Γ yields $l^2(\Gamma/\Lambda) \prec \oplus l^2(\Gamma/\Lambda_i)$. Since $1_\Gamma \prec l^2(\Gamma/\Lambda)$ we deduce $1_\Gamma \prec \oplus l^2(\Gamma/\Lambda_i)$, so tensoring with π and combining with the previous conclusion completes the proof. □

Remark. As is clear from the proof, the normality of Λ in Γ is not necessary; assuming that $1_\Gamma \prec l^2(\Gamma/\Lambda)$ would suffice for our purposes.

The property that every finite index subgroup of Λ is closed in the profinite topology of Γ is not easily achieved. The next Lemma gives a situation in which it is satisfied:

LEMMA 2.6. *Assume Γ is a discrete group, and Λ is a finitely generated normal subgroup of Γ such that $Z(\hat{\Lambda}) = \{1\}$ (when $Z(\hat{\Lambda})$ denotes the center of the profinite completion $\hat{\Lambda}$ of Λ). Then the profinite topology of Γ induces the profinite topology of Λ. In particular, if in addition Γ/Λ is residually finite, then every finite index subgroup of Λ is the intersection of finite index subgroups of Γ.*

PROOF. The conclusion of the first (main) statement is equivalent to the assertion that the natural map i from $\hat{\Lambda}$ to $\bar{\Lambda}$ (= the closure of Λ in $\hat{\Gamma}$) is an isomorphism (it is always continuous and onto; the point is its injectivity).

As Λ is finitely generated, $Aut(\hat{\Lambda})$ is a profinite group (see below). Hence we have the natural map $\hat{\jmath} : \hat{\Gamma} \to Aut(\hat{\Lambda})$ extending the map $j : \Gamma \to Aut(\Lambda)$ given by conjugation within Γ. Now, $\hat{\jmath}$ restricted to $\bar{\Lambda}$ (in $\hat{\Gamma}$) is just the extension of the map sending each element of Λ to the inner automorphism determined by it. It follows that $\bar{\Lambda}$ is mapped under $\hat{\jmath}$ onto the image of $\hat{\Lambda}$ in $Aut(\hat{\Lambda})$, which by the center freeness assumption, may be identified with $\hat{\Lambda}$. Hence $\psi = \hat{\jmath} \circ i$ is an epimorphism from the finitely generated profinite group $K = \hat{\Lambda}$ onto itself, which must therefore be a topological isomorphism, since every epimorphism from a finitely generated profinite group K onto itself is an isomorphism. Indeed, as K is finitely generated (namely, it has a dense finitely generated subgroup), for any n it has finitely many closed subgroups of index at most n, so their intersection $K_n \lhd K$ has finite index. Because ψ is onto, for every finite index subgroup $K_0 < K$ one has $[K : K_0] = [K : \psi^{-1}(K_0)]$, so $\psi(K_n) \subseteq K_n$. Thus ψ induces a homomorphism, $\tilde{\psi} : K/K_n \to K/K_n$, which is onto (as is ψ). Since K/K_n is finite, $\tilde{\psi}$ is injective. Now, given any $e \neq k \in K$, there is some n for which $k \notin K_n$, and by the above $\tilde{\psi}[k] \neq [e]$ in K/K_n, so $\psi(k) \neq e$ thus showing the injectivity of ψ. (Note also that with these notations, $Aut(K)$ is the inverse limit of $Aut(K/K_n)$, hence it is indeed a profinite group, as claimed at the beginning of the proof.) Thus the first statement is established, whereas the second follows easily since the assumption made there implies that Λ itself, being the intersection of finite index subgroups of Γ, is closed in the profinite topology of Γ. □

We remark that the finite generation assumption on Λ is necessary. Indeed, as shown below, all other assumptions hold when Γ is a finitely generated free group and Λ – the kernel of its abelianization – is infinitely generated. However, in this case the profinite topology on Λ does not have a countable basis, hence it cannot be induced by that of Γ.

The following result provides a class of examples where the condition in the previous Lemma can be verified:

PROPOSITION 2.7. *If Λ is a group presented by d generators and e relations with $d - e \geq 2$, then $Z(\hat{\Lambda}) = 1$. In particular, any non abelian free or surface group has this property.*

Proof. Suppose by contradiction that there exists a non-trivial element z in $Z(\hat{\Lambda})$, and take $\overline{N}$ normal open in $\hat{\Lambda}$ so that $z \notin \overline{N}$. Then the action of $\hat{\Lambda}/\overline{N}$ on $\overline{N}/[\overline{N},\overline{N}]$ is not faithful. But $\hat{\Lambda}/[\overline{N},\overline{N}]$ is the profinite completion of the residually finite group $\Lambda/[N,N]$, when $N = \overline{N} \cap \Lambda$. Hence $\Lambda/N \simeq \hat{\Lambda}/\hat{N}$ acts non-faithfully on $N/[N,N]$. However this is impossible in our case. Indeed, by [JR, Theorem 3], for every finite index normal subgroup N of Λ, with G denoting the quotient Λ/N, there exists $T \triangleleft \Lambda$ with $T \subseteq N$ and $N/T \simeq \mathbb{Z}^m$ for some m (in particular, $T \supseteq [N,N]$), such that $(\mathbb{Q}G)^{d-e-1}$ appears as a direct summand of the $\mathbb{Q}G$-module $\mathbb{Q} \otimes_{\mathbb{Z}} N/T$. In particular, by our assumption, $G = \Lambda/N$ itself must act faithfully on N/T, hence also on $N/[N,N]$, a contradiction. □

Theorem 2.8. *The groups Γ defined below all have property FD:*

(1) $\Gamma = \Lambda \rtimes \mathbb{Z}$ *– a cyclic extension of the finitely generated free group Λ. In particular, the Picard group $\Gamma = SL_2(\mathbb{Z}[\sqrt{-1}])$ and the group $SL_2(\mathbb{Z}[\sqrt{-3}])$ have property FD.*

(2) Γ *is a surface group.*

(3) *The same as in (1), where Λ denotes now a surface group. In particular, the fundamental group of any closed hyperbolic 3-manifold which fibers over the circle has property FD.*

Remark. As mentioned in the Introduction, for the Picard and surface groups, the Theorem answers questions raised by Bekka and Louvet in [BL].

Proof. Part (1) follows immediately from Theorem 2.2 after applying the three preceding results. The fact that (a finite index subgroup of) the Picard group fits in the form treated by (1) is well known, by realizing the fundamental group of the whitehead link as its finite index subgroup (cf. [Wi]), and using the fact that this link is fibered (cf. [Ro] p. 338 Ex. 5). The case of $SL_2(\mathbb{Z}[\sqrt{-3}])$ can be dealt with similarly, this time using the figure 8 knot (see [Sh1] Section 5 and the references therein for an explicit matrix realization of the claimed free and cyclic subgroups). For part (2) we wish to apply the second part of Corollary 2.5 in the case where $\Lambda = F_\infty$ is the kernel of the abelianization of Γ, again using Theorem 2.2 above for finitely generated free groups. The fact that the second condition in the statement of the Corollary is satisfied as well, is not trivial; it is given by the following result of Peter Scott [Sc] which, interestingly, is established by geometric methods: A surface group Γ is LERF (= locally extended residually finite), i.e., every finitely generated subgroup of it is closed in the profinite topology of Γ. Finally, part (3) follows now exactly as in (1), from (2).

□

We conclude this section by remarking that since property FD passes to subgroups, parts (1) and (3) yield many other new Kleinian groups having property FD (compare also with the discussion in Section 6 below, and in particular Conjecture 6.4).

3. Groups without property FD

Unlike the case of amenable and free groups, property FD turns out not to be typical in the class arithmetic groups. The following aims to place the Theorem (and method) of Bekka [Be] within its largest natural framework:

Theorem 3.1. *Let k be a global field, $\mathcal{O}$ its ring of integers, S a finite set of valuations of k containing all the archimedean ones, $\mathcal{O}_S = \{x \in k \mid v(x) \geq 0$ for* *all* $v \notin S\}$ *and let* $\Gamma = G(\mathcal{O}_S)$ *where* G *is a simple, simply-connected* k*-algebraic group. Assume that both*

1) k*-rank*$(G) \geq 1$*, and*
2) S*-rank*(G) $\big(= \sum_{v\in S} k_v{-}rank(G)\big) \geq 2.$

Then Γ *does not have property FD.*

Lemma 3.2. *Let* Γ *be a discrete group and* Λ *a subgroup of* Γ*. If* F *is a dense subset of* $\tilde{\Gamma}$ *then* $F|_\Lambda$ *(= the representations in* F *restricted to* Λ*) is dense in* $\tilde{\Lambda}$*.*

Proof. If $\rho \in \tilde{\Lambda}$ then by assumption $Ind_\Lambda^\Gamma(\rho) \in \tilde{\Gamma}$ is the limit of a subset of representations $\{\pi_i\}$ in F. Since $\rho \subseteq Ind_\Lambda^\Gamma(\rho)\big|_\Lambda = \lim_i \pi_i\big|_\Lambda$, the Lemma follows. □

Notice that this Lemma in particular shows that property FD passes to subgroups. The following provides some kind of "converse" in the special framework of a lattice subgroup:

Lemma 3.3. *Let* Λ *be a lattice in a second countable locally compact group* G*. If* F *is a dense subset of* $\tilde{\Lambda}$ *then* $F_G = \{Ind_\Gamma^G(\rho)\big|\rho \in F\}$ *is dense in* $\tilde{G}$*.*

Proof. Let $\rho \in \tilde{G}$. Then $\rho|_\Gamma = \lim_i \pi_i$ where $\{\pi_i\} \subseteq F$. Hence $\rho \otimes L^2(G/\Gamma) \cong Ind_\Gamma^G(\rho|_\Gamma) = \lim_i Ind_\Gamma^G(\pi_i)$. The first contains ρ since Γ is a lattice in G, so ρ is indeed in the closure of F_G. □

Lemma 3.4. *Let* $k, \mathcal{O}$ *and* S *as in Theorem 3.1 and* $\Lambda = SL_2(\mathcal{O}_S)$*. Let* $\mathcal{C}(\Lambda)$ *be the set of congruence representations of* Λ*, i.e., the representations factoring through a congruence quotient of* Λ*. Then* $\mathcal{C}(\Lambda)$ *is not dense in* $\tilde{\Lambda}$*.*

Proof. The group Λ is a lattice in $\prod_{v\in S} SL_2(k_v)$. The well known Theorem of Selberg ("$\lambda_1 \geq \frac{3}{16}$") and its extensions ("the Selberg property" or "property τ with respect to congruence subgroups" - cf. [Lu1], [Lu2], [Cl]) implies in the characteristic 0 case that some of the complementary series representations of $SL_2(k_v)$ are not in the closure of $C_0 = \{Ind_\Gamma^G(\rho)\big|\rho \in \mathcal{C}(\Lambda)\}$. The same applies in characteristic $p > 0$ due to the Theorem of Drinfeld who proved the Ramanujan conjecture in this case. Lemma 3.3 now finishes the Proof. □

Proof of Theorem 3.1 . Let G and Γ be as in the Theorem. As k-rank$(G) \geq 1$, G has a k-algebraic subgroup H of type A_1. This implies that Γ has a subgroup $\Lambda = H(\mathcal{O}_S)$ which is commensurable to $SL_2(\mathcal{O}_S)$ and the congruence topology of Γ induces the congruence topology of Λ. Now, Γ has the congruence subgroup property (cf. [PR]), thus all its finite representations factor through congruence subgroups, and $\tilde{\Gamma}_F$ restricted to Λ gives $C(\Lambda)$ in the notations of Lemma 3.4. As $C(\Lambda)$ is not dense in $\tilde{\Gamma}$, Lemma 3.2 implies that $F(\Gamma) \neq \tilde{\Gamma}$. □

Remark: Note that $\Gamma = G(\mathcal{O}_S)$ is a lattice in $\prod_{v\in S} G(k_v)$, which is a semisimple group of rank ≥ 2 by our assumptions. Γ is a non-uniform lattice since we assume k-rank $(G) \geq 1$, and the theorem covers all the non-uniform lattices by Margulis' arithmeticity theorem. The Theorem and the proof are still valid for **some** uniform arithmetic lattices in higher rank semi-simple groups but not for all. The same proof will apply for G of k-rank zero (i.e. anisotropic) if (a) Γ satisfies the congruence subgroup property and (b) G has a k-subgroup of type A_1. Note that Condition

(a) is currently known for all groups whose type is different from A_n. See Section 6 below for more in this direction.

4. Ramanujan groups

In this section we place all the examples of Ramanujan groups under one unified construction, which also simplifies (conceptually and technically) the work of Lubotzky-Phillips-Sarnak [LPS].

Let G be a locally compact second countable group, $\Gamma < G$ a discrete co-compact torsion free subgroup, and $K < G$ a compact subgroup. Let $X \subseteq G$ be a bounded K-invariant fundamental domain for Γ, i.e. $G = X\Gamma$, and assume that the Haar measure m of G is normalized on X. Associated naturally to this X is a cocycle $\alpha_X : G \times X \to \Gamma$, defined by $\alpha_X(g,x) = \gamma$ iff $gx\gamma \in X$, and a measure preserving G-action on it, denoted $(g,x) \to g \cdot x$ (distinguished from the usual multiplication in G), which is obtained by identifying X with G/Γ. Finally, to any compactly supported probability measure μ on G, we may define the probability measure $\nu = \alpha_X(\mu)$ on Γ, by pushing the measure $\mu \times m$ to Γ via the map $\alpha_X : G \times X \to \Gamma$. Notice that because μ is compactly supported and X is bounded, $\alpha_X(\mu)$ is finitely supported. To simplify notation, given a unitary G-representation π we shall denote by $\pi(\mu)$ the μ-averaging (or "convolution") operator acting in the representation π, and by $||\pi(\mu)||$ its norm (similarly for measures on Γ).

In the rest of the paper we denote the regular representation of a group H by ρ_H. Our main tool for constructing Ramanujan groups is the following general result:

Proposition 4.1. *Retain the above notations for $G, \Gamma, \alpha_X, \mu, \nu$. Let σ be a unitary Γ-representation and $\pi = Ind_\Gamma^G \sigma$.*
(1) *We have $||\sigma(\nu)|| \leq ||\pi(\mu)||$, and if μ is bi-K-invariant then $||\pi(\mu)||$ may be replaced by $sup||\pi_x^K(\mu)||$, where π_x^K is the set of those unitary G-representations π_x having a non-zero K-fixed vector in some direct integral decomposition $\pi = \int \pi_x dx$.*
(2) *If μ is bi-K-invariant and $X = K$, (i.e. $G = K\Gamma$), then equality holds: $||\sigma(\nu)|| = ||\pi(\mu)||$ (and $||\pi(\mu)||$ may be replaced by $sup||\pi_x^K(\mu)||$ as in (1)). In particular, (taking $\sigma = \rho_\Gamma$) we have: $||\rho_\Gamma(\nu)|| = ||\rho_G(\mu)||$.*

The point of the Theorem is that it enables one to deduce that certain groups are Ramanujan without making any explicit numerical calculations of norms, as the following shows:

Corollary 4.2. *Retain the above notations and assume that μ is bi-K-invariant and $X = K$, as in (2).*

1) *If σ is a Γ-representation for which $\pi = Ind_\Gamma^G \sigma \prec \rho_G$ then $||\sigma(\nu)|| \leq ||\rho_\Gamma(\nu)||$. In fact, the same conclusion holds for any σ for which we assume the weak containment only for the π_x^K above in place of π.*
2) *Consequently, if $\mathcal{L} = \{\Gamma_i\}$ is a family of finite index subgroups of Γ such that for any i one has: $L_0^2(G/\Gamma_i) \prec \rho_G$, then the group Γ is $(T_\nu, \mathcal{L})$-Ramanujan, where the operator $T_\nu \in \mathbb{C}\Gamma$ is the one corresponding to ν. In fact, the same conclusion holds if we assume the above weak containment in ρ_G only for the "K-spherical part" of the representations $L_0^2(G/\Gamma_i)$ (as in (1) of 4.1).*
3) *In particular, if $\mathcal{L}$ is as in (2) above, then for any $g \in G$ the Γ-averaging operator over the set $KgK \cap \Gamma$, or more generally, the averaging operator*

over $F \cap \Gamma$ for **any** *bi-K-invariant compact subset $F \subseteq G$, satisfies the conclusion of (2) above.*

Indeed, (1) follows immediately since by (1) and (2) of the Proposition we have: $||\sigma(\nu)|| \leq ||\pi(\mu)|| \leq ||\rho_G(\nu)|| = ||\rho_\Gamma(\nu)||$.

Parts (2) and (3) of the Corollary make abstract a large part of the proof of the existence of Ramanujan graphs (or groups). All that remains to be done is to plug in the (deep) number theoretic tools which ensure that its condition is satisfied, and in the list of examples below we recall various cases where it can indeed be done.

Finally, before proving the Proposition, we remark that when applied to appropriate finite dimensional but infinite Γ-representations, it immediately gives a unified proof also for the part of work of Lubotzky-Phillips-Sarnak pertaining to uniform points distribution on the sphere (cf. [Lu1] Ch. 9). Moreover, using Margulis superrigidity and conditional on the conjectural congruence subgroup property, one could deduce that some of the examples below satisfy the Ramanujan property with respect to **all** finite dimensional representations, not only the finite ones.

Proof of Proposition 4.1: Retain all the notations preceding the statement of the Proposition. Consider part (1). Recall that the representation space for $Ind_\Gamma^G \sigma$ may be identified with $V_\pi = L^2(X, V_\sigma)$, where G operates on $f \in L^2(X, V_\sigma)$ by $[\pi(g)f](x) = \sigma \circ \alpha(g,x)[f(g^{-1} \cdot x)]$. For unit vectors $v, w \in V_\pi$, consider the constant functions $f_v, f_w \in L^2(X, V_\sigma)$, taking the constant value v,w resp. Note that because X is K-invariant, so are f_v, f_w. For any $g \in G$ we have by definition of the scalar product in V_π: $\langle \pi(g) f_v, f_w \rangle_{V_\pi} = \int_X \langle \sigma \circ \alpha(g,x) v, w \rangle_{V_\sigma} dm(x)$, hence integrating both sides over g with respect to μ and recalling the definition of the measure ν on Γ, gives:

$$(*) \qquad \langle \pi(\mu) f_v, f_w \rangle_{V_\pi} = \int_G \int_X \langle \sigma \circ \alpha(g,x) v, w \rangle_{V_\sigma} dm(x) d\mu(g) = \langle \sigma(\nu) v, w \rangle_{V_\sigma}$$

Since v and w are arbitrary, we can choose them so that the right hand side of $(*)$ is arbitrarily close to $||\sigma(\nu)||$, so the first part of (1) follows. The second part would follow from the assertion that in $(*)$ one can replace in the very left hand side π by π^K, where $\pi^K \subseteq \pi$ is the subrepresentation supporting the π_x^K's. Indeed, by K-invariance of μ, $\pi(\mu)$ vanishes on the orthogonal complement to π^K, and f_v, f_w are K-invariant, so the whole computation goes through with π^K in place of π.

For part (2) we wish to establish an inequality in the opposite direction. If we knew that in $(*)$, **any** functions $f_1, f_2 \in V_\pi$ can be obtained in the form f_v, f_w for some $v, w \in V_\sigma$, then the very same argument as in the previous proof could be reversed to show, by appropriately choosing f_1, f_2, that the opposite inequality holds. Notice that the assumption that μ is bi-K-invariant shows, as before, that it is actually enough to consider only f_1, f_2 which are K-invariant. However, since we now assume $X = K$, a K-invariant function is (essentially) constant on X, so it is obvious that the K-invariant functions are exactly those functions of the form f_v, thereby completing the proof.

Examples:

(1) Let H be the standard quaternion algebra, let p be a prime with $p \equiv 1(4)$ and $H(\mathbb{Z}[\frac{1}{p}])$ the $\mathbb{Z}[\frac{1}{p}]$-points of H , i.e.

$$\alpha = x_0 + x_1 i + x_2 j + x_3 k \ \text{ with } \ x_i \in \mathbb{Z}\left[\frac{1}{p}\right].$$

For such α denote $\|\alpha\|^2 = \alpha \cdot \bar{\alpha} = \sum_{i=0}^{3} x_i^2$. Let $\Gamma = H(\mathbb{Z}[\frac{1}{p}])^*/Z$, i.e., the invertible elements of $H(\mathbb{Z}[\frac{1}{p}])$ (i.e., those α with $\|\alpha\|^2 \in p^{\mathbb{Z}}$) modulo the center. Γ is a cocompact lattice in $G = PGL_2(\mathbb{Q}_p)$. It is shown in [Lu1] that $\Gamma(2) = \{\alpha \in \Gamma | \alpha \equiv 1(mod 2)\}$ acts simply transitively on the Bruhat-Tits building of $PGL_2(\mathbb{Q}_p)$, which is a $(p+1)$-regular tree. Thus $G = K\Gamma(2)$ with $K = PGL_2(\mathbb{Z}_p)$. The Ramanujan-Petterson conjecture (proved by Deligne) together with the Jacquet-Langlands correspondence, implies that for every congruence subgroup $\Gamma(2m)$ in $\Gamma(2)$, the induced representation $Ind_{\Gamma(2)}^{G} l_0^2(\Gamma(2)/\Gamma(2m))$ is tempered, and is thus weakly contained in $L^2(G)$ (see [Lu1]). Therefore the assumptions of Prop. 4.1 and Cor. 4.2 are satisfied, showing that $\Gamma(2)$ is a Ramanujan group with respect to every spherically symmetric operator T and $\mathcal{L}$ being the family of congruence subgroups.

(2) In [Mor] Morgenstern showed that for every positive characteristic local field F, $G = PGL_2(F)$ has a suitable arithmetic group Γ such that Γ acts simply transitive on the Bruhat-Tits building of G which is a $(q+1)$-regular tree, where $q = |\mathcal{O}/\pi\mathcal{O}|$. Here $\mathcal{O}$ is the valuation ring of F and π a uniformizer in $\mathcal{O}$. Moreover, the analogue Ramanujan-Petterson conjecture was proved in this case by Drinfeld. Similarly to the previous example, this now shows in particular that a free group F_r with $r = \frac{1}{2}(p^n + 1)$ and p an odd prime, is a Ramanujan group.

(3) Let $\mathcal{S} = \{p_1, \ldots, p_\ell\}$ be a set of ℓ-primes with $p_i \equiv 1(mod 4)$ for every $i = 1, \ldots, \ell$. Let $\mathbb{Z}_{\mathcal{S}} = \mathbb{Z}\left[\frac{1}{p_1}, \ldots, \frac{1}{p_\ell}\right]$, $H(\mathbb{Z}_{\mathcal{S}})$ - the $\mathbb{Z}_{\mathcal{S}}$-points of the standard quaternion algebra H (see example 1). Let $\Gamma = H(\mathbb{Z}_{\mathcal{S}})^*/Z$ the invertible elements of $H(\mathbb{Z}_{\mathcal{S}})$ modulo its center. This is a lattice in the product $G = \prod_{i=1}^{\ell} G_i$ with $G_i = PGL_2(\mathbb{Q}_{p_i})$. The congruence subgroup $\Gamma(2)$ acts simply-transitive on the vertices of $X = \prod_{i=1}^{\ell} T_i$ where T_i is the (p_i+1)-regular tree which is the Bruhat-Tits building of G_i. Thus $G = K\Gamma$, where $K = \prod_{i=1}^{\ell} K_i$, $K_i = PGL_2(\mathbb{Z}_{p_i})$. As shown by Jordan and Livne [JL], Deligne's solution of the Ramanujan-Petterson conjecture implies that the assumptions of (Prop 4.1 and) Cor 4.2 are satisfied for $\Gamma(2)$, and $\mathcal{L}$ being its congruence subgroups. The reader is referred to [JL] for more on examples of this kind.

(4) Let F be a local field of positive characteristic, $\mathcal{O}$ and q as in example 2. In [CS1], Cartwright and Steger constructed, for every $d \geq 2$, a cocompact lattice in $G = PGL_d(F)$, which acts simply-transitively on the vertices of the Bruhat-Tits building $\mathcal{B}_d(F)$, which is a building of type $\tilde{A}_{d-1}$. So $G = K\Gamma$ when $K = PGL_d(\mathcal{O})$. Let Γ be the Cartwright-Steger lattice. In [LSV1], Lubotzky, Samuels and Vishne used Lafforgue solution of the Ramanujan-Peterson conjecture for GL_d in positive characteristic to show: if $\Gamma(m)$ is a congruence subgroup of Γ, then the representation $\tilde{\pi}$, which is the K-spherical part of $\pi = Ind_{\Gamma}^{G}(l_0^2(\Gamma/\Gamma(m))$, is weakly contained in the regular representation of G provided at least one of the following two conditions is satisfied: either (a) d is a prime number or (b) m is prime to at least one of the ramification primes of the division algebra defining Γ (in the notations of [CS2], this means that m is a polynomial in $\mathbb{F}_q[y]$ which is prime to $z = y + 1$; see [LSV1] and [LSV2] for a detailed discussion).

We mention in passing that unlike examples 2, 3 and 4, in the case d is not a prime, for some (in fact, for infinitely many) congruence subgroups, the "Ramanujan Conjecture" is not satisfied. Still for infinitely many – it does. In any event, whenever condition (a) or (b) is satisfied, we get (by Proposition 4.1 and Corollary 4.2) quotients making Γ Ramanujan. Since these constructions are very recent, here are some further details for the convenience of the reader. First we describe sets S, for which according to part (3) of the Corollary, $T = T_S$ satisfies the condition of the Ramanujan property:

For $1 \le k \le d-1$, let $g_k = diag(\pi, \pi, \ldots, \pi, 1, \ldots, 1)$ when π appears k times. Let $S_k = Kg_kK \cap \Gamma$ and $A_k = \sum_{s \in S_k} s$ be the corresponding element of the group algebra $\mathbb{C}\Gamma$. In general, A_k is not a self adjoint operator (when acting in unitary Γ-representations), but (i) it is a normal operator, and (ii) $A_k + A_{d-k}$ is self-adjoint. The spectrum of A_k on $l^2(\Gamma)$ (which can be identified with $l^2(B_n^0(F))$, $B_n^0(F)$ being the 0-skeleton of the building $B_n(F)$), is a subset of $\mathbb{C}$ which is described in [CS2] as:

$$C_{d,k} = \{q^{k(d-k)/2}\sigma_k(z_1, \ldots, z_k) \Big| For\ i = 1, \ldots, d,\ \ z_i \in \mathbb{C}, |z_i| = 1 \text{ and } \prod_{i=1}^{d} z_i = 1\}$$

Here σ_k is the k-symmetric function i.e.,

$$\sigma_k(z_1, \ldots, z_d) = \sum_{i_1 < i_2 < \cdots < i_k} z_{i_1} z_{i_2} \cdot \cdots \cdot z_{i_k}.$$

In [LSV1] it is shown that for the congruence subgroups satisfying either (a) or (b) the "non-trivial" eigenvalues of A_k on $l^2(\Gamma/\Gamma(m))$ are in $C_{d,k}$. Moreover, for $A_k + A_{d-k}$, it means that its norm on $l_0^2(\Gamma/\Gamma(m))$ is bounded by $2\binom{d}{k}q^{k(d-k)/2}$. This last part follows also directly from Corollary 4.2 above. The A_k's commute with each other and we may take $S = \bigcup_{k=1}^{d-1} S_k$ and $A = \sum_{k=1}^{d-1} A_k$. This is the usual adjacency operator with respect to S. Thus, for the family $\mathcal{L} = \Gamma(m)$, the group Γ is $(T_S, \mathcal{L})$-Ramanujan whenever (a) or (b) is satisfied.

5. Non-Ramanujan groups

Lemma 2.1 above is often useful in showing that a group Γ is **not** Ramanujan with respect to a given family of subgroups $\mathcal{L}$. This is reminiscent of the Burger-Li-Sarnak method [BLS] to show the non-validity of the naive Ramanujan conjecture, of which the Lemma may be viewed as a "discrete" analogue. Here is a useful illustration:

Proposition 5.1. *Let $\Gamma = H_1 \times H_2$ be a direct product of two non-amenable groups H_1 and H_2. Then Γ is not Ramanujan (i.e.,* **no** *$T, \mathcal{L}$ as in the Definition make it Ramanujan).*

Proof. Assume to the contrary that Γ is Ramanujan with respect to some $\mathcal{L} = \{\Gamma_i\}$ and some positive $T \in \mathbb{C}\Gamma$. We claim that for $j = 1, 2$, $\overline{H_j}$ is of finite index in Γ, where $\overline{H_j}$ is the closure of H_j in the profinite topology on Γ defined by $\mathcal{L}$. Indeed, otherwise by Lemma 2.1 we have: $||T||_{l^2(\Gamma/\overline{H_j})} \leq sup\, ||T||_{l_0^2(\Gamma/\Gamma_i)} \leq ||T||_{\rho_\Gamma}$. However, since $\overline{H_j}$ are non-amenable and still normal in Γ, by a well known result of Kesten [Ke]: $||T||_{\rho_\Gamma} < ||T||_{l^2(\Gamma/\overline{H_j})}$, which is a contradiction. Thus $\overline{H_j}$ must be of finite index in Γ for both $j = 1, 2$, and the two must commute, as do the H_j's. This proves that Γ is virtually abelian, in contradiction to the assumption that the H_j's are non-amenable. □

Let $\Gamma_2 = F_{\frac{p_1+1}{2}} \times F_{\frac{p_2+1}{2}}$ be the product of two free groups on $\frac{p_1+1}{2}$ and $\frac{p_2+1}{2}$ generators, with set of free generators S' and S'', respectively. Here p_1 and p_2 are two primes with $p_1 \equiv p_2 \equiv 1 (mod 4)$. Let $S_2 = S' \cup S''$. Then $Cay(\Gamma_2; S_2)$ is the direct product of the $(p_1 + 1)$-regular tree and the $(p_2 + 1)$-regular tree. By Proposition 5.1, Γ_2 is not Ramanujan. Compare this to Γ_1- the congruence subgroup mod 2 of $H^*(\mathbb{Z}[\frac{1}{p_1}, \frac{1}{p_2}])/Z$ discussed in Example 3 of §4. This group acts simply transitive on the product of the trees associated to $PGL_2(\mathbb{Q}_{p_1})$ and $PGL_2(\mathbb{Q}_{p_2})$. Hence, its Cayley graph with respect to the generators which take any fixed "origin" to its neighbours (of distance 1) is also the product of (p_1+1)-regular tree and the $(p_2 + 1)$-regular tree (for more on these groups see [Moz]). We thus obtained two groups Γ_1 and Γ_2 with systems of generators S_1 and S_2, respectively, such that $Cay(\Gamma_1; S_1)$ is isometric to $Cay(\Gamma_2; S_2)$. Now, Γ_1 is Ramanujan with respect to the averaging operator T_{S_1} (this certainly holds with respect to the family of congruence subgroups, although notice that by the yet conjectural congruence subgroup property of this group, there should be no others), while $(\Gamma_2; S_2)$ is not Ramanujan.

Finally, by direct considerations we can get some information about the outstanding example of an arithmetic group, namely, the group $SL_n(\mathbb{Z})$:

Proposition 5.2. *Let* $\Gamma_n = SL_n(\mathbb{Z})$*, and* S_n *be the set of elementary matrices and their inverses, i.e.* $S_n = \{I \pm E_{ij} \big| 1 \leq i \neq j \leq n\}$. *Then for* n *large enough,* Γ_n *is not* $(T_n, \mathcal{L})$ *Ramanujan with respect to any* $\mathcal{L}$*, where* T_n *is the averaging operator over* S_n.

Proof. Denote by ρ_n the regular representation of Γ_n. Our strategy is to show that there exists some constant $C < 1$ such that $||T_n||_{\rho_n} < C$ for all n, whereas for any n we can find a finite representation π_n of Γ_n with no invariant vectors, such that $||T_n||_{\pi_n} \to 1$ as $n \to \infty$.

To establish the first claim, we divide the set S_n into $(n^2 - n)/2$ subsets of 4 elements, each consisting of the elements $I \pm E_{ij}, I \pm E_{ji}$ for all different choices of $1 \leq i < j \leq n$. Notice that every such subset generates a copy of a subgroup isomorphic to $SL_2(\mathbb{Z})$, in a way that maps the 4 elements to the $(\pm)$ upper and lower unit elementary matrices of $SL_2(\mathbb{Z})$. Let C denote the norm of the averaging operator on the latter 4-element subset of $SL_2(\mathbb{Z})$, acting in its $(SL_2(\mathbb{Z})$-) regular representation. By non-amenability of $SL_2(\mathbb{Z})$ we have $C < 1$. We claim that this C is the one required in our first claim. Indeed, grouping the elements of S_n in $(n^2 - n)/2$ 4-element subsets as above, and then iterating the triangle inequality $(n^2-n)/2$ times on the ρ_n-norm, gives the bound $||T_n||_{\rho_n} \leq \frac{1}{2}(n^2 - n) \cdot (C/\frac{1}{2}(n^2 - n)) = C$, as soon as we show that the norm of the (normalized by 4) averaging operator on each 4-element subset acting in ρ_n is C. This, however, is obvious from the definition of C, as the restriction of ρ_n to the subgroup $(\cong SL_2(\mathbb{Z}))$ generated by each 4-element subset is a multiple of its regular representation.

We now establish the second ingredient of our strategy. Fix n, and consider a "maximal parabolic" subgroup $\Lambda_n < \Gamma_n$, i.e., the stabilizer in Γ_n of the vector $(1, 0, \cdots, 0)$ for the standard linear action of Γ_n on $\mathbb{R}^n$. Consider the natural unitary Γ_n-representation τ_n on $l^2(\Gamma_n/\Lambda_n)$, and the Dirac function $f = 1_e$ at the identity coset. Since all but $2(n-1)$ elements of S_n are contained in Λ_n, they fix f, while the other $2(n-1)$ elements take f to an orthogonal vector. Therefore we have $\langle T_n f, f\rangle_{\tau_n} = \frac{1}{|S_n|}(2(n^2-n) - 2(n-1)) = \frac{2}{|S_n|}(n^2 - 2n + 1) = \frac{n-1}{n} = 1 - \frac{1}{n}$. Consequently, $||T_n||_{\tau_n} \geq 1 - \frac{1}{n} \to 1$. Thus it only remains to show that for every n, the representation τ_n can be approximated in the Fell topology by finite representations of the form $l_0^2(\Gamma_n/\Gamma_n^i)$ for a sequence of finite index subgroups $\Gamma_n^i < \Gamma_n$. However, since it is easy to see that Λ_n is closed in the congruence topology of Γ_n, i.e., it is the intersection of finite index (congruence) subgroups of Γ_n, this follows now from Lemma 2.1.

We have therefore proved the result when $\mathcal{L}$ is the family of all congruence subgroups. It is not difficult to see that the groups Λ_n used in the proof are in fact closed in any profinite topology on $SL_n(\mathbb{Z})$ which is weaker than the congruence one. Together with the congruence subgroup property, this establishes the result in general. We leave the verification of some missing details here to the reader.

□

Question: Is $SL_n(\mathbb{Z})$ Ramanujan for some $n \geq 3$?

6. Concluding remarks and open questions

Since all the groups we studied in this paper are S-arithmetic, it is natural to discuss a relation between their Ramanujan and FD properties, and those of the ambient algebraic groups, while giving a natural meaning to the latter. For simplicity, let us concentrate here only on the arithmetic case, i.e., we let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{Q}$, and consider the group $\Gamma = \mathbf{G}(\mathbb{Z})$. Recall that the **Automorphic Spectrum** Aut $\mathbf{G}$ of $\mathbf{G}$ is defined to be the closure in $\widetilde{\mathbf{G}(\mathbb{R})}$, of the set of all representations of the form $L^2(\mathbf{G}(\mathbb{R})/\Gamma)$, where Γ varies over all the congruence subgroups of $\mathbf{G}(\mathbb{Z})$. Denote also by $\mathrm{Aut}_0\mathbf{G}$ the same space, but with the trivial representation deleted. Much efforts have been devoted to identifying Aut $\mathbf{G}$, which is one of the most interesting objects of number theory. Analogues to the Ramanujan and FD properties for the arithmetic groups, one can now define them for the ambient $\mathbb{Q}$-group by saying that $\mathbf{G}$ has the Ramanujan property if $\mathrm{Aut}_0\mathbf{G}$ is contained in the tempered spectrum (i.e. it is weakly contained in the regular representation), and has property FD if it is the whole dual. These definitions make sharpest the marked difference between the two properties. Indeed, $\mathbf{G}$ is Ramanujan iff it satisfies the naive Ramanujan conjecture (e.g., Selberg's 1/4 conjecture in the case of $\mathbf{G} = SL_2$), that is to say, the automorphic spectrum is as restricted as it can possibly be, and it has property FD if nothing can be said about the "location" of automorphic representations. Although the naive Ramanujan conjecture is of course known to be false in general (cf. [BLS] and the references therein), one certainly expects that the automorphic representations should not be spread out over the whole unitary dual, namely, the following seems plausible:

6.1 Conjecture. Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{Q}$ and $\mathbb{R}$-isotropic. Then $\mathbf{G}$ does not have property FD.

For example, this is known for $\mathbf{G} = SL_2$. Of course, one can expect property FD of the algebraic group to be analogous to that of the arithmetic group only where the family of (finite) **congruence** representations are involved. Indeed, we have:

Observation: If the family of finite representations of $\Gamma = \mathbf{G}(\mathbb{Z})$ which factor through some congruence quotient is dense in $\tilde{\Gamma}$, then $\mathbf{G}$ has property FD. In particular, if Γ has the congruence subgroup property and property FD, then $\mathbf{G}$ has property FD.

This follows from Lemma 3.3. Of course, it is a natural question whether the converse (of the first statement) holds as well, although as we expect that no $\mathbf{G}$ has property FD, this may be of limited interest. However, combined with Conjecture 6.1, the observation gives:

6.2 Conjecture. Let Γ be an arithmetic group. If Γ satisfies the congruence subgroup property then it does not have property FD.

In fact, by our reasoning above, the Conjecture becomes a Theorem whenever the ambient group $\mathbf{G}$ is known not to have property FD. Since there are numerous groups for which the latter can be shown (using known results on the automorphic spectrum), the observation can be made useful in showing that certain arithmetic groups do not have property FD (in a way independent of Section 3 above and Bekka's method [Be]).

Unfortunately, the relation between the Ramanujan property for the algebraic and arithmetic group seems less understood at the present. Even though it appears that the Ramanujan property for the algebraic groups is stronger, we were able to deduce from it the Ramanujan property of the lattice subgroup only under a strong restriction on the fundamental domain. Thus the following still remains:

6.3 Question: Does $\mathbf{G}$ being Ramanujan implies that (some finite index subgroup of) $\Gamma = \mathbf{G}(\mathbb{Z})$ is as well, at least in the uniform case, i.e., when $\mathbb{Q}$-rank$\mathbf{G} = 0$?

Of particular interest seems the case of a surface group.

To conclude this direction, we remark that among free groups, we currently have the Ramanujan property only for some of them, even though we do not require the relevant operator to be an average over a free generating subset. However, notice that the list of Ramanujan groups consists not only of those obtained by Morgenstern [Mor], but also of **every index two subgroup** of these. This is obtained by using Proposition 4.1 and Corollary 4.2 for spherical operators which are supported on spheres of **even** radius, whose elements generate (though not freely !) a subgroup of index two. At any rate, one would like to know for which values of r, the free group F_r is Ramanujan, which may be viewed also as a motivation for the general problem of understanding to what extent the Ramanujan property passes to finite index subgroups and overgroups.

Turning to a different point of view, the results in §3 show that "most" lattices in higher rank simple Lie groups, do not have property FD. On the other hand from Theorems 2.2 and 2.8 it follows that every lattice Γ in $SL_2(\mathbb{R})$ does, and moreover, the same conclusion holds for some lattices in $SL_2(\mathbb{C})$. A well known conjecture of Thurston asserts that every irreducible 3-dimensional hyperbolic manifold is fibered over a circle. If true, it would imply that every lattice in $SL_2(\mathbb{C})$ has a finite index subgroup of the form $\Lambda \rtimes \mathbb{Z}$, when Λ is a surface or a free group. In either case, Theorem 2.8 shows that such a group has property FD. In fact, more might be expected:

6.4 Conjecture: Every finitely generated discrete subgroup of $SO(n,1)$ has property FD.

Observe that from the definition it follows that any group Γ which has both property τ and FD, has Kazhdan's property (indeed, the family of finite representations is then both dense and isolated from 1_Γ). Consequently, if an arithmetic lattice in $SO(n,1)$ or $SU(n,1)$ has property FD, it cannot have τ, and hence it has a negative solution to the congruence subgroup problem. Indeed, it is known that Γ has the Selberg property (i.e., property (τ) with respect to the congruence subgroups – see [Lu1], [Lu2] and [Cl]). This means that among the non-trivial finite representations of Γ, the **congruence** yet **not all** ones are bounded away from 1_Γ. Thus, Conjecture 6.4 implies Serre's conjecture [Se] for the arithmetic groups of $SO(n,1)$, while making an abstract group theoretical statement. For the current status of Serre's conjecture for lattices in $SO(n,1)$ see [Lu2].

It may be quite reasonable to include $SU(n,1)$ as well in Conjecture 6.4. On the other hand we conjecture that the opposite holds for lattices in higher rank simple Lie groups (a conjecture which is strongly supported by the results in §3). One would hesitate regarding lattices in $Sp(n,1)$ and $F_4^{(-20)}$. The lattices in these rank one Lie groups sometimes behave as the other rank one groups $SO(n,1)$ and $SU(n,1)$, and sometimes as higher rank lattices. In particular, the following seems very intriguing:

6.5 Question: Does there exist an infinite discrete Kazhdan group with property FD ?

Recall that by a result of Wang [Wa], for Kazhdan groups not only the trivial, but in fact **any** finite dimensional unitary representation is isolated in the unitary dual. This may suggest a negative answer to the question. Such an answer would also account for many of the foregoing results (and questions).

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Institute of Mathematics, The Hebrew University of Jerusalem, 91904 Israel, School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978 Israel.