FINITE GROUPS AND HYPERBOLIC MANIFOLDS

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ABSTRACT. The isometry group of a compact n-dimensional hyperbolic manifold is known to be finite. We show that for every $n \geq 2$, every finite group is realized as the full isometry group of some compact hyperbolic n-manifold. The cases $n = 2$ and $n = 3$ have been proven by Greenberg [G] and Kojima [K], respectively. Our proof is non constructive: it uses counting results from subgroup growth theory to show that such manifolds exist.

1. INTRODUCTION

Let $\mathcal{H}^n$ denote the hyperbolic n-space, that is the unique connected, simply connected Riemanian manifold of constant curvature -1. By a compact hyperbolic n-manifold we mean a quotient space $M = \Gamma\backslash\mathcal{H}^n$ where Γ is a cocompact torsion-free discrete subgroup of the group $H = \mathrm{Isom}(\mathcal{H}^n)$ of the isometries of $\mathcal{H}^n$. The group $\mathrm{Isom}(M)$ of the isometries of M is finite and it is isomorphic to $N_H(\Gamma)/\Gamma$ where $N_H(\Gamma)$ denotes the normalizer of Γ in H.

In 1972, Greenberg [G] showed that if $n = 2$, then for every finite group G there exists a compact 2-dimensional hyperbolic manifold M (equivalently, cocompact Γ in H) such that $\mathrm{Isom}(M) \cong G$ (equivalently, $N_H(\Gamma)/\Gamma \cong G$). A similar result for $n = 3$ was proved in 1988 by Kojima [K], who also mentioned the general conjecture. The methods of Greenberg and Kojima are very much of low dimensional geometry (Teichmüller theory and Thurston's Dehn surgery, respectively).

The long standing problem of realizing every finite group as the isometry group of some n-dimensional compact hyperbolic manifold is in flavor of the inverse Galois problem and other questions of such kind (see e.g. [F]). What makes our problem quite delicate is that even when we solve it for a group G, it is still not settled neither for the subgroups nor for the factor groups of G. In particular, our problem is not trivial even for the case of the trivial group, for which it means the existence of asymmetric hyperbolic n-manifolds. Recently, Long and Reid [LR] showed that for every n there exists a compact hyperbolic n-dimensional manifold M with $\mathrm{Isom}^+(M) = \{e\}$. Here, $\mathrm{Isom}^+(M)$ is the group of orientation preserving isometries, it is a subgroup of index at most two in $\mathrm{Isom}(M)$. They asked separately ([LR], §4.3 and §4.4) whether such an M exists with $\mathrm{Isom}(M) = \{e\}$ as well as for the general G. In this paper we give a complete solution to the problem. It turns out, indeed, that the proof is somewhat different for the cases $G = \{e\}$ and $G \neq \{e\}$.

Our main result is the following

Theorem 1.1. *For every $n \geq 2$ and every finite group G there exist infinitely many compact n-dimensional hyperbolic manifolds M with* $\mathrm{Isom}(M) \cong G$.

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Let us describe the line of the proof. We start with the Gromov and Piatetski-Shapiro construction [GPS] of a non-arithmetic lattice Γ_0 in $\mathrm{Isom}(\mathcal{H}^n)$. These lattices are obtained by interbreeding two arithmetic lattices and the construction, in particular, implies that Γ_0 is represented as a non-trivial free product with amalgam. By Margulis theorem [Mr, Theorem 1, p. 2] the non-arithmeticity of the lattice implies that its commensurator $\Gamma = \mathrm{Comm}_{\mathrm{H}}(\Gamma_0)$ is a maximal discrete subgroup of $\mathrm{Isom}(\mathcal{H}^n)$, so for every finite index subgroup B of Γ, $N_H(B) = N_\Gamma(B)$. It therefore suffices to find such a torsion-free B with $N_\Gamma(B)/B \cong G$.

We begin the search for B inside Γ_0 which enables us to use its amalgamated structure. To this end we modify the argument of [L2] to show that Γ has a suitable finite index torsion-free normal subgroup Δ which is mapped onto a free group $F = F_r$ on $r \geq 2$ generators with kernel M. We then apply ideas and results from subgroup growth theory [LS] to prove that Δ has a finite index subgroup A with $N_\Delta(A)/A$ isomorphic to G. A crucial point here is that F has at least $k!$ subgroups of index k but at most $k^{cr\log_2 k}$ of them are normal in F, for some absolute constant c. (This result was proved in [L3] using the classification of the finite simple groups, but the version we need is somewhat weaker and can be proved without the classification. So the current paper is classification free!) Another interesting group theoretic aspect is the use along the way of a result from [L1] asserting that an automorphism of a free group preserving every normal subgroup of a p-power index must be inner. (The only known proof of this result relies on the theory of pro-p groups.)

Now another problem has to be fixed. While $N_\Delta(A)/A \cong G$, $N_\Gamma(A)$ can be (and in fact in many cases it is) larger than $N_\Delta(A)$. To deal with this issue we modify A by replacing it by a somewhat larger subgroup B of Γ for which indeed $N_\Gamma(B)/B \cong G$. Two delicate points have to be overcome on the way: First is controlling the normalizer; what makes the whole proof difficult is the fact that "normalizer is not continuous"; even a small change from A to B can change the normalizer dramatically. The second point is to keep B torsion-free just as A. This is achieved by keeping B inside a suitable principal congruence subgroup.

The paper is organized as follows: In §2 we collect a number of group theoretic results to be used in the later sections. In §3 we bring the main group theoretical method to find finite index subgroups B in a group Γ with $N_\Gamma(B)/B \cong G$. Then in §4, a group Γ in $\mathrm{Isom}(\mathcal{H}^n)$ is constructed which satisfies all the needed assumptions. So only §4 contains some geometry. We end with remarks and suggestions for further research in §5.

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2. Group theoretic preliminaries

In this section we will present a number of group theoretic results which we will use later. We begin with the *free groups*.

2.1. Let $F = F_r$ be a free group on $r \geq 2$ generators. For a prime p denote by $\mathcal{L}_p(F)$ the family of all normal subgroups of F of p-power index.

Theorem 2.1. [L1] *(a) F is residually-p, i.e., $\bigcap_{N\in\mathcal{L}_p(F)} N = \{e\}$.*
(b) If α is an automorphism of F such that $\alpha(N) = N$ for any $N \in \mathcal{L}_p(F)$ then α is inner.

While part (a) is well known and easy, it is interesting to remark that the proof of part (b) in [L1] is based on the work of Jarden - Ritter [JR] which combines pro-p groups and relation modules.

Proposition 2.2. (Schreier's theorem, cf. [S, Theorem 5, p. 29]) *If H is a subgroup of $F = F_r$ of index k then H is a free group on $1 + k(r-1)$ generators.*

For a finitely generated group Γ we denote by $a_n(\Gamma)$ (resp., $a_n^{\triangleleft}(\Gamma)$, $a_n^{\triangleleft\triangleleft}(\Gamma)$) the number of subgroups (resp., normal subgroups, subnormal subgroups) of index n in Γ, and let $s_n^{\bullet}(\Gamma) = \sum_{i=1}^{n} a_i^{\bullet}(\Gamma)$.

Theorem 2.3. *(a)* (cf. [LS, Corollary 1.1.2, p. 13 and Corollary 2.1.2, p. 41])

$$(n!)^{r-1} \le a_n(F) \le n(n!)^{r-1}.$$

(b) (cf. [L3], see also [LS, Theorem 2.6]) *There exists a constant c such that*

$$a_n^{\triangleleft}(F) \le n^{cr\log_2 n}.$$

(c) (cf. [LS, Theorem 2.3]) $s_n^{\triangleleft\triangleleft}(F) \le 2^{rn}$.

2.2. We now turn to *free products with amalgam* and *HNN-constructions*. Let Q be a finite group of order ≥ 3 and T be a subgroup of Q satisfying the following:

$(*)$ If $N \le T \le Q$ and $N \lhd Q$ then $N = \{e\}$.

In particular, $(*)$ implies that $[Q : T] > 2$.

Let $R = Q *_T Q$ be the free product of Q with itself amalgamated along T or $R = Q*_T$ – the HNN-construction. There is a natural projection $\bar{\pi} : R \to Q$ whose kernel will be denoted by $F = \mathrm{Ker}(\bar{\pi})$. As Q is also a subgroup of R we have $R = F \rtimes Q$ – a semi-direct product.

Proposition 2.4. *(a) F is a non-abelian free group.*
(b) $C_R(F) = \{e\}$, i.e., the centralizer of F in R is trivial.
(c) If α is an automorphisms of R satisfying $\alpha(F) = F$ and $\alpha|_F = id$ then $\alpha = id$.

Proof. (a) This is a known fact, let us briefly recall the argument. By definition $F \setminus \{e\}$ does not meet any conjugate of Q, so it acts trivially on the tree associated to R by the Bass-Serre theory [S, Section 4.2]. This implies that F is a free group [S, Theorem 4, p. 27]. The rank r of F can be computed using the formula from [S, Exercise 3, p. 123], in particular, the condition $[Q : T] > 2$ implies that $r \ge 2$ and so F is a non-abelian free group.

(b) The group $C = C_R(F)$ is a normal subgroup of R. As F is a non-abelian free group with a trivial center, $C \cap F = \{e\}$ and hence C is finite. We claim that $C \le T$. If R is the free product with amalgam then by [S, Theorem 7, p. 32], R acts on a tree with a fundamental domain consisting of an edge e with two vertices v_1 and v_2 such that $\mathrm{Stab}_R(v_i)$, $i = 1, 2$, are two copies of Q in R which we denote by Q_1 and Q_2. Since C is finite it is conjugate into one of the Q_i's, say Q_1. But since C is normal in R it is contained in Q_1 and in all the conjugates of Q_1 in R. This implies that C fixes all the vertices in the R-orbit of v_1. The later fact implies also that C fixes v_2 (and all the vertices in its orbit). Hence C is also in Q_2 and so $C \le Q_1 \cap Q_2 = T$. The case of HNN-construction is even easier. Again R acts on

a tree and this time the fundamental domain contains a single vertex. We get that C is contained in the stabilizers of all the vertices and edges, so C is contained in T. In both cases condition $(*)$ implies that $C = \{e\}$.

(c) Let $f \in F$ be an arbitrary element of F and $q \in Q$ an arbitrary element of Q. We have $q^{-1}fq \in F$, so $\alpha(q^{-1}fq) = q^{-1}fq$ and $\alpha(f) = f$. This implies that

$$q^{-1}fq = \alpha(q^{-1}fq) = \alpha(q)^{-1}f\alpha(q),$$

and hence

$$\alpha(q)^{-1}q \in C_R(F) = \{e\}.$$

Thus $\alpha(q) = q$ and so α is the identity on Q. Since $R = F \rtimes Q$, α is the identity automorphism of R. □

2.3. Finally, we will need some facts on the *finite orthogonal groups* (see e.g. [A]). Let f be an m-dimensional quadratic form over a finite field $\mathbb{F}$ of characteristic $p > 2$. If m is odd, there is a unique, up to isomorphism, orthogonal group $\mathrm{O}(f) = \mathrm{O}_m(\mathbb{F}) = \mathrm{O}_m$. If m is even there are two groups O_m^+ and O_m^- corresponding to the cases when f splits and does not split over $\mathbb{F}$, respectively. Let $\mathrm{SO}(f)$, $\mathrm{PSO}(f)$ denote the corresponding special orthogonal and projective special orthogonal groups, and let $\Omega(f) = [\mathrm{O}(f), \mathrm{O}(f)]$ be the commutator subgroup of $\mathrm{O}(f)$. The projective group $\mathrm{P}\Omega(f) = \mathrm{P}\Omega_m^{\pm}(\mathbb{F})$ is generally simple and is contained in $\mathrm{PSO}(f)$ with index at most 2. More precisely, $\mathrm{P}\Omega_m$ is simple if $m \geq 5$ or $m = 3$ and $p > 3$; in case $m = 4$, $\mathrm{P}\Omega_4^-$ is simple but $\mathrm{P}\Omega_4^+ = \mathrm{P}\Omega_3 \times \mathrm{P}\Omega_3$ is a direct product of two groups which are simple if $p > 3$; the cases $m = 1, 2$ will not be used in this paper. We sometimes omit the $\pm$ sign is the notations. For the future reference note also that the centralizer of $\mathrm{P}\Omega(f)$ in $\mathrm{PO}(f)$ is always trivial.

3. The main algebraic result

In this section we prove a purely group theoretic result. In Section 4 we will show that it can be implemented for suitable non-arithmetic lattices in $\mathrm{PO}(n, 1)$.

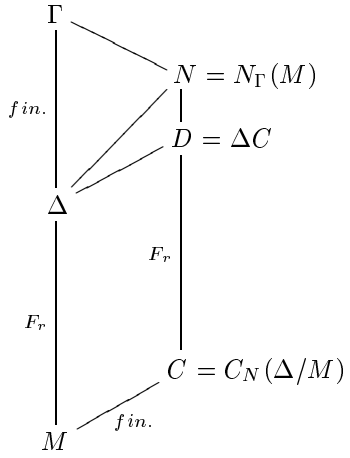
Throughout this section Γ is a finitely generated group, Δ a finite index normal subgroup of Γ, and M is a normal subgroup of Δ with Δ/M being isomorphic to a free group $F = F_r$ on $r \geq 2$ generators. We denote by $N = N_\Gamma(M)$ the normalizer of M in Γ, so $\Delta \leq N \leq \Gamma$. The group N acts by conjugation on $F = \Delta/M$. Denote by $C = C_N(\Delta/M)$ the kernel of this action and by D the subgroup of all elements of N which induce inner automorphisms of F. Both C and D are normal in N and clearly

$$D = \Delta C.$$

Moreover, M is normal in D, Δ and in C. As $F = \Delta/M$ has a trivial center and hence intersects C/M trivially, $\Delta \cap C = M$. So, taking mod M we get

$$D/M = \Delta/M \times C/M.$$

Moreover, Δ/M is of finite index in D/M and hence C/M is a finite group.



This section is devoted to the proof of the following result.

Theorem 3.1. *Let Γ, Δ and D be as above. For every finite group G there exist infinitely many finite index subgroups B of D with $N_\Gamma(B)/B$ isomorphic to G.*

Proof. Denote the order of G by $g = |G|$, and $g' = |\operatorname{Aut}(G)|$. Also let $d+1 = [\Gamma : D]$ and $e, \gamma_1, \ldots, \gamma_d$ be representatives of the right cosets of D in Γ, i.e. $\Gamma \setminus D = \bigcup_{i=1}^{d} D\gamma_i$.

Let $x > max\{g, g'\}$ be a very large integer, to be determined later. Choose $d+1$ primes $p_0 < p_1 < \ldots < p_d$ with $p_0 \geq x^2$.

Now, if $d = 0$, choose a normal subgroup of Δ of index p_0 containing M and call it K. If $d > 0$ the definition of K will be more delicate: We claim that for every $i = 1, \ldots, d$ there exists a normal subgroup $K_i \supset M$ of index $p_i^{\alpha_i}$ in Δ for some $\alpha_i \in \mathbb{N}$, such that $K_i^{\gamma_i} \neq K_i$ (where $K_i^{\gamma_i} = \gamma_i^{-1} K_i \gamma_i$). Indeed, if not then conjugation by γ_i stabilizes all normal subgroups of Δ containing M and of index p_i-power in Δ. As $\Delta/M \cong F$ is residually-p_i (Theorem 2.1(a)), M is the intersection of these normal subgroups and hence γ_i normalizes M and so $\gamma_i \in N_\Gamma(M)$. Moreover, γ_i acting on $F = \Delta/M$ is now an automorphism of F which preserves any normal subgroup of F of p_i-power index. By Theorem 2.1(b), γ_i induces on F an inner automorphism and hence $\gamma_i \in D$ in contradiction to the way γ_i was chosen. We define $K := \bigcap_{i=1}^{d} K_i$.

In both cases denote the index of K in Δ by k. Observe that $k \geq x^2$ and K/M is a subgroup of $\Delta/M = F = F_r$ of index k so by Proposition 2.2, K/M is a free group on $1 + k(r-1)$ generators.

The proof now splits into two cases.

Case 1: $G \neq \{e\}$. We claim that there are at least $g^{1+k(r-1)-\log_2 g}$ different epimorphisms from K onto G whose kernels contain M. Indeed, it follows from an easy argument that a finite group of order g is generated by at most $\log_2 g$ elements (cf. [LS, Lemma 1.2.2, p. 14]), so we can send the first $\log_2 g$ generators of $K/M \cong F_{1+k(r-1)}$ to fixed generators of G and all the rest generators of K/M can be sent arbitrarily into G. Two such epimorphisms ϕ_1 and ϕ_2 have the same

kernel if and only if there exists $\beta \in \mathrm{Aut}(G)$ such that $\phi_1 = \beta \circ \phi_2$. Again, as G is generated by $\log_2 g$ elements, $g' = |\mathrm{Aut}(G)| \leq g^{\log_2 g}$. Hence there exist at least

$$\frac{1}{g'}\, g^{1+k(r-1)-\log_2 g} \geq g^{k(r-1)-2\log_2 g} =: z$$

normal subgroups A of K containing M with $K/A \cong G$.

Let $\mathfrak{M}$ denote the set of these subgroups A. We claim that for every $A \in \mathfrak{M}$, $N_\Gamma(A) \leq D$. If $d = 0$, $\Gamma = D$ and there is nothing to prove. If $d > 0$, let $\gamma \in N_\Gamma(A)$ and assume $\gamma \in \Gamma \smallsetminus D$, so $\gamma = \delta\gamma_i$ for some $\delta \in D$ and some $i \in \{1, \ldots, d\}$. Now, γ normalizes Δ and A, hence γ normalizes K_i since K_i is the only normal subgroup of Δ of index $p_i^{\alpha_i}$ containing A. Moreover, δ being an element of D induces an inner automorphism on $F = \Delta/M$ hence it also normalizes K_i. Thus $\gamma_i = \delta^{-1}\gamma$ normalizes K_i, but this contradicts the way K_i was chosen. So for every $A \in \mathfrak{M}$, $N_\Gamma(A) \leq D$.

Let us observe that $KC \leq N_\Gamma(A)$ (since K and C are normal subgroups of D, their product KC is indeed a subgroup). We will show next that for *some* $A \in \mathfrak{M}$, $N_\Gamma(A) = KC$. This will be done by a counting argument.

We already know that $N_\Gamma(A) = N_D(A) \supseteq KC$. As $D/M \cong \Delta/M \times C/M$, $N_D(A)/M \cong N_\Delta(A)/M \times C/M$, so we can project everything to Δ/M and it suffices to show that there exists $A \in \mathfrak{M}$ with $N_\Delta(A) = K$.

Fix one $A \in \mathfrak{M}$ and denote $L := N_\Delta(A)$, $l := [\Delta : L]$. The subgroup L contains M and L/M is a subgroup of $\Delta/M \cong F_r$ of index l, so L/M is a free group on $1 + l(r-1)$ generators. Also, A is a normal subgroup of L of index gk/l, so A/M is a normal subgroup of L/M of the same index. By Theorem 2.3(b), L/M has at most

$$(\frac{gk}{l})^{c(1+l(r-1))\log_2(gk/l)} \leq 2^{crl\log_2^2(gk/l)} =: y$$

normal subgroups of index gk/l, where c is an absolute constant. Since $L \geq K$, l divides k and it is a proper divisor if $L \neq K$. So $L \gneqq K$ implies that l is at most $k/p_o \leq k/x^2 < k/x$. Note also that as $k \geq x^2$, $k/x \geq x$. Running over all l in the range between 1 and k/x, for big enough x, the maximal value of y is obtained when $l = k/x$. Thus, there are at most $2^{cr(k/x)\log_2^2(gx)}$ subgroups in $\mathfrak{M}$ which are normalized by a given $L \neq K$.

Now note that Δ/K is a finite nilpotent group (of order p_0 if $d = 0$ and of order $\Pi_{i=1}^d p_i^{\alpha_i}$ otherwise), L/K is a subgroup of Δ/K and any subgroup of a nilpotent group is subnormal. This implies that L is a subnormal subgroup of Δ of index less than k/x (if $L \neq K$), hence by Theorem 2.3(c) there are at most $2^{rk/x}$ possibilities for such L's.

Putting all this together we see that there are at least $z = g^{k(r-1)-2\log_2 g}$ possibilities for A and out of them at most

$$w := 2^{rk/x} 2^{cr(k/x)\log_2^2(gx)} = 2^{(rk/x)(1+c\log_2^2(gx))}$$

have their normalizer bigger then KC. For x large enough, $z > w$ and so there exist $A \in \mathfrak{M}$ with $N_\Gamma(A) = KC$. In fact, most $A \in \mathfrak{M}$ do satisfy this.

Note however, that for these A

$$N_\Gamma(A)/A = KC/A \cong K/A \times C/M \cong G \times C/M.$$

So, we have not achieved the goal of Theorem 3.1 yet. For this we will have to enlarge A, but first let us consider the second case.

Case 2: $G = \{e\}$. Let q be a prime close to k and different from $p_0, \ldots, p_d$, such that $q > p_0 \geq x^2$. Let $\mathfrak{M}$ be the set of all subgroups of K of index q containing M. By Theorem 2.3(a) there are at least

$$(q!)^{k(r-1)+1} > (q!)^{k(r-1)} =: z$$

such subgroups. As before we claim that for every $A \in \mathfrak{M}$, $N_\Gamma(A) \leq D$. If $d = 0$ there is nothing to prove. If $d > 0$, the argument is exactly the same: write $\gamma \in \Gamma \smallsetminus D$ as $\gamma = \delta\gamma_i$ for some $\delta \in D$ and $i \in \{1, \ldots, d\}$, if γ normalizes A then it also normalizes K_i, which implies that γ_i normalizes K_i – a contradiction.

So, again for every $A \in \mathfrak{M}$, $N_\Gamma(A) \leq D$. This time we want to prove that for some (in fact, for most) $A \in \mathfrak{M}$, $N_\Gamma(A) = AC$ (note: not KC as for $G \neq \{e\}$). As in the previous case we will project everything to Δ/M and show that there exists $A \in \mathfrak{M}$ with $N_\Delta(A) = A$.

To this end, note that if $L := N_\Delta(A)$ is strictly larger than A, then L is a subgroup of index, say, l in Δ, l being a proper divisor of kq and so $l \leq kq/x$. Similarly to the previous case L/M is a free group on $1 + l(r-1) < lr$ generators and A/M is its normal subgroup of index kq/l. It follows from Theorem 2.3(b) that L can normalize at most

$$(\frac{kq}{l})^{clr \log_2(kq/l)} = 2^{crl \log_2^2(kq/l)} =: y$$

subgroups from $\mathfrak{M}$. Now, l can get values between 1 and kq/x, and clearly the maximum of y is attained when $l = kq/x$ (if x is big enough) corresponding to $y = 2^{cr(kq/x)\log_2^2 x}$.

Continue working mod M, L/M is a subgroup of index at most kq/x of the free group $\Delta/M \cong F_r$. There are, therefore, at most $((kq/x)!)^r$ such subgroups by Theorem 2.3(a). The later number is bounded by $(kq/x)^{c'(kq/x)r}$ for a suitable constant c'. Altogether, at most

$$w := (kq/x)^{c'(kq/x)r}\, 2^{cr(kq/x)\log_2^2 x} = 2^{r(kq/x)(c'\log_2(kq/x) + c\log_2^2 x)}$$

of $A \in \mathfrak{M}$ have normalizer $N_\Delta(A)$ larger then A. Recall that q was chosen to be approximately k and $z = (q!)^{(r-1)k}$. An easy estimate shows that $z > w$ provided x is large enough. Thus, there exists A in $\mathfrak{M}$ (in fact, most $A \in \mathfrak{M}$) for which $N_\Delta(A) = A$ and $N_\Gamma(A) = N_D(A) = AC$.

Let us now treat both cases $G \neq \{e\}$ and $G = \{e\}$ together: The above arguments show that we can always find a subgroup A of a finite index in Δ such that $N_\Delta(A)/A \cong G$, $N_\Gamma(A) = N_D(A)$ and $N_\Gamma(A)/A \cong G \times C/M$. Let us replace A by $B = AC$. Since A and C are both in D and $C \lhd D$, B is indeed a subgroup and it is contained in D. It is also clear that $B \cap \Delta = A$ (look at everything mod M!). We claim that $N_\Gamma(B)/B \cong G$. This will finish the proof of the theorem.

First, note that if $\gamma \in N_\Gamma(B)$ then γ also normalizes Δ (since $\Delta \lhd \Gamma$) and hence:

$$A^\gamma = (B \cap \Delta)^\gamma = B^\gamma \cap \Delta^\gamma = B \cap \Delta = A,$$

so $\gamma \in N_\Gamma(A)$. On the other hand every $\gamma \in N_\Gamma(A)$ also normalizes C, since $N_\Gamma(A) \leq D$ and C is normal in D. This shows that $N_\Gamma(B) = N_\Gamma(A)$ and so

$$N_\Gamma(B)/B = N_\Gamma(A)/AC \cong N_\Delta(A)/A \cong G$$

as claimed.

We finally mention that by choosing infinitely many different x's (and hence also the p_i's) we will get infinitely many subgroups B of Γ with $N_\Gamma(B)/B \cong G$. □

4. Geometric realization

In this section we will show that for every $n \geq 2$ there exist non-arithmetic lattices in $H = \mathrm{Isom}(\mathcal{H}^n)$ satisfying the assumptions of Theorem 3.1 and then deduce the main result of this paper. Recall that H can be identified with $\mathrm{O}_0(n,1)$ – the subgroup of the orthogonal group $\mathrm{O}(n,1)$ which preserves the upper-half space, it is isomorphic to the projective orthogonal group $\mathrm{PO}(n,1) = \mathrm{O}(n,1)/\{+1,-1\}$. The subgroup $\mathrm{SO}_0(n,1)$ of $\mathrm{O}_0(n,1)$ of all the elements of H with determinant 1, is the group of orientation preserving isometries.

Proposition 4.1. *For every $n \geq 2$ there exist a maximal cocompact non-arithmetic lattice Γ in H with subgroups M, Δ and D satisfying the following:*

(i) $\Delta \lhd \Gamma$ *and* $[\Gamma : \Delta] < \infty$.
(ii) $M \lhd \Delta$, Δ/M *is a non-abelian free group.*
(iii) $[\Gamma : D] < \infty$, $D \leq N_\Gamma(M)$ *and*
$D = \{\delta \in N_\Gamma(M) \mid \delta$ *induces an inner automorphism on* $\Delta/M\}$.
(iv) D *is torsion-free.*

Proof. Let Γ_0 be a cocompact non-arithmetic lattice in H obtained by Gromov - Piatetski-Shapiro construction [GPS]. Recall that Γ_0 is constructed as follows: One starts with two non-commensurable torsion-free arithmetic lattices L_1 and L_2 in H, such that each of the corresponding factor manifolds $W_i = L_i \backslash \mathcal{H}^n$ admits a totaly geodesic hypersurface Z_i $(i = 1,2)$ and Z_1 is isometric to Z_2. Assume that Z_i $(i = 1,2)$ separates W_i into two pieces $X_i \cup Y_i$ (the non-separating case can be treated in a similar way). Then a new manifold W is defined by gluing X_1 with Y_2 along Z_1 (which is isomorphic to Z_2). In particular, W itself has a properly embedded totally geodesic hypersurface Z (isomorphic to Z_1 and Z_2) and so $\pi_1(W) = \pi_1(X_1) *_{\pi_1(Z)} \pi_1(Y_2)$. As explained in [GPS], $\Gamma_0 = \pi_1(W)$ is a non-arithmetic lattice in $H \cong \mathrm{O}_0(n,1)$ which can be supposed to be contained in $\mathrm{SO}_0(n,1)$ (so W is orientable) and $\pi_1(Z)$ is a subgroup (in fact, a lattice) in a conjugate of $\mathrm{SO}_0(n-1,1)$.

Let $\Gamma = \mathrm{Comm}_H(\Gamma_0) = \{g \in H \mid [\Gamma_0 : \Gamma_0 \cap g^{-1}\Gamma_0 g] < \infty\}$ be the commensurability group of Γ_0. Since Γ_0 is non-arithmetic, Margulis Theorem [Mr, Theorem 1, p. 2] implies that Γ is also a lattice, a maximal lattice in $\mathrm{Isom}(\mathcal{H}^n)$.

Let now Λ be a finite index normal subgroup of Γ which is contained in Γ_0. So Λ is the fundamental group of a finite sheeted cover W' of W. Pulling back the hypersurface Z to W', we deduce that W' also admits a properly embedded totally geodesic hypersurface and hence $\Lambda = \Lambda_1 *_{\Lambda_3} \Lambda_2$ or $\Lambda = \Lambda_1 *_{\Lambda_3}$ is a non-trivial free product with amalgam or an HNN-construction.

By [GPS, Corollary 1.7.B] the groups Λ_1 and Λ_2 are Zariski dense in $\mathrm{SO}(n,1)$. By the construction these groups are contained in $\Lambda \leq \Gamma$. Let $\mathcal{O}$ (resp., $\mathcal{O}_i$, for $i = 1,2$) be the minimal ring of definition of Γ (resp., Λ_i) in the sense of [V]. As Λ is finitely generated group, $\mathcal{O}$ is a finitely generated ring. In fact, it is contained in some number field k. This last claim is true for all lattices in $\mathrm{O}(n,1)$ if $n \geq 3$ by the local rigidity of these lattices (see [R, Proposition 6.6, p. 90]). But it also follows (for every n, including $n = 2$) for the Gromov - Piatetski-Shapiro lattices directly from their construction.

Thus $\mathcal{O}$ is a ring of S-integers in some (real) number field k. For $i = 1, 2$, $\mathcal{O}_i$ is a subring of $\mathcal{O}$, so it is the ring of S_i-integers of some subfield k_i of k for a suitable finite set S_i of primes in k_i. We can assume $\Gamma \leq \mathrm{SO}(n,1)(\mathcal{O})$ and $\Lambda_i \leq \mathrm{SO}(n,1)(\mathcal{O}_i)$ for $i = 1, 2$.

Now, the strong approximation for linear groups ([We, Theorem 1.1], [P, Theorem 0.2], see also [LS, Window 9]) implies that for almost every maximal ideal $\mathcal{P}$ of $\mathcal{O}$ with finite quotient field $\mathbb{F}_q = \mathcal{O}/\mathcal{P}$, $q = |\mathcal{O}/\mathcal{P}|$, the image of Γ in $\mathrm{PO}_{n+1}(\mathbb{F}_q)$ contains $\mathrm{P}\Omega_{n+1}(\mathbb{F}_q)$ which is of index at most two in $\mathrm{PSO}_{n+1}(\mathbb{F}_q)$. The same also applies to Λ since Λ is of finite index in Γ. Moreover, Λ_i are also Zariski dense, so a similar statement holds for Λ_i with respect to the ring $\mathcal{O}_i$. By Chebotarev density theorem, there exist infinitely many primes l in $\mathbb{Q}$ which split completely in k (and hence also in k_i). Thus for every prime ideal $\mathcal{P}$ of $\mathcal{O}$ which lies above such l, $\mathcal{O}/\mathcal{P} = \mathcal{O}_i/\mathcal{O}_i \cap \mathcal{P} \cong \mathbb{F}_l$. Moreover, if we replace Λ by the intersection of all the index 2 subgroups in it (note that this intersection is characteristic in Λ and so normal in Γ), we can assume that for infinitely many rational primes l, the images of Λ, Λ_1 and Λ_2 are exactly the groups $\mathrm{P}\Omega_{n+1}(\mathbb{F}_l)$.

Choose such a prime l. We obtain a homomorphism

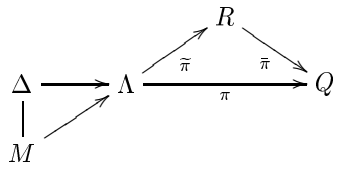
$$\pi : \Lambda \to Q = \mathrm{P}\Omega_{n+1}(\mathbb{F}_l)$$

with $\pi(\Lambda_1) = \pi(\Lambda_2) = Q$ while $T = \pi(\Lambda_3) \leq \mathrm{PSO}_n(\mathbb{F}_l)$ is a proper subgroup of Q (and by choosing l sufficiently large we can assume that the index of T in Q is as large as we want). For later use we observe that if T contains a normal subgroup N of Q then $N = \{e\}$. Indeed, if $n \neq 3$ or $n = 3$ and $Q = \mathrm{P}\Omega_4^-$, Q is a finite simple group and there is nothing to prove. Suppose $n = 3$, $Q = \mathrm{P}\Omega_4^+ \cong \mathrm{P}\Omega_3 \times \mathrm{P}\Omega_3$. The only possibility for $N \neq \{e\}$ is $N = \mathrm{P}\Omega_3$. The image T of Λ_3 in Ω_4 is equal to $\mathrm{Stab}_{\Omega_4}(U)$, the stabilizer of a 3-dimensional subspace U of $V = \mathbb{F}_l^4$ and hence indeed it is isomorphic to Ω_3, but it cannot be a normal subgroup of Ω_4. If it would be, then for every $g \in \Omega_4$, $\mathrm{Stab}_{\Omega_4}(gU) = gTg^{-1} = T$. This implies $gU = U$ (otherwise Ω_3 would preserve a 2-dimensional subspace). Now, this means that U is Ω_4 invariant, which is a contradiction. (We recall that $\mathrm{P}\Omega_4^+ \cong \mathrm{P}\Omega_3 \times \mathrm{P}\Omega_3 \cong \mathrm{PSL}(2) \times \mathrm{PSL}(2)$ via the action of $\mathrm{SL}(2) \times \mathrm{SL}(2)$ on the 2×2 matrices by $(g,h)(A) = gAh^{-1}$, with $\det(A)$ being the invariant quadratic form. But, the action on the 4-dimensional space is irreducible.)

The universal property of free products with amalgam and HNN-constructions implies that there exist a homomorphism

$$\tilde{\pi} : \Lambda \to R = Q *_T Q \ (\text{or } = Q*_T)$$

depending on $\Lambda = \Lambda_1 *_{\Lambda_3} \Lambda_2$ or $\Lambda = \Lambda_1 *_{\Lambda_3}$. The group R is mapped by $\bar{\pi}$ onto Q with a kernel F which is a non-abelian free group by Proposition 2.4(a). Let $M = \mathrm{Ker}(\tilde{\pi})$ and $\Delta = \mathrm{Ker}(\bar{\pi} \circ \tilde{\pi})$. We have:



It is easy to see that $M \lhd \Delta$ and $\Delta/M \cong F$. Also, $\Delta = \Lambda \cap \Gamma(l)$ where the congruence subgroup $\Gamma(l)$ is the kernel of the projection of Γ to $\mathrm{PO}_{n+1}(\mathbb{F}_l)$. Thus

Δ is a finite index normal subgroup of Γ. We therefore have the properties (i) and (ii) of the proposition.

Let now $D = \{\delta \in N_\Gamma(M) \mid \delta$ induces an inner automorphism on $F \cong \Delta/M\}$. Since D contains Δ, we are left only with proving that D is torsion-free. Denote $C = \{\delta \in N_\Gamma(M) \mid \delta|_{\Delta/M} = id\}$. Then $D = \Delta C$. We will show that $C \leq \Gamma(l)$ which will prove that $D \leq \Gamma(l)$. As $\Gamma(l)$ is torsion-free (when l is large enough), we will deduce that D has no torsion.

Let $c \in C$. The element c acts on $\Lambda/M \cong R$ with restriction to $\Delta/M \cong F$ being trivial. Such an automorphism of R is trivial by Proposition 2.4(c), so c acts trivially on Λ/M, i.e. $[c, \Lambda] \subseteq M$. Taking this mod $\Gamma(l)$ we deduce that c centralizes $Q = \mathrm{P}\Omega_{n+1}(\mathbb{F}_l)$ in $\mathrm{PO}_{n+1}(\mathbb{F}_l)$, so c is trivial there by Section 2.3. This implies that $c \in \Gamma(l)$ and Proposition 4.1 is now proved. □

We can now prove the main Theorem from the Introduction:

Let $n \geq 2$, G is a finite group, Γ and D are lattices as in Proposition 4.1. By Theorem 3.1, there exist infinitely many finite index subgroups B of Δ with $N_\Gamma(B)/B \cong G$. Note that $N_H(B) \leq \mathrm{Comm}_H(\Gamma) = \Gamma$, so $N_H(B) = N_\Gamma(B)$ and hence $N_H(B)/B \cong G$. As explained in the introduction $N_H(B)/B \cong \mathrm{Isom}(B\backslash\mathcal{H}^n)$. The theorem is proved.

5. Remarks

5.1. As it was already pointed out in the Introduction although in Theorem 2.3(b) we make use of the results of [L3], which rely on the classification of the finite simple groups, what we really need in this paper does not require the classification. Indeed, Theorem 2.3(b) says that $a_n^{\triangleleft}(F_r) \leq n^{cr\log_2 n}$, where $a_n^{\triangleleft}(F_r)$ denotes the number of index n normal subgroups N of the free group $F = F_r$ and c is a constant. But when we use it in the proof of Theorem 3.1, we need such an upper bound only for those $N \triangleleft F$, for which F/N has a normal subgroup G_0 isomorphic to the fixed finite group G and $(F/N)/G_0$ is nilpotent. In [Mn], Mann showed implicitly that if $\mathcal{S}$ is a family of finite simple groups such that for every $S \in \mathcal{S}$, S has a presentation with at most $c_0(\mathcal{S})\log_2|S|$ relations, then there is a constant $c_1(\mathcal{S})$ such that the number of index n normal subgroups N of F_r with all composition factors of F/N being from $\mathcal{S}$ is bounded by $n^{c_1(\mathcal{S})r\log_2 n}$. Mann's argument is elementary. We could use this result instead of Theorem 2.3(b). Since all the composition factors of our groups are either those of G or abelean, clearly, such a $c_0(\mathcal{S})$ exists.

5.2. The finite volume non-compact case can be treated in an entirely similar way. So, for every finite group G there also exist infinitely many finite volume non-compact n-dimensional hyperbolic manifolds M with $\mathrm{Isom}(M) \cong G$.

5.3. The proof of Proposition 4.1 actually shows that the subgroup D constructed there is contained in a principle congruence subgroup $\Gamma(l)$. Now, if $l > 2$ (which was indeed an assumption), then $\Gamma(l) \subset \mathrm{SO}(n,1)$. So we actually provided infinitely many M's with $\mathrm{Isom}(M) = \mathrm{Isom}^+(M) \cong G$.

P. M. Neumann suggested the following generalization for the problem: Let G be a finite group with a subgroup G^+ of index 2. For every $n \geq 2$ does there exist a compact hyperbolic n-manifold M with an isomorphism $\psi : \mathrm{Isom}(M) \to G$ such that $\psi(\mathrm{Isom}^+(M)) = G^+$?

5.4. Our argument is close in spirit to Greenberg's proof for $n = 2$: while he counts the dimensions of certain subspaces of the Moduli spaces we use counting results on subgroups growth which also allow to detect the existence of the manifolds with the prescribed groups of symmetries. The method of Long and Reid is constructive in a sense.

Our method is not constructive, still the proof says something about its effectiveness. For a fixed n, one has to find Γ_0 – the Gromov - Piatetski-Shapiro lattice in $\mathrm{O}(n,1)$ as in §4. Then, with the notations of §4, we need to find a prime l for which the image of Λ (and also of its subgroups Λ_1 and Λ_2) to $\mathrm{PO}_{n+1}(\mathbb{F}_l)$ contains $\mathrm{P}\Omega_{n+1}(\mathbb{F}_l)$ (all but finitely many primes which split in the ring of definition of Λ have this property). Once this is done the proof gives an explicit estimate for the index of B in Λ for which $N_H(B)/B \cong G$.

One can define for a finite group G, $f(n,G)$ to be the minimal volume of a compact n-dimensional manifold M with $\mathrm{Isom}(M) \cong G$. It may be of interest to give some bounds on $f(n,G)$.

We mention by passing that for $n \geq 4$ and a given $r > 0$, there are only finitely many n-dimensional hyperbolic manifolds of volume at most r [Wa]. In [BGLM], it is shown that the growth rate of the number of manifolds is like $\exp(c(n)r \log r)$. One may ask, whether for *most* of them $\mathrm{Isom}(M) \cong \{e\}$ (our proof gives a partial support to believe that this is the case).

5.5. Another natural question is if a result like Theorem 1.1 will hold if we replace $\mathcal{H}^n$ by other space X, say X is H/K where H is a simple Lie group and K is a maximal compact subgroup of H.

In case $\mathbb{R}\text{-rank}(H) \geq 2$ one cannot expect this to be true. Indeed, by Margulis Theorem [Mr, Theorem 1, p. 2] every lattice Γ in H is arithmetic, moreover, by Serre conjecture (cf. [PR, Section 9.5]) we expect Γ to have the congruence subgroup property (in fact, Serre's conjecture has by now been established for most of the cases). This gives a strong restriction on the finite groups that can appear as quotients of finite index subgroups of such Γ's. For example, their Lie type composition factors should have a bounded Lie rank depending only on H and not on Γ.

An analogue of Theorem 1.1 might hold for the complex hyperbolic spaces $\mathcal{H}\mathbb{C}^n = \mathrm{SU}(n,1)/K$. Unfortunately, very little is currently known here. In [Li], Livne produced an example of a cocompact lattice in $\mathrm{SU}(2,1)$ which is mapped onto a non-abelean free group. This implies that for every finite group G there exists a compact manifold M covered by $\mathcal{H}\mathbb{C}^2$, with $G \subset \mathrm{Isom}(M)$. For $n > 2$ we can not prove even this weak result.

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