DIVISION ALGEBRAS AND NON-COMMENSURABLE ISOSPECTRAL MANIFOLDS

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ABSTRACT. A. Reid [R] showed that if Γ_1 and Γ_2 are arithmetic lattices in $G = \mathrm{PGL}_2(\mathbb{R})$ or in $\mathrm{PGL}_2(\mathbb{C})$ which give rise to isospectral manifolds, then Γ_1 and Γ_2 are commensurable (after conjugation). We show that for $d \geq 3$ and $\mathcal{S} = \mathrm{PGL}_d(\mathbb{R})/\mathrm{PO}_d(\mathbb{R})$, or $\mathcal{S} = \mathrm{PGL}_d(\mathbb{C})/\mathrm{PU}_d(\mathbb{C})$, the situation is quite different: there are arbitrarily large finite families of isospectral non-commensurable compact manifolds covered by $\mathcal{S}$.

The constructions are based on the arithmetic groups obtained from division algebras with the same ramification points but different invariants.

1. INTRODUCTION

Let X be a compact Riemannian manifold and $\Delta = \Delta_X$ its Laplacian. The spectrum of X, spec(X), is the multiset of eigenvalues of Δ acting on $\mathrm{L}^2(X)$. This is a discrete subset of $\mathbb{R}$. If Y is another such object, we say that X and Y are *isospectral* (or, sometimes, cospectral) if spec$(X) =$ spec(Y). The problem of finding isospectral non-isomorphic objects has a long history, starting with Marc Kac's seminal paper [Ka], "Can one hear the shape of a drum?" (see [Go] and the references therein).

The most powerful and general method of obtaining such pairs is due to Sunada [Su]: he showed that if X_0 covers X with a finite Galois group H, i.e. $X = X_0/H$, and if H_1 and H_2 are subgroups of H such that for every conjugacy class $C \subset H$, $|C \cap H_1| = |C \cap H_2|$, then $X_1 = X_0/H_1$ and $X_2 = X_0/H_2$ are isospectral. Of course, one still has to determine whether X_1 and X_2 are isomorphic (for example, X_1 and X_2 will be isomorphic if H_1 is conjugate to H_2, but not only in such a

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case). Sunada's method has been implemented in many situations (e.g. [Br1], [Br2]), and, in particular, has led to the solution of the original problem of M. Kac (see [GWW]).

A common feature to all the isospectral pairs, X_1 and X_2, constructed by Sunada's method, is that they are commensurable, that is, X_1 and X_2 have a common finite cover X_0. So, while X_1 and X_2 are not isomorphic, they are 'virtually isomorphic'.

Sunada implemented his method to obtain non-isomorphic isospectral Riemann surfaces. He gave examples of torsion free cocompact lattices $\Gamma \triangleleft \Gamma_0$ in $G = \mathrm{PGL}_2(\mathbb{R})$ with $\Gamma_0/\Gamma \cong H$ and $H_1 = \Gamma_1/\Gamma$, and $H_2 = \Gamma_2/\Gamma$, as above. His method implies that the G-representations $\mathrm{L}^2(\Gamma_1 \backslash G)$ and $\mathrm{L}^2(\Gamma_2 \backslash G)$ are isomorphic, which demonstrates that $\Gamma_1 \backslash \mathbb{H}^2$ is isospectral to $\Gamma_2 \backslash \mathbb{H}^2$. Here $\mathbb{H}^2 = \mathrm{PGL}_2(\mathbb{R})/K$ is the associated symmetric space, and K is the maximal compact subgroup $\mathrm{PO}_2(\mathbb{R})$ of G.

Vigneras was the first to present examples of isospectral Riemann surfaces in [V], where she used the theory of quaternion algebras. Her examples are also commensurable to each other. In fact, it is still an open problem whether every two isospectral Riemann surfaces are commensurable, or equivalently whether every two cocompact lattices Γ_1 and Γ_2 in $G = \mathrm{PGL}_2(\mathbb{R})$, such that $\Gamma_1 \backslash G/K$ and $\Gamma_2 \backslash G/K$ are isospectral, are commensurable (after conjugation, i.e. for some $g \in G$, $[\Gamma_2 : g^{-1}\Gamma_1 g \cap \Gamma_2] < \infty$). But, Alan Reid [R] showed that if Γ_1 and Γ_2 are such *arithmetic* lattices, then indeed Γ_1 and Γ_2 are commensurable. He also showed a similar result for $G = \mathrm{PGL}_2(\mathbb{C})$.

Our main result shows that the situation is quite different for $G = \mathrm{PGL}_d(\mathbb{R})$ or $\mathrm{PGL}_d(\mathbb{C})$ if $d \geq 3$.

Theorem 1. *Let $F = \mathbb{R}$ or $\mathbb{C}$, $G = \mathrm{PGL}_d(F)$ where $d \geq 3$, $K \leq G$ a maximal compact subgroup, and $\mathcal{S} = G/K$ the associated symmetric space. Then for every $m \in \mathbb{N}$, there exists a family of m torsion free cocompact arithmetic lattices $\{\Gamma_i\}_{i=1,\ldots,m}$ in G, such that $\Gamma_i \backslash \mathcal{S}$ are isospectral and not commensurable.*

Taking $K = \mathrm{PO}_d(\mathbb{R})$ in the real case and $K = \mathrm{PU}_d(\mathbb{C})$ in the complex case, the quotients $\Gamma_i \backslash G/K$ are isospectral compact manifolds

covered by the symmetric space $\mathcal{S} = \mathrm{PGL}_d(F)/K$ which is equal to $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R})$ or $\mathrm{SL}_d(\mathbb{C})/\mathrm{SU}_d(\mathbb{C})$, respectively. The covering map $\mathcal{S} \to \Gamma_i \backslash \mathcal{S}$ is a local isomorphism, since the Γ_i are torsion free.

This result should be compared with Spatzier's construction [Sp] of isospectral locally symmetric spaces of high rank. However, his isospectral examples are always commensurable to each other.

Before outlining the proof, let us make some remarks putting our results in a more general framework (see [Go] for more details). There are isospectral compact manifolds whose universal covers are not isometric, but whether this is possible for locally symmetric manifolds seems to be still open.

Given a symmetric space X, the isospectral families of manifolds covered by X are finite, although they can be as large as one wishes. Isospectral manifolds have the same volume. It will be interesting to bound the size of such isospectral families as a function of the volume. See [GLNP] and the references therein for a count of all manifolds covered by X as a function of the volume.

A remark and a question in a different direction: many of the isospectral manifolds are obtained by Sunada's method (see [Go, Section 4.5]) for a discussion). Our examples are not, but they share with Sunada pairs the property that they come from lattices which are 'representation theoretical isospectral', i.e. $\mathrm{L}^2(\Gamma_1 \backslash G)$ and $\mathrm{L}^2(\Gamma_2 \backslash G)$ are isomorphic as G-modules (see below). In some cases it is known that geometric isospectrality and representation isospectrality are equivalent (see [Pe], [Go] and [D]). But for general simple Lie groups this seems to be an open problem.

We can now outline the proof of Theorem 1. The following proposition is well known, see for example [Pe] and [Go].

Proposition 2. *Let G be a semisimple group, $K \leq G$ a maximal compact subgroup, and $\Gamma_1, \Gamma_2 \leq G$ discrete cocompact subgroups. If*

$$(1) \qquad \qquad \mathrm{L}^2(\Gamma_1 \backslash G) \cong \mathrm{L}^2(\Gamma_2 \backslash G)$$

as (right) G-representations, then $\Gamma_1\backslash G/K$ and $\Gamma_2\backslash G/K$ are isospectral quotients of G/K (indeed, they are even strongly isospectral, i.e. with respect to the higher dimensional Laplacians).

To prove Theorem 1 for $G = \mathrm{PGL}_d(F)$ ($F = \mathbb{R}$ or $\mathbb{C}$), we choose the discrete subgroups Γ_i to be arithmetic lattices of inner forms, as follows. Let k be a global field and $\{k_\nu\}$ the completions with respect to its valuations. Recall that $\mathrm{Br}(k_\nu) \cong \mathbb{Q}/\mathbb{Z}$ for a non-archimedean valuation, while $\mathrm{Br}(\mathbb{R}) \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ and $\mathrm{Br}(\mathbb{C}) = 0$, where $\mathrm{Br}(\cdot)$ denotes the Brauer group of a field. For a division algebra D, $[D] \in \mathrm{Br}(k)$, the value associated to $D\otimes_k k_\nu$ is called the 'local invariant' at ν. By Albert-Brauer-Hasse-Noether theorem, $[D] \mapsto ([D\otimes_k k_\nu])_\nu$ defines an injective map for the Brauer groups

$$(2) \qquad\qquad \mathrm{Br}(k) \longrightarrow \oplus\,\mathrm{Br}(k_\nu),$$

and the image of this map is composed of vectors whose sum is zero. Over local (and thus also global) fields, the degree of a division algebra is equal to its exponent (which is the order of its equivalence class in the Brauer group).

Fix $d > 1$ and let $T = \{\theta_1, \ldots, \theta_t\}$ be a finite non-empty set of non-archimedean valuations of k. Let $a_1, \ldots, a_t, b_1, \ldots, b_t \in \mathbb{N}$ be such that for every $j = 1, \ldots, t$, $0 \leq a_j, b_j < d$, $(a_j, d) = (b_j, d) = 1$, and $\sum a_j \equiv \sum b_j \equiv 0 \pmod{d}$. Let D_1 (resp. D_2) be the unique division algebra of degree d (i.e. dimension d^2) over k which ramifies exactly at T, and whose invariant at k_{θ_j} is a_j/d (resp. b_j/d). Thus D_1 and D_2 split at every $\nu \notin T$. Let G'_i be the algebraic group $D_i^\times/Z^\times$ ($i = 1, 2$) where Z is the center, and $G = \mathrm{PGL}_d$. Since $G'_i(k_\nu)$ splits for every $\nu \notin T$, the groups $G'_1(\mathbb{A}_{T^c})$ and $G'_2(\mathbb{A}_{T^c})$ are equal where $\mathbb{A}_{T^c} = \prod'_{\nu \notin T} k_\nu$. We identify these groups with $G_0 = G(\mathbb{A}_{T^c})$. Here, $\mathbb{A}$ is the ring of adèles and $\prod'$ denotes the restricted product, namely the vectors (x_ν) such that $\nu(x_\nu) \geq 0$ for almost every ν.

A basic result is

Theorem 3. *Let $G_0 = \mathrm{PGL}_d(\mathbb{A}_{T^c})$ be the group defined above, and let $\Delta_i = G'_i(k)$ ($i = 1, 2$), cocompact lattices in G_0. The spaces $\mathrm{L}^2(\Delta_1\backslash G_0)$ and $\mathrm{L}^2(\Delta_2\backslash G_0)$ are isomorphic as G_0-representations.*

The result follows by comparing trace formulas. This was done explicitly for the case where d is prime in [GJ, Theorem 1.12] and in [Bu, Theorem 54]. It seems that the general case is also known to experts, but we were unable to locate a reference. For the sake of completeness we show in Section 4 how the result can be deduced (for arbitrary d) from the global Jacquet-Langlands correspondence, as described in [HT]. (The proof there is given for the characteristic zero case, and we refer the reader to [LSV1, Remark 1.6] for some details about the positive characteristic case).

We should stress that in the current paper, we are using only the characteristic zero case. We prefer to give here the more general formulation, preparing for a subsequent paper [LSV2], in which we use analogous ideas over a local field of positive characteristic to construct isospectral simplicial complexes and isospectral Cayley graphs of some finite simple groups.

One deduces the isospectrality in Theorem 1 by applying strong approximation to Theorem 3. This is done in Section 2. To prove the non-commensurability, we first show (Theorem 9) that two division k-algebras D_1 and D_2 give rise to commensurable arithmetic lattices in $\mathrm{PGL}_d(F)$ iff $D_1(k)$ and $D_2(k)$ are isomorphic or anti-isomorphic (as rings, rather than as k-algebras). We then analyze when two such division algebras are isomorphic (or anti-isomorphic) as rings, and show that we can produce as many examples as we need of non-isomorphic division rings. This is done in Section 3.

Finally, we mention that our work is very much in the spirit of the paper of Vigneras [V], who used a similar idea to find isospectral Riemann surfaces. However, as mentioned above, her examples, which are lattices in $\mathrm{PGL}_2(\mathbb{R})$, are commensurable, as they should be by Reid's theorem. The case $d = 2$ differs from those of $d \geq 3$ in that a global division algebra of degree 2 is determined by its ramification points, while for $d \geq 3$ there are non-isomorphic division algebras with the same ramification points.

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2. Isospectrality

Although for the proof of Theorem 1 we may assume characteristic zero, we are stating and proving the results of this section for positive characteristic as well. Therefore, we obtain Theorem 1 together with:

Theorem 4. *Let F be a local field of positive characteristic, $G = \mathrm{PGL}_d(F)$ where $d \geq 3$, K a maximal compact subgroup and $\mathcal{B}_d(F) = G/K$ the associated Bruhat-Tits building. Then for every $m \in \mathbb{N}$ there exists a family of m torsion free cocompact arithmetic lattices $\{\Gamma_i\}_{i=1,\ldots,m}$ in G, such that the finite complexes $\Gamma_i \backslash \mathcal{B}_d(F)$ are isospectral and not commensurable.*

Notice that Theorems 1 and 4 cover the archimedean local fields, and the non-archimedean local fields of positive characteristic. Our methods do not apply to non-archimedean local fields F of zero characteristic, where there are no cocompact lattices of inner type in $\mathrm{PGL}_d(F)$ ($d \geq 3$). There are cocompact lattices of outer type in these groups; however we leave open the following problem:

Question 5. *Let F be a non-archimedean local field of zero characteristic, $G = \mathrm{PGL}_d(F)$ where $d \geq 3$, K a maximal compact subgroup and $\mathcal{B}_d(F) = G/K$ the associated Bruhat-Tits building. Do there exist torsion free cocompact arithmetic lattices $\{\Gamma_i\}_{i=1,2}$ in G, such that the finite complexes $\Gamma_i \backslash \mathcal{B}_d(F)$ are isospectral and not commensurable?*

Let k be a global field, and assume it has at most one archimedean valuation. Let ν_0 denote the archimedean valuation if char $k = 0$, and a valuation of degree 1 if k is a function field (for $k = \mathbb{F}_q(t)$, ν_0 is either the minus-degree valuation or is determined by a linear prime). Let $F = k_{\nu_0}$. (For $F = \mathbb{R}$ or $F = \mathbb{C}$, we take $k = \mathbb{Q}$ or $k = \mathbb{Q}[i]$, respectively).

Let $\mathcal{V}$ denote the set of all valuations of k other than ν_0. For a valuation $\nu \in \mathcal{V}$, $\mathcal{O}_\nu$ is the ring of integers in the completion k_ν, and P_ν is the valuation ideal.

Let $T \subseteq \mathcal{V}$ be a finite subset, and define D_i and G'_i as in the intro-duction ($i = 1, 2$). Let G denote the algebraic group PGL_d. Notice that $\nu_0 \notin T$, so in particular $G'_i(F) = G(F)$ ($i = 1, 2$). Let $\mathbb{A}_T$ denote the direct product $\prod_{\theta \in T} k_\theta$, so that $\mathbb{A} = \mathbb{A}_T \times \mathbb{A}_{T^c}$ is the ring of adèles over k.

Let $R_0 = \{x \in k \colon \nu(x) \geq 0 \quad \text{for every } \nu \neq \nu_0\}$. (In the cases $k = \mathbb{Q}$ or $k = \mathbb{Q}[i]$, with ν_0 as the archimedean valuation, we obtain $R_0 = \mathbb{Z}$ or $R_0 = \mathbb{Z}[i]$, respectively.) Then, for every $\theta \in T$, we can choose a uniformizer ϖ_θ such that $\nu(\varpi_\theta) = 0$ for every $\nu \notin \{\nu_0, \theta\}$ (so in particular, $\varpi_\theta \in R_0$). In this situation,

$$(3) \qquad R_{0,T} = \{x \in k \colon \nu(x) \geq 0 \quad \text{for every } \nu \notin T \cup \{\nu_0\}\}$$

is equal to $R_0[\{\varpi_\theta^{-1}\}_{\theta \in T}]$. Fix $i \in \{1, 2\}$. Since division algebras over global fields are cyclic, there is a Galois field extensions k_i/k of dimension d contained in $D_i(k)$. If $\phi \in \mathrm{Gal}(k_i/k)$ is a generator, there is an element $b_i \in k^\times$ such that $D_i = k_i[z \mid zaz^{-1} = \phi(a), z^d = b_i]$. Taking an integral basis, one finds an order $O_i \subseteq k_i$ which is a cyclic extension (of rings) of $R_{0,T}$. The constant b_i can be chosen to be in the multi-plicative group generated by the uniformizers ϖ_θ. Thus, b_i is invertible in $R_{0,T}$. So $O_i[z]$ (with the relations as above) is an Azumaya algebra of rank d over the center $R_{0,T}$, and $D_i = k \otimes_{R_{0,T}} O_i[z]$. The group of invertible elements in $O_i[z]$ (again with the above relations), modulo its center, is $G'_i(R_{0,T})$.

If an algebraic group G' is defined over a ring R and $0 \neq I \triangleleft R$ is an ideal, we have the principal congruence subgroups

$$(4) \qquad G'(R, I) = \mathrm{Ker}(G'(R) \rightarrow G'(R/I)).$$

The congruence subgroups $G'_i(R_{0,T}, I)$ are discrete in $G(F)$, and co-compact if $T \neq \emptyset$ [PlR, Theorem 5.5].

Let $0 \neq I \triangleleft R_{0,T}$. Define a function $r \colon \mathcal{V} - (T \cup \{\nu_0\}) \rightarrow \mathbb{N} \cup \{0\}$ by setting

$$(5) \qquad r_\nu = \min \{\nu(a) \colon a \in I\},$$

and let

$$(6) \qquad U^{(r)} = \prod_{\nu \in \mathcal{V} - (T \cup \{\nu_0\})} G(\mathcal{O}_\nu, P_\nu^{r_\nu}),$$

an open compact subgroup of $G(\mathbb{A}_{T^c - \{\nu_0\}})$. For almost every ν, $r_\nu = 0$, and then $P_\nu^0 = \mathcal{O}_\nu$ and $G(\mathcal{O}_\nu, P_\nu^{r_\nu}) = G(\mathcal{O}_\nu)$ by definition. Let

$$(7) \qquad U_i = U^{(r)} G_i'(\mathbb{A}_T),$$

an open compact subgroup of $G_i'(\mathbb{A})$.

Lemma 6. *The quotient $G_i'(\mathbb{A})/(G_i'(k)G(F)U_i)$ is a d-torsion finite abelian group, which is independent of i.*

Proof. Recall the reduced norm map $D_i(k)^\times \to k^\times$, which coincides with the determinant over splitting fields of k. Let $\mathcal{G}_i$ denote the simply connected cover of G_i', i.e. $\mathcal{G}_i$ is the subgroup of elements of reduced norm 1 in $D_i^\times$. For every field $k_1 \supseteq k$, the image of the covering map $\Psi : \mathcal{G}_i(k_1) \to G_i'(k_1)$ is co-abelian in $G_i'(k_1)$. The strong approximation theorem [PlR] (see also [LSV1, Subsection 3.2]) applies to $\mathcal{G}_i$, and so $\mathcal{G}_i(k)\mathcal{G}_i(F)U = \mathcal{G}_i(\mathbb{A})$ where

$$U = \prod_{\nu \in \mathcal{V} - (T \cup \{\nu_0\})} \mathcal{G}_i(\mathcal{O}_\nu, P_\nu^{r_\nu}) \times \mathcal{G}_i(\mathbb{A}_T),$$

and $\mathcal{G}_i(\mathcal{O}_{\nu_0})U$ is open compact in $\mathcal{G}_i(\mathbb{A})$.

The reduced norms $G_i'(k_\nu) \to k_\nu^\times / k_\nu^{\times d}$ is onto whenever G_i' splits at k_ν (as $G_i'(k_\nu) = \mathrm{PGL}_d(k_\nu)$), and also whenever k_ν is non-archimedean (since over local fields the norm map in a division algebra is onto, [Pi, Chapter 17]). In the second case, the restriction to $G_i'(\mathcal{O}_\nu) \to \mathcal{O}_\nu^\times / \mathcal{O}_\nu^{\times d}$ is also onto. Thus, the norm map $G_i'(\mathbb{A}) \to \mathbb{A}^\times / \mathbb{A}^{\times d}$ is onto, and induces an isomorphism $\Phi : G_i'(\mathbb{A})/\Psi(\mathcal{G}_i(\mathbb{A})) \to \mathbb{A}^\times / \mathbb{A}^{\times d}$.

Since Ψ maps $\mathcal{G}_i(k)$, $\mathcal{G}_i(F)$ and U into $G_i'(k)$, $G(F)$ and U_i, respectively, we have $\Psi(\mathcal{G}_i(\mathbb{A})) = \Psi(\mathcal{G}_i(k)\mathcal{G}_i(F)U) \subseteq G_i'(k)G(F)U_i$. We need to compute the image of $G_i'(k)G(F)U_i$ under Φ. For a valuation ring $\mathcal{O}$ with maximal ideal P and $r \geq 0$, let $\mathcal{O}^{(r)}$ denote the subgroup $\{a \in \mathcal{O}^\times : a \equiv 1 \pmod{P^r}\}$. Now, modulo d powers, the determinant maps $G(F)$ to $F^\times$ and $G(\mathcal{O}_\nu, P_\nu^{r_\nu})$ to $\mathcal{O}_\nu^{(r_\nu)}$. It follows

that Φ takes $G_i'(k)$ to $k^\times \mathbb{A}^{\times d}$, $G(F)$ to $F^\times \mathbb{A}^{\times d}$, and U_i to the product $(\prod_{\nu \in \mathcal{V}-T} \mathcal{O}_\nu^{(r_\nu)} \times \prod_{\theta \in T} k_\theta^\times)\mathbb{A}^{\times d}$. We can now compute that

$$
\begin{aligned}
G_i'(\mathbb{A})/(G_i'(k)G(F)U_i) & \cong \mathbb{A}^\times/(k^\times F^\times \mathbb{A}^{\times d} \prod_{\nu \in \mathcal{V}-(T \cup \{\nu_0\})} \mathcal{O}_\nu^{(r_\nu)} \prod_{\theta \in T} k_\theta^\times) \\
& \cong \mathbb{A}_{T^c}^\times/(k^\times F^\times \mathbb{A}_{T^c}^{\times d} \prod_{\nu \in \mathcal{V}-(T \cup \{\nu_0\})} \mathcal{O}_\nu^{(r_\nu)}).
\end{aligned}
$$

Recall that $k_\nu^\times \cong \mathbb{Z} \times \mathcal{O}_\nu^\times$, where the $\mathbb{Z}$ summand is the value group. Dividing the numerator and denominator in the last expression by $\prod_{\nu \in \mathcal{V}-(T \cup \{\nu_0\})} \mathcal{O}_\nu^{\times d}$, the quotient is

$$
{\prod}' k_\nu^\times/(k^\times {\prod}' \mathcal{O}_\nu^{(r_\nu)} k_\nu^{\times d}) \cong
$$

$$
\begin{aligned}
& \cong {\prod}'(\mathbb{Z} \times \mathcal{O}_\nu^\times)/(k^\times {\prod}'(d\mathbb{Z} \times \mathcal{O}_\nu^{(r_\nu)} \mathcal{O}_\nu^\times)) \\
& \cong {\prod}'(\mathbb{Z} \times \mathcal{O}_\nu^\times/\mathcal{O}_\nu^{\times d})/(k^\times {\prod}'(d\mathbb{Z} \times \mathcal{O}_\nu^{(r_\nu)} \mathcal{O}_\nu^\times/\mathcal{O}_\nu^{\times d})) \\
& \cong ({\prod}' \mathbb{Z} \prod \mathcal{O}_\nu^\times/\mathcal{O}_\nu^{\times d})/(k^\times {\prod}' d\mathbb{Z} \prod \mathcal{O}_\nu^{(r_\nu)} \mathcal{O}_\nu^\times/\mathcal{O}_\nu^{\times d}) \\
& \cong ({\prod}'(\mathbb{Z}/d \times \mathcal{O}_\nu^\times/\mathcal{O}_\nu^{(r_\nu)} \mathcal{O}_\nu^{\times d}))/k^\times,
\end{aligned}
$$

where all products are over $\nu \in \mathcal{V} - (T \cup \{\nu_0\})$ and $\prod'$ denotes the restricted product (such that $\nu(a_\nu) \geq 0$ almost always). The quotients $\mathcal{O}_\nu^\times/\mathcal{O}_\nu^{(r_\nu)}$ are always finite, and $r_\nu = 0$ for almost all ν, hence $\mathcal{O}_\nu^\times/\mathcal{O}_\nu^{(r_\nu)} \mathcal{O}_\nu^{\times d} = 0$ for all but finitely many valuation. In the last expression, $k^\times$ embeds in the ν component by $a \mapsto (\nu(a), a\varpi_\nu^{-\nu(a)} \mathcal{O}_\nu^{(r_\nu)} \mathcal{O}_\nu^{\times d})$, and in particular the components with $\mathcal{O}_\nu^\times/\mathcal{O}_\nu^{(r_\nu)} \mathcal{O}_\nu^{\times d} = 0$ vanish in the quotient. Therefore we have a finite product of finite groups, and the result follows. $\qquad\square$

Proposition 7. *Let $0 \neq I \lhd R_{0,T}$. There is an isomorphism*

$$
\text{(8)} \qquad \mathrm{L}^2(G_1'(R_{0,T}, I)\backslash G(F)) \cong \mathrm{L}^2(G_2'(R_{0,T}, I)\backslash G(F)),
$$

as representations of $G(F) = \mathrm{PGL}_d(F)$.

Proof. We prove this statement by transferring it to its adèlic analogue. Let U_i be the group defined in Equation (7). Then $G(\mathcal{O}_{\nu_0})U_i$ is open compact in $G_i'(\mathbb{A})$, and

$$(9) \qquad G_i'(R_{0,T}, I) = G_i'(k) \cap G(F)U_i,$$

the intersection taken in $G_i'(\mathbb{A})$ and then projected to the $G(F)$ component. Therefore $G_i'(R_{0,T}, I)\backslash G(F)$ is related to $G_i'(k)\backslash G_i'(\mathbb{A})/U_i$.

Since $G_0 = G(\mathbb{A}_{T^c})$ by definition, $G_i'(\mathbb{A})/U_i = G_i'(\mathbb{A}_{T^c})/U^{(r)} = G_0/U^{(r)}$, and

$$G_i'(k)\backslash G_i'(\mathbb{A})/U_i \cong G_i'(k)\backslash G_0/U^{(r)}.$$

Consequently,

$$\mathrm{L}^2(G_i'(k)\backslash G_0)^{U^{(r)}} \cong \mathrm{L}^2(G_i'(k)\backslash G_0/U^{(r)}) \cong \mathrm{L}^2(G_i'(k)\backslash G_i'(\mathbb{A})/U_i).$$

Now let $H_i = G_i'(k)G(F)U_i$. From Equation (9) it follows that

$$G_i'(R_{0,T}, I)\backslash G(F) \cong G_i'(k)\backslash H_i/U_i.$$

By the lemma, $C = G_i'(\mathbb{A})/H_i$ is finite (d-torsion) abelian, which is independent of i. Since $G(F)$ commutes with U_i and is contained in H_i, we have the decomposition

$$\mathrm{L}^2(G_i'(k)\backslash G_i'(\mathbb{A})/U_i) \cong \oplus_{|C|} \mathrm{L}^2(G_i'(k)\backslash H_i/U_i)$$

as $G(F)$-representations. Thus

$$\mathrm{L}^2(G_i'(k)\backslash G_0)^{U^{(r)}} \cong \oplus_{|C|}\mathrm{L}^2(G_i'(k)\backslash H_i/U_i) \cong \oplus_{|C|}\mathrm{L}^2(G_i'(R_{0,T}, I)\backslash G(F)).$$

By Theorem 3, $\mathrm{L}^2(G_1'(k)\backslash G_0) \cong \mathrm{L}^2(G_2'(k)\backslash G_0)$ as representation spaces, so the result follows. $\qquad \square$

Now, let $0 \neq I \lhd R_{0,T}$, and take $\Gamma_i = G_i'(R_{0,T}, I)$. By Proposition 2 and Proposition 7, we see that $\Gamma_1\backslash G(F)/K$ and $\Gamma_2\backslash G(F)/K$ are isospectral, where K is a fixed maximal compact subgroup of $G(F)$. If I is small enough, then Γ_1 and Γ_2 are torsion free, and so the projection $G(F)/K \to \Gamma_i\backslash G(F)/K$ is a local isomorphism.

Remark 8. *When T is fixed, the choice of $U^{(r)}$ determines the family of quotients. When $U^{(r)'} \subseteq U^{(r)}$, the quotients corresponding to $U^{(r)}$ are covered by those corresponding to $U^{(r)'}$. In this sense, we find not*

only families of isospectral structures, but infinite "inverse limits" of such families.

3. Non-commensurability

As in the previous section, we state and prove the results of this section for arbitrary characteristic, proving Theorems 1 and 4 together.

To show that $\Gamma_i \backslash G(F)/K$ are not commensurable, we need the following theorem. Recall that the maximal compact subgroup K of $\mathrm{PGL}_d(F)$ is taken to be $K = \mathrm{PO}_d(\mathbb{R})$ if $k = \mathbb{R}$, $K = \mathrm{PU}_d(\mathbb{C})$ if $k = \mathbb{C}$, and $K = \mathrm{PGL}_d(\mathcal{O})$ (where $\mathcal{O}$ is the ring of integers of $F = k_{\nu_0}$) if $\operatorname{char} k > 0$. Let D_1 and D_2 be two central division algebras over k, as in the previous section, and G_i' the corresponding algebraic groups. Let T_1 and T_2 be the ramification points of D_1 and D_2 (this time not necessarily equal), and let R_{0,T_i} be the subrings of k defined in Equation (3) (for T_i rather than T). Note that if $\sigma : D_1(k) \rightarrow D_2(k)$ is an isomorphism or anti-isomorphism of rings, then σ induces an automorphism of the center k which acts on the set of (non-archimedean) valuations of k. This maps T_1 to T_2. For Theorem 1 we only need the implication (1) $\implies$ (3) of the next theorem, however we give the full picture, which seems to be of independent interest.

Theorem 9. *Let $\mathcal{S} = G(F)/K$ be the building or symmetric space corresponding to $G(F)$. The following are equivalent:*

1. There exists finite index torsion-free subgroups Ω_1 of $G_1'(R_{0,T_1})$ and Ω_2 of $G_2'(R_{0,T_2})$ such that the manifolds or complexes $\Omega_1 \backslash \mathcal{S}$ and $\Omega_2 \backslash \mathcal{S}$ are isomorphic.

2. For every finite index torsion-free subgroup $\Omega_1 \leq G_1'(R_{0,T_1})$, there exists a finite index torsion-free subgroup $\Omega_2 \leq G_2'(R_{0,T_2})$ such that $\Omega_1 \backslash \mathcal{S}$ and $\Omega_2 \backslash \mathcal{S}$ are isomorphic.

3. There is a ring isomorphism or anti-isomorphism $\sigma : D_1(k) \rightarrow D_2(k)$, such that the restriction of σ to the center k fixes ν_0 and maps T_1 to T_2.

Proof. The second assertion trivially implies the first. We prove (1) $\implies$ (3). Since $\mathcal{S}$ is the universal cover of $\Omega_i \backslash \mathcal{S}$ and Ω_i is the fundamental group of $\Omega_i \backslash \mathcal{S}$, the given isomorphism lifts to an automorphism

$\psi : \mathcal{S} \rightarrow \mathcal{S}$ such that $\psi \Omega_1 \psi^{-1} = \Omega_2$ in $\mathrm{Isom}(\mathcal{S})$, the group of isometries of $\mathcal{S}$.

By Cartan's theorem [H, Chapter IV] in characteristic zero and Tits' theorem [T, Corollaries 5.9 and 5.10] in positive characteristic, $\mathrm{Isom}(\mathcal{S}) =$ the continuous automorphisms of $\mathrm{PGL}_d(F)$.

It is well known that $\mathrm{Aut}(\mathrm{PGL}_d(F))$ is a semidirect product of the subgroup generated by (continuous) field automorphisms and inner automorphisms, by the subgroup of order 2 generated by the 'diagram automorphism' $a \mapsto (a^{\mathrm{t}})^{-1}$. Replacing D_2 with D_2^{op} if necessary, we may ignore the last option. Therefore ψ extends to an automorphism of the matrix algebra $\mathrm{M}_d(F)$.

Recall that $\Omega_i \subseteq G_i'(R_{0,T_i}) \subseteq G_i'(k) \subseteq \mathrm{PGL}_d(F)$. The quotient $\mathrm{PGL}_d(F)/\mathrm{PSL}_d(F)$ is abelian torsion, and Ω_i are finitely generated (as the fundamental groups of compact objects). Therefore, we can replace the Ω_i by finite index subgroups, which will be contained in $\mathrm{PSL}_d(F)$, still satisfying the conjugation property above. Moreover, as $\mathrm{Aut}(\mathrm{PGL}_d(F)) = \mathrm{Aut}(\mathrm{PSL}_d(F)) = \mathrm{Aut}(\mathrm{SL}_d(F)) = \mathrm{Aut}(\mathrm{GL}_d(F))$, we can lift Ω_i to subgroups $\Omega_i^{(1)}$ of $\mathrm{SL}_d(F)$, with $\psi \in \mathrm{Aut}(\mathrm{GL}_d(F))$ mapping $\Omega_1^{(1)}$ to $\Omega_2^{(1)}$.

Let H_i' denote the group of elements of norm one in $D_i^{\times}$; in particular, $H_i'(F) = \mathrm{SL}_d(F)$. Then $\Omega_i^{(1)} \subseteq H_i'(R_{0,T_i}) \subseteq H_i'(k) \subseteq \mathrm{SL}_d(F)$. Let N_i be the commensurator of $\Omega_i^{(1)}$ in $\mathrm{GL}_d(F)$. We claim that N_i is $D_i^{\times}(k)$ times the scalar matrices. Indeed, an element $a \in N_i$ conjugates a finite index subgroup A of $\Omega_i^{(1)}$ into $\Omega_i^{(1)}$. Since $\Omega_i^{(1)}$ is Zariski dense, so is A. Therefore, the $\bar{F}$-subalgebra of $\mathrm{M}_d(\bar{F})$ generated by A is equal to $\mathrm{M}_d(\bar{F})$ (where $\bar{F}$ is the algebraic closure). This algebra is the scalar extension of the k-subalgebra generated by A, which is therefore a d^2-dimensional subalgebra of $D_i(k)$ (as $A \subseteq D_i^{\times}(k)$) and thus equal to $D_i(k)$. Therefore, conjugation by a is an automorphism of $D_i(k)$. But by Skolem-Noether there is an element of $D_i^{\times}(k)$ inducing the same automorphism, so up to scalar matrices, $a \in D_i^{\times}(k)$.

Since ψ maps $\Omega_1^{(1)}$ to $\Omega_2^{(1)}$, it maps N_1 to N_2. It also maps the commutator subgroup of N_1, which is the commutator subgroup of $D_1^{\times}(k)$, to that of $D_2^{\times}(k)$. It follows that ψ maps the division subring generated by commutators of $D_1(k)$ to the division subring generated

by commutators of $D_2(k)$, but by Lemma 10 below, these are $D_1(k)$ and $D_2(k)$, respectively. Hence we proved that ψ extends to an isomorphism (of rings) from $D_1(k)$ to $D_2(k)$.

The restriction to k of this isomorphism preserves ν_0 since it is induced by a continuous automorphism of $F = k_{\nu_0}$. Also, it takes the ramification points T_1 of D_1 to the ramification points T_2 of D_2 (note that the opposite algebra has the same set of ramification points).

We remark that in characteristic zero we could bypass the argument with the commensurator, by observing that up to complex conjugation, ψ must be an algebraic map and hence maps $G_1'(k)$ to $G_2'(k)$.

To show that (3) $\implies$ (2), we first assume $D_2(k)$ is isomorphic to $D_1(k)$. By tensoring with F, we obtain an automorphism of $D_i(F) = \mathrm{M}_d(F)$, which is continuous by assumption, and so induces an automorphism of $\mathcal{S}$. Moreover it maps the set T_1 to T_2, inducing an isomorphism from the subring R_{0,T_1} to R_{0,T_2}. This extends to an isomorphism from $G_1'(R_{0,T_1})$ to $G_2'(R_{0,T_2})$. Now, if $\Omega_1 \leq G_1'(R_{0,T_1}) \leq \mathrm{GL}_d(F)$ is any finite index torsion-free subgroup, then its image under the automorphism, Ω_2, is a finite index torsion-free subgroup of $G_2'(R_{0,T_2})$, with $\Omega_1 \backslash \mathcal{S}$ isomorphic to $\Omega_2 \backslash \mathcal{S}$.

If $D_2(k)$ is isomorphic to $D_1(k)^{\mathrm{op}}$, then by tensoring with F we obtain an anti-automorphism of $\mathrm{M}_d(F)$, and the argument follows throughout. $\square$

Lemma 10. *Let D be a non-commutative division ring. Then the smallest division subring of D containing all multiplicative commutators is D itself.*

Proof. First note that a division subring of D containing all commutators is invariant under conjugations (as $xyx^{-1} = [x, y]y$). By the Cartan-Brauer-Hua theorem [J, p. 186] such a subring is either central or equal to D.

Assume all commutators in D are central, and let $x \in D$ be a non-central element. Then the field generated by x over $\mathrm{Cent}(D)$ is invariant and therefore equal to D, so D is commutative, contrary to assumption. $\square$

Theorem 9 gives the condition for two manifolds to be commensurable when they are defined from two division k-algebras. To complete the proof of Theorem 1, we need to construct arbitrarily many division algebras with the same set of ramification points, which are not isomorphic as rings.

The classification of division algebras over k by their invariants is crucial for that matter, so we add some (well known) details on this topic (see [Pi, Chapsters 17–18]). Let D be a division algebra, with center k, a global field. Let T denote the set of valuations θ of k such that $D \otimes k_\theta$ is not a matrix algebra, where k_θ is the completion with respect to θ. Then T is a finite set. Recall that by Grunwald-Wang theorem [AT, Chapter 10], there exists a cyclic field extension k_1 of dimension d over k, which is unramified with respect to every valuation $\theta \in T$. From Albert-Brauer-Hasse-Noether theorem it then follows that D contains an isomorphic copy of k_1.

Let ν be a non-archimedean valuation of k. Let ϕ denote the Frobenius automorphism of k_1/k (i.e. the generator of $\mathrm{Gal}(k_1/k)$ which induces the Frobenius automorphism on the residue field of k_1 over that of k). By Skolem-Noether theorem, there is an element $z \in D$ such that conjugation by z induces ϕ on k_1. Then z^d commutes with $k_1[z] = D$, so z^d is central, i.e. $z^d \in k$. $D = k_1[z \mid zaz^{-1} = \phi(a), z^d = b \in k]$. If $k_2 \subseteq D$ is an isomorphic copy of k_1, then again by Skolem-Noether there is an element $w \in D$ such that $wk_1w^{-1} = k_2$. Now wzw^{-1} induces ϕ (or more precisely its isomorphic image) on k_2, and the d-power of this element is $wz^dw^{-1} = z^d$. Of course, z is not unique; if z' is another element inducing ϕ on k_1, then $z'z^{-1}$ commutes with k_1. So $z' = az$ for some $a \in k_1$, since k_1 is its own centralizer. Then, $(z')^d = \mathrm{N}_{k_1/k}(a)z^d$. However, $\nu(\mathrm{N}_{k_1/k}(a))$ is divisible by d since k_1/k is unramified, so $\nu(z^d)$ (mod d) is well defined. This number, divided by d, is called the local invariant of D at ν (the invariant is viewed as an element of $\mathbb{Q}/\mathbb{Z}$). The completion k_ν is a splitting field of D iff the invariant is zero.

We remark that D is determined (up to isomorphism as a k-algebra) by its local invariants. For every finitely supported function $f : \mathcal{V} \cup \{\nu_0\} \to \{0, 1/d, \ldots, (d-1)/d\} \subset \mathbb{Q}/\mathbb{Z}$ such that $\sum f(\nu) \equiv 0 \pmod 1$

and $f(\nu_0) = 0$, there is a central simple k-algebra of degree d which is matrices over $F = k_{\nu_0}$ and with $f(\nu)$ as the local invariant at ν.

Proposition 11. *Let D and D' be k-central division algebras which are ramified at places $(\nu_1, \ldots, \nu_t)$ with invariants $(a_1, \ldots, a_t)$ and $(\nu_1', \ldots, \nu_{t'}')$ with invariants $(a_1', \ldots, a_{t'}')$, respectively. Then D and D' are isomorphic as rings if and only if $t = t'$ and there exists a k-automorphism σ such that (after reordering) $\nu_i' = \nu_i \circ \sigma$, and $a_i = a_i'$.*

Moreover, if $\bar\sigma$ is a fixed isomorphism $D \to D'$ extending σ, then every isomorphism from D to D' is equal to $\bar\sigma$ composed with an inner automorphism.

Proof. Assume D' and D are isomorphic as rings. The ring isomorphism induces an isomorphism of the centers, which is an automorphism σ of k. Let k_1 denote an cyclic maximal subfield of D as before. If conjugation by z induces ϕ on k_1, then conjugation by its image z' induces an automorphism ϕ' on the image k_1' of k_1, and of course $(z')^d = \sigma(z^d)$. Therefore, the invariant of D' at $\nu \circ \sigma^{-1}$ is $(\nu \circ \sigma^{-1})((z')^d) = \nu(z^d)$, namely the invariant of D at the valuation ν. In the other direction, the given σ extends to an isomorphism of the rings translating the ramification data of D to that of D'.

Finally, if two isomorphisms $\psi, \psi' : D \to D'$ agree on the center of D, then $\psi^{-1} \circ \psi'$ is an automorphism of D as a central division algebra. By Skolem-Noether, it is an inner automorphism. $\square$

Corollary 12. *Let D_1 and D_2 be k-division algebras of degree d with the same ramification points, such that for every automorphism of k, the vector $a = (a_1, \ldots, a_t)$ of ramified invariants of D_1 is not permuted to the vector of ramified invariants $b = (b_1, \ldots, b_t)$ of D_2, nor to $-b$.*

Let $\Omega_i \subseteq G_i'(R_{0,T})$ be finite index subgroups (where G_i' are defined as before). Then $\Omega_1 \backslash \mathcal{S}$ and $\Omega_2 \backslash \mathcal{S}$ are not commensurable (where $\mathcal{S} = G(F)/K$). Equivalently, Ω_1 and Ω_2 are not commensurable (after conjugation) as subgroups of $\mathrm{Aut}(\mathrm{PGL}_d(F))$ (and in particular not as subgroups of $\mathrm{PGL}_d(F)$).

Proof. Obviously the two claims are equivalent, for if $\Omega_0 = \Omega_1 \cap g\Omega_2 g^{-1}$ is a finite index subgroup, then $\Omega_0\backslash\mathcal{S} \cong g\Omega_0 g^{-1}\backslash\mathcal{S}$ is a joint cover of finite index, and vice versa.

If such a joint cover exists, then D_2 must be isomorphic or anti-isomorphic to D_1 as rings by Theorem 9, and this is prohibited by the previous proposition. $\qquad\square$

Before we continue, we first prove a Lemma.

Lemma 13. *A global field k has infinitely many orbits of valuations under the action of its automorphism group.*

Proof. Indeed, in zero characteristic there are infinitely many valuations (covering the p-adic valuations for p the rational primes). Since the Galois group over $\mathbb{Q}$ is finite, there are finitely many valuations in each orbit.

In the case of a function field, k is finite over $\mathbb{F}_q(t)$ where $\mathbb{F}_q$ is the maximal finite subfield. The degree of a valuation is the dimension of its residue field over $\mathbb{F}_q$. If $p \in \mathbb{F}_q[t]$ is a prime, then the degree of the p-adic valuation on $\mathbb{F}_q(t)$ is equal to the degree of p (as a polynomial), and serves as a lower bound for the degree of an extension of this valuation to k. Therefore, the degrees of valuations of k are unbounded. However, the automorphism group, when acting on the valuations, does not change the degree. $\qquad\square$

We can now finish the proof of Theorems 1 and 4. Given the number m, let t be an even integer such that $2^t/t \geq 2m$. Choose a set $T \subseteq \mathcal{V}$ composed of t non-archimedean valuations of k, which belong to different orbits of the automorphism group of k. For every $e = (e_1, \ldots, e_t)$ with e_i prime to d which sum to zero modulo d, let D_e be the division algebra over k with Brauer-Hasse-Noether invariants $\frac{e_1}{d}, \ldots, \frac{e_t}{d}$ in the valuations of T, which is unramified outside of T. Since t is even we can choose half of the invariants to be 1 and half to be -1 ($1 \not\equiv -1$ as $d > 2$). There are at least $\binom{t}{t/2} \geq 2^t/t \geq 2m$ options. In each case, the sum of e_i is zero modulo d. Furthermore we set $e_1 = 1$, so there are at least m vectors e which are not only different, but also not opposite to each other.

Since our valuations in T are from different orbits and we have chosen distinct invariants, there is no automorphism σ of k which permutes the ramified valuations and their invariants. This guarantees that the D_e are not isomorphic to each other (nor their opposites) as rings. Having chosen the division algebras, take $U = U^{(r)}$ an open compact subgroup of $G(\mathbb{A}_{T^c - \{\nu_0\}})$ as in Equation (6). Let G'_e denote the k-algebraic group $D_e^\times / Z^\times$ (where Z is the center), and take $\Gamma_e = G'_e(R_{0,T}, I) = G'_e(k) \cap G(F) U G'_e(\mathbb{A}_T)$ (see Equation (9)). We saw above that $X_e = \Gamma_e \backslash G(F) / K$ are all isospectral, where K is a fixed maximal compact subgroup of $G(F)$. As a congruence subgroup, Γ_e is of finite index in $G'_e(R_{0,T})$, so the X_e are non-commensurable by Theorem 9.

4. Proof of Theorem 3

Let D be a division algebra of degree d over the global field k, and let G' and G denote the algebraic groups $D^\times / Z^\times$ and PGL_d, respectively. Let T denote the set of ramification points of D, and assume as before that $D \otimes k_\theta$ is a division algebra for each $\theta \in T$ (so the invariants of D at T are prime to d). Now let $\theta \in T$. The *local Jacquet-Langlands correspondence* is a bijective correspondence, which maps every irreducible, unitary representation ρ' of $G'(k_\theta)$ to an irreducible, unitary square-integrable representation $\rho = \mathrm{JL}_\theta(\rho')$ of $G(k_\theta)$ (see [HT, p. 29] or [LSV1] for details). Every such representation of $G(k_\theta)$ is either super-cuspidal (i.e. the matrix coefficients are compactly supported modulo the center), or is induced from a super-cuspidal representation of a subgroup of smaller rank: $\rho = \mathrm{Sp}_s(\psi)$ where ψ is a super-cuspidal representation of $\mathrm{PGL}_{d/s}(k_\theta)$, and Sp_s is the construction of a 'special representation'. See [HT] or [LSV1, Subsection 2.5]. We note that $\mathrm{Sp}_1(\psi) = \psi$.

Let $\mathbb{A}$ denote the ring of adèles over k. The automorphic representations of $G(\mathbb{A})$ are the irreducible sub-representations of $\mathrm{L}^2(G(k) \backslash G(\mathbb{A}))$ (as right $G(\mathbb{A})$-modules), and likewise for $G'(\mathbb{A})$. Such a representation decomposes as a tensor product of *local components* π_ν ($\pi = \otimes \pi_\nu$), which are representations of $G(k_\nu)$ (respectively $G'(k_\nu)$) for the various

valuations ν of k. Moreover, π is determined by all but finitely many local components (this is 'strong multiplicity one' for PGL_d.) .

The global Jacquet-Langlands correspondence maps an irreducible automorphic representation $\pi' = \otimes \pi'_\nu$ of $G'(\mathbb{A})$ to an irreducible automorphic representation $\pi = \mathrm{JL}(\pi') = \otimes \mathrm{JL}(\pi')_\nu$ of $G(\mathbb{A})$ which occurs in the discrete spectrum (see [HT, p. 195]). If $\nu \notin T$, then $\mathrm{JL}(\pi')_\nu \cong \pi'_\nu$. For the valuations $\theta \in T$, the local correspondence takes $(\pi')_\theta$ to $\mathrm{JL}_\theta((\pi')_\theta) = \mathrm{Sp}_s(\psi)$ for a suitable $s \mid d$ and ψ super-cuspidal. Now in the global map, the local component $\mathrm{JL}(\pi')_\theta$ is isomorphic to either $\mathrm{Sp}_s(\psi)$ or to a certain representation which we denote by $\mathrm{C}_s(\psi)$, which is never square-integrable. In both cases, ψ and s are uniquely determined by π'_θ.

By part 3 of [HT, Theorem VI.1.1], the image of the global JL consists of the automorphic representations π in the discrete spectrum, such that for each $\theta \in T$, the local component π_θ is either of the form $\mathrm{Sp}_s(\psi)$ or $\mathrm{C}_s(\psi)$, a condition which depends on T but is independent of the invariants of D. Therefore the image of JL only depends on the set T, and not on the invariants.

Let $G_0 = G'(\mathbb{A}_{T^c})$ and $\Delta = G'(k) \subseteq G_0$ (with the diagonal embedding). Recall that $G_0 = G(\mathbb{A}_{T^c})$, since $G'(k_\nu) = G(k_\nu)$ for every $\nu \notin T$. First, we show that the only finite-dimensional sub-representation of $\mathrm{L}^2(\Delta \backslash G_0)$ is the trivial one. It is well known that all finite-dimensional sub-representations are one-dimensional, and so they come from the quotient $\mathrm{PGL}_d(\mathbb{A}_{T^c})/\mathrm{PSL}_d(\mathbb{A}_{T^c})$. However, since T is not empty, $\mathrm{PGL}_d(k) \cdot \mathrm{PSL}_d(\mathbb{A}_{T^c}) = \mathrm{PGL}_d(\mathbb{A}_{T^c})$, and so all such representations are trivial as in Lemma 6.

Let us now define a map JL^0, which takes an irreducible infinite-dimensional sub-representation π'' of $\mathrm{L}^2(\Delta \backslash G_0)$ to an irreducible infinite-dimensional sub-representation π of $\mathrm{L}^2(G(k) \backslash G(\mathbb{A}))$, which occurs in the discrete spectrum. $\mathrm{JL}^0(\pi'')$ is defined as follows: Since $G_0 = G'(\mathbb{A})/G'(\mathbb{A}_T)$ and $\mathrm{L}^2(G'(k) \backslash G'(\mathbb{A})/G'(\mathbb{A}_T)) \cong \mathrm{L}^2(G'(k) \backslash G'(\mathbb{A}))^{G'(\mathbb{A}_T)} \subseteq \mathrm{L}^2(G'(k) \backslash G'(\mathbb{A}))$, π'' lifts to a sub-representation π' of $\mathrm{L}^2(G'(k) \backslash G'(\mathbb{A}))$ on which $G'(\mathbb{A}_T)$ acts trivially. Namely, $(\pi')_\theta$ is trivial for every $\theta \in T$. By the global Jacquet-Langlands correspondence, there is an automorphic representation $\pi = \mathrm{JL}(\pi')$ of $G(\mathbb{A})$ such that $(\pi')_\nu \cong \pi_\nu$ for every

$\nu \notin T$. We set $\mathrm{JL}^0(\pi'') = \pi$. Now, for $\theta \in T$, $\mathrm{JL}_\theta((\pi')_\theta) = \mathrm{JL}_\theta(1) = Sp_d(\delta)$ is the Steinberg representation, for $\delta = |\cdot|^{(1-d)/2}$. Moreover $\mathrm{C}_d(\delta)$ is the trivial representation, and hence π_θ is either trivial or Steinberg. But if π is infinite-dimensional, π_θ cannot be trivial. Therefore, it is the Steinberg representation.

The strong multiplicity one theorem for $D^\times$ (see part 4 of [HT, Theorem VI.1.1]) implies that π' has multiplicity one in $\mathrm{L}^2(G'(k)\backslash G'(\mathbb{A}))$ and hence π'' has multiplicity one in $\mathrm{L}^2(\Delta\backslash G_0)$. Moreover, it implies that $\mathrm{JL}^0(\pi'')$ determines π''. By part 3 of [HT, Thmeorem VI.1.1] and the discussion above, the image of JL^0 is composed of the discrete automorphic representations π of $\mathrm{PGL}_d(\mathbb{A})$ such that for every $\theta \in T$, the local component π_θ is a Steinberg representation. In particular, the image depends on T but not on the specific invariants of D.

Theorem 3 follows immediately. Since $\mathrm{L}^2(\Delta_1\backslash G_0)$ and $\mathrm{L}^2(\Delta_2\backslash G_0)$ have the same isomorphic images via the JL^0 map in $\mathrm{L}^2(G(k)\backslash G(\mathbb{A}))$, they have isomorphic irreducible sub-representations, each appearing with multiplicity one.

We should remark that [HT] deals with representations of GL_d or $D^\times$ with a fixed central character (and not PGL_d and $D^\times/Z^\times$), but this does not change the argument as we can take the character to be trivial.

5. Generalization of Theorem 3 and Proposition 7

In this section we generalize Theorem 3 and its main corollary. The generalizations are not needed here, but we use them in a subsequent paper on isospectral Cayley graphs [LSV2].

As before, let k be a global field, T a fixed finite set of valuations, and D_1, D_2 central division algebras of degree d over k, which are unramified outside of T, and remain division algebras over the completions k_θ for $\theta \in T$. Let G_i' denote the multiplicative group of D_i modulo center. Recall that $G_i'(\mathbb{A}_{T^c})$ can be identified with $G_0 = G(\mathbb{A}_{T^c})$. Let $J_i = G_i'(\mathbb{A}_T)$, so that $G_i'(\mathbb{A}) = G_i'(\mathbb{A}_{T^c})J_i = G_0 J_i$. Again $\Delta_i = G_i'(k)$ is embedded diagonally in $G_i'(\mathbb{A})$.

Theorem 14. *Let J_i^0 be finite index subgroups of J_i ($i = 1, 2$), with $[J_1 : J_1^0] = [J_2 : J_2^0]$.*

Then $\mathrm{L}^2(\Delta_1 \backslash G_1'(\mathbb{A}))^{J_1^0}$ and $\mathrm{L}^2(\Delta_2 \backslash G_2'(\mathbb{A}))^{J_2^0}$ (the spaces of J_i^0-invariant functions on $\Delta_i \backslash G_i'(\mathbb{A})$) are isomorphic as G_0-representations.

Since $\mathrm{L}^2(\Delta_i \backslash G_i'(\mathbb{A}))^{J_i} = \mathrm{L}^2(\Delta_i \backslash G_i'(\mathbb{A}_{T^c}))$, we obtain Theorem 3 as a special case by taking $J_i^0 = J_i$. In fact, Theorem 14 follows at once from Theorem 3: the natural projection

$$\Delta_i \backslash G_i'(\mathbb{A}) \to \Delta_i \backslash G_i'(\mathbb{A}_{T^c})$$

has fiber J_i over every point. Therefore

$$\begin{aligned}
\mathrm{L}^2(\Delta_i \backslash G_i'(\mathbb{A}))^{J_i^0} &= (\mathrm{L}^2(\Delta_i \backslash G_i'(\mathbb{A}))^{J_i})^{[J_i : J_i^0]} \\
&= \mathrm{L}^2(\Delta_i \backslash G_i'(\mathbb{A}_{T^c}))^{[J_i : J_i^0]}
\end{aligned}$$

where the first equality (for G_0-modules) follows from the fact that J_i^0 and G_0 commute elementwise. As the indices are equal by assumption, this completes the proof of Theorem 14.

Fix a valuation $\nu_0 \notin T$, and let $F = k_{\nu_0}$ be the completion with respect to this valuation. Recall the rings $R_{0,T}$ defined in Equation (3), and

(10) $$R_0 = \{x \in k : \nu(x) \geq 0 \quad \text{for every } \nu \neq \nu_0\}.$$

Of course, $R_0 \subset R_{0,T}$, and every ideal $I \triangleleft R_{0,T}$ induces the ideal $I \cap R_0$ of R_0. On the other hand not every ideal of R_0 has this form (for example when $I \triangleleft R_0$ contains an element $a \in I$ with $\theta(a) = 0$ for some $\theta \in T$).

Fix embeddings $G_i'(k) \hookrightarrow \mathrm{GL}_t(k)$ (same t for both $i = 1, 2$) such that over the completions k_θ, $\theta \in T$, $G_i'(k_\theta) = G_i'(\mathcal{O}_\theta)$. This is possible since $G'(k_\theta)$ is compact for every θ in the finite set T. Set $G_i'(R_0) = G_i'(k) \cap \mathrm{GL}_t(R_0)$, and define the congruence subgroups $G_i'(R_0, I)$ as in Equation (4). We obtain the following extension of Proposition 7.

Proposition 15. *Let $0 \neq I \triangleleft R_{0,T}$ be an ideal, and let J_i^0 be finite index subgroups of $J_i = G_i'(A_T)$ with $[J_1 : J_1^0] = [J_2 : J_2^0]$. Let $\Lambda_i = G'(k) \cap G(F) U^{(r)} J_i^0$ be the corresponding discrete subgroup, where $U^{(r)}$*

is defined as in (6). Then

$$\mathrm{L}^2(\Lambda_1 \backslash G(F)) \cong \mathrm{L}^2(\Lambda_2 \backslash G(F))$$

as representations of $G(F) = \mathrm{PGL}_d(F)$.

Proof. Let $\Delta_i = G_i'(k)$ as before. Theorem 14 implies that

$$\mathrm{L}^2(\Delta_1 \backslash G_1'(\mathbb{A}))^{U^{(r)} J_1^0} \cong \mathrm{L}^2(\Delta_2 \backslash G_2'(\mathbb{A}))^{U^{(r)} J_2^0},$$

so

$$\mathrm{L}^2(\Delta_1 \backslash G_1'(\mathbb{A})/U^{(r)} J_1^0) \cong \mathrm{L}^2(\Delta_2 \backslash G_2'(\mathbb{A})/U^{(r)} J_2^0).$$

On the other hand, combining Lemma 6 and the fact that $[J_i : J_i^0]$ is independent of i, we see that

$$G_i'(\mathbb{A})/\Delta_i G(F) U^{(r)} J_i^0$$

is independent of i. As in the proof of Proposition 7, this implies that

$$\mathrm{L}^2(\Delta_i \backslash \Delta_i G(F) U^{(r)} J_i^0/U^{(r)} J_i^0)$$

is independent of i. But this space is isomorphic to $\mathrm{L}^2(\Lambda_i \backslash G(F))$ by the choice of Λ_i. $\qquad\square$

This proposition generalizes Proposition 7, providing isospectrality for principal congruence subgroups with respect to ideals of R_0, rather than only ideals of $R_{0,T}$. (However it also covers non-principal congruence subgroups).

Corollary 16. *Let $0 \neq I \lhd R_0$. There is an isomorphism*

$$\mathrm{L}^2(G_1'(R_0, I) \backslash G(F)) \cong \mathrm{L}^2(G_2'(R_0, I) \backslash G(F)),$$

as representations of $G(F) = \mathrm{PGL}_d(F)$.

Proof. Define a function $r : \mathcal{V} - \{\nu_0\} \to \mathbb{N} \cup \{0\}$ as in Equation (5), and let $U^{(r)}$ be the group defined in Equation (6). Take $J_i^0 = \prod_{\theta \in T} G_i'(\mathcal{O}_\theta, P_\theta^{r_\theta})$, so that $G_i'(R_0, I) = G_i'(k) \cap G(F) U^{(r)} J_i^0$. From the detailed description of the quotients $G'(k_\theta)/G_i'(\mathcal{O}_\theta, P_\theta^{r_\theta})$ in [PrR, Proposition 1.5], one sees that the index of J_i^0 in $G_i'(\mathbb{A}_T)$ does not depend on i, so the proposition applies. $\qquad\square$

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