On property (τ)

-preliminary version-

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Introduction

Property (T) was introduced in a seminal paper of Kazhdan [104] in 1967. A group G has this property if the trivial one dimensional representation of G is "bounded away" from all the other irreducible unitary representations of G. Kazhdan property (T) turned out to be a powerful representation theoretic method to study discrete subgroups of Lie groups.

The current notes are about a baby version of property (T), which is called property (τ). It asserts, for a discrete group Γ, that trivial representation is bounded away fom the non-trivial irreducible finite representations (i.e., those with finite images). In many applications, it is even useful to look at a smaller subclass of representations: Let $\mathcal{L} = \{N_i\}_{i \in I}$ be a family of finite index subgroups of Γ. Γ is said to have τ with respect to $\mathcal{L}$ ($\tau(\mathcal{L})$ for short) if the non-trivial irreducible Γ subrepresentations of $l^2(\Gamma/N_i)$, $i \in I$, are bounded away from the trivial representation. An important case which will be dealt in details is when Γ is an arithmetic group and $\mathcal{L}$ is its family of congruence subgroups.

It turns out that (τ) - being weaker than (T) - is sometimes even more useful as it holds for a wider class of groups. Moreover it can be presented in various equivalent forms (for simplicity, we assume that $\mathcal{L}$ is large enough to define a topology on Γ):

(a) Representation theoretical - the original definition;

(b) Combinatorial - Γ has ($\tau(\mathcal{L})$) if and only if the quotient Schreier graphs form a family of expanding graphs ("expanders");

(c) Measure theoretic - Γ has ($\tau(\mathcal{L})$) if and only if the Haar measure is the unique Γ-invariant finitely additive measure on $\hat{\Gamma}_{\mathcal{L}}$ (the profinite group obtained from Γ by completing it with respect to the topology determined by $\mathcal{L}$).

(d) Cohomological - Γ has ($\tau(\mathcal{L})$) if and only if $H^1(\Gamma, L^2(\hat{\Gamma}_{\mathcal{L}})) = 0$.

If in addition $\Gamma = \pi_1(M)$ where M is a compact Riemannian manifold and M_i is the finite sheeted covering of M corresponding to N_i, $i \in I$ then we also have

(e) Analytic - Γ has $(\tau(\mathcal{L}))$ if and only if there is an $\varepsilon > 0$ such that $\lambda_1(M_i) > \varepsilon$, for every $i \in I$, where $\lambda_1(M_i)$ is the smallest positive eigenvalue of the Laplace-Beltrami operator on M_i.

(f) Geometric - Γ has $(\tau(\mathcal{L}))$ if and only if there is an $\varepsilon > 0$ such that $h(M_i) \geq \varepsilon$ for every i, where $h(M_i)$ is the Cheeger constant (the isoperimetric constant) of the manifold M_i.

The fact that (τ) can be expressed in so many different ways opens the door to applications in several directions. The main goal of these notes is to describe these applications which look quite unrelated, from a unified perspective. Some of these applications are, by now, quite well known and some of them are more recent. There are only few new results in this book (e.g. Sections 2.4, 5.2 and 8.2).

These applications include:

(i) The constructions of expanders. These graphs are of fundamental importance in computer science and combinatorics;

(ii) The analysis of a popular algorithm in computational group theory called "the product replacement algorithm" (PRA - for short). This algorithm provides pseudo random elements from a finite group given by its generators. In practice, it turns out to have outstanding performances, but its theoretical analysis does not, as yet, explains why. Property (τ) or more precisely a "non-commutative Selberg Theorem" can give the proper explanation. This connection suggests some problems and conjectures regarding (τ) for the automorphism group of the free group.

(iii) The uniqueness of the Haar measure as the only finitely additive invariant measure of some local and adélic profinite groups.

(iv) Applications to C^* algebras and in particular to the question when the C^* algebra of a discrete group Γ is seperated by its finite dimensional representations.

The most surprising applications are for

(v) Hyperbolic manifolds. In this regard we present a proof for arithmetic manifolds of Thurston's conjecture on non-vanishing of the first Betti number of finite covers of such manifolds. Moreover the recent work of Lackenby [113] suggests a path how the Lubotzky-Sarnak conjecture (asserting that fundamental groups of compact hyperbolic 3-manifolds do not have (τ)) can lead to a proof of the famous "virtual Haken conjecture" for hyperbolic 3-manifolds.

As said the main goal of these notes is to present these applications from a unified point of view. Along the way we discuss the connections with questions on automorphic forms, finite groups, discrete subgroups in Lie groups, 3-manifolds, pro-p groups and more. A number of open problems for further research are also presented.

In a way the current book is a sequel to [122]. Here and there, the reader will find some overlap, but the two books can be read independently. The book is organized as follows:

The first part deals with property (τ) for its own sake. More specifically:

In Chapter 1, we present (T) and (τ) and their basic properties. Chapter 2 deals with the various equivalent forms of (τ) (as some of the material here overlaps with [122], we are sometimes quite brief, but we give full details for the cohomological form which is new). In Chapter 3 we relate a "quantitative version of (τ)" to the existence of nontrivial first cohomology. This will come up again in Chapter 7, where we consider the Thurston conjecture. Chapter 4 deals with the connection of (τ) to the Selberg Theorem ($\lambda_1 \geq \frac{3}{16}$) and the Ramanujan conjecture.

In the second part of the book, we turn to the applications.

In Chapter 5, we describe the applications to finite graphs and finite groups. These applications are mainly throught the notion of expanders which play an important role in computer science and combinatorics. Some of this material is covered in [122], so here we will concentrate more on its implications to the theory of finite groups: We study the "finitary τ", i.e., considering (τ) as a property of an infinite family of finite groups rather than as a property of one infinite group. We present some recent results on the dependence of this property on the choice of generators. Some of this material will be used in Chapter 8 when the measure theoretic applications will be discussed.

Chapter 6 deals with the application to the product replacement algorithm. The reader's attention is called to some conjectures about (τ) and "Selberg property" for $Aut(F_k)$ (F_k being the free group on k generators) whose proof or disproof would imply interesting corollaries.

In Chapter 7 we describe the applications to hyperbolic manifolds, starting with the Thurston conjecture. We then pass to the connections between the Heegaard splitting, (τ) and the virtual Haken conjecture. Some group theoretical conjectures are presented whose proof can lead to proofs of the above mentioned conjectures on hyperbolic 3-mainfolds. We will explain why pro-p theory in general and Golod-Shafarevitch in particular can be relevant. The recent work of Zelmanov [196], gives some hope that this "pro-p" approach to this 3-manifolds problem is not so absurd as may be thought at first sight.

Chapter 8 presents the measure theoretic applications and Chapter 9 the ones to the C^* algebras.

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Part I

Properties and examples

Chapter 1

Property (T) and property (τ)

In this chapter we introduce properties (T) and (τ), give some examples (more to come in Chapter 4) and study some of their properties.

1.1 Fell topology

Throughout this section G is a locally compact group. Let $\widetilde{G}$, $\widetilde{G}_0$ denote the space of equivalence classes of all continuous unitary representations of G and those without invariant vectors, respectively.

The Fell topology on $\widetilde{G}$ is defined as follows. Consider a representation $(\mathcal{H}, \pi)$ in $\widetilde{G}$. For a compact subset K of G, $\varepsilon > 0$ and $v \in \mathcal{H}$ of norm one we define the neighborhood $W(K, \varepsilon, v)$ as those representations $(\mathcal{H}', \pi')$ in $\widetilde{G}$ for which there exists $v' \in \mathcal{H}'$ of norm one such that for every $g \in K$

$$|\langle v, \pi(g)v\rangle - \langle v', \pi'(g)v'\rangle| < \varepsilon.$$

If H is a closed subgroup of G then the induction of representations is a continuous map from $\widetilde{H}$ to $\widetilde{G}$ with respect to this topology. We say that a representation $(\pi, \mathcal{H}) \in \widetilde{G}$ is weakly contained in the representation $(\pi', \mathcal{H}') \in \widetilde{G}$ and we write $\pi \prec \pi'$ if π is contained in the closure of π' in the Fell topology (see for instance [56]).

If 1_G denotes the trivial representation of G, then a representation $(\pi, \mathcal{H})$ of G weakly contains 1_G, if and only if π has almost invariant vectors, i.e. for every $\varepsilon > 0$ and for every compact subset K of G there exists $v \in \mathcal{H}$ of norm 1 such that for every $g \in K$

$$\|\pi(g)v - v\| < \varepsilon.$$

1.2 Property (T)

In 1967 Kazhdan introduced the notion of property (T) which plays an important role in the theory of semi simple Lie groups and their discrete subgroups.

Definition 1.1 ([104]) *Let G be a locally compact group. We say that G has* **property (T)** *or G is a* **Kazhdan group** *if the trivial representation is isolated from $\widetilde{G}_0$ in $\widetilde{G}$, i.e. 1_G is not in the closure of $\widetilde{G}_0$.*

In Section 1.2.1 one can find a description of Lie groups which have property (T). Their lattices, i.e. discrete subgroups of Lie groups of finite covolume, have property (T) as well. This enabled Kazhdan (and in fact was the main motivation to introduce property (T)) to prove a conjecture of Siegel stating that these lattices are finitely generated. Indeed if Γ is a discrete group which is Kazhdan, then Γ is finitely generated (see Proposition 1.24). For finitely generated groups one can give the following formulation of property (T).

Proposition 1.2 *Let Γ be a group generated by a finite set S. Γ has property (T) if and only if there exists $\varepsilon(S) > 0$ such that for every $(\pi, \mathcal{H}) \in \widetilde{\Gamma}_0$ and every $v \in \mathcal{H}$*

$$\max_{s \in S} \|\pi(s)v - v\| \geq \varepsilon(S)\|v\|.$$

The constant $\varepsilon(S)$ is called a **Kazhdan constant** with respect to the set S. Property (T) is independent on the generating set but $\varepsilon(S)$ does: for some Kazhdan groups it is not possible to find $\varepsilon > 0$ which would be a Kazhdan constant for all finite sets of generators as was shown by Gelander and Żuk [71]. For instance if $\Gamma = SO_5(\mathbb{Z}[1/5])$ then Γ has property (T) and in [71] it was shown that there exists a sequence of generating subsets S_n of size 5 such that $\varepsilon(S_n)$ tends to zero. For many applications of property (T) it is interesting to estimate the value of Kazhdan constants (for examples of such estimates see [182], [183], [200], [36]).

1.2.1 Examples of groups with property (T)

A fundamental remark of Kazhdan was that for lattices (i.e. discrete subgroups of Lie groups of finite covolume) property (T) is inherited from Lie groups.

Theorem 1.3 *A lattice* Γ *in a locally compact group* G *has property (T) if and only if* G *has property (T).*

Corollary 1.4 *Let* Γ_1 *and* Γ_2 *be lattices in a locally compact group* G*. Then* Γ_1 *has (T) if and only if* Γ_2 *has.*

By results of Kazhdan [104], Delaroche-Kirillov [49], Kostant [111], Vaserstein [187] and Wang [191] it is known which Lie groups are Kazhdan (see [148], [125] and [197] for an exposition of these results). For example,

Theorem 1.5 *Let* F *be a local field and* G *the group of* F*-points of a simple algebraic group defined over* F *and of* F*-rank* ≥ 2*. Then* G *is a Kazhdan group.*

For F non-archemedian, rank one groups never have (T) as they act on trees (see Chapter 2 below). Among $\mathbb{R}$-rank one real simple Lie groups some have property (T) and some do not.

Theorem 1.6 *The real Lie groups* $Sp(n,1)$ *and* $F_4^{(-20)}$ *are Kazhdan groups while* $SO(n,1)$ *and* $SU(n,1)$ *are not.*

There are other examples of groups with property (T) ([82], [36], [159], [198], [13]). In fact in some sense "most" discrete groups have (T) ([200]). But it is not clear how many of those are residually finite, so property (τ) for them is not so significant.

1.3 Combinatorial approach to property (T)

We present a simple combinatorial condition which enables one to prove property (T).

Let Γ be a group generated by a finite set S such that S is symmetric, i.e. $S = S^{-1}$, and the identity element e does not belong to S.

Definition 1.7 *Let* $L(S)$ *be a finite graph defined in the following way:*

1. *vertices of* $L(S) = \{s; s \in S\}$,
2. *edges of* $L(S) = \{(s,s'); s, s', s^{-1}s' \in S\}$.

Let us suppose that the graph $L(S)$ is connected. This condition is not restrictive, because for a finitely generated group Γ one can always find a finite, symmetric generating set S, not containing e, such that $L(S)$ is connected (for instance $S \cup S^2 \setminus e$ will do). This can be seen in the simple case of $\Gamma = \mathbb{Z}$; if $S = \{-1, 1\}$ then the graph $L(S)$ is not connected but if we add to the set of generators $\{-2, 2\}$ the graph becomes connected.

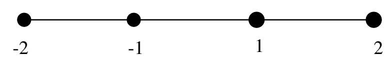

Figure 1.1: The graph $L(S)$ for $\Gamma = \mathbb{Z}$ and $S = \{-2, -1, 1, 2\}$

For a vertex $s \in L(S)$ let $deg(s)$ denote its degree, i.e. the number of edges adjacent to s. Let Δ be a discrete Laplace operator acting on functions defined on vertices of $L(S)$, i.e. for $f \in l^2(L(S), deg)$

$$\Delta f(s) = f(s) - \frac{1}{deg(s)} \sum_{s' \sim s} f(s'),$$

where $s' \sim s$ means that the vertex s' is adjacent to the vertex s.

The operator Δ is a non-negative, self-adjoint operator on $l^2(L(S), deg)$. If $L(S)$ is connected then the zero is a simple eigenvalue of Δ. Let $\lambda_1(L(S))$ be the smallest non-zero eigenvalue of Δ acting on $l^2(L(S), deg)$.

Theorem 1.8 (Żuk [200]) *Let Γ be a group generated by a finite subset S, such that S is symmetric and $e \notin S$. If the graph $L(S)$ is connected and*

$$\lambda_1(L(S)) > \frac{1}{2} \tag{1.1}$$

then Γ has Kazhdan's property (T). Moreover

$$\frac{2}{\sqrt{3}} \left(2 - \frac{1}{\lambda_1(L(S))} \right)$$

is a Kazhdan constant with respect to the set S.

Remark The condition stated in Theorem 1.8 is optimal. In order to see this, let us consider the group $\Gamma = \mathbb{Z}$ with the set of generators $S = \{\pm 1, \pm 2\}$. Then the graph $L(S)$ consists of four vertices and three edges (see Figure 1.1). For this graph $\lambda_1(L(S)) = \frac{1}{2}$ and the group $\mathbb{Z}$ does not have property (T).

The condition (1.1) can be easily checked. One can imagine a computer program which for a group given by a presentation checks (1.1) and thus can prove that this group has property (T).

The above condition applies to several groups. For instance one can give a new proof of property (T) for some lattices. Let us see this for lattices in $SL(3, \mathbb{Q}_p)$. Recall that $SL(3, \mathbb{Q}_p)$ acts on an affine building of type $\widetilde{A}_2$.

In [36], a family of groups acting co-compactly on buildings of type $\widetilde{A}_2$ was constructed. These groups are parameterized by an integer q which is a power of a prime number. They admit a presentation such that $L(S)$ is the incidence graph of the projective plane $\mathbb{P}^2(\mathbb{F}_q)$ over the finite field $\mathbb{F}_q$, i.e.

$$\begin{aligned} \text{vertices of } L(S) &= \{\text{points } p \text{ and lines } l \text{ such that } p, l \in \mathbb{P}^2(\mathbb{F}_q)\}, \\ \text{edges of } L(S) &= \{(p, l); p \in l\}. \end{aligned}$$

Figure 1.2 shows the graph $L(S)$ for $q = 2$.

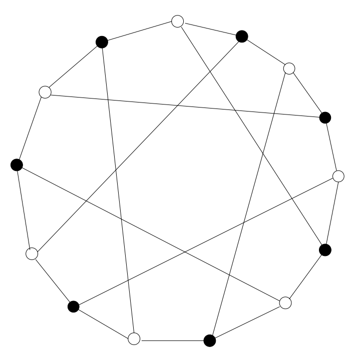

Figure 1.2: The graph $L(S)$ for a lattice in $SL(3, \mathbb{Q}_2)$

In [66], Feit and Higman computed the spectrum of the Laplace operator on graphs which are incidence graphs of finite projective planes.

Proposition 1.9 (Feit-Higman [66]) *Let L be the incidence graph of $\mathbb{P}^2(\mathbb{F}_q)$. Then*

$$\lambda_1(L) = 1 - \frac{\sqrt{q}}{q+1}.$$

For any $q \geq 2$ we have $\lambda_1(L) = 1 - \frac{\sqrt{q}}{q+1} > \frac{1}{2}$ and thus these groups have property (T).

The combinatorial condition (Theorem 1.8) enables to show that in several settings generic presentations define groups with with property (T) ([84], [200]).

We consider the following model $\mathcal{M}$ for random groups, which is related to Gromov's model ([82]) and was studied in [200]. Let us consider presentations with relations of length 3. Let d (called density as before) be between 0 and 1. Let $P_{\mathcal{M}}(m,d)$ be a set of presentations with m generators, relations of length 3 and density d, i.e. the number of relations is between $c^{-1}(2m-1)^{3d}$ and $c(2m-1)^{3d}$, where $c > 1$ is any fixed constant. Denote by $\Gamma(P)$ the group given by presentation P.

Theorem 1.10 (Żuk [200]) *For $d > \frac{1}{3}$ one has*

$$\lim_{m\to+\infty} \frac{\#\{P \in P_{\mathcal{M}}(m,d); \Gamma(P) \text{ has property (T)}\}}{\#P_{\mathcal{M}}(m,d)} = 1.$$

For $d < \frac{1}{2}$ most presentations in the above model define infinite hyperbolic groups ([82], [200]).

1.4 Property (τ)

Let Γ be a finitely generated group and $\mathcal{L} = \{N_i\}$ be a family of finite index subgroups of Γ. Let $R = R(\mathcal{L}) = \{\phi \in \widetilde{\Gamma}; \phi$ appears as a subrepresentation of the action of Γ on $L^2(\Gamma/N_i)$ for some $i\}$. If the N_i's are normal, as it is in most applications, this simply means that $R(\mathcal{L})$ is the set of all representations of Γ which factors through Γ/N_i for some $N_i \in \mathcal{L}$. Let $R_0 = R_0(\mathcal{L}) = R \cap \widetilde{\Gamma}_0$. The following weaker form of (T) was introduced by Lubotzky [122] and coined (τ):

Definition 1.11 *We say that Γ has* **property (τ) with respect to the family** $\mathcal{L}$*, or for short, Γ has $\tau(\mathcal{L})$, if the trivial representation is isolated from the set R_0 in $\widetilde{\Gamma}$. We say that Γ has property (τ) if it has this property with respect to the family of all finite index subgroups.*

Note that property (τ) with respect to $\mathcal{L}$ means that there exists a finite set S in Γ and $\varepsilon > 0$ such that for any representation $(V, \phi) \in R_0(\mathcal{L})$ and $0 \neq v \in V$ there exists $s \in S$ such that

$$\|\phi(s)v - v\| > \varepsilon \|v\|.$$

Such a pair (S, ε) is called a (τ) constant of Γ with respect to $\mathcal{L}$. Note that Γ is not necessarily finitely generated (see Section 1.5.1).

For a compact space X with a finite measure μ, let $l_0^2(X)$ be the set of complex valued functions f such that $\int_X |f(x)|^2 d\mu(x) < \infty$ and $\int_X f(x) d\mu(x) = 0$.

The following follows from the definition of $\tau(\mathcal{L})$.

Proposition 1.12 *A group Γ has property $\tau(\mathcal{L})$ if and only if the representation of Γ on $\widehat{\bigoplus} l_0^2(\Gamma/N_i)$ does not weakly contain the trivial representation.*

1.4.1 Profinite topologies

Definition 1.13 *Let $\mathcal{L} = \{N_i\}_{i \in I}$ be family of finite index subgroups. We define a topology on Γ for which the following system of subgroups serves as a fundamental system of neighborhoods:*

$$Top(\mathcal{L}) = \{\cap_{j=1}^k N_{i_j}^{g_{i_j}}; N_{i_j} \in \mathcal{L}, g_{i_j} \in \Gamma\}.$$

Question 1.14 *Assume the group Γ has property (τ) with respect to $\mathcal{L} = \{N_i\}_{i \in I}$. Does it have property (τ) with respect to $Top(\mathcal{L})$?*

We do not know the answer even if $\mathcal{L} = \{N_i\}$ is a family of normal subgroups.

For $\mathcal{L} = \{N_i\}$ as before with $\mathcal{L} = Top(\mathcal{L})$, let $\widehat{\Gamma}_{\mathcal{L}} = \lim_{\leftarrow} \Gamma/N_i$ denote the completion of Γ with respect to the topology $Top(\mathcal{L})$. $\widehat{\Gamma}_{\mathcal{L}}$ is a profinite group. There is a natural homomorphism $i : \Gamma \to \widehat{\Gamma}_{\mathcal{L}}$ whose image is dense in $\widehat{\Gamma}_{\mathcal{L}}$. It is an embedding if and only if the topology defined by $Top(\mathcal{L})$ is Hausdorff, i.e. if and only if $\bigcap N_i = \{id\}$. We denote by μ the Haar measure on $\widehat{\Gamma}_{\mathcal{L}}$.

Proposition 1.15 *Assume $\mathcal{L} = Top(\mathcal{L})$. The group Γ has property (τ) with respect to $\mathcal{L}$ if and only if $L_0^2(\widehat{\Gamma}_{\mathcal{L}}, \mu)$ does not weakly contain the trivial representation.*

Proof By the Peter-Weyl theorem this is just a reformulation of property (τ) using one representation $L_0^2(\widehat{\Gamma}_{\mathcal{L}}, \mu)$ (compare with Proposition 1.12). □

1.4.2 Lattices in semi-simple groups

Property (T) implies (τ) so every group with property (T) has property (τ). Among lattices this applies to lattices in "most" simple groups (Theorems 1.5 and 1.6).

The converse is not true. There are lattices with property (τ) and without property (T).

Let G be a semi simple group, by which we mean $G = \Pi_{i=1}^{r} G_i(K_i)$ where K_i are local fields and G_i is a simple K_i-algebraic group. By $rank(G)$ we mean $\sum_{i=1}^{r} K_i - rank(G_i)$. A lattice in G has (T) if and only if G has (T) and G has (T) if and only if each factor of G has (T). Theorems 1.5 and 1.6 give the complete picture on the simple factors G_i. A lattice in a product of locally compact separable groups is irreducible if its projections on every factor are dense. An irreducible lattice Γ in G may have (τ) even if G does not have (T).

Theorem 1.16 (Lubotzky-Zimmer [140]) *Let H_1 and H_2 be locally compact separable groups. Let Γ be an irreducible lattice in a product $H_1 \times H_2$, where both factors are not compact. If H_1 has property (T) then Γ has property (τ).*

Corollary 1.17 *Let $G = \Pi_{i=1}^{r} G_i(K_i)$ is a semi-simple group as before and let $\Gamma < G$ be an irreducible lattice. If one of the non-compact factors $G_i(K_i)$ has (T) then Γ has (τ). If in addition one of the factors does not have (T) then Γ has (τ) but not (T).*

Definition 1.18 *A lattice $\Gamma < G = \Pi G_i(K_i)$ is called* **arithmetic** *if there exist a global field k with the ring of integers $\mathcal{O}$, a finite subset S of V, the set of valuations of k, containing V_∞-the archemeadian valuations and $\mathcal{G}$ a connected absolutely simple k group and an homomorphism*

$$\pi : \Pi_{v \in S} \mathcal{G}(k_v) \to G$$

such that $ker(\pi)$ and $coker(\pi)$ are compact and $\pi(\mathcal{G}(\mathcal{O}_S))$ is commensurable to Γ, where $\mathcal{O}_S = \{x \in k; v(x) \geq 0 \text{ for } v \notin S\}$.

A remarkable theorem of Margulis [149] asserts that all irreducible lattices in semi-simple Lie groups of rank at least 2 are arithmetic.

Example 1.19 *Let $n \geq 3$ and $\Gamma < SO(n,2) \times SO(n+1,1)$ be an irreducible lattice. Then Γ has (τ) but does not have (T). This is because $rk(SO(p,q)) = \min(p,q)$ and follows from Theorem 1.5 and Corollary 1.17.*

An example of an irreducible lattice in $SO(n,2) \times SO(n+1,1)$ is as follows. Let $k = \mathbb{Q}(\sqrt{2})$, $\mathcal{O} = \mathbb{Z}(\sqrt{2})$ and let f be the quadratic form

$$f(x_0, \ldots, x_{n+1}) = x_0^2 + \ldots + x_{n-1}^2 - x_n^2 + \sqrt{2}x_{n+1}^2.$$

Let σ be the non-trivial element of the Galois group $Gal(k/\mathbb{Q})$. Then f^σ is of the form $x_0^2 + \ldots + x_{n-1}^2 - x_n^2 - \sqrt{2}x_{n+1}^2$. Let G be the k-group of $(n+1) \times (n+1)$ matrices preserving the form f. Then $\Gamma = G(\mathbb{Z}(\sqrt{2}))$ is an irreducible lattice in $G(\mathbb{R}) \times G^\sigma(\mathbb{R}) \simeq SO(n,2) \times SO(n+1,1)$.

We will see later more examples of groups with property (τ) and not (T) (see Chapter 4).

Unfortunately Corollary 1.4 does not hold for property (τ). We have just seen that the group $SO(n,2) \times S(n+1,1)$ has a lattice with (τ), but it clearly has a lattice without property (τ): take a lattice Δ_1 in $SO(n,2)$ (which has property (T)) with a lattice Δ_2 in $SO(n+1,1)$ which has in infinite abelianization (such a lattice exists - see Chapter 7 below). Then $\Delta_1 \times \Delta_2$ is a lattice in $SO(n,2) \times SO(n+1,1)$ without (τ). But we do not know

Question 1.20 *Let Γ_1 and Γ_2 be irreducible lattices in a semi-simple Lie group G or lattices in a simple group G. Does Γ_1 have (τ) if and only if Γ_2 has?*

An affirmative answer to this problem will be of great importance. For example it will imply Conjecture 7.1 (see Chapter 7).

We should mention that we do not know any counterexample to the following

Question 1.21 *If Γ is an irreducible lattice in $H_1 \times H_2$, where H_1 and H_2 are semi-simple and non-compact. Does Γ have property (τ)?*

An interesting challenge to this or similar cases are products of rank one Lie groups or the Burger-Mozes lattices [32], i.e. irreducible lattices in $Aut(X_p) \times Aut(X_q)$ where X_p and X_q are regular trees of degree p and q respectively. We do not know if they have (τ). They clearly do not have

(T) as they act without fixed points on a tree. Recall that the group with property (T) always fixes a point when it acts on a tree [4].

In Chapter 4 we will show some more examples of groups with property (τ) (or $\tau(\mathcal{L})$ for some $\mathcal{L} = \{N_i\}$). But all our non-trivial examples, here and there, are arithmetic subgroups (or a slight modification of them, e.g. $\mathbb{Z}^n \rtimes SL_n(\mathbb{Z})$ has (T) and (τ) for $n \geq 3$. It is not a lattice in a semi-simple group but in another Lie group $\mathbb{R}^n \rtimes SL_n(\mathbb{R})$). It is known, due to [200] that in some sense most groups have (T) (and so (τ)) but the probabilistic methods of [200] do not reveal whether they have infinitely many finite quotients, to make (τ) of some interest.

Among linear groups we do not know any finitely generated group with (τ) which is not "of arithmetic type".

In this context it is worth mentioning a specific candidate and an open problem. But first a related story:

For some time, there were no known examples of linear rigid groups which were not of arithmetic type. A rigid group is a finitely generated group which for every n has only finitely many equivalent classes of irreducible representations of dimension n. In fact, a well known conjecture of Platonov asserts that every finitely generated rigid linear group is of arithmetic type.

This conjecture was disproved by Bass and Lubotzky [15]. For every lattice Λ in $G = \mathbb{F}_4^{(-20)}$ the exceptional $\mathbb{R}$-rank one simple Lie group, they provided a subgroup Γ of $L \times L$ of infinite index there, such that the inclusion map i from Γ to $L \times L$ induces an isomorphism $\hat{i} : \hat{\Gamma} \to \widehat{L \times L} = \hat{L} \times \hat{L}$ between the profinite completions. They appeal then to [87] to deduce that every finite dimensional linear representation of Γ can be extended to $L \times L$. Now by [48] and [86] L is superrigid in G, i.e., every finite dimensional representation of L can be extended, up to a finite index subgroup to a representation of G. The same holds therefore for $L \times L$ in $G \times G$. One can now deduce that every finite dimensional representation of Γ can be extended, up to finite index, to $G \times G$. So Γ is a super-rigid group in $G \times G$, from which one can easily deduce that Γ is a rigid group and so a counterexample to the Platonov conjecture.

It is very tempting to believe that the above Γ has property (τ) (and maybe even (T)). Note that as $\hat{\Gamma} \simeq \hat{L} \times \hat{L}$, Γ has exactly the same finite quotients as $L \times L$. The latter has (T) and so its finite quotients are expanders. But, they are expanders with respect to the generators coming from $L \times L$. We do not know if they are expanders with respect to the generators of Γ.

This issue will come up again (see Section 5.2 and especially Theorem

5.16 and Theorem 5.17).

Let us ask:

Question 1.22 *(i) Do the Bass-Lubotzky groups have (τ)?*

(ii) Is there any finitely generated linear group with (τ) (or (T)) which is not of arithmetic type?

(iii) Let Γ be a subgroup of $SL_3(\mathbb{Z})$ with (τ) (or even (T)). Is Γ of finite index in $SL_3(\mathbb{Z}$?

For later reference, let us recall the **congruence subgroup problem**. In the notations from Definition 1.18 we say that $\mathcal{G}(\mathcal{O}_S)$ has the **congruence subgroup property** (CSP, for short) if $Ker\left(\widehat{\mathcal{G}(\mathcal{O}_S)} \to \mathcal{G}(\widehat{\mathcal{O}_S})\right)$ is finite. Necessary conditions for the CSP to hold are (a) $\mathcal{G}$ is simply connected and (b) $(S \setminus V_\infty) \cap T = \emptyset$ when $T = \{v \in V; \mathcal{G}(k_v) \text{ is compact}\}$. If Γ is an arithmetic lattice of G as in Definition 1.18 we may, and will, chose $\mathcal{G}$ and S which satisfy conditions (a) and (b). We will say that Γ has the **congruence subgroup property** (CSP) if $\mathcal{G}(\mathcal{O}_S)$ has CSP. (This is independent of choice of $\mathcal{G}$, k and S).

Conjecture 1.23 (Serre) *Assume Γ is infinite (i.e. G is not compact, or equivalently $rank(G) \geq 1$). Then Γ has the congruence subgroup property if and only if $rank(G) \geq 2$.*

The positive part of Serre's conjecture has been proved in most (but not all!) cases. Less is known about the negative part, i.e., the case $rank(G) = 1$. See Section 4.4 and Section 7.6 below for further discussion.

1.5 Properties of groups with (τ)

Property (T) was introduced by Kazhdan in order to prove two properties of lattices Γ in semi-simple Lie groups: finite generation and vanishing of the first Betti numbers, i.e. finiteness of the commutator quotients.

1.5.1 Finite generation

Proposition 1.24 *If Γ be a countable discrete group which is Kazhdan, then Γ is finitely generated.*

Proof Say $\Gamma = \{g_1, g_2, \dots\}$ and $H_i = \langle g_1, \dots, g_i\rangle$. Then the Γ representation on $\widehat{\bigoplus}_i l^2(\Gamma/H_i)$ weakly contains the trivial representation. Hence by (T) it contains it. This implies that one of the H_i is of finite index in Γ and hence Γ is finitely generated. $\square$

The above proof does not work for property (τ). In fact we will see below (see Chapter 4), that if $\mathcal{P}$ is an infinite set of primes not containing all primes, then for any $n \geq 2$, $\Gamma = SL_n(\mathbb{Z}[\frac{1}{\mathcal{P}}])$ has (τ) but it is not finitely generated ($\mathbb{Z}[\frac{1}{\mathcal{P}}]$ is the set of all rational numbers whose denominator is divisible only by primes from $\mathcal{P}$). The profinite completion of $SL_n(\mathbb{Z}[\frac{1}{\mathcal{P}}])$ is actually $\prod_{p\notin\mathcal{P}} SL_n(\widehat{\mathbb{Z}}_p)$ which is a quotient of $SL_n(\widehat{\mathbb{Z}})$ ($\widehat{\mathbb{Z}}_p$ be the ring on p-adic integers). The latter is a finitely generated profinite group, so is the first. Thus $\widehat{\Gamma}$ is finitely generated. This is the general case:

Proposition 1.25 *Assume that $\mathcal{L} = Top(\mathcal{L})$ and Γ has $\tau(\mathcal{L})$. Then $\widehat{\Gamma}$ is finitely generated. Moreover it is generated by finitely many elements of Γ (i.e. Γ has a finitely generated subgroup Δ which is dense in the topology $\mathcal{L}$ of Γ).*

Proof Let S be the finite set in the definition of $\tau(\mathcal{L})$. Let $\Delta = \langle S\rangle$. Then we can continue in a similar way as in the proof that discrete groups with property (T) are finitely generated to deduce that the closure of Δ is of finite index. Hence, $\widehat{\Gamma}$ is finitely generated. $\square$

Corollary 1.26 *If Γ has (τ) then for every n, $s_n(\Gamma) < \infty$ where*

$$s_n(\Gamma) = \#\{H \leq \Gamma | [\Gamma : H] \leq n\}.$$

The growth rate of $s_n(\Gamma)$ is called the subgroup growth rate of Γ. It has been studied extensively for finitely generated groups (see [133]). If Γ is a finitely generated group, then its subgroup growth rate is at most n^n. The residually finite groups of polynomial subgroup growth are all virtually solvable. Thus if Γ is a finitely generated residually finite infinite group with property (τ), then its subgroup growth is more than polynomial. In all examples where we know the subgroup growth of groups with property (τ) the growth is either $n^{\log n/\log\log n}$ (for arithmetic groups in characteristic 0) or $n^{\log n}$ (for arithmetic groups in characteristic $p > 0$). It will be interesting to know if (τ) implies any nontrivial upper bound (or a better lower bound) on the subgroup growth.

Another interesting question is:

Question 1.27 *Let Γ be a finitely generated group with property (τ). Is $\widehat{\Gamma}$ a finitely presented profinite group?*

Note that $SL_3(\mathbb{F}_p[t])$ is not finitely presented [18] but has (T) and $\widehat{SL_3(\mathbb{F}_p[t])} = SL_3(\widehat{\mathbb{F}_p[t]})$ is finitely presented [134].

Also $SL_2(\mathbb{F}_p[t])$ is not finitely generated and has the Selberg property (i.e. has (τ) with respect to the congruence subgroups - see Chapter 4 below). $\widehat{SL_2(\mathbb{F}_p[t])}$ is not finitely generated but $\widehat{SL_2(\mathbb{F}_p[t])}_{\mathcal{L}}$, where $\mathcal{L}$ is the congruence subgroup is finitely generated and finitely presented [134].

1.5.2 Abelian quotients

We first note that if Δ is a finite index subgroup of Γ then Γ has (T) (respectively (τ)) if and only if Δ does. Now, it is easy to see that $\mathbb{Z}$ does not have (T), neither it has (τ). Indeed the one dimensional representations ρ_n of $\mathbb{Z}$: $\rho_n(k) = \left(e^{2\pi i/n}\right)^k$, clearly converge to the trivial representation since $e^{2\pi i/n}$ converges to 1. So a finitely generated group with property (τ), and its finite index subgroups have finite abelianization.

Definition 1.28 *Let Γ and $\mathcal{L} = \{N_i\}_{i\in I}$ be as before. We say that Γ has $FAb(\mathcal{L})$ if for every i, $|N_i/[N_i,N_i]| < \infty$. If $\mathcal{L}$ is a family of all finite index subgroups than we say that Γ has property FAb.*

Corollary 1.29 *If Γ is a finitely generated group with property (τ) then it has property FAb.*

Note however, that $\tau(\mathcal{L})$ does not imply $FAb(\mathcal{L})$. We will see below that the non-abelian free groups can have (τ) with respect to some $\mathcal{L}$ and clearly they do not have $FAb(\mathcal{L})$. What can be shown is that if $\mathcal{L} = Top(\mathcal{L})$ then $\tau(\mathcal{L})$ for Γ implies that $\widehat{\Gamma}_{\mathcal{L}}$ has $FAb(\mathcal{L})$.

Let us mention in passing few facts

Proposition 1.30 *A finitely generated group Γ has property FAb if and only if for every finite representation ρ over $\mathbb{C}$*

$$H^1(\Gamma,\rho) = 0.$$

Proof This follows from Shapiro's lemma. Namely, let H be a finite index subgroup of Γ and let $(\pi, \mathcal{H})$ be a unitary representation of H. The induced representation $ind_H^\Gamma \pi$ of Γ is

$$ind_H^\Gamma \pi = \{f : \Gamma \to \mathcal{H}; f(h\gamma) = \pi(h)f(\gamma) \text{ for } h \in H, \gamma \in \Gamma\}.$$

The following is a particular case of Shapiro's Lemma

$$H^1(\Gamma, ind_H^\Gamma(\pi)) = H^1(H, \pi).$$

Thus if H has an infinite abelianization, Γ has a notrivial first cohomology group with coefficients in the induced representation from the trivial representation on H. And if Γ has a notrivial first cohomology group in a finite representation, then the kernel of this representation, a finite index subgroup H, has $H^1(H, \mathbb{R}) \neq 0$, i.e. H has infinite abelianization. □

Proposition 1.31 *Property FAb does not imply property (τ).*

To see this, let us first observe

Proposition 1.32 *Let Γ be a finitely generated, residually finite group with property (τ). Then:*

1. *If Γ is finitely generated and infinite then Γ has exponential growth.*

2. *If Γ is amenable then Γ is finite.*

Proof 1 is a consequence of 2. To see 2 we consider Γ which has property (τ) with respect to a finite subset S. If Γ is amenable let us consider a Følner sequence A_n of finite subsets of Γ such that for every $s \in S$

$$|A_n \triangle sA_n|/|A_n| \to 0 \qquad (1.2)$$

when n tends to infinity.

Let Γ_n be a sufficiently large quotient of Γ so that the projection of A_n into Γ_n is injective and there exists $\gamma_n \in \Gamma_n$ such that A_n and $B_n = \gamma_n A_n$ are disjoint.

The function $v_n = \chi_{A_n} - \chi_{B_n}$ is in $l_0^2(\Gamma_n)$ and by property (1.2) we have for the corresponding unitary representation ρ_n of Γ in $l_0^2(\Gamma_n)$ for every $s \in S$

$$\|\rho_n(s)v_n - v_n\|/\|v_n\| \to 0$$

which contradicts property (τ). □

Proof of Proposition 1.31 The Grigorchuk group G [80] has property FAb and does not have property (τ). The group G has property FAb because it is a torsion group. It does not have property (τ) because it is infinite, amenable, residually finite group (see Proposition 1.32). □

We do not know any example of a finitely presented group with property FAb and without property (τ).

In Section 4.6 we mention a quantitative version for FAb which is valid for groups with property (τ) (or for the $\mathcal{L}$ completion $G = \Gamma_{\widehat{\mathcal{L}}}$ of Γ with $\tau(\mathcal{L})$ if $\mathcal{L} = Top(\mathcal{L})$).

Chapter 2

Various equivalent forms of (τ)

What makes property (τ) so useful is that it can be expressed in various equivalent forms related to different subjects. In this chapter it is shown that property (τ) can be reformulated in various forms: combinatorially, analytically, geometrically, cohomologically and even measure theoretically. The following theorem summarizes some of these reformulations (for explanations of notations see the various sections).

Theorem 2.1 *Let Γ be a fundamental group of a compact Riemanniann manifold M, with finite set of generators S. Then the following are equivalent:*

1. *Γ has property (τ);*
2. *The Cayley graphs $Cay(\Gamma/N, S)$ form a familly of expanders when N runs over the finite index subgroups of Γ;*
3. *$\lambda_1(M')$ is bounded away from 0 when M' runs over the finite sheeted covers of M;*
4. *The isoperimetric (Cheeger) constants $h(M')$ are bounded away from 0;*
5. *$H^1(\Gamma, L^2(\hat{\Gamma})) = 0$;*
6. *The Haar measure of $\hat{\Gamma}$ is the unique finitely additive Γ-invariant measure on $\hat{\Gamma}$.*

2.1 Combinatorial reformulation

2.1.1 Expanders and isoperimetric inequalities

Let X be a finite graph of degree k, i.e. for every vertex $x \in X$ there are k edges adjacent to x. For a finite subset of vertices $A \subset X$ we define its boundary ∂A as the set of edges with one extremity in A and another one in $X \setminus A$. We define the Cheeger isoperimetric constant $h(X)$ as

$$h(X) = \min \left\{ \frac{|\partial A|}{|A|}; A \subset X \text{ and } 1 \leq |A| \leq \frac{1}{2}|X| \right\}.$$

Definition 2.2 *A family of finite graphs $\{X_n\}$ of a fixed degree is called a family of expanders if there exists a constant $c > 0$ such that $h(X_n) \geq c$ for every n.*

Let Γ be a group generated by a finite set S, with $S = S^{-1}$ and $|S| = k$. Let $\mathcal{L} = \{N_i\}$ be a family of finite index subgroups. The graphs $X(\Gamma/N_i, S)$ are defined to be the k-regular graphs whose vertices are the left cosets of Γ/N_i and for every $s \in S$ and a coset aN_i ($a \in \Gamma$) we put an edge from aN_i to saN_i. These are called Schreier graphs of Γ with respect to $\{N_i\}$ and S. If $N_i \lhd \Gamma$ this is simply the Cayley graph of Γ/N_i.

We will see below that $\tau(\mathcal{L})$ is equivalent to $X(\Gamma/N_i, S)$ being a family of expanders.

2.1.2 The spectral gap

For a connected, regular graph X of degree k, let Δ be the discrete Laplace operator acting on $l^2(X)$ as follows

$$\Delta f(x) = f(x) - \frac{1}{k} \sum_{y \sim x} f(y),$$

where $f \in l^2(X)$, x is a vertex of X and $y \sim x$ means that y and x are connected by an edge. The operator Δ is self-adjoint and non-negative. For an infinite graph X, let $\lambda_0(X)$ denote the bottom of l^2 spectrum of Δ and for a finite graph X, let $\lambda_1(X)$ denote the first non-zero eigenvalue of Δ.

Large λ_1 translates into fast convergence of random walks. Namely, assume also that X is not bipartite, for example it has at least one loop. Fix a vertex $v \in X$ and denote by Q_v^t the probability distribution of the nearest neighbor random walk on X starting at v after t steps. Since X is connected and non–bipartite, the random walk has a stationary distribution, which is uniform since X is regular:

$$Q_v^t(w) \to \frac{1}{|X|}, \quad \text{as} \quad t \to \infty, \quad \text{for all} \quad w \in X.$$

The total variation distance is defined as

$$\|P - Q\|_{TV} = \max_{B \subset X} |P(B) - Q(B)| = \frac{1}{2} \sum_{w \in X} |P(w) - Q(w)|,$$

where P, Q are two probability distributions on X. By U denote the uniform distribution. Now define a **mixing time** mix_v of the random walk as follows:

$$mix_v = \min \left\{ t;\ \|Q_v^t(w) - U\|_{TV} < \frac{1}{e} \right\}$$

A classical and easy bound on the variation distance gives:

$$\|Q_v^t - U\|_{TV} \leq \frac{\sqrt{|X|}}{2} (1 - \lambda_1(X))^t$$

From here we immediately have $mix_v < C(\lambda_1(X)) \log |X|$. More precisely:

Proposition 2.3 *For the total variation distance* $\|Q_v^t - U\|_{TV}$, $v \in X$ *of the random walk on* X *we have:*

$$\|Q_v^t - U\|_{TV} \leq e^{-c}, \quad \text{for} \quad t \geq \frac{\log |X| + c}{\lambda_1(X)}.$$

In particular, for the mixing time mix_v *we have:*

$$mix_v \leq \frac{1}{\lambda_1} (\log |X| + 1).$$

2.1.3 Equivalent definitions

There is a relation between the isoperimetric constant $h(X)$ and the eigenvalues $\lambda_0(X)$ and $\lambda_1(X)$ of Δ acting on $l^2(X)$. The following results are due to Dodziuk [58], Alon [1], Dodziuk-Kendal [59], Alon-Milman [3] and Mohar [152].

Proposition 2.4 *For a finite graph X of degree k one has*

$$h(X) \geq \frac{1}{2}k \cdot \lambda_1(X)$$

and

$$h(X) \leq k\sqrt{\lambda_1(2-\lambda_1)}.$$

This leads to the following characterization of property (τ).

Proposition 2.5 *Let S be a finite set of generators of the group Γ and $\mathcal{L} = \{N_i\}$ as before. The following conditions are equivalent*

1. *Γ has property (τ) with respect to the family $\mathcal{L}$,*

2. *the graphs $X(\Gamma/N_i, S)$ form a family of expanders, i.e. there exists a constant $\varepsilon > 0$ such that*

$$h(X(G/N_i, S)) \geq \varepsilon$$

   for every N_i,

3. *there exists $\varepsilon' > 0$ such that*

$$\lambda_1(X(G/N_i, S)) \geq \varepsilon'$$

   for every N_i.

Just to see the connection, let us show that 1 implies 2. Let A be any subset of $X_i = X(\Gamma/N_i, S)$ of size $|A| \leq \frac{|X|}{2}$. Let $f \in l^2(\Gamma/N_i)$ be a function defined by

$$f(x) = \begin{cases} |X| - |A| & \text{if} \quad x \in A \\ -|A| & \text{if} \quad x \notin A \end{cases}$$

So $f \in l_0^2(\Gamma/N_i)$ as $\sum_{x\in X} f(x) = 0$. Now, for any $s \in S$, by the definition of $|\partial A|$ we have

$$\|f - \pi(s)f\|^2 \leq (|X| - |A| + |A|)^2 |\partial A| = |X|^2 |\partial A|.$$

As Γ has property (τ) there exists $\varepsilon > 0$ independent of N_i such that

$$\max_{s\in S} \|f - \pi(s)f\|^2 \geq \varepsilon \|f\|^2.$$

Because $\|f\|^2 \geq |A||X|^2/2$ we get

$$|\partial A| \geq \frac{\varepsilon}{2}|A|.$$

For a group Γ generated by a finite set S one can also define the Markov operator M in the representation π of Γ in $\hat{\oplus} l_0^2(\Gamma/N_i)$ as

$$Mf = \frac{1}{|S|}\sum_{s\in S}\pi(s)f \tag{2.1}$$

for $f \in \hat{\oplus} l_0^2(\Gamma/N_i)$.

Proposition 2.6 *Let S be a finite set of generators of the group Γ and $\mathcal{L} = \{N_i\}$ as before. The following conditions are equivalent*

1. *Γ has property (τ) with respect to the family $\mathcal{L}$,*

2. *for the operator M defined by (2.1) we have $\|M\| < 1$.*

Proof The condition 2 here is equivalent to condition 3 from Proposition 2.5. □

2.2 Analytic and geometric reformulation

2.2.1 Isoperimetric inequalities

For a Riemannian manifold M^n of dimension n and finite volume we define its Cheeger isoperimetric constant $h(M^n)$ as follows

$$h(M^n) = \inf\left\{\frac{vol_{n-1}(\partial A)}{vol_n(A)}; A \subset M^n, 0 < vol_n(A) \leq \frac{vol_n(M^n)}{2}\right\},$$

where $vol_{n-1}(\partial A)$ and $vol_n(A)$ are the measures with respect to the Riemannian metric.

2.2.2 The Laplace operator

For a connected, smooth manifold M with a Riemannian metric g let Δ be the Laplace-Beltrami operator associated to g, i.e.

$$\Delta = -div(grad).$$

Explicitly, for $\mathbb{R}^2$ with the standard metric

$$\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$$

and for the upper half-plane $\mathbb{H}^2 = \{x + iy; x, y \in \mathbb{R}, y > 0\}$ with the metric $dy^2 = \frac{1}{y^2}(dx^2 + dy^2)$

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

The operator Δ is self-adjoint and non-negative. For a manifold M of infinite volume, let $\lambda_0(M)$ denote the bottom of L^2 spectrum of Δ and for a manifold M of finite volume, let $\lambda_1(M)$ denote the bottom of the spectrum of Δ on $L^2(M)$.

2.2.3 Discretization

For a Riemannian manifold M we can consider its simplicial subdivision. There is a relation between λ_0 and λ_1 for the Laplace-Beltrami operator and for the Laplace operator associated to the simplicial division.

Let $\widetilde{M}$ be the universal cover of M. Then Γ acts on $\widetilde{M}$ and $M = \widetilde{M}/\Gamma$. Let $\mathcal{F}$ be a connected, closed fundamental domain for Γ, for instance the Dirichlet fundamental domain, i.e.

$$\mathcal{F} = \{x \in \widetilde{M}; dist(x_0, x) \leq dist(x_0, \gamma(x)) \text{ for all } \gamma \in \Gamma\},$$

where x_0 is some fixed point in $\widetilde{M}$. Let us suppose that $\mathcal{F}$ is compact and that the N_i's are normal in Γ. Thus $\mathcal{F}$ has a finite number of faces $S_1, \ldots, S_r$. Then take $s_i \in \Gamma$ such that $s_i(\mathcal{F}) \cap \mathcal{F} = S_i$. It is well known that $S = \{s_1, \ldots, s_s\}$ generates Γ. Now every one of the finite sheeted coverings M_i of M is also covered by $\widetilde{M}$ with $\pi_i : \widetilde{M} \to M_i$ the covering map. Then $\pi_i(\mathcal{F})$ is a fundamental domain for the action of Γ/N_i on M_i

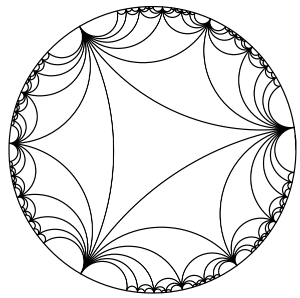

Figure 2.1: A tessellation of $\mathbb{H}^2$

and $M_i/(\Gamma/N_i) = M$. We claim that the Cayley graph $X_i = X(\Gamma/N_i, S)$ can be "drawn" in a natural way on M_i as follows: The Γ/N_i-translations of $\pi_i(\mathcal{F})$ will be the vertices of the graph, and two vertices are adjacent if and only if they have a common face. Now any subset A of the vertices of X of size at most $\frac{|X|}{2}$ gives rise to a subset of M_i of area at most $vol(M_i)/2$ and whose boundary is the union of the faces corresponding to the edges in ∂A. Since the volume of $\mathcal{F}$ is fixed and the areas of faces are bounded for all i we can conclude that a lower bound on $h(M_i)$ gives rise to a lower bound on $h(X(\Gamma/N_i), S)$. The converse is also true [24].

Actually when the approximation becomes more and more refined, under some assumption (the simplices should not be too much distorted) the discrete Laplace operator converges to the Riemannian Laplace operator. Instead of stating the general condition let us illustrate this on a simple example. Let $\varepsilon\mathbb{Z}^2$ be a square greed of size ε in $\mathbb{R}^2$ and let Δ_ε be the associated Laplace operator. Then for any $f \in C^2(\mathbb{R}^2)$

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\Delta_\varepsilon f(x,y)}{\varepsilon^2} &= \lim_{\varepsilon\to 0}\frac{f(x,y) - \frac{f(x+\varepsilon,y)+f(x-\varepsilon,y)+f(x,y+\varepsilon)+f(x,y-\varepsilon)}{4}}{\varepsilon^2}\\ &= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f(x,y).\end{aligned}$$

2.2.4 Equivalent definitions

Let M be an n dimensional compact manifold and and $R(M)$ its Ricci curvature. Theorems of Cheeger and Buser express a relation between λ_1 and the Cheeger constant h.

Proposition 2.7 (Cheeger [40]) *For any manifold M*

$$\lambda_1(M) \geq h^2(M)/4.$$

Proposition 2.8 (Buser [35]) *If $R(M) \geq -(n-1)a^2$ for some $a \geq 0$ where $n = \dim M$, then*

$$\lambda_1 \leq 2a(n-1)h(M) + 10h^2(M).$$

Property (τ) can be expressed using both quantities:

Proposition 2.9 (Brooks [24]) *Let Γ be the fundamental group of a compact manifold M with the universal cover $\widetilde{M}$ and let $\mathcal{L} = \{N_i\}$ be as before. The following conditions are equivalent:*

1. *Γ has property (τ) with respect to the family $\mathcal{L}$,*

2. *there exists a constant $\varepsilon > 0$ such that*

$$h(\widetilde{M}/N_i) \geq \varepsilon$$

   for every N_i,

3. *there exists $\varepsilon'' > 0$ such that*

$$\lambda_1(\widetilde{M}/N_i) \geq \varepsilon''$$

   for every N_i.

Proposition 2.9 can be obtained from Proposition 2.5 by the discretization method described in Section 2.2.3.

Remark Proposition 2.9 is still true for non-compact, finite volume manifolds M under additional assumption. For instance it is enough to suppose that $\lambda_1(M) > 0$ [26]. This is the case for quotients of symmetric spaces and more generally of manifolds with bounded curvature ([26]).

2.3 Measure theoretic reformulation

2.3.1 Uniqueness of invariant measures

A representation theoretic formulation of the problem of the uniqueness of invariant measures was discovered in [170].

Proposition 2.10 *Let G be a compact group. The Haar measure is the only finitely additive G-invariant measure on G if and only if G has a finitely generated subgroup whose action on $L_0^2(G)$ does not weakly contain the trivial representation.*

As a consequence we obtain

Corollary 2.11 *Assume $\mathcal{L}$ is a topology, i.e. $\mathcal{L} = Top(\mathcal{L})$. Then the group Γ has property (τ) with respect to $\mathcal{L} = \{N_i\}_{i\in I}$ if and only if the Haar measure is the only finitely additive Γ invariant measure on $\Gamma_{\widehat{\mathcal{L}}}$.*

Proof This follows from Proposition 1.15 and Proposition 2.10. □

2.4 Cohomological interpretation

There exists a cohomological characterization of property (T) (see Theorem 2.12 below) in terms of the first cohomology group with coefficients in unitary representations. In this section we present such a characterization for property (τ). As these results are new, we will give complete proofs.

Let $(\pi, \mathcal{H})$ be a unitary representation of Γ. Let $Z^1(\Gamma, \pi)$ be the space of 1-cocycles, i.e.

$$Z^1(\Gamma, \pi) = \{b : \Gamma \to \mathcal{H}; b(g_1 g_2) = \pi(g_1)b(g_2) + b(g_1)\}$$

and let $B^1(\Gamma, \pi)$ be the space of 1-coboundaries, i.e.

$$B^1(\Gamma, \pi) = \{b : \Gamma \to \mathcal{H};\ \text{there exists } v \in \mathcal{H},\ \ b(g) = \pi(g)v - v\}\,.$$

Of course we have $B^1 \subset Z^1$. The first cohomology group $H^1(\Gamma, \pi)$ is

$$H^1(\Gamma, \pi) = \frac{Z^1(\Gamma, \pi)}{B^1(\Gamma, \pi)}.$$

Suppose that Γ is countable ($\Gamma = \{g_1, g_2, \ldots\}$). We can endow Z^1 and thus B^1 with a metric as follows:

The distance $d(b_1, b_2)$ between the cocycles b_1 and b_2 is defined as

$$d(b_1, b_2) = \sum_{i=1}^{\infty} 2^{-i} \min\{2, \|b_1(g_i) - b_2(g_i)\|\}.$$

The space Z^1 is complete with respect to this distance but B^1 does not need to be a closed subspace. The space H^1 endowed with the quotient topology is Hausdorff if and only if B^1 is closed. The following result is due to Guichardet [88] and Delorme [52].

Theorem 2.12 *Let Γ be a countable group. The following conditions are equivalent:*

1. *Γ has property (T);*
2. *$H^1(\Gamma, \pi)$ is Hausdorff for every $\pi \in \widetilde{\Gamma}$.*
3. *$H^1(\Gamma, \pi) = 0$ for every $\pi \in \widetilde{\Gamma}$.*

We shall prove a similar characterization of property (τ) for a countable group Γ.

Proposition 2.13 *Let $(\pi, \mathcal{H})$ be unitary representation of Γ without invariant vectors. The following conditions are equivalent:*

1. *$H^1(\Gamma, \pi)$ is Hausdorff;*
2. *π does not have almost invariant vectors.*

Proof If π does not have almost invariant vectors then for some finite set S there exists $\varepsilon > 0$ such that for any $v \in \mathcal{H}$

$$\sum_{s \in S} \|\pi(s)v - v\| \geq \varepsilon \|v\|.$$

Because S is finite there exists $\varepsilon' > 0$ such that

$$\varepsilon' \sum_{s \in S} \|\pi(s)v - v\| \leq \sum_{i=1}^{\infty} 2^{-i} \|\pi(g_i)v - v\|$$

for every $v \in \mathcal{H}$. This implies that there exists ε'' (depending only on S) such that for any coboundaries b_1, b_2 associated to v_1, v_2 we have

$$\begin{aligned} d(b_1, b_2) &\geq \min\{\varepsilon'', \varepsilon' \sum_{s \in S} \|\pi(s)(v_1 - v_2) - (v_1 - v_2)\|\} \\ &\geq \min\{\varepsilon'', \varepsilon' \varepsilon \|v_1 - v_2\|\}. \end{aligned}$$

This implies that B^1 is closed.

Conversely let us suppose that B^1 is closed. Consider the map $b : \mathcal{H} \to B^1$ given by

$$b(v)(g) = \pi(g)v - v.$$

There are no invariant vectors in $\mathcal{H}$, so the map b is injective. Moreover b is continuous and linear. As $\mathcal{H}$ and B^1 are Fréchet spaces and b is bounded, by the closed graph theorem, the map b admits a bounded inverse b^{-1}. This means that there exists $\varepsilon > 0$ such that for any $v \in \mathcal{H}$ with $\|v\| = 1$,

$$\varepsilon \|v\| \leq d(b(v), 0). \tag{2.2}$$

Choose $n(\varepsilon)$ sufficiently large such that

$$\sum_{i=n(\varepsilon)}^{\infty} 2^{-i+1} \leq \frac{\varepsilon}{2}.$$

As π is unitary, for every $v \in \mathcal{H}$

$$\sum_{i=n(\varepsilon)}^{\infty} 2^{-i} \|\pi(g_i)v - v\| \leq \sum_{i=n(\varepsilon)}^{\infty} 2^{-i+1} \|v\| \leq \frac{\varepsilon}{2} \|v\|,$$

which together with (2.2) gives

$$\sum_{i=1}^{n(\varepsilon)-1} \|\pi(g_i)v - v\| \geq \frac{\varepsilon}{2} \|v\|.$$

The above inequality means that for the set $S = \{g_1, \ldots, g_{n(\varepsilon)-1}\}$ there are no almost invariant vectors. □

Proposition 2.14 *Let $\mathcal{L} = \{N_i\}_{i \in I}$ be a family of finite index subgroups of Γ. The following conditions are equivalent:*

1. *the group* Γ *has property* (τ) *with respect to* $\mathcal{L} = \{N_i\}_{i\in I}$*;*

2. $H^1(\Gamma, \widehat{\bigoplus} l^2(\Gamma/N_i))$ *is Hausdorff.*

   If, in addition, $\mathcal{L} = Top(\mathcal{L})$ *then also*

3. $H^1(\Gamma, L^2(\widehat{\Gamma}_{\mathcal{L}}))$ *is Hausdorff.*

Proof For $\pi = \widehat{\bigoplus} l_0^2(\Gamma/N_i)$ this follows from Proposition 2.13 and Proposition 1.12 as π has no invariant vectors. If instead of l_0^2 we consider l^2 the cohomology with coefficients in the trivial representation can appear. This corresponds to the homomorphisms in $\mathbb{R}$ and in this case B^1 is always closed. Similarly with $\pi = L^2(\widehat{\Gamma}_{\mathcal{L}})$. □

Comparing with Theorem 2.12, one may tempt to believe that $\tau(\mathcal{L})$ is equivalent also to the vanishing of $H^1(\Gamma, \widehat{\bigoplus} l^2(\Gamma/N_i))$. This is not quite true. We will see below that the free group $F_n = \{x_1, \dots, x_n\}$, $n \geq 2$ has (τ) with respect to some sequence $\mathcal{L} = \{N_i\}$ with $\bigcap N_i = 1$. Now clearly $H^1(F_n, V) \neq 0$ for every $V \neq 0$ as every map $\phi : \{x_1, \dots, x_n\} \to V$ can be extended to a cocycle on V.

The vanishing of $H^1(\Gamma, \widehat{\bigoplus} l^2(\Gamma/N_i))$ is somewhat stronger than $\tau(\mathcal{L})$. For the statement we will need property $FAb(\mathcal{L})$ (see Definition 1.28).

Proposition 2.15 *Let* Γ *and* $\mathcal{L} = \{N_i\}_{i\in I}$ *be as before. The following conditions are equivalent:*

1. Γ *has property* $\tau(\mathcal{L})$ *and* $Hom(N_i, \mathbb{R}) = 0$ *for every* $i \in I$*,*

2. $H^1(\Gamma, \widehat{\bigoplus} l^2(\Gamma/N_i)) = 0$*.*

   If in addition $\mathcal{L} = Top(\mathcal{L})$ *then also*

3. $H^1(\Gamma, L^2(\widehat{\Gamma}_{\mathcal{L}})) = 0$*.*

Before proving Proposition 2.15, note that in general property $\tau(\mathcal{L})$ does not imply $FAb(\mathcal{L})$, but if Γ is a finitely generated and has property (τ) (with respect to all finite index subgroups) then Γ has FAb. Thus we have

Theorem 2.16 *Let* Γ *be a finitely generated group. The following conditions are equivalent:*

1. Γ *has* (τ)*;*

2. *$H^1(\Gamma, \widehat{\bigoplus} l^2(\Gamma/N_i))$ is Hausdorff for the family $\{N_i\}$ of all finite index subgroups of Γ;*

3. *$H^1(\Gamma, \widehat{\bigoplus} l^2(\Gamma/N_i)) = 0$ for the family $\{N_i\}$ of all finite index subgroups of Γ;*

4. *$H^1(\Gamma, L^2(\widehat{\Gamma}))$ is Hausdorff;*

5. *$H^1(\Gamma, L^2(\widehat{\Gamma})) = 0$.*

Let us recall that a cocycle $b \in Z^1(\Gamma, (\rho, \mathcal{H}))$ vanishes in $H^1(\Gamma, (\rho, \mathcal{H}))$ if and only if it is bounded. Indeed, b is a cocycle if and only if

$$\alpha = \rho + b$$

is an affine action on $\mathcal{H}$. Now as ρ is an isometry, b is bounded if and only if α has a bounded orbit. The latter is equivalent to the fact that α has a fixed point, which means that b is a coboundary.

Proof of Proposition 2.15 The equivalence of (2) and (3) in case $\mathcal{L} = Top(\mathcal{L})$ is clear. So we need to prove only the equivalence of (1) and (2). We start with the following lemma, which just a special case of the Shapiro Lemma. We bring the proof to exhibit the role played by $Hom(N, \mathbb{R})$.

Lemma 2.17 *Let N be a normal subgroup of Γ of finite index. Then*

$$H^1(\Gamma, l^2(\Gamma/N)) = 0$$

if and only if $Hom(N, \mathbb{R}) = 0$.

Proof Suppose that $Hom(N, \mathbb{R}) = 0$. Let b be a cocycle, $b : \Gamma \to l^2(\Gamma/N)$. For any $\gamma_1, \gamma_2 \in N$ we have

$$b(\gamma_1\gamma_2) = b(\gamma_1) + b(\gamma_2).$$

As N has no non-trivial homomorphism into $\mathbb{R}$, b is trivial on N. Because N is of finite index in Γ this implies that b is bounded and thus H^1 vanishes.

Now suppose that $Hom(N, \mathbb{R}) \neq 0$, i.e. there exists a non-trivial homomorphism $b : N \to \mathbb{R}$. This can be extended to a cocycle on Γ (by induced representation) which has to be nontrivial because it is unbounded. This finishes the proof of Lemma 2.17. $\square$

Let us now return to the proof of Proposition 2.15. First we show that 1 implies 2. Consider a cocycle $b : \Gamma \to \widehat{\oplus} l^2(\Gamma/N_i)$ and let b_i denote its restriction to $l^2(\Gamma/N_i)$, i.e. $b = (b_1, b_2, \ldots)$. By Lemma 2.17 for every i there exists $v_i \in l^2(\Gamma/N_i)$ (in fact we can suppose that $v_i \in l^2_0(\Gamma/N_i)$) such that

$$b_i(g) = \pi(g)v_i - v_i.$$

So b is "locally" a coboundary and we should show that it is also a couboundary "globally". As Γ has property (τ), there exists a finite subset S of Γ such that for every i,

$$\max_{s\in S} \|b_i(s)\| \geq \varepsilon \|v_i\|$$

for some $\varepsilon > 0$ independent of i. Thus for some $\varepsilon' > 0$,

$$\sum_{s\in S} \|b(s)\|^2 \geq \varepsilon' \sum_{i=1}^{\infty} \|v_i\|^2,$$

which means that $v = (v_1, \ldots) \in \widehat{\bigoplus} l^2(\Gamma/N_i)$ and $b = \pi(g)v - v$ is a coboundary.

Now let us show that 2 implies 1. By Lemma 2.17, 2 implies that $Hom(N, \mathbb{R}) = 0$ for every $i \in I$. Suppose that Γ does not have property (τ) with respect to the family $\mathcal{L}$. Then by Proposition 2.14, $H^1(\Gamma, \widehat{\bigoplus} l^2(\Gamma/N_i))$ is not Hausdorff and in particular is not zero.

If the group Γ is generated by a finite set S we can even explicitly build a cocycle which is not a coboundary. Namely if Γ does not have (τ), then for some infinite subsequence $\{i_j\}_{j=1}^{\infty}$, there exists a sequence of vectors $v_{i_j} \in l^2_0(\Gamma/N_{i_j})$ of norm 1 such that

$$\sum_{j=1}^{\infty} \sum_{s\in S} \|\pi(s)v_{i_j} - v_{i_j}\|^2 < \infty. \tag{2.3}$$

Let b be a cocycle, $b : \Gamma \to \widehat{\oplus} l^2(\Gamma/N_i)$ and as before $b = (b_1, b_2, \ldots)$ where

$$\begin{aligned} b_{i_j}(g) &= \pi(g)v_{i_j} - v_{i_j}, \\ b_i(g) &= 0 \text{ for other indices } i. \end{aligned}$$

By (2.3) b is indeed a cocycle in $\widehat{\oplus} l^2(\Gamma/N_i)$. But it is not a coboundary because if there is $v' = (v'_1, v'_2, \ldots)$ such that $b(g) = \pi(g)v' - v'$, then $\pi(g)(v -$

$v') - (v - v') = 0$. This means that for every $i \in I$ we have $v' = v + cosnt$, where *const* is a constant function in $l^2(\Gamma/N_i)$. In particular for indexes i_j, $\|v'_{i_j}\|^2 = const^2 + \|v_{i_j}\|^2 \geq \|v_{i_j}\|^2$ as $v_{i_j} \in l_0^2(\Gamma/N_i)$. Thus $\sum_{j=1}^{\infty} \|v'_{i_j}\|^2 \geq \sum_{j=1}^{\infty} \|v_{i_j}\|^2 = \infty$. □

As a consequence we obtain the following reformulation of the cohomological characterizations of property (τ).

Proposition 2.18 *The group Γ has property (τ) with respect to $\mathcal{L} = \{N_i\}_{i \in I}$ if and only if*

$$\bigcap_i \ker \left(H^1(\Gamma, \widehat{\bigoplus} l^2(\Gamma/N_i)) \to H^1(\Gamma, l_0^2(\Gamma/N_i)) \right) = 0.$$

Proposition 2.19 *If $\mathcal{L} = Top(\mathcal{L})$, the group Γ has property (τ) with respect to $\mathcal{L} = \{N_i\}_{i \in I}$ if and only if*

$$\bigcap_i \ker \left(H^1(\Gamma, L^2(\Gamma_{\widehat{\mathcal{L}}})) \to H^1(\Gamma, l_0^2(\Gamma/N_i)) \right) = 0.$$

2.5 Fixed point property

There is a very nice characterization of property (T) in terms of a fixed point property. Namely

Theorem 2.20 *A group Γ has property (T) if and only if every affine action by isometries of G on a Hilbert space has a global fixed point.*

Proof An affine action on a Hilbert space $\mathcal{H}$ gives a cocyle (and vice versa). A cocycle $b \in Z^1(\Gamma, (\rho, \mathcal{H}))$ which vanishes in $H^1(\Gamma, (\rho, \mathcal{H}))$ if and only if it is bounded, Now, b is bounded if and only if the action has a global fixed point. Thus Theorem 2.20 follows from Theorem 2.12. □

To every tree one can associate a Hilbert space so that to any action without a fixed point on a tree corresponds an action without a fixed point on this Hilbert space. Thus from the above one can deduce

Theorem 2.21 (Alperin [4]) *If a group Γ with property (T) acts on a tree, it has a fixed point.*

A similar characterization as Theorem 2.20 holds also for property (τ).

Proposition 2.22 *The group Γ has property (τ) if and only if for every affine action α of Γ on a Hilbert space H such that $H = \widehat{\oplus} H_i$, where H_i is finite dimensional and Γ invariant with a finite Γ-action, α has a global fixed point.*

Proof This follows from the cohomological characterization of property (τ) and from the fact that $H^1(\Gamma, (\pi, \mathcal{H})) = 0$ if and only if every affine action on a Hilbert space $\mathcal{H}$ with the linear part π has a fixed point. □

However Theorem 2.21 does not hold for property (τ). We will see below, that $SL_2(\mathbb{Z}[\frac{1}{p}])$ has property (τ) but it acts without a fixed point on the Bruhat-Tits tree of $SL_2(\mathbb{Q}_p)$.

Note that for general $\mathcal{L}$, H^1 may not vanish if Γ does not have the property that $Hom(N, \mathbb{R}) = 0$ for every $N \in \mathcal{L}$. So there might be an affine action of Γ without fixed points, even if Γ has $\tau(\mathcal{L})$.

Chapter 3

Quantitative (τ) and abelian quotients

In Chapter 2, we saw various different ways to characterize property (τ) for a group Γ with respect to a family $\mathcal{L} = \{N_i\}$ of finite index subgroups. We have used the invariants $\lambda_1(\Gamma/N_i)$, $h(\Gamma/N_i)$, $\lambda_1(M_i)$ and $h(M_i)$ when M is a compact Riemannian manifold with $\pi_1(M) = \Gamma$ and M_i is the covering of M corresponding to N_i. The failure of (τ) asserts that $\lambda_1(\Gamma/N_i)$, $h(\Gamma/N_i)$, $\lambda_1(M_i)$ and $h(M_i)$ all have subsequences that tend to zero. In this chapter we show that if this convergence is fast enough, the positive virtual β_1 can be deduced, i.e., one of the N_i's has an infinite abelianization. This issue of the positive virtual β_1 is of fundamental importance in geometry - see Chapter 7 below.

We will also see in this chapter that property (τ) for Γ implies a quantitative bound on the size of abelian quotients of the index n subgroups of Γ and on the number of finite n-dimensional representations of Γ.

3.1 Quantitative (τ)

The following theorem is a quantitative version of Theorem 2.1.

Theorem 3.1 (Lackenby [113]) *Let Γ be a* **finitely presented** *group, and let S be a finite set of generators. Let $\{N_i\}$ be its finite index subgroups. Let X_i be the Schreier coset graph of Γ/N_i induced by S. Then the following are equivalent and independent of the choice of S:*

1. *Some* N_i *has an infinite abelianization;*
2. *There is a subsequence with bounded abnormility and bounded* $\lambda_1(X_i)|X_i|^2$*;*
3. *There is a subsequence with bounded abnormility, and with* $\lambda_1(X_i)|X_i|$ *having zero infimum;*
4. *There is a subsequence with bounded abnormility and bounded* $h(X_i)|X_i|$*;*
5. *There is a subsequence with bounded abnormility, and with* $h(X_i)\sqrt{|X_i|}$ *having zero infimum.*

   Furthermore, if Γ *is the fundamental group of some closed orientable Riemannian manifold* M*, and* M_i *is the cover of* M *corresponding to* N_i*, then the above are also equivalent to each of the following:*
6. *There is a subsequence with bounded abnormility and bounded* $\lambda_1(M_i)vol(M_i)^2$*;*
7. *There is a subsequence with bounded abnormility, and with* $\lambda_1(M_i)vol(M_i)$ *having zero infimum;*
8. *There is a subsequence with bounded abnormility and bounded* $h(M_i)vol(M_i)$*;*
9. *There is a subsequence with bounded abnormility, and with* $h(M_i)\sqrt{vol(M_i)}$ *having zero infimum.*

Proof First of all we only need to consider conditions 1-5 as conditions 6-9 are equivalent to their discrete analogues. The implications $1 \Rightarrow 2$ and $1 \Rightarrow 4$ follow from following

Lemma 3.2 *Consider the family of Cayley graphs of* $\mathbb{Z}/n\mathbb{Z}$ *with respect to fixed generators of* $\mathbb{Z}$*. The isoperimetric constant* h *in this case behaves like* $\frac{const}{n}$ *and* λ_1 *behaves like* $\frac{const}{n^2}$*.*

Proof Consider the standard generators $\{\pm 1\}$. By taking $A = \{1, \dots, n/2\}$ we can see that $h \leq \frac{n}{2}$ and of course every subset of size at most $\frac{n}{2}$ has at least two edges in the boundary. This shows the first assertion.

By the second statement in Proposition 2.7. $\lambda_1 \geq const \cdot h^2 \geq \frac{const'}{n^2}$. To see the upper bound consider the function $f(k) = e^{\frac{2\pi ik}{n}}$ for $k \in \mathbb{Z}/n\mathbb{Z}$. Then $f \in l_0^2(\mathbb{Z}/n\mathbb{Z})$. We get

$$\Delta f(k) = f(k) - \left(\frac{e^{\frac{2\pi i}{n}} + e^{\frac{-2\pi i}{n}}}{2}\right) f(k) = \left(1 - \cos\frac{2\pi}{n}\right) f(k)$$

and $1 - \cos \frac{2\pi}{n} \leq \frac{const}{n^2}$. This shows that $\lambda_1 \leq \frac{const}{n^2}$. □

The implications 2 ⇒ 3 and 4 ⇒ 5 are clear. The implications 2 ⇒ 4 and 3 ⇒ 5 follow from the second statement in Proposition 2.7. The link between the statements for graphs and manifolds is explained in Sections 2.1 and 2.2. What remains to show is 5 ⇒ 1. This is done in Section 3.2. □

3.2 Small isoperimetric constant implies positive β_1

The main new idea in the proof of 5 ⇒ 1 of Theorem 3.1 is the assertion that if the 1-skeleton of a triangulation of a 2-complex K whose 1-skeleton is a Cayley graph, has small isoperimetric Cheeger constant then $H^1(K) \neq 0$. The following proposition is implicit in [113].

Proposition 3.3 *Let K be a triangulated 2-complex and let X be its 1-skeleton. Assume X has two subsets A_1 and A_2 such that*

1. *$A_1 \cap A_2 \neq \emptyset$ and $A_1 \cup A_2 \neq X$;*
2. *$\partial A_1 \cap \partial A_2 = \emptyset$;*
3. *A_1, A_2, A_1^c and A_2^c are connected;*
4. *$A_1 \setminus A_2 \neq \emptyset$ and $A_2 \setminus A_1 \neq \emptyset$.*

   Then $H^1(K) \neq 0$.

Proof Let χ_A be the characteristic function of A and $C_+ = d\chi_{A_1}$, i.e., C_+ is a function on the edges of X, such that if $e = (x, y)$ then $C_+(e) = \chi_{A_1}(x) - \chi_{A_1}(y)$. So, $C_+(e)$ is non-zero if and only if one end of e is in A_1 and the other is not.

Let C be the function of the edges of X, which agrees with C_+ on edges whose both end points are in A_2 and is zero otherwise. So $C(x, y) \neq 0$ if and only if both x and y are in A_2 and exactly one of them is in A_1.

We claim

1. C is a 1 cocycle;
2. C is not a coboundary.

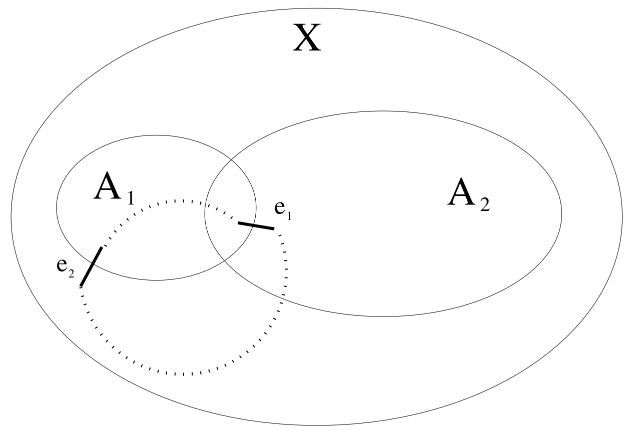

Figure 3.1: Construction of the loop

So 1 and 2 will finish the proof of the proposition.

To prove 1 we need to show that for every triangle (x, y, z) the sum of C along the edges is zero. We need only to consider the situation when C is not zero on some of the edges of the triangle (x, y, z), i.e. either exactly one vertex is in A_1 or exactly one vertex is in A_1^c. By symmetry (note that all conditions of Proposition 3.3 are preserved if we replace A_1 by A_1^c and A_2 by A_2^c) it is enough to consider only the first case, so assume $x \in A_1$ and $y, z \notin A_1$. Thus $(x, y), (x, z) \in \partial A_1$. The condition $\partial A_1 \cap \partial A_2 = \emptyset$ implies that if $x \in A_2$ then $y, z \in A_2$ and if $x \notin A_2$ then $y, z \notin A_2$. Thus either $x, y, z \in A_2$ or $x, y, z \notin A_2$. In the first case C coincides with C_+ on the edges of the triangle and in the second case it is zero. In both cases the sum of C along the edges of the triangle (x, y, z) is zero.

To prove 2 we construct a loop in X along which the sum of C is not zero. Let e_1 and e_2 be the edges in ∂A_1 such that $e_1 \in A_2$ and $e_2 \in A_2^c$, existence of which follows from the assumptions of the proposition. Because A_1 and A_1^c are connected we can connect the endpoints of e_1 and e_2 which are in A_1 by a path inside A_1, and the endpoints of e_1 and e_2 which are in A_1^c by a path inside A_1^c as in Figure 3.1. On this loop, C is not zero only on the edge e_1. In particular it does not sum to zero along this loop. □

In order to prove Theorem 3.1 we will also need the following lemma which describes properties of the subsets realizing the isoperimetric constant.

Lemma 3.4 *If A is a subset of a Cayley graph X such that $|A| \leq \frac{|X|}{2}$ and $\frac{|\partial A|}{|A|} = h(X)$ then $|A| > \frac{1}{4}|X|$, and the subgraphs induced on A and A^c are connected.*

Proof Let A be the smallest subset realizing the isoperimetric constant and suppose $|A| \leq \frac{1}{4}|X|$. Consider a translation B of A by any element of the group. Then

$$|\partial A| + |\partial B| - |\partial(A \cap B)| = |\partial(A \cup B)| + 2e(A \setminus B, B \setminus A),$$

where e denotes the number of edges connecting two subsets.

By the minimality of A,

$$|\partial(A \cap B)| \geq h(X)|A \cap B|, \tag{3.1}$$

and the equality holds if and only if $A \cap B = \emptyset$ or $A = B$. Therefore

$$\begin{aligned} |\partial(A \cup B)| &= |\partial A| + |\partial B| - |\partial(A \cap B)| - 2e(A \setminus B, B \setminus A) \\ &\leq h(X)(|A| + |B| - |A \cap B|) = h(X)|A \cup B|. \end{aligned}$$

This must be an equality since $A \cup B$ has size at most $|X|/2$. This implies equality in (3.1), and hence A and B concide or are disjoint and thus there are no edges between A and B. Since this applies to every translate of A, the connectivity of the graph X is contradicted. This shows that $|A| > \frac{1}{4}|X|$.

Now suppose that A is not connected. Each of its connected components also realizes the isoperimetric constant but some of its components has size at most $|X|/4$ which gives a desired contradiction.

If A^c were not connected, consider its non empty components B_1 and B_2. Their sizes must be at most $|X|/2$ as otherwise we might add the smaller set to A and obtain a bigger subset than (but still of size at most $|X|/2$) with a smaller boundary, contradicting the choice of A. For one of the components B_i we have

$$\frac{|\partial B_i|}{|B_i|} \leq \frac{|\partial(B_1 \cup B_2)|}{|B_1 \cup B_2)|}$$

Now $|\partial(B_1 \cup B_2)| = |\partial A|$, thus if $|A| < |B_1 \cup B_2|$ the above inequality would contradict the definition of the isoperimetric constant. Thus $|B_1 \cup B_2| \leq$

$|A| \leq |X|/2$ and one of the components, say B_1, has size at most $|X|/4$. Then $|\partial B_1| > h(X)|B_1|$ and thus

$$|A| = |\partial A|/h(X) = (|\partial B_1| + |\partial B_2|)/h(X) > |B_1| + |B_2| = |X| - |A|,$$

which implies $|A| > |X|/2$ and provides a desired contradiction. □

Using the above lemma one shows that if the isoperimetric constant $h(X)$ of a Cayley graph is small enough, a set realizing the $h(X)$ and a translation of it provide the sets A_1 and A_2 for Proposition 3.3. Indeed

Lemma 3.5 *Let X be a Cayley graph of a finite group G arising from a finite presentation of G with all relators of length 3. Suppose that $h(X) < \sqrt{\frac{2}{3}|X|}$. Then there exist the subsets A_1 and A_2 of X with properties 1-4 as in Proposition 3.3.*

Proof Fix a subset A with $\frac{|\partial A|}{|A|} = h(X)$ and $|A| \leq \frac{|X|}{2}$. Hence by Lemma 3.4 $|A| > |X|/4$. Then a counting argument shows that among 4-tuples of the elements $(g_1, \ldots, g_4) \in G^4$ there is a quadruple with $g_i(\partial A) \cap g_j(\partial A) = \emptyset$ for every $i \neq j$.

Indeed, pick an orientation on the edges of X that is preserved by the action of G. Let C denote initial vertices of ∂A. For $1 \leq i < j \leq 4$ define

$$p_{ij} : G^4 \to G$$

$$(g_1, \ldots, g_4) \to g_j^{-1} g_i.$$

If there is no quadruple with the desired property then the sets $p_{ij}^{-1}(CC^{-1})$ cover G^4. Each set has size $|G|^3|CC^{-1}|$ and so

$$|G|^4 \leq \binom{4}{2} |G|^3|C|,$$

which implies that

$$|G| \leq 6|C|^2 \leq 6|\partial A|^2 = 6(h(X)|A|)^2 < 6\left(\sqrt{\frac{2}{3|X|}}\frac{|X|}{2}\right)^2 = |G|.$$

This is a contradiction which shows the existance of the desired quadruple $(g_1, \ldots, g_4)$.

As $|A| > \frac{1}{4}|X|$, in the quadruple $(g_1, \dots, g_4)$ there exist i and j such that $g_i(A) \cap g_j(A) \neq \emptyset$. One takes $A_1 = g_i(A)$ and $A_2 = g_j(A)$ and checks that indeed they satisfy the assumptions of Proposition 3.3. □

Now, we can prove the implication 5 ⇒ 1 of Theorem 3.1. Suppose that condition 5 holds. It still holds if we change the generators. So we can assume we are given a presentation with all relators of length three. Now consider X_i such that $h(X_i) < \sqrt{\frac{2}{3}|X_i|}$. By Lemma 3.5 there exist the subsets A_1 and A_2 of X_i with properties 1-4 as in Proposition 3.3. By Proposition 3.3 this implies that $N_i = \pi_1(X_i)$ has infinite abelianization.

3.3 Quantitative bounds on abelian quotients

We saw in Corollary 1.29 that (τ) imples FAb. In fact we have a quantitive bound:

Theorem 3.6 (Lubotzky-Weiss [139]) *If the group Γ has property (τ) then there exists a constant c such that for every subgroup H of Γ of index n*

$$|H/H'| < c^n.$$

Proof of Theorem 3.6 We start with

Proposition 3.7 *Let Γ be a group generated by $S = \{x_1, \dots, x_d\}$. Assume Γ has (τ) with (S, ε) as (τ)-constant. Let H be a subgroup of Γ of index n. Then H has a subset of generators S' with $|S'| \leq nd$ such that (S', ε) is a (τ) constant for H.*

Proof Let $\gamma_1, \dots, \gamma_n$ be a set of representatives for the right H cosets in Γ. For $i \in \{1, \dots, d\}$ and $j \in \{1, \dots, n\}$ let $\gamma_{r(i,j)}$ be the representative of the coset $\gamma_j x_i$ and let $h_{ij} = \gamma_j x_i \gamma_{r(i,j)}^{-1}$. Clearly $h_{ij} \in H$ and it is not difficult to see that they generate H. We claim that $(S' = \{h_{ij}\}, \varepsilon)$ is a (τ) constant for H.

Let (ρ, V) be a finite unitary representation of H which does not contain the trivial representation. Let $(\widetilde{\rho}, \widetilde{V})$ be the induced representation from H to Γ. So $\widetilde{V}$ is the space of all functions $f : \Gamma \to V$, such that

(1) $f(h\gamma) = \rho(h) f(\gamma)$ for all $h \in H$, $\gamma \in \Gamma$.

Every such f is completely determined by its values on $\gamma_1,\dots,\gamma_n$ and $\|f\|^2_{\widetilde{V}}=\sum_{i=1}^n\|f(\gamma_i)\|^2_V$ and Γ acts by: $\widetilde{\rho}(\gamma)f(\gamma')=f(\gamma'\gamma)$.

Assume there exists $v_0\in V$ of norm 1 such that $\|\rho(h_{ij})v_0-v_0\|<\varepsilon$ for every $h_{ij}\in S'$. Look at $f_0\in\widetilde{V}$ defined by $f_0(\gamma_j)=\frac{v_0}{\sqrt{n}}$, $j=1,\dots,n$. Then $\|f_0\|^2=1$ and for every $i=1,\dots,d$

$$\begin{aligned}\|\rho(x_i)f_0-f_0\|^2 &= \sum_{j=1}^n\|f_0(\gamma_j x_i)-f_0(\gamma_j)\|^2\\ &= \sum_{j=1}^n\|\rho(h_{ij})f_0(\gamma_{r(i,j)})-f_0(\gamma_j)\|^2\\ &= \frac{1}{n}\sum_{j=1}^n\|\rho(h_{ij})v_0-v_0\|^2\le\varepsilon^2.\end{aligned}$$

Thus $(\widetilde{\rho},\widetilde{V})$ should contain the trivial representation. But this is impossible by the Frobenius reciprocity or by checking directly that for every fixed vector $w_0\in V$ the constant function $f(\gamma)=w_0$ is not in $\widetilde{V}$. □

Proposition 3.8 *Let A be an abelian group generated by a set S of size k. If (S,ε) is a Kazhdan constant for A, then*

$$|A|\le 2(2/\varepsilon^2+1)^k.$$

We will give two proofs.

Proof A Let $s_1,\dots,s_k$ be the elements of S and let $B=\{s_1^{i_1}\cdot\ldots\cdot s_k^{i_k};1\le i_j\le n\}$ where n is the largest possible so that $|B|\le\frac{|A|}{2}$. Let $C=A\setminus B$ be the complement of B in A so that $|C|\ge|B|$. We set $v=\frac{1}{\sqrt{|B|}}\chi_B-\frac{1}{\sqrt{|C|}}\chi_C$. Then $\|v\|=1$ and $v\in l_0^2(A)$. For any $s\in S$

$$\|\chi_B-s\chi_B\|\le\frac{|B|}{n}.$$

Indeed, for a given $s_t\in S$ we define by induction the sets $B_1^t,\dots,B_n^t$ as follows

$$B_1^t=\{s_1^{i_1}\cdot\ldots\cdot s_j^{i_j}\cdot\ldots\cdot s_k^{i_k};1\le i_j\le n\text{ for }j\ne t,\ i_t=1\}$$

and

$$B_i^t = s_t B_{i-1}^t \setminus (B_0^t \cup \ldots \cup B_{i-1}^t)$$

for $i = 1, \ldots, n$.

Then $B = B_1^t \,\dot{\cup}\, \ldots \,\dot{\cup}\, B_n^t$. By definition $|B_n^t| \leq \ldots \leq |B_i^t| \leq \ldots \leq |B_1^t|$. Thus $|B_n^t| \leq \frac{|B|}{n}$.

Now

$$|B \setminus s_t B| \leq |B_n^t| \leq \frac{|B|}{n}$$

as desired.

Therefore we have for the representation ρ of A in $l_0^2(A)$

$$\|v - \rho(s)v\| \leq \sqrt{\frac{2}{n} + \frac{2}{n}} = \frac{2}{\sqrt{n}}.$$

By definition of (S, ε) this implies

$$\frac{2}{\sqrt{n}} > \varepsilon$$

and by definition of n we get $|A| \leq 2(n+1)^k$, which shows the desired inequality. □

Proof B Say $|A| = n$. Then by Pontryagin duality A has n different irreducible one dimensional representations $\rho_i : A \to SU_1(\mathbb{C}) \simeq S^1$, $i = 1, \ldots, n$.

Let $r = \left[\frac{2}{\varepsilon^2}\right] + 1$ and divide S^1 into r segments between $\eta \cdot \xi^j$ to $\eta \cdot \xi^{j+1}$ where $\eta = e^{\pi i/r}$, $\xi = e^{2\pi i/r}$ and $j = 0, \ldots, r-1$. This induces a division of $(S^1)^k$ to r^k boxes. Now, for every representation ρ_i of A, $(\rho_i(s_1), \ldots, \rho_i(s_k))$ is in one of the boxes when $S = \{s_1, \ldots, s_k\}$ is the set of generators of S. If two representation ρ_i and $\rho_{i'}$ are in the same box then for every $l = 1, \ldots, k$ $\rho_i(s_l)^{-1}\rho_{i'}(s_l)$ is in the segment (η^{-1}, η) around 1. This means that the representation $\rho_i^{-1} \otimes \rho_{i'}$ is close to the identity within $|\eta - 1|^2 = 2 - 2\cos\frac{\pi}{r} < \frac{2}{r} < \varepsilon^2$. This is a contradiction so every two representations should be in different boxes. The pigeon hole principle implies that $n \leq \left(\left[\frac{2}{\varepsilon^2}\right] + 1\right)^k$. □

Now the Lubotzky-Weiss theorem follows from Proposition 3.7 and Proposition 3.8 □

Remark We will see some examples in Theorem 5.10 (as well as in the proof of Theorem 5.13) which show that the exponential bound in Theorem 3.6 is the best possible.

For groups which are finitely presented, Lackenby shows that it possible to obtain the conclusion of Theorem 3.6 assuming only property FAb.

Theorem 3.9 (Lackenby [113]) *If a finitely presented group Γ has property FAb then there exists a constant c such that for every subgroup H of Γ of index n*

$$|H/H'| < c^n.$$

Proof The crucial argument is the following proposition which may be of independent interest.

Proposition 3.10 *Let $\Gamma = \langle X; R\rangle$ be a group with a finite presentation with $|X|$ generators and $|R|$ relators. Denote by m the maximal length of relators in R. Let H be a subgroup of Γ of index n. Then H admits a finite presentation $\langle X'; R'\rangle$ with $|X'| \leq n|X|$ generators and $|R'| \leq n|R|$ relators whose length (in the generators from X') is at most m.*

We will prove the proposition in two different ways, or more precisely in two languages as the proofs are really equivalent.

Topological proof Let K be the 2-complex associated to the presentation $\langle X; R\rangle$, obtained as follows: K has one vertex. Put an oriented edge for every generator. Then for every relator of length r in R we glue an r-gon to the r edges correspong to the sequence of generators in this relator according to the orientation. For a finite index subgroup H of Γ, let K' be a cover of K such that $\pi_1(K') = H$. Collapse a maximal tree in K' to get a complex with just one vertex, which gives a presentation $H = \langle X'; R'\rangle$, where X' is the set of generators and R' the set of relators satisfying the properties stated in the proposition. □

Algebraic proof Recall the Reidemeister-Schreier method for writing a presentation for H from a presentation for Γ ([142] page 103): Let T be a Schreier transversal for H in Γ, i.e. T is a set of words in X which give coset representatives for H in Γ and such that a subsegment of an element in T is also in T (such a Schreier transversal always exists, since every graph has a spanning tree). Denote $\gamma(t, x) = tx(\overline{tx})^{-1}$ for $t \in T$ and $x \in X$ where $\overline{y}$ is the unique representative of y in T and for $w = y_1 \dots y_k$, where $y_i \in X \cup X^{-1}$ let $\tau(w) = \gamma(1, y_1) \dots \gamma(\overline{y_1 \dots y_{i-1}}, y_i) \dots \gamma(\overline{y_1 \dots y_{k-1}}, y_k)$. The well-known Reidemeister-Schreier algorithm asserts that H admits a presentation $\langle X'; R'\rangle$ where the generators X' are $\gamma(t, x)$ for $t \in T$, $x \in X$ and the relators R' are $\tau(twt^{-1})$, for $t \in T$ and $w \in R$. As T is a Schreier transversal one can check that the length of $\tau(twt^{-1})$ with respect to X' is at most the length of w with respect to X. □

To prove Theorem 3.9 we consider the abelianization of H which is a quotient of $\mathbb{Z}^{|X'|}$. As it is finite, the number of relators $|R'|$ is at least the number of generators $|X'|$. Take $R'' \subset R'$ with $|R''| = |X'|$ so that $\langle X'; R'' \rangle$ has finite abelianization. The order of the later is at least the order of the abelianization of H and can be estimated as follows. It is equal to the determinant of the square $|X'| \times |R''|$ matrix M whose i-th row v_i is the image in $\mathbb{Z}^{|X'|}$ of the i-th element of R''. By Proposition 3.10 the the length of the longest relator in R'' can be bounded by m, which implies that the ℓ^1 and thus ℓ^2 norm of every v_i can be bounded by m. Hence $\det(M) \leq m^{|X'|}$. As $|X'|$ can be bounded by $n|X|$, this gives a desired bound $m^{n|X|}$. □

3.4 Bounds on the number of representations

Property (τ) gives some information on the growth of the number of irreducible finite dimensional representations.

Theorem 3.11 (de la Harpe, Robertson, Valette [92]) *If the group Γ has property (τ) then there exists c such that the number of finite representations of Γ of dimension n is bounded by c^{n^2}.*

The proof of Theorem 3.11 goes as follows. Let $\mathcal{H} = \mathbb{C}^n$ be a fixed finite dimensional Hilbert space. Denote by $\|x\|_2 = (tr(x^*x))^{\frac{1}{2}}$ the Hilbert-Schmidt norm of an operator x on $\mathcal{H}$.

Fixing a set of generators $s_1, \dots, s_k$ for Γ, we can identify a representation $\rho : \Gamma \to U_n(\mathbb{C})$ with the k unitary matrices $(\rho(s_1), \dots, \rho(s_k)) \in U_n(\mathbb{C})^k$. Now a matrix in $U_n(\mathbb{C})$ has n columns, each in the sphere of radius one in $\mathbb{C}^n$, so a unitary matrix can be considered as an element of the sphere of radius $\sqrt{n}$ in $\mathbb{C}^{n^2}$. Finally the $\sqrt{n}$ sphere in $\mathbb{C}^{n^2}$ is inside the $\sqrt{n}$ sphere of $\mathbb{R}^{2n^2}$. So all together, $U_n(\mathbb{C})$ with its Hilbert-Schmidt norm can be thought as sitting in the $\sqrt{n}$ sphere of $\mathbb{R}^{2n^2}$ with its usual l^2 norm.

Now if ρ, σ are n dimensional irreducible unitary representations of Γ and $\|\rho(s_i) - \sigma(s_i)\|_{SU_n(\mathbb{C})} \leq \varepsilon\sqrt{n}$ for every i, then a simple computation shows that $\rho^* \otimes \sigma$ is ε-closed to the identity and hence contains the identity. But $\rho^* \otimes \sigma \simeq Hom(\rho, \sigma)$ which implies, therefore, that ρ and σ are isomorphic.

To summerize:

Lemma 3.12 ([193]) *Let Γ be a discrete group with property (τ) generated by a finite set S. There exists $\varepsilon > 0$ such that if π_1, π_2 are finite irreducible representations of Γ on $\mathcal{H} = \mathbb{C}^n$ such that $\|\pi_1(s) - \pi_2(s)\|_2 \leq \varepsilon\sqrt{n}$ for all $s \in S$ then π_1 is equivalent to π_2.*

The estimate on the number of representations in Theorem 3.11 follows from an estimate on the number of balls of the Hilbert-Schmidt radius $\varepsilon\sqrt{n}/2$ needed to cover the unitary group $U_n(\mathbb{C})$.

Lemma 3.13 ([195]) *There is a positive constant α such that for every n, $e^{\alpha n^2}$ balls of radius $\varepsilon/2$ can cover the unit sphere in $\mathbb{R}^{2n^2}$.*

Thus, $e^{\alpha n^2}$ balls of radius $\frac{\varepsilon\sqrt{n}}{2}$ can cover the $\sqrt{n}$ sphere of $\mathbb{R}^{2n^2}$. Applying it now for $\mathbb{R}^{2n^2}$ contaning $U_n(\mathbb{C})$, we deduce:

Proposition 3.14 *For every $\varepsilon > 0$, there exists a constant $\alpha > 0$ depending only on ε, such that if Γ is generated by a set S of size k and (S, ε) is a (τ) constant for Γ, then the number of n dimensional representations of Γ is at most $e^{\alpha k n^2}$.*

Indeed if Γ has more than $e^{\alpha k n^2}$ non-equivalent unitary representations of dimension n, then at least two of them ρ and σ satisfy $\|\rho(s_i) - \sigma(s_i)\| < \varepsilon\sqrt{n}$ for every $i = 1, \ldots, k$. But this implies by Lemma 3.12 that they are equivalent which gives a desired contradiction. This proves Proposition 3.14 and hence Theorem 3.11.

In some cases one can improve the above bounds.

Theorem 3.15 (Meshulam-Widgerson [150]) *If the group Γ has property (τ) and any complex irreducible representation of Γ is induced from a representation of a uniformly bounded dimension m of some subgroup $H < \Gamma$ then there exists c such that the number of finite representations of Γ of dimension n is bounded by $s_n(\Gamma) \cdot c^n$ where $s_n(\Gamma)$ is the number of subgroups of index at most n in Γ.*

Proof Every irreducible n-dimensional representation of Γ is induced from some irreducible representation σ of degree $m' \leq m$ of some subgroup H. Now $n = m'[\Gamma : H]$ so $[\Gamma : H] = \frac{n}{m'} \leq \frac{n}{m} \leq n$ and thus the number of possibilities for H is bounded by $s_n(\Gamma)$. Given such an H, it has by

Proposition 3.7, a generating set S_H of size at most $n \cdot k$ so that (S_H, ε) is a (τ)-constant for Γ. Now apply Proposition 3.14 to H, which has kn generators and (S_H, ε) as (τ) constant to deduce that the number of m'-dimensional representations of H is at most $e^{\alpha k n m'^2}$. As $m' < m$ is constant and k is a constant, the theorem is proved. □

Remark It is known - see Section 1.5.1 above and [133] that for finitely generated group the subgroup growth $s_n(\Gamma)$ grows at most as $e^{cn \log n}$ for every n, so in any event Theorem 3.15 gives the bound $e^{cn \log n}$ but under the strong assumption that every representation is induced from a low dimensional representation. This holds for a virtually pro-solvable group, such as $SL_d(\mathbb{Z}_p)$ or $SL_d(\mathbb{F}_p[[x]])$. But in the latter examples much better bounds hold (in fact the number of n dimensional representations is polynomialy bounded [127]).

Chapter 4

The Selberg property

In this chapter we present the main examples of groups which have property (τ) with respect to some subclass $\mathcal{L}$ of finite index subgroups. These are the arithmetic groups and $\mathcal{L}$ is the family of congruence subgroups (see Section 1.4.2). All started with the Selberg theorem on congruence subgroups of $SL_2(\mathbb{Z})$ so we call it the Selberg property. We will also show how this property is related to the Ramanujan conjecture and its various generalizations.

4.1 Selberg's theorem

A seminal result of Selberg asserts:

Theorem 4.1 (Selberg [176]) *Let* $\Gamma = SL_2(\mathbb{Z})$ *and* $\Gamma(m) = Ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/m\mathbb{Z}))$. *Then*

$$\lambda_1(\Gamma(m) \setminus \mathbb{H}^2) \geq \frac{3}{16}$$

where $\mathbb{H}^2 = SL_2(\mathbb{R})/SO(2)$ *is the upper half plane on which* $G = SL_2(\mathbb{R})$ *acts.*

By Proposition 2.9 Selberg's theorem indeed implies that $SL_2(\mathbb{Z})$ has (τ) with respect to the congruence subgroups $\mathcal{L} = \{\Gamma(m)\}_{m \in \mathbb{N}}$.

The constant $\frac{3}{16}$ has been improved in [141] to $\frac{21}{100}$ and recently in [109] to $\frac{66}{289} \sim 0.22837$. An important conjecture says that the right bound is $\frac{1}{4} = \lambda_0(\mathbb{H}^2)$. This is called Selberg's conjecture.

The behaviour of λ_1 for infinite families of surfaces is very much related to the behaviour of their isoperimetric constants h (see Section 2.2). Let us mention in passing that the analogue of the Selberg conjecture for the Cheeger constant would be that $h(\Gamma(m) \setminus \mathbb{H}^2) \geq \frac{1}{2}h(\mathbb{H}^2) = \frac{1}{2}$. However

Theorem 4.2 (Brooks-Żuk [27]) *There exists a constant $c < \frac{1}{2}$ such that for m sufficiently large*

$$h(\Gamma(m) \setminus \mathbb{H}^2) < c.$$

There are other examples for which Selberg's theorem holds (with different lower bounds) . Let k be a number field with r_1 real embeddings and r_2 complex (non-real) embeddings and $\mathcal{O}$ its ring of integers. Let $\Gamma = SL_2(\mathcal{O})$. Then Γ is a lattice in $G = SL_2(\mathbb{R})^{r_1} \times SL_2(\mathbb{C})^{r_2}$. By [73] and [171], Γ has property (τ) with respect to congruence subgroups. If $r_1 + r_2 \geq 2$ then Γ has the congruence subgroup property ([177]) and hence Γ also has property (τ) ([177]). On the other hand, if $r_1 + r_2 = 1$, i.e., $\Gamma = SL_2(\mathbb{Z}[\sqrt{-d}])$ for some $d \geq 0$, then Γ has a finite index subgroup which is mapped onto $\mathbb{Z}$. Hence Γ does not have property (τ) if $r_1 + r_2 = 1$. More generally, if S is a finite set of valuations of $\mathcal{O}$ including all the archemedian ones V_∞, then $\Gamma = SL_2(\mathcal{O}_S)$ is a lattice in $G = SL_2(\mathbb{R})^{r_1} \times SL_2(\mathbb{C})^{r_2} \times \prod_{p \in S \setminus V_\infty} SL_2(\mathbb{Q}_p)$. Recall that $rank(G) = r_1 + r_2 + |S \setminus V_\infty|$. Then Γ has property (τ) with respect to the congruence subgroups in all cases and has the congruence subgroup property (CSP) if and only if $rank(G) > 1$. None of these groups has (T). So those with $rank(G) > 1$ provide many examples of groups with (τ) and without (T). These include $SL_2(\mathbb{Z}[\frac{1}{p}])$ and $SL_2(\mathbb{Z}[\sqrt{p}])$ for any prime p.

Let Γ be a lattice in $SL_2(\mathbb{R})$. Γ is either virtually free or virtually a surface group. In either case it has a finite index subgroup mapped onto $\mathbb{Z}$ and so does not have property (τ). If Γ is arithmetic then Γ is commensurable with the group of units of an order in a quaternion algebra D defined over a totally real field K (such that D splits over $\mathbb{R}$ for one embedding of K into $\mathbb{R}$ and ramifies for all others). The Jacquet-Langlands correspondence ([96], see also [125]) implies that Γ has the property (τ) with respect to the congruence subgroups (see also [173] for an elementary proof).

Let Γ is an arithmetic lattice in $SL_2(\mathbb{C})$. If Γ is non-uniform then it is commensurable to one of the Bianchi groups $SL_2(\mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers in $\mathbb{Q}[\sqrt{-d}]$). These have the Selberg property as explained earlier.

If Γ is cocompact in $SL_2(\mathbb{C})$ it is commensurable with the units of an order of a quaternion algebra D defined over a number field L with a unique complex embedding and such that D ramifies for all real embeddings of L.

Again, the Jacquet-Langlands correspondence enables one to deduce that Γ has the Selberg property.

Let us look at the general case. We use the notation as in Definition 1.18 and fix an embedding $\mathcal{G} \to GL_m$. Denote

$$\Gamma = \mathcal{G}(\mathcal{O}_S) := \mathcal{G} \cap GL_m(\mathcal{O}_S).$$

We assume that Γ is infinite (equivalently $\prod_{v \in S} \mathcal{G}(k_v)$ is non-compact). For every ideal $I \lhd \mathcal{O}_S$ denote:

$$\Gamma(I) = Ker(\mathcal{G}(\mathcal{O}_S) \to GL_m(\mathcal{O}_S/I)).$$

$\Gamma(I)$ is a **principal congruence subgroup** of the (S-)arithmetic group Γ. This is a finite index subgroup as $[\mathcal{O}_S : I] < \infty$.

Definition 4.3 Γ *has the* **Selberg property** *if* Γ *has* (τ) *with respect to the congruence subgroup, i.e. with respect to*

$$\mathcal{L} = \{\Gamma(I); 0 \neq I \lhd \mathcal{O}_S\}.$$

If Γ is an arithmetic lattice in a semi-simple group (see Section 1.2.1) we still say that Γ has the Selberg property if $\mathcal{G}(\mathcal{O}_S)$ (see Definition 1.18) has the Selberg property. One can show that this is independent of $\mathcal{G}, \mathcal{O}$ and S.

4.2 S-arithmetic groups

Selberg's theorem has been extended to many arithmetic groups (see [125] as a reference for these results and much more). Moreover it has been proven ([44]) for all arithmetic groups $\Gamma = \mathcal{G}(\mathcal{O}_S)$ as above, when $char(k) = 0$ (confirming a conjecture posed in [140] and [139]).

Before elaborating on how this was done, let us show how to prove a somewhat weaker result by some easier methods.

If there exists $v \in S$ such that $\mathcal{G}(k_v)$ is non-compact and has (T), then $\Gamma = \mathcal{G}(\mathcal{O}_S)$ being an irreducible lattice in $H = \Pi_{v \in S} G(k_v)$ has (τ) by Theorem 1.16 and in particular it has the Selberg property.

Theorem 4.4 *Let k and $\mathcal{G}$ be as before. Then there exists $v \in V_k$ such that if $v \in S$, then $\Gamma = \mathcal{G}(\mathcal{O}_S)$ has the Selberg property.*

Proof Given $\mathcal{G}$ and k, then if for some $v_0 \in V_k$, $\mathcal{G}(k_{v_0})$ is a non-compact factor and has (T) then by a remark preceding the theorem we can take $v = v_0$ and the theorem is proved. Assume therefore that for every $v \in V_k$ either $\mathcal{G}(k_v)$ is compact or does not have (T). We claim that this implies that $\mathcal{G}$ is a form SL_2. If $\overline{k}$ is the separable closure of k, we claim that $rk_{\overline{k}}(\mathcal{G}) = 1$, otherwise, there are infinitely many valuations $v \in V_k$ for which $rk_{\overline{k}}(\mathcal{G}) \geq 2$. For any of these valuations $\mathcal{G}(k_v)$ is not compact and has (T) by Theorem 1.5, thus contradicting our assumption.

So $rk_{\overline{k}}(\mathcal{G}) = 1$. This implies that $\mathcal{G}$ is a form of SL_2. Now the forms of SL_2 are $SL(1, D)$ when D is a quaternion algebra over k (see [180]). For these the Selberg property is known: by Selberg's theorem for $SL_2(\mathbb{Z})$, by Gelbart-Jacquet [73] for the general SL_2 in characteristic 0 and by Drinfeld theorem in characteristic p [60]. For non-split quaternion algebras it follows from the result for SL_2 using the Jacquet-Langlands correspondence (see [125]). □

Note that the proof of Theorem 4.4 shows more: it shows that if $\mathcal{G}$ is not a form of SL_2, then there exists $v \in V_k$ such that if $v \in S$, then $\Gamma = \mathcal{G}(\mathcal{O}_S)$ has property (τ). It also shows that v can be chosen in Theorem 4.4 in infinitely many ways.

The above theorem is in any characteristic. Of course, one believes that that this holds unconditionally for every S. As said before this was indeed proved for many cases, and Clozel confirmed if for all cases in characteristic 0.

A mile stone in the proof was the following result of Burger and Sarnak [33] .

Proposition 4.5 ([33],[125]) *Let $H \leq G$ be two non-compact semi-simple groups. Assume Γ is an irreducible arithmetic lattice of G and assume $\Delta := H \cap \Gamma$ is an arithmetic lattice of H. Then*

(i) If Δ has the Selberg property then so does Γ;

(ii) If Δ has (τ) then Γ has (τ).

Proof In order to prove Proposition 4.5, first we prove the following general representation theoretical result whose proof we omit and refer the reader to [33].

Proposition 4.6 ([33]) *Let G be a locally compact group, Γ a lattice in G and $C = \mathrm{Comm}(\Gamma)$ the commensurability group of Γ in G (i.e, the set of $g \in G$ such that $g^{-1}\Gamma g \cap \Gamma$ is of finite index in Γ). Assume C is dense in G.*

Let H be a closed subgroup of G and assume $\Delta = H \cap \Gamma$ is a lattice in H. For $x \in C$ denote $\Delta_x = \Delta \cap x^{-1}\Gamma x$, so Δ_x is of finite index in Δ. Then

$$\mathrm{Res}_H \rho_{\Gamma\backslash G} \in \overline{\bigcup_{x \in C} \rho_{\Delta_x \backslash H}}$$

where $\rho_{\Gamma\backslash G}$ and $\rho_{\Delta_x\backslash H}$ denote the regular representations on $L^2(\Gamma \setminus G)$ and $L^2(\Delta_x \setminus H)$ respectively, and $\mathrm{Res}_H \rho_{\Gamma\backslash G}$ is the representation $\rho_{\Gamma\backslash G}$ restricted to H.

Now in order to prove (i) of Proposition 4.5 we should show that the trivial representation of G is not weakly contained in $\oplus_{\Gamma'} L^2_0(\Gamma' \setminus G)$ where Γ' runs over all the congruence subgroups of Γ. Assume the contrary. Then by restricting to H by Proposition 4.6 we get that the trivial representation of H is weakly contained in $\oplus_{\Gamma'} \oplus_x L^2_0(\Delta'_x \setminus H)$ where $\Delta'_x = x^{-1}\Gamma' x \cap \Delta$ for $x \in \mathrm{Comm}(\Gamma)$. (Note that a non-trivial representation of G restricted to H does not strongly contain the trivial representation, by our assumptions and the Howe-Moore Theorem).

Now, if Γ' is a congruence subgroup of Γ, then for $x \in \mathrm{Comm}(\Gamma)$, $x^{-1}\Gamma' x \cap \Gamma$ is also a congruence subgroup of Γ and thus Δ'_x is a congruence subgroup of Δ. As Δ has the Selberg property, the H-representations in $\oplus_{\Gamma'} \oplus_x L^2_0(\Delta'_x \setminus H)$ are bounded away from the trivial representation. Thus the same applies for the G-representations in $\oplus_{\Gamma'} L^2_0(\Gamma' \setminus G)$.

The proof of (ii) is exactly the same: This time Γ' is an arbitrary finite index subgroup of Γ and so is Δ'_x in Δ. □

The above result shows that it suffices to prove the Selberg property for "minimal" semi-simple groups. Moreover, if such a minimal semi-simple group is of rank at least 2, it has (T) and so (τ) for Γ follows immediately. One thus has to consider only rank one minimal groups. These are $SL(2)$, $SL(1, D)$ where D is a quaternion algebra or $SU(D, *)$ when D is a division algebra of prime degree over a quadratic extension E of a finite extension K of k and $*$ is an E/K involution of the second type.

The case of $SL(2)$ is covered by Selberg's theorem for $k_\infty = \mathbb{R}$ and by Gelbart-Jacquet [73] for all other completions. $SL(1, D)$ follows from $SL(2)$ using the Jacquet-Langlands correspondence (see Rogawski's appendix in [122]). The case of $SU(D, *)$ is treated by Clozel in [44]. Thus the Selberg property holds for all arithmetic groups in characteristic 0.

4.3 An equivalent formulation

Let us mention an equivalent formulation for the Selberg property. Let $\mathbb{A} = \Pi_v^* k_v$ be the ring of adèles of k, so $\mathcal{G}(k)$ is a lattice in $\mathcal{G}(\mathbb{A})$.

Proposition 4.7 *Let $\Gamma = \mathcal{G}(\mathcal{O}_S)$ as before and $H = \Pi_{v\in S}\mathcal{G}(k_v)$. The following conditions are equivalent:*

1. *Γ has the Selberg property;*
2. *The action of H on $L^2(\mathcal{G}(k)\setminus\mathcal{G}(\mathbb{A}))$ does not weakly contain the trivial representation;*
3. *There exists $v \in S$ such that $\mathcal{G}(k_v)$ is not compact and the $\mathcal{G}(k_v)$ action on $L^2(\mathcal{G}(k)\setminus\mathcal{G}(\mathbb{A}))$ does not weakly contain the trivial representation.*

Proof Let $v \in S$ be such that $\mathcal{G}(k_v)$ is not compact. By the Strong Approximation Theorem (see [125])

$$\mathcal{G}(\mathbb{A}) = \mathcal{G}(k)\mathcal{G}(k_v)\mathcal{G}(I, \widehat{\mathcal{O}})$$

where $\mathcal{G}(I, \widehat{\mathcal{O}}) = Ker(\mathcal{G}(\widehat{\mathcal{O}}) \to \mathcal{G}(\widehat{\mathcal{O}}/I\widehat{\mathcal{O}}))$ and I is any non-zero ideal in $\mathcal{O}$.
So

$$\mathcal{G}(\mathbb{A}) = \mathcal{G}(k)H\mathcal{G}(I, \widehat{\mathcal{O}})$$

and

$$\Gamma(I)\setminus H = \mathcal{G}(k)\setminus\mathcal{G}(\mathbb{A})/\mathcal{G}(I, \widehat{\mathcal{O}}).$$

Hence

$$\lim_{\leftarrow}\Gamma(I)\setminus H = \lim_{\leftarrow}\mathcal{G}(k)\setminus\mathcal{G}(\mathbb{A})/\mathcal{G}(I, \widehat{\mathcal{O}}) = \mathcal{G}(k)\setminus\mathcal{G}(\mathbb{A}).$$

Thus the closure in the Fell topology of the representations of H on $L^2(\Gamma(I)\setminus H)$ is the same as the closure of the representation of H on $L^2(\mathcal{G}(k)\setminus\mathcal{G}(\mathbb{A}))$. This shows that 1 and 2 are equivalent. The equivalence of 2 and 3 is easy. □

As a consequence we obtain

Corollary 4.8 *If $S_1 \subset S_2$ and $G(\mathcal{O}_{S_1})$ is infinite and has the Selberg property then $G(\mathcal{O}_{S_2})$ has the Selberg property.*

This can be also deduced directly from Proposition 4.5.

4.4 Property (τ) and the congruence subgroup property

We have seen that every arithmetic lattice Γ in a semisimple group G (at least over fields of characteristic 0 and conjecturally always) has the Selberg property. If in addition Γ has the congruence subgroup property, then Γ has also (τ). (A remark is in order here: Formally speaking CSP does not say that every finite index subgroup is a congruence subgroup, as the congruence kernel can be finite non-trivial. Still one can easily see that CSP plus the Selberg property imply (τ). In most cases for which the CSP has been proven to fail, it was also proved that Γ does not have FAb, and so Γ does not have (τ). This leads to the following conjecture:

Conjecture 4.9 *Let Γ be a lattice in a semisimple group. Then Γ has CSP if and only if Γ has (τ).*

We should say right away that this conjecture is not compatible with Serre's conjecture. They are compatible in most cases but differ in their prediction for the CSP for lattices in the $\mathbb{R}$-rank one groups $G = Sp(n, 1)$ and $G = \mathbb{F}_4^{(-20)}$. According to Serre's conjecture lattices in these groups are not supposed to have the CSP as G is of rank one. On the other hand Conjecture 4.9 predicts that such Γ's do have the CSP as G and Γ has (T) (see Theorem 1.5) and hence Γ has (τ).

One can give good arguments in both directions: On one hand lattices Γ in $Sp(n, 1)$ and $\mathbb{F}_4^{(-20)}$ behave in many ways as rank one lattices. For example, if Γ is cocompact, it is a hyperbolic group and has therefore plenty of infinite normal subgroups of infinite index. Recall, that in all cases where CSP has been proved, it was also shown that every normal subgroup is of finite index. So this may be an indication in favor of Serre's conjecture.

On the other hand, lattices Γ in $Sp(n, 1)$ and $\mathbb{F}_4^{(-20)}$ have a lot in common with higher rank lattices. For example property (T) and (τ). More important, they have super-rigidity. Now, super-rigidity is a corollary of the congruence subgroup property (see [16] and [161]). Moreover, every lattice in $Sp(n, 1)$ and $\mathbb{F}_4^{(-20)}$ is arithmetic. In vague terms it means that the only flexibility the geometry of $Sp(n, 1)$ and $\mathbb{F}_4^{(-20)}$ allows is the one enforced by the number theory. The congruence subgroup problem is also a kind of an arithmeticity question; it asks, whether, within one given arithmetic lattice, the sub-lattices (i.e., the subgroups of finite index) are all coming from number theoretical considerations, i.e. are all congruence subgroups. So, the fact that all lattices are arithmetic gives also some "moral" support to believe in the CSP for the arithmetic lattices.

If we would be enforced to bet, we would incline to go for Conjecture 4.9. In 1970, when Serre made his conjecture [177], none of the above information on these lattices in $Sp(n,1)$ and $\mathbb{F}_4^{(-20)}$ was available. Anyway lattices behave like lattices of "rank $1\frac{1}{2}$", half the way between rank one and higher rank. It will be extremely interesting to answer CSP even for one of these lattices - see Chapter 7 for more discussion and potential applications.

4.5 The Ramanujan conjecture

The classical Ramanujan conjecture asserts that the Ramanujan tau function $\tau(n)$ defined by

$$q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n$$

satisfies

$$\tau(p) \leq 2p^{\frac{11}{2}}$$

for every prime p.

This innocent looking conjecture of combinatorial nature, led to far reaching extensions which are usually expressed in representation theoretic terms and are relevant to our topic. Let us explain briefly why. For a more detailed treatment see Rogawski's appendix of [122].

The point is that if we substitute $q = e^{2\pi iz}$ we can think of $f = \sum \tau(n)q^n$ as a function on $\mathbb{H}^2 = \{z = x + iy | y > 0\}$ the upper half plane. It turns out that $\Delta(z) = \sum_{n\geq 1}\tau(n)e^{2\pi iz}$ is a "cusp form of weight 12 for the group $SL_2(\mathbb{Z})$", in fact, it is even an "Hecke eigenform", i.e., it is a common eigenfunction for the Hecke operator T_p acting on $S_{12}(SL_2(\mathbb{Z}))$ - the space of cusp forms of weight 12 over $SL_2(\mathbb{Z})$. Well, for this case, $dim(S_{12}(SL_2(\mathbb{Z}))) = 1$, so this is not surprising.

Let us explain briefly and vaguely what the above words mean - referring the reader again to [169] for an excellent (fairly short) treatment.

Let us recall the definition of automorphic and cusp forms. First, for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$$

and $z = x + iy \in \mathbb{H}^2$ denote

$$j(g, z) = cz + d.$$

One can check directly the cocycle condition:

$$j(hg, z) = j(h, gz)j(g, z) \quad \text{for } g, h \in G, z \in \mathbb{H}^2 \tag{4.1}$$

and

$$j(gr(\theta), i) = e^{i\theta} j(g, i) \tag{4.2}$$

where

$$r(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in K = SO(2), \theta \in [0, 2\pi].$$

Now an holomorphic automorphic form of weight k with respect to $\Gamma = SL_2(\mathbb{Z})$ is a holomorphic function satisfying two conditions:

$$(a) \qquad f(\gamma(z)) = j(\gamma, z)^k f(z)$$

for $\gamma \in \Gamma$ and $z \in \mathbb{H}^2$. This implies that

$$f(z+1) = f\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z\right) = f(z)$$

and hence f can be viewed as a function in the variable $q = e^{2\pi i z}$ and has the Fourier expansion as

$$f(q) = \sum_{n \in \mathbb{Z}} a_n q^n.$$

The second condition is

$$(b) \qquad a_n = 0 \quad \text{for } n < 0$$

and if in addition it satisfies

$$(c) \qquad a_0 = 0$$

then f is a **cusp form**.

We say that f is a **Hecke eigenform** if

$$(d) \qquad T_p f = \lambda_p f$$

for every prime p, where T_p is the Hecke operator acting on the space of cusp forms of weight k by

$$T_p(f) = p^{k-1} f(pz) + p^{-1} \sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right).$$

A computation shows that if f is normalized so that $a_1 = 1$, then $\lambda_p = a_p$.

These definitions can be extended to the case when Γ is any congruence subgroup. The regularity conditions ((b) and (c)) should be adapted with respect to every cusp and condition (d) with respect to almost all primes (the exceptional set depends on the congruence subgroups).

The Peterson-Ramanujan conjecture generalizes the original Ramanujan conjecture about (τ) and asserts that $\lambda_p = a_p \leq 2p^{k-1}$.

A seminal paper of Satake [174] showed that this conjecture has a representation theoretic formulation. It basically goes like that: Given such a Hecke eigenform f (satisfying all conditions (a)-(d)), one can think of it as a function on $G(\mathbb{R} = SL_2(\mathbb{R})$ and "normalize" it by taking

$$\widetilde{f}(g) = f(g(i))j(g,i)^{-k}.$$

The cocycle condition (4.1) implies that $\widetilde{f}$ is $\Gamma = G(\mathbb{Z})$-invariant ($G(\mathbb{Z}) = SL_2(\mathbb{Z})$) and (4.2) implies that

$$\widetilde{f}(gr(\theta)) = \widetilde{f}(g)e^{-ik\theta}.$$

Recall now that by the Strong Approximation Theorem (see [160], [122], [133])

$$G(\mathbb{Q}) \setminus G(\mathbb{A})/G(\hat{\mathbb{Z}}) \simeq G(\mathbb{Z}) \setminus G(\mathbb{R}).$$

So we can consider $\widetilde{f}$ as a function on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})$ which is $G(\hat{\mathbb{Z}})$-invariant. (If $\widetilde{f}$ comes from a cusp form with respect to a proper congruence subgroup of $G(\mathbb{Z})$ then $G(\hat{\mathbb{Z}})$ will be replaced by its appropriate congruence subgroup.)

Now if (ρ, V) is the minimal $G(\mathbb{A})$ submodule containing $\widetilde{f}$, then it is irreducible and $V = \otimes_{p\leq\infty} V_p$, $\rho = \otimes_{p\leq\infty}\rho_p$ and by construction, every (ρ_p, V_p) has $K_p = SL_2(\mathbb{Z}_p)$ fixed point and $K_\infty = SO(2)$ acts by the character $\chi_k(r(\theta)) = e^{ik\theta}$.

The representations of $G(\mathbb{Q}_p)$ with K_p-fixed points are of two kinds: the principal series and the complementary series.

Then comes the miracle - by a small computation: the Peterson-Ramanujan conjecture holds for a_p if and only if ρ_p is in the principal series. This miracle comes from the fact that the K_p-fixed vectors is an eigenvector for the Hecke algebra of G_p with respect to K_p, and the eigenvector is related directly to λ_p (see [169]).

All this long process is reversible, i.e., if v is a $G(\hat{\mathbb{Z}})$ fixed vector in $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$ upon which K_∞ acts via χ_k, then this defines a function in $L^2(\Gamma \setminus G(\mathbb{R}))$ which can be "renormalized" to give a Hecke eigenform on Γ.

Thus the Peterson-Ramanujan conjecture has now a purely group theoretical equivalent form: if $(\rho, V) = (\otimes_p \rho_p, \otimes_p V_p)$ is infinite dimensional irreducible subrepresentation of $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$ such that ρ_∞ is in the discrete series (this amounts to be equivalent to the fact that $SO(2)$ acts by the character χ_k). Then for every finite p, either ρ_p has no K_p-fixed point (in which case it is in the discrete series and it does not matter for us), or if it has a K-fixed point, then it is in the principal series and not in the complementary series. So to put all this together: The Peterson-Ramanujan conjecture holds (for every congruence subgroup Γ) if and only if for every $(\rho, V) = (\otimes_p \rho_p, \otimes_p V_p)$ in $L^2(G(\mathbb{Q} \setminus G(\mathbb{A}))$, with ρ_∞ in the discrete series, ρ_p is not in the complementary series. This conjecture was proved by Deligne [50].

What all this has to do with our (τ)? Well, the point is that the only way to converge to the trivial representation in $\widetilde{G}(\mathbb{Q}_p)$ - the unitary dual of $G(\mathbb{Q}_p)$ is via the complementary series. So, to say that some representations are not in the complementary series, says in particular that they are bounded away from the trivial representation (and even with an explicit bound), i.e. a form of relative property (T) or (τ). Moreover, as we saw above, the Selberg property is also a statement about the subrepresentations of $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$. (In fact, the Selberg conjecture $\lambda_1 \geq \frac{1}{4}$ would follow if the Deligne theorem would be proved without the assumption that ρ_∞ is in the discrete series).

The Peterson-Ramanujan conjecture was generalized in various directions. One of them is a Drinfeld theorem that asserts that if (ρ, V) is an irreducible infinite dimensional subrepresentation of $L^2(SL_2(k) \setminus SL_2(\mathbb{A}))$ when k is a global field of $char > 0$, then all the local components are not in the complementary series. This implies

Corollary 4.10 *The group $SL_2(\mathbb{F}_q[x])$ has the Selberg property, i.e., property (τ) with respect to the congruence subgroups.*

Note that this group is not finitely generated.

Anyway, the various generalizations of the Ramanujan conjecture from GL_2 to general groups implies some strong form of the Selberg property. The reader should be aware that some of the naive generalizations are not correct (see [94] and [30]) but results in this direction can in principle be interpreted as the Selberg property with an explicit constant.

A conjecture for general semi-simple groups is given by Arthur [6]. This is by far stronger than what we call the Selberg property.

4.6 The spectrum: from the infinite to the finite

The Ramanujan conjecture in its various representation theoretical generalizations provide some non-trivial bound on the eigenvalues of the adjacency operators of some finite k-regular graphs (see Chapter 5 and [122], [154]). The next result shows a limitation to such bounds.

Theorem 4.11 (Alon-Boppana [1]) *Let X_n be a sequence of finite graphs of degree k such that $|X_n| \to_{n\to\infty} \infty$. Then*

$$\limsup_{n\to\infty} \lambda_1(X_n) \leq 1 - \frac{2\sqrt{k-1}}{k}.$$

A stronger form of Alon-Boppana was proved by Greenberg [79] (we will not quote it here as Theorem 4.14 below is even stronger).

before continuing, let us mention two seemingly unrelated results. We then show how all of them can be seen as special cases of one theorem of Grigorchuk and Żuk.

Theorem 4.12 (Lubotzky-Weiss [139]) *If Γ is an amenable group then there is no infinite family $\mathcal{L}$ of finite index subgroups such that Γ has $\tau(\mathcal{L})$.*

Note that this is a strong form of Proposition 1.32 above.

Theorem 4.13 (Shalom [181]) *Let $H \leq \Gamma$ and suppose that there are infinitely many finite index subgroups N_i of Γ such that $H \leq N_i$ and $\cap N_i = H$. Then*

$$l^2(\Gamma/H) \prec \widehat{\oplus} l^2_0(\Gamma/N_i).$$

Before we continue, we observe that Shalom's result implies both Lubotzky-Weiss and Alon-Boppana theorems. Indeed to see the first: If Γ is amenable, and $\mathcal{L} = \{N_i\}$, let $H = \cap N_i$. Then the spectral radius of the Markov operator on Γ/H, $\rho(\Gamma/H)$ is at least as large as $\rho(\Gamma)$ and $\rho(\Gamma) = 1$ as Γ is amenable. So $\rho(\Gamma/H) = 1$ which means that $l^2(\Gamma/H)$ weakly contains the trivial representation. By Theorem 4.13 it implies that $\widehat{\oplus} l_0^2(\Gamma/N_i)$ weakly contains the trivial representation and so Γ does not have (τ) with respect to $\mathcal{L}$.

Similarly from Theorem 4.13 we obtain Theorem 4.11: The k-regular tree is the Cayley graph of a suitable group Γ (e.g., Γ is the free product of k cyclic groups of order 2, or if k is even, Γ is the free group on $\frac{k}{2}$ elements). Every k-regular graph X_i gives rise to a finite index subgroup N_i of Γ. Let $H = \cap N_i$. Then again $\rho(\Gamma/H) \geq \rho(\Gamma)$. The by the well known result of Kesten $\rho(\Gamma) = \frac{2\sqrt{k-1}}{k}$. Thus the same holds for the $liminf$ of $\lambda_1(X_i = \Gamma/N_i)$.

Theorem 4.13 is also what stands behind the method of Burger, Li and Sarnak [30] when they define "subgroup spectrum" and show that "Ramanujan spectrum" must include it. We will not elaborate on these aspects but refer the reader to their paper as a beautiful example how quite elementary considerations from representation theory (or combinatorics) can lead to existence results for automorphic forms.

A general form of the above three results is given in [81]. Grigorchuk and Żuk consider the space of "marked graphs" of uniformly bounded degree. They showed that the space is compact and

Theorem 4.14 (Grigorchuk-Żuk [81]) *The spectral measure is a continuous (measure valued) function on the space of marked graphs.*

Proof Let us consider a family $\{(X_n, v_n)\}$ of marked graphs, i.e. graphs with chosen vertices $v_n \in X_n$.

On the space of marked graphs there is a metric $Dist$ defined as follows

$$Dist((X_1, v_1), (X_2, v_2)) = \inf \left\{ \frac{1}{n+1}; B_{X_1}(v_1, n) \text{ is isometric to } B_{X_2}(v_2, n) \right\}$$

where $B_X(v, n)$ is the ball of radius n in X centered on v.

For a sequence of marked graphs (X_n, v_n) we say that (X, v) is the limit graph if

$$\lim_{n\to\infty} Dist((X, v), (X_n, v_n)) = 0.$$

The limit graph is unique up to the isometry.

We will consider locally finite graphs, i.e. the degree $deg(v)$ of each vertex v is finite and we will always assume that the graphs are connected. Now we prove:

Lemma 4.15 *Let $\{(X_n, v_n)\}_{n=1}^{\infty}$ be a sequence of marked graphs whose degrees are uniformly bounded, i.e. there exists $k > 0$, such that* $\deg(X_n) \leq k$ *for all $n \in \mathbb{N}$. Then there exists a subsequence $\{(X_{n_i}, v_{n_i})\}_{i=1}^{\infty}$ which converges to some marked graph (X, v).*

Proof Because the degrees of the graphs are uniformly bounded, we can use the diagonal argument. □

Lemma 4.15 has as a corollary the following:

Proposition 4.16 *The space of marked graphs of uniformly bounded degree is compact.*

Proposition 4.17 *For any regular marked graph (X, v) there exists a sequence of finite marked regular graphs (X_n, v_n) converging to (X, v).*

Proof First of all let us suppose that the degree of X is even and equal to $2n$. Then X can be represented as the Schreier graph of F_n/H where F_n is a free group on n generators, H some subgroup of F_n and as generators for F_n/H we take the images of standard generators of F_n. We can suppose that the vertex v in X is the image of the identity element e in F_n. Now $H = \bigcup_{i=1}^{\infty} H_i$, where H_i is a sequence of subgroups of F_n such that for every i we have $H_i \subset H_{i+1}$ and H_i is finitely generated. By a theorem of M. Hall [90] every finitely generated subgroup of H_i can be represented as the intersection $\bigcap_{j=1}^{\infty} H_{ij}$ where H_{ij} are subgroups of F_n of finite index. By a diagonal process we can choose a sequence $\{H_{i_k j_k}\}_{k=1}^{\infty}$ such that the finite marked Schreier's graphs $\{F_n/H_{i_k j_k}, e\}_{k=1}^{\infty}$ converge to (X, v).

In the case when the degree of X is odd, the proof is similar but we have to use the version of Hall's theorem for $\mathbb{Z}_2 * \ldots * \mathbb{Z}_2$. □

On the locally finite, connected graph $X = (V_X, E_X)$ we can consider a random walk operator M acting on functions $f \in l^2(X, deg)$ as follows:

$$Mf(v) = \frac{1}{deg(v)} \sum_{w \sim v} f(w),$$

where $w \sim v$ means that w is a neighbor of v.

Let $\rho(M)$ be the spectral radius of M, i.e.

$$\rho(M) = \|M\| = \lim_{n\to\infty} \sqrt[n]{\|M^n\|} = \lim_{n\to\infty} \sqrt[2n]{p_{2n}(x,x)}$$

for any $x \in X$ where $p_n(x,x)$ is the number of closed loops of length n which start at x, divided by the number of all loops which start at x. In other words, $p_n(x,x)$ is the probability that if we start a simple random walk on X at x, after n steps we return to x.

Lemma 4.18 *Let $f : X_1 \to X_2$ be a covering between two graphs X_1 and X_2. Then*

$$\rho(X_2) \geq \rho(X_1).$$

Proof By definition of the cover, different loops in X_1 are projected onto different loops in X_2. The loops in X_1 which start and finish in v are projected onto loops in X_2 which start and finish in $f(v)$. Thus

$$p^n(v,v) \leq p^n(f(v), f(v)),$$

which implies that $\rho(X_1) \leq \rho(X_2)$. □

Since M is a bounded ($||M|| \leq 1$) and self-adjoint operator, it has the spectral decomposition

$$M = \int_{-1}^{1} \lambda E(\lambda),$$

where E is the spectral measure. This spectral measure is defined on Borel subsets of the interval $[-1,1]$ and takes the values in projections on the Hilbert space $l^2(X, deg)$. The matrix μ^X of measures μ^X_{xy} for vertices $x, y \in X$ can be associated with E as follows:

$$\mu^X_{xy}(B) = \langle E(B)\delta_x, \delta_y\rangle,$$

where B is a Borel subset of $[-1,1]$ and δ_x is the function which equals 1 at x and 0 elsewhere.

Now, in general, $\lambda \in Sp(M)$ if and only if for every $\varepsilon > 0$ there exists μ^X_{xy} such that $|\mu^X_{xy}((\lambda-\varepsilon, \lambda+\varepsilon))| > 0$. But we also have the following result (see [106]):

Lemma 4.19 *$\lambda \in Sp(M)$ if and only if for every $\varepsilon > 0$ there exists $x \in X$ such that $\mu^X_{xx}((\lambda - \varepsilon, \lambda + \varepsilon)) > 0$.*

Proof We need only show that if, for $B = (\lambda - \varepsilon, \lambda + \varepsilon)$, we have $|\mu^X_{xy}(B)| > 0$ then $\mu^X_{xx}(B) > 0$. As $E(B)$ is a projection, one has

$$\begin{aligned} 0 &< (\mu^X_{xy}(B))^2 = \langle E(B)\delta_x, \delta_y\rangle^2 \leq \langle E(B)\delta_x, E(B)\delta_x\rangle\langle\delta_y, \delta_y\rangle \\ &= \langle E(B)\delta_x, \delta_x\rangle deg(y) = \mu^X_{xx}(B) deg(y), \end{aligned}$$

which ends the proof. □

Our main tool will be the weak convergence of the measures $\mu^{X_n}_{v_n v_n}$ to the measure μ^X_{vv} (provided that the sequence of marked graphs (X_n, v_n) converges to the marked graph (X, v)), i.e.

$$\lim_{n\to\infty} \int_{-1}^{1} f \mu^{X_n}_{v_n v_n} = \int_{-1}^{1} f \mu^X_{vv}$$

for any $f \in C[-1, 1]$. The weak convergence implies (see for instance [67]) that for any open interval $B \subset [-1, 1]$:

$$\liminf_{n\to\infty} \mu^{X_n}_{v_n v_n}(B) \geq \mu^X_{vv}(B).$$

Lemma 4.20 *Let us suppose that the sequence of marked graphs (X_n, v_n) converges to the marked graph (X, v). Then the measures $\mu^{X_n}_{v_n v_n}$ converge weakly to the measure μ^X_{vv}.*

Proof We are going to prove that the moments of the measures $\mu^{X_n}_{v_n v_n}$ converge to the moments of the measure μ^X_{vv}. As it is easy to see, in our situation this implies weak convergence of corresponding measures (see for example [67]). The l-th moment of the measure μ^Y_{yy} for a graph Y and $y \in Y$ is given by

$$(\mu^Y_{yy})^{(l)} = \int_{-1}^{1} \lambda^l \mu^Y_{yy}(\lambda) = \int_{-1}^{1} \lambda^l \langle E(\lambda)\delta_y, \delta_y\rangle = \langle M^l \delta_y, \delta_y\rangle.$$

Thus the l-th moment of the measure μ^Y_{yy} is equal to the probability of going from y to y in l steps. But for n sufficiently large, the balls $B_{X_n}(v_n, l)$ and $B_X(v, l)$ are isometric and the l-th moment of the measure $\mu^{X_n}_{v_n v_n}$ is the same as the l-th moment of the measure μ^X_{vv}. □

Now Theorem 4.14 is a consequence of Lemma 4.20 and the fact that the space of graphs that we are considering is a metric space. □

This theorem implies Theorem 4.13 and thus Theorem 4.12 and Theorem 4.11. Indeed, if Γ, H and $\{N_i\}$ are as in Theorem 4.13, then for any fixed set of generators S, the graphs $\{Cay(\Gamma/N_i, S)\}$ have a convergent subsequence by the compactness of the space of marked graphs. So without loss of generality we can suppose that the sequence $Cay(\Gamma/N_i, S)$ converges. It clearly converges to $Cay(\Gamma/L, S)$ where L is a subgroup containing H. Thus we obtain the inequality between the norms of the corresponding Markov operators. As the set S was chosen arbitrary, by a result of Eymard [65], it implies the weak containment of the representations.

Corollary 4.21 *Let N_i be an infinite family of finite index subgroups of Γ. Then*

$$\lambda_0(\Gamma) \geq \limsup_{n\to\infty} \lambda_1(\Gamma/N_i).$$

Let us recall

Theorem 4.22 *Let N be a subgroup of a finitely generated group Γ.*

(a) Then

$$\lambda_0(\Gamma) \geq \lambda_0(\Gamma/N).$$

(b) If N is normal then

$$\lambda_0(\Gamma) = \lambda_0(\Gamma/N)$$

if and only if N is amenable.

(c) If Γ is finite and N is normal then

$$\lambda_1(\Gamma) \leq \lambda_1(\Gamma/N).$$

The proof of (a) follows for instance from the following characterization of λ_0: this is infimum of the spectrum of the Laplace operator Δ on the positive functions. As a positive eigenfunction of Δ on Γ/N gives rise to a positive eigenfunction of Δ on Γ (with the same eigenvalue) we get $\lambda_0(\Gamma) \geq \lambda_0(\Gamma/N)$.

Part (b) is due to Kesten [106] and the assumption that N is normal is necessary.

As far as (c) is concerned this follows from the fact the the eigenfunctions in $l^2_0(\Gamma/N)$ give rise to the functions in $l^2_0(\Gamma)$.

Part II

Applications of (τ)

Chapter 5

Expanders

The first and the main application of property (τ) has been the explicit construction of expanders initiated by Margulis [145]. Expanders play an important role in computer science and combinatorics.

This led to vast amount of work, some of this is described in [122] and we will not repeat it here. In this chapter we concentrate on the question to what extent the expansion properties of a Cayley graph of a group depend on the group structure or on the choice of generators. Similar question are of interest also concerning the diameters and other combinatorial invariants of the Cayley graphs.

5.1 Expanders and Ramanujan graphs

As shown in Section 2.1, property (T) and (τ) give explicit constructions of expanders. Such graphs are very much needed in graph theory and computer science for theoretical and applied applications. As this topic has been extensively covered in the literature we will not repeat the vast applications here, we rather refer reader to [108], [68] and [125].

Till recently, the only known constructions of expanders were through property (τ). Recently Reingold, Vadhan and Widgerson [166] found a direct combinatorial method ("Zig-Zag") to construct expanders, which does not require the mathematical machinery needed to prove (τ). Still the expanders coming from (τ) are Cayley graphs and hence enjoy additional symmetric properties which are not achieved by the methods of [166]. An interesting challenge is to construct Cayley graphs which are expanders by the Zig-Zag methods. First steps in this direction are taken in [150].

As explained in Section 4.5, the Ramanujan conjecture is an explicit strong form of (τ) with respect to congruence subgroups and so it can be used to give expanders with an explicit estimate on the expansion constant and the eigenvalues.

Definition 5.1 *A finite k-regular graph X is called Ramanujan if*

$$\lambda_1(X) \geq 1 - \frac{2\sqrt{k-1}}{k}.$$

By a theorem of Alon-Boppana (Theorem 4.11) the bound on Ramanujan graphs is asymptotically the best possible.

Theorem 5.2 (Lubotzky-Phillips-Sarnak [130], Margulis [148]) *Let Γ be a congruence subgroup of an arithmetic lattice in $PGL_2(\mathbb{Q}_p)$. Then the graph*

$$X = \Gamma \setminus PGL_2(\mathbb{Q}_p)/PGL_2(\mathbb{Z}_p)$$

is a finite $p+1$ regular Ramanujan graph.

Some of these graphs can be explicitly constructed as follows. Let p, q be two different primes which are congruent to 1 modulo 4 and and assume $\left(\begin{array}{c} p \\ q \end{array}\right) = -1$. Let i be an integer satisfying $i^2 = -1 \pmod q$. By a theorem of Jacobi there are $8(p+1)$ quadruples $(a_0, a_1, a_2, a_3) \in \mathbb{Z}^4$ which are solutions to

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = p.$$

Among these solutions there are $p+1$ with $a_0 > 0$ and odd and a_1, a_2, a_3 even. Let us associate to them the following matrices in $PGL_2(\mathbb{Z}/q\mathbb{Z})$:

$$\left(\begin{array}{cc} a_0 + ia_0 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{array}\right).$$

The Cayley graphs of $PGL_2(\mathbb{Z}/q\mathbb{Z})$ with respect to the above generators are $p+1$-regular Ramanujan graphs.

For more Ramnaujan graphs see [154], for higher dimensional analogues see [37], [116], [135], [136].

5.2 Dependence on generators

Let Γ be an infinite group with two finite systems of generators S and S'. Let $\mathcal{L} = \{N_i\}_{i\in I}$ be a family of finite index normal subgroups of Γ. Then the family $\{Cay(\Gamma(N_i, S)\}_{i\in I}$ is a family of expanders if and only if $\{Cay(\Gamma(N_i, S')\}_{i\in I}$ is. On the other hand if $\{G_i\}_{i\in I}$ is a family of finite groups and two systems of generators S_i and S_i' of G_i are chosen independently for every i, it is not clear whether being a family of expanders for the first implies the same for the second.

Let us make the following definitions:

Definition 5.3 *Let $\mathcal{G} = \{G_i\}_{i\in I}$ be a family of finite groups. We say that $\mathcal{G}$ is:*

1. *An expanding family (or a family of expanders) if there exist $\varepsilon > 0$, $k \in \mathbb{N}$ and generating subsets $S_i \subset G_i$ of size at most k such that*

$$h(Cay(G_i, S_i)) \geq \varepsilon$$

for every i, i.e. G_i are expanders with respect to some uniformly bounded number of generators;

2. *A family of uniformly expanding groups (or a uniform family) if there exist $\varepsilon > 0$ and $k \in \mathbb{N}$ such that for every i and every generating subset $S_i \subset G_i$ of size at most k*

$$h(Cay(G_i, S_i)) \geq \varepsilon,$$

i.e. G_i are expanders with respect to any *choice of k generators;*

3. *Non-expanding if $\mathcal{G}$ is not a family of expanders.*

The most obvious example of an expanding family is when Γ has (τ) with respect to a family of normal subgroups $\mathcal{L} = \{N_i\}$, $G_i = \Gamma/N_i$ and one can take S_i to be the projection of S to G_i, where S is a finite set of generators of G.

Definition 5.4 *We say that the group Γ has uniform property (τ) with respect to a family of normal subgroups $\mathcal{L} = \{N_i\}$ if there exist $\varepsilon > 0$ and $k \in \mathbb{N}$ such that for every i and every generating subset $S \subset \Gamma$ of size at most k*

$$h(Cay(\Gamma/N_i, S)) \geq \varepsilon.$$

At this point we do not know any example of group with property (τ) whith or without uniform property (τ) with respect to an infinite family of normal finite index subgroups.

Analogous question concerning uniform Kazhdan constants was raised in [122]. We say that a group Γ has uniform property (T) if Kazhdan constants with respect to all finite sets of generators have uniform positive lower bounded.

Theorem 5.5 (Gelander-Żuk [71]) *Let Γ be a Kazhdan group densely embedded (or more generally, which has a dense homomorphic image) in a connected topological group G. Assume that there exists a continuous unitary representation $(\pi_G, \mathcal{H})$ of G without invariant vectors. Then Γ does not have uniform property (T).*

If moreover G is a connected Lie group, then Γ does not have uniform property (T) even with respect to generating sets of bounded size.

The assumption on the existence of a continuous unitary representation without invariant vectors is automatically satisfied if G is locally compact. One can take the action by left multiplication on $L^2(G)$, if G is not compact, and on $L^2_0(G)$ (the orthogonal complement to the constant functions) if G is compact.

There are two main families of examples of Kazhdan groups densely embedded in connected simple Lie groups:

1. Any uniform arithmetic lattice Γ in a Kazhdan semi-simple Lie group is densely embedded in a compact simple Lie group. For this compact group we can take any factor of K where Γ is (commensurable to) the group of integral points $H_{\mathbb{Z}}$ where $H = G \times K$.

2. Any S-arithmetic lattice in a product of two or more Kazhdan simple groups for which at least one of the places is archimedean, non-compact, is densely embedded in a connected Lie group (e.g. each of those that lie in the archimedean places). In some examples this connected group cannot be compact: e.g. $SL_3(\mathbb{Z}[\frac{1}{p}])$ is naturally densely embedded in $SL_3(\mathbb{R})$, but every homorphism of $SL_3(\mathbb{Z}[\frac{1}{p}])$ into a compact connected group has finite image.

The methods of [71] however do not extend to property (τ).

If the family $G_i = \Gamma/N_i$ is also a uniform family of expanders, then one can deduce that Γ is of uniform exponential growth, at least with respect to generating sets of size at most k, for some k. In fact, if S is a set of generators for Γ such that $h(Cay(G_i, S)) > \varepsilon$ for every i, then the word growth of Γ with respect to S is of exponential base at least $1+\varepsilon$ (i.e. there are at least $(1+\varepsilon)^l$ elements in Γ of length at most l). We mention in passing that recently Eskin, Mozes and Oh [64] showed that every linear group in characteristic 0 of exponential growth is of uniform exponential growth. This gives some support to an affirmative answer to Problem 5.21 below. At this point anyway, we do not know any family of uniformly expanding groups.

Here are some examples of groups which are never expanders:

Example 5.6 *Let $\mathcal{G} = \{G_i\}_{i\in I}$ be a family of d generated solvable groups of derived length at most l. Then $\mathcal{G}$ is a non expanding family. Indeed, assume $X(G_i, S_i)$ is a family of expanders and $|S_i| = k$ for every $i \in I$. Let Γ be the free solvable group on k generators $\{x_1, \ldots, x_k\}$ and derived length l. Then for every $i \in I$ there exists an epimorphism $\phi_i : \Gamma \to G_i$ with $\phi_i(\{x_1, \ldots, x_k\}) = S_i$. It follows that Γ has $\tau(\mathcal{L})$ with respect to $\mathcal{L} = \{Ker\phi_i\}_{i\in I}$ which is a contradiction with Theorem 4.12.*

It is certainly possible for a family of solvable groups to be expanders if the derived length is unbounded. Here is an example: Fix $2 \leq d \in \mathbb{N}$ and p prime. Let $\Gamma_d(p^n) = Ker(SL_d(\mathbb{Z}) \to SL_d(\mathbb{Z}/p^n))$. Then $\Gamma_d(p)$ has the Selberg property with respect to $\mathcal{L} = \{\Gamma_d(p^n); n \in \mathbb{N}\}$. In fact for $d \geq 3$ it even has (T). At the same time $\Gamma_d(p)/\Gamma_d(p^n)$ is a finite p group of order $p^{(n-1)(d^2-1)}$, hence nilpotent and thus solvable.

Example 5.7 *Let H_i be any family of finite groups and p_i be primes with $p_i \to \infty$ as $i \to \infty$. Let $G_i = \mathbb{F}_{p_i}[H_i] \rtimes H_i$. Then G_i are never expanders. Indeed, $\mathbb{F}_{p_i}[H_i]$ is an abelian subgroup of G_i of index $|H_i|$ and of order $p_i^{|H_i|}$ which is greater than $c^{|H_i|}$ for any constant c, when $p_i \to \infty$. Hence by Proposition 3.6 they cannot be expanders.*

For some time it was not known if there is a family of groups G_i with two bounded systems of generators S_i and S_i' such that $Cay(G_i, S_i)$ are expanders while $Cay(G_i, S_i')$ are not. This can happen:

Theorem 5.8 (Alon-Lubotzky-Widgerson [2]) *There exists an infinite family of finite groups G_i which are expanders with one choice of generators (of constant size) and are not with another such choice.*

The proof in [2] was based on the Zig-Zag product of graphs. This product allows the construction of large expanding graphs from smaller ones (without enlarging the degree). It can be applied for semi-direct product of groups. But for groups one can give also a direct proof. Let us, therefore, present the main core of the proof in the language of groups and representation theory.

Proposition 5.9 *Let $\{H_i\}_{i\in I}$ and $\{A_i\}_{i\in I}$ be two families of finite groups such that for every $i \in I$, H_i acts on A_i and let $G_i = A_i \rtimes H_i$. Let $S_i \subset H_i$ and $B_i \subset A_i$ be subsets such that:*

1. *There exist $t, k \in \mathbb{N}$ such that for every $i \in I$, $|S_i| \leq k$ and $|B_i| \leq t$;*
2. *For every $i \in I$, S_i generates H_i and $Cay(H_i, S_i)$ are expanders with some $\varepsilon > 0$ (independent of i);*
3. *For every $i \in I$, let $\widetilde{B}_i$ be the multiset which is the union of the orbits $\{b^h; b \in B_i, h \in H_i\}$ so $|\widetilde{B}_i| = |B_i||H_i|$. Then $\widetilde{B}_i$ generates A_i and $\lambda_1(Cay(A_i, \widetilde{B}_i)) > \varepsilon'$ for some $\varepsilon' > 0$ (independent of i).*

Then the Cayley graphs $Cay(G_i; \{S_i \cup B_i\})$ form a family of expanders.

Remark Note that $Cay(A_i, \widetilde{B}_i)$ are expanders with respect to an unbounded number of generators, but G_i are with respect to a bounded number. This is essentially what the semi-direct (or the Zig-Zag product) does for us.

Proof What we need to prove is that there exists $\delta > 0$ such that if (ρ, V) is a non-trivial irreducible unitary representation of G_i, then for every $v \in V$ with $\|v\| = 1$, for some $y \in S_i \cup B_i$,

$$\|\rho(y)v - v\|^2 > \delta.$$

Let $W_0 = \{w \in V; \rho(a)w = w, a \in A_i\}$. As $A_i \triangleleft G_i$, W_0 is G_i invariant. Thus either $W_0 = V$ or $W_0 = \{0\}$. If $W_0 = V$, then the representation ρ factors through H_i and the claim follows from assumption 2.

Assume therefore that $W_0 = \{0\}$. Let V_0 be the subspace of V of the H_i fixed points, i.e. $V_0 = \{w \in V; \rho(h)w = w, h \in H_i\}$. Let $V_1 = V_0^{\perp}$ be the orthogonal complement of V_0.

Now, given $v \in V$ write it as $v = v_0 + v_1$ with $v_i \in V_i$. The representation of H_i on V_1 does not have any H_i fixed vector. Thus, by assumption 2, there exists $s \in S_i$ such that $\|\rho(s)v_1 - v_1\|^2 > \delta'\|v_1\|^2$, for some δ' (independent of i). Now

$$\|\rho(s)v - v\|^2 = \|\rho(s)v_0 - v_0\|^2 + \|\rho(s)v_1 - v_1\|^2 \geq \delta'\|v_1\|^2.$$

Thus if $\|v_1\|^2 \geq \frac{1}{2}\|v\|^2$, we are done with $\delta = \frac{1}{2}\delta'$.

Assume therefore $\|v_0\|^2 > \frac{1}{2}\|v\|^2$. Now, by assumption 3 and since $W_0 = \{0\}$ we know that

$$\left\| \frac{1}{|\widetilde{B}_i|} \sum_{y \in \widetilde{B}_i} \rho(y)v_0 - v_0 \right\|^2 \geq \delta''\|v_0\|^2$$

for some δ'' independent of i.

Now as $\rho(h)v_0 = v_0$ we can deduce

$$\begin{aligned} \max_{b \in B_i} \|\rho(b)v_0 - v_0\|^2 &\geq \left\| \frac{1}{|B_i|} \sum_{b \in B_i} \rho(b)v_0 - v_0 \right\|^2 \\ &= \left\| \frac{1}{|B_i|} \frac{1}{|H_i|} \sum_{h \in H_i} \sum_{b \in B_i} \rho(bh)v_0 - \rho(h)v_0 \right\|^2 \\ &= \left\| \frac{1}{|\widetilde{B}_i|} (\sum_{y \in \widetilde{B}_i} \rho(y)v_0 - v_0) \right\|^2 \\ &\geq \delta''\|v_0\|^2 \geq \frac{\delta''}{2}\|v\|^2 \end{aligned}$$

and the proposition is now proved. □

So to prove now Theorem 5.8, one takes for example $H_p = SL_2(p)$, p prime, acting on $A_p = \mathbb{F}_q^{p+1} = \{f : \mathbb{P}^1(\mathbb{F}_p) \to \mathbb{F}_q\}$, for a fixed prime q.

Now, a counting argument [2] shows that for t sufficiently large (but fixed in terms of q) almost all the t-tuple of elements of $\mathbb{F}_q^{p+1}$ satisfy the assumption 3 of Proposition 5.9. As $SL_2(p)$ are expanders (with respect to $s_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $s_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ for example) the groups $G_p = \mathbb{F}_q^{p+1} \rtimes SL_2(p)$ can be made into a family of expanders. At the same time G_p are not expanders with respect to $\{t_1, s_1, s_2\}$ if $t_1 : \mathbb{P}^1(\mathbb{F}_p) \to \mathbb{F}_q$ given by

$$t_1(i) = \begin{cases} 1 & i = \infty \\ 0 & i \neq \infty \end{cases}$$

for $i \in \mathbb{P}^1(\mathbb{F}_p)$.

The last assertion can be easily seen by looking at the subset of G_p, $\{(f,h); f \in A_p, h \in H_p \text{ with } f(h(\infty)) = 0\}$. This set is invariant under multiplication by H_p from the left and "almost invariant" under t_1, which gives Theorem 5.8.

Remark Another way to see that $\{t_1, s_1, s_2\}$ do not give expanders is by proving that in Proposition 5.9 the condition 2 and 3 are equivalent to G_i being expanders with respect to $S_i \cup B_i$. Clearly $Cay(A_p, H_p \cdot t_1)$ are not expanders as this is exactly the $(p+1)$ dimensional cube over $\mathbb{F}_q$.

A more general scheme was develloped by Meshulam and Wigderson [150] which gives many more examples of families of groups which are expanders with respect to one choice of generators and not with respect to the another such choice. They concentrated (in response to a question asked in [139]) on families G_i of the form $G_i = \mathbb{F}_p[H_i] \rtimes H_i$, when H_i is a finite group, p a fixed prime and $\mathbb{F}_p[H_i]$ the group algebra of H_i over $\mathbb{F}_p$ and G_i is the natural semi-direct product.

In Example 5.7 above we saw that if p is changed with i and goes to infinity then the groups G_i are not expanders with respect to any choice of (a uniformly bounded number of) generators. So the real question is for a fixed p.

Rather than repeating here the method of [150], we will twist their method to get a more general statement.

The starting point is [2] which in order to prove Theorem 5.8 above actually proves the following:

Assume that $G = A \rtimes H$ where A is a finite dimensional $\mathbb{F}_p$ vector space and H a finite group. We say that an element $\alpha \in A$ is of rank r if $dim(Span\{H\alpha\}) = r$. Let $\rho(r)$ denote the number of elements or $rank \leq r$ in A.

Proposition 5.10 ([2]) *Assume $S_1 \subset H$ is a set of generators such that $h(Cay(H, S_1)) \geq \varepsilon$ and assume that there exists a constant c such that $\rho(r) \leq c^r$. Then G has a subset S of size $s = f(|S_1|, c, p)$ generating G such that $h(Cay(G, S)) \geq \delta = \delta(|S_1|, c, \varepsilon, p) > 0$.*

Remark In fact the proof in [2] implies that there exists $t = f(|S_1|, c, p)$ such that a random choice of a subset T of t elements from A, will satisfy $h(Cay(G, T \cup S_1)) > \delta$ with probability at least $q = q(p, c) > 0$.

The proposition implies that for a fixed prime p and H_i a family of expanders the groups $G_i = A_i \rtimes H_i$ can be made into expanders if we have an exponential bound on the number of elements of rank r in A_i.

Fix i, take $H = H_i$, look at the case $A = \mathbb{F}_p[H]$ and assume $p \not| \, |H|$. In this case the group algebra is semi-simple and by Wedderburn theorem $\mathbb{F}_p[H] = \oplus_{j=1}^{t} M_{d_j}(\mathbb{F}_{p^{e_j}})$ such that $|H| = \sum_{j=1}^{t} e_j d_j^2$.

The Galois group $\Gamma = Gal(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ acts naturally on the set $Irr(H; \overline{\mathbb{F}_p})$ of equivalent classes of irreducible representations of H over $\overline{\mathbb{F}_p}$, where $\overline{\mathbb{F}_p}$ is the algebraic closure of $\mathbb{F}_p$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_t$ denote the orbits of $Irr(H; \overline{\mathbb{F}_p})$ under Γ and for each $1 \leq i \leq t$ choose a representative $\eta_i \in \mathcal{F}_i$ of dimension d_i. Let $\Gamma_i < \Gamma$ denote the stabilizer of η_i and let σ denote the Frobenius automorphism $\sigma(x) = x^p$. For $e_i = [\Gamma : \Gamma_i]$ the direct sum $\oplus_{j=1}^{e_i - 1} \eta_i^{\sigma^j}$ is a $d_i e_i$ dimensional irreducible $\mathbb{F}_p$ representation ρ_i of H. All irreducible representations of H arise this way.

Now, $\alpha \in A = \mathbb{F}_p[H]$ has rank r if and only if the left ideal generated by α is of dimension r. Write $\alpha = (A_1, \ldots, A_t)$ where A_j is a $d_j \times d_j$ matrix over $\mathbb{F}_{p^{e_j}}$. Then

$$dim(\mathbb{F}_p[H]\alpha) = \sum_{j=1}^{t} e_j d_j rank_{\mathbb{F}_{p^{e_j}}}(A_j)$$

where this time $rank_{\mathbb{F}_{p^{e_j}}}(A_j)$ is the rank of the matrix in the usual sense.

Proposition 5.11 *If $\mathbb{F}$ is a finite field of order q, then the number of matrices of rank r in $M_d(\mathbb{F})$ is bounded by q^{2dr}.*

Proof See [150]. □

Denote

$$K = \left\{ (r_1, \ldots, r_t) \in \mathbb{N}^t ; 0 \leq r_j \leq d_j, \sum_{j=1}^{t} r_j d_j e_j = r \right\}.$$

Then the number of elements of degree r in $\mathbb{F}_p[H]$ is bounded by

$$\sum_{(r_1,\ldots,r_t)\in K} \prod_{j=1}^{t} p^{2r_j d_j e_j} = |K| p^{2r}.$$

We need therefore to bound $|K|$. The following lemma follows from elementary considerations.

Lemma 5.12 *Let $\pi_d(H)$ be the number of inequivalent irreducible representations of H of dimension d over $\mathbb{F}_p$. If $\pi_d(H)$ is exponentially bounded as a function of d, so is $|K|$ as a function of r.*

We can now prove the following theorem, which also implies Theorem 5.10 above as promised. Also part 1 of it gives a strong contrast to Example 5.7 above.

Theorem 5.13 *Let $\{H_i\}_{i\in I}$ be an expanding family of groups and p a prime such that for every $i \in I$, $p \not| \, |H_i|$. Then*

1. *The groups $\mathbb{F}_p[H_i] \rtimes H_i$ also form a family of expanders;*
2. *The groups $\mathbb{F}_p[H_i] \rtimes H_i$ are not uniformly expanding.*

Proof By the above, to prove 1 we only need to show that the number of irreducible $\mathbb{F}_p$ representations of the H_i's of dimension d is exponentially bounded as a function of d. This is indeed the case. To see this let $\mathcal{G}_k$ be the class of all finite groups not involving the alternating group Alt_k as a section. Thus $\mathcal{G}_k$ satisfies the well known Babai-Cameron-Palfy restriction and by [8] and [23] we have: there exists a constant $c = c(k)$ such that $GL_d(p)$ has at most p^{cd} conjugacy classes of maximal irreducible $\mathcal{G}_k$ subgroups, and each one is of order at most p^{cd}.

Now all the H_i are generated by, say l, generators. So given a maximal irreducible $\mathcal{G}_k$ subgroup M of $GL_d(p)$, there are at most p^{cdl} possible homomorphisms from H_i to M and there are therefore at most $p^{cd}p^{cdl} = p^{c(l+1)d}$ irreducible representations of H_i into $GL_d(p)$.

Altogether this shows that the groups $\mathbb{F}_p[H_i] \rtimes H_i$ can be made a family of expanders with respect to a bounded number of generators.

On the other hand, to prove 2, look at the following set of generators for $G_i = \mathbb{F}_p[H_i] \rtimes H_i$: $1 \cdot e$ as an element of $\mathbb{F}_p[H_i]$ and S_i a set of l generators for H_i. Together this gives $l+1$ generators for G_i, but $h(G_i)$ tends to zero as i tends to infinity. Indeed, one can check that the following sets $A_i \subset G_i$ are almost invariant

$$A_i = \left\{ \left(\sum_{h \in H_i} a_h h, h' \right) \in G_i; a_{h'} = 0 \right\}.$$

This set is of size $\frac{1}{p}|G_i|$, it is invariant under multiplication from the left by S_i and almost invariant under multiplication by $1 \cdot e$. $\square$

Here are some concrete examples (to be used later in Chapter 8) to illustrate Theorem 5.13.

Example 5.14 *Fix a prime $p \geq 5$ and $\{H_q\}$ to be the family*

$$\{H_q\} = \{SL_2(\mathbb{F}_q) | q \equiv \frac{p+1}{2} (mod\, p)\}.$$

Since $|SL_2(q)| = (q+1)q(q-1)$, we have that $(|H_q|, p) = 1$. Then $\{\mathbb{F}_p[H_q] \rtimes H_q\}$ is a family of expanders by Theorem 5.13 and the Selberg property for $SL_2(\mathbb{Z})$.

Example 5.15 *Let p and q be fixed different primes and r a fixed integer ≥ 2. For $i \geq 1$, denote*

$$\Gamma(q^i) = Ker(SL_r(\mathbb{Z}) \to SL_r(\mathbb{Z}/q^i\mathbb{Z})).$$

Then:

(i) $\Gamma(q)$ has (τ) with respect to $\mathcal{L} = \{\Gamma(q^i) | i \in \mathbb{N}\}$. Indeed, if $d \geq 3$, $\Gamma(q)$ has (T) and if $d = 2$, it follows from the Selberg Theorem

and

(ii) $\Gamma(q)/\Gamma(q^i)$ is a q-group of order $q^{(i-1)(r^2-1)}$, as can be seen by a direct computation.

Thus the family $H_i = \Gamma(q)/\Gamma(q^i)$ satisfies the assumptions of Theorem 5.13.

Incidently, proving Theorem 5.13 for a family like in Example 5.15 is especially easy: If all the H_i are q groups (and $q \neq p$) then a maximal q-group of $GL_d(p)$ is simply a q-Sylow. So there is just one conjugacy class. Elementary arguments show that the order of the q-Sylow subgroup of $GL_d(p)$ is bounded by p^{cd} for some constant c. Thus to prove Theorem 5.13 (and hence also Theorem 5.10) for such a family, one does not need to appeal to [8] and [23] which are based on the classification of the finite simple groups.

5.3 Finite simple groups as expanders

At this stage of knowledge we do not know any infinite family of uniform expanders. The most likely candidate is the family $\mathcal{G} = \{SL_2(p)\}$ where p is prime. It can be made a family of expanders in various different ways. The following results suggest that at least with respect to generators which are chosen "uniformly" from $SL_2(\mathbb{Z})$ the quotients $SL_2(p)$ are expanders

Theorem 5.16 (Shalom [181]) *Let $H < SL_2(\mathbb{Z})$ be a subgroup which is normal in some congruence subgroup. If $\lambda_0(\mathbb{H}^2/H) < 0.22$, then there exists $N < \infty$ such that the projection of H to $SL_2(\mathbb{F}_p)$ is onto for $p > N$ and there exists a finite set $S \subset H$ such that $Cay(SL_2(\mathbb{F}_p), S)$ for $p > N$ is a family of expanders.*

Let us note that there are many normal subgroups H of congruence subgroups with $\lambda_0(\mathbb{H}^2/H) < 0.22$. This happens for example if $H = [\Delta, \Delta]$ for some congruence subgroup Δ, or more generally if Δ/H is amenable, in which case $\lambda_0(\mathbb{H}^2/H) = 0$. The 0.22 in the above theorem comes from the current state of knowledge regarding Selberg's conjecture ($\lambda_1 \geq \frac{1}{4}$, see Chapter 4). Assuming Selberg's conjecture, Theorem 5.16 would say that every non-trivial normal subgroup of $SL_2(\mathbb{Z})$ would contain S as in Theorem 5.16 (see also [181] for related results when $SL_2(\mathbb{Z})$ is replaced by a suitable lattice in $SL_2(\mathbb{Q}_p)$. In this case one also gets many "uniform" generators for $SL_2(p)$ which give rise to expanders).

Theorem 5.17 (Gamburd [69]) *Let $H < SL_2(\mathbb{Z})$ be a subgroup such that the Hausdorff dimension of its limit set is at least $\frac{5}{6}$. Then there exists $N < \infty$ such that the projection of H to $SL_2(\mathbb{F}_p)$ is onto for $p > N$ and there exists a finite set $S \subset H$ such that $Cay(SL_2(\mathbb{F}_p), S)$ for $p > N$ is a family of expanders.*

But at this point the following problems are open.

Problem 5.18 *Are $\{SL_2(p); p \text{ prime} > 3\}$ expanders with respect to*

$$\left\{ \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right\}?$$

Problem 5.19 *Let S be a finite subset of $SL_2(\mathbb{Z})$ which generates a non-solvable subgroup Δ of $SL_2(\mathbb{Z})$ (equivalently, it generates a subgroup which is Zariski dense in SL_2). It follows from the Strong Approximation Theorem for linear groups [194] that there exists $l \in \mathbb{N}$ such that Δ is mapped onto $SL_2(p)$ for any $p > l$. Is $\{X(SL_2(p); S); p > l\}$ a family of expanders?*

Problem 5.20 *Is* $\{SL_2(p); p \text{ prime}\}$ *a family of uniformly expanding groups?*

An even more ambitious is to ask:

Problem 5.21 *Let* G *be a Chevaley group defined over* $\mathbb{Z}$. *Is* $\{G(\mathbb{F}_p); p \text{ prime}\}$ *a family of uniformly expanding groups? Is for a fixed prime* p, $\{G(\mathbb{F}_{p^l}); l \in \mathbb{N}\}$ *a family of uniformly expanding groups?*

In both cases, it follows from Theorem 4.4 above, that they are expanders with respect to some choice of generators.

While Problem 5.21 suggests that for a fixed n the family

$$\mathcal{G}_n = \{SL_n(p); p \text{ prime}\}$$

may be uniformly expanding (this family is certainly expanding by Theorem 4.4) the "vertical family"

$$\mathcal{G}^p = \{SL_n(p); n \geq 2\}$$

is certainly not uniformly expanding. In [139] it was shown that $SL_n(p)$ for a fixed p and $n \to \infty$ has a bounded set of generators S_n for which $Cay(SL_n(p), S_n)$ are not expanders. In fact, it was even shown there that the compact group $K^p = \Pi_{n\geq 2} SL_n(p)$ has a finitely generated dense amenable subgroup. So the family $\mathcal{G}_n$ behaves like "property (T)" while $\mathcal{G}^p$ behaves as an "amenable" family. We do not know however the answer to the following problems:

Problem 5.22
1. *Is* $\{Sym(n)\}$ *a family of expanders ?*
2. *Is some infinite subset of* $\{Sym(n)\}$ *a family of expanders ?*

Recall that $Sym(n)$ are not expanders with respect to the generators $\{\tau = (1, 2), \sigma = (1, 2, \ldots, n)\}$ (see [122] page 51).

Problem 5.23 *Fix a prime* p. *Is* $\mathcal{G}^p = \{SL_n(p); n = 2, 3, \ldots\}$ *a family of expanders? It will be interesting to understand which families of finite simple groups are expanding families.*

5.4 Diameter of finite simple groups

On a connected graph there is a natural metric on the set of vertices. Namely the distance between two vertices is the minimal number of edges needed to connect them. A **diameter** of a graph X, denoted $diam(X)$, is the maximum of the distances between two vertices.

An easy property of expanders is

Proposition 5.24 *If X is a finite graph of degree k then*

$$diam(X) \leq -\frac{\ln(2|X|)}{\ln(\lambda_1(X))}.$$

Thus whenever we get a family of expanders we also get a logarithmic bound on their diameters.

For Cayley graphs the diameter has a group theoretic meaning. This is smallest l such that every element in the group can be written as a word of length at most l, using the generators.

The Selberg theorem (Theorem 4.1) implies

Corollary 5.25

$$diamCay\left(SL_2(p); \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right) = O(\ln(p)).$$

In spite of the easy down to earth formulation we do not know an elementary proof for it. Larsen managed to get close to it by elementary arguments and with a probabilistic algorithm.

Theorem 5.26 ([114]) *There exist constants c_1 and $c_2 > 0$ and a polynomial P such that one trial of the given algorithm will find a word of length $\leq c_1 \ln p \ln\ln p$ in time $\leq P(\ln p)$ representing a given element $\alpha \in SL_2(\mathbb{F}_p)$ with probability $\geq c_2$.*

To illustrate the non-triviality of Corollary 5.25 let us mention that

$$diamCay\left(SL_2(p); \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\right) = O(\ln(p))$$

for $p > 2$. But the question from [123] whether this holds for the generators $\left\{\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\right\}$ (for $p > 3$) is still open. This is also related to Problem 5.18.

Corollary 5.25 was the starting point for the following theorem, though it turned out to have a proof which does not need the Selberg Theorem.

Theorem 5.27 (Babai-Kantor-Lubotzky [10]) *There exists a constant C such that every non-abelian finite simple group G has a set of seven generators such that $diam(Cay(G,S)) \leq C\ln(|G|)$.*

It is still an open question whether this holds for two generators (see [99] and [100]). Some interesting results concerning the case when S is a normal subset of G were obtained by Liebeck and Shalev.

Theorem 5.28 ([119]) *There exists a constant C such that if G is a finite simple non-abelian group and $S \subset G$ a nontrivial normal subset, then*

$$diam(X(G,S)) \leq C\ln(|G|)/\ln(|S|).$$

But of course, as S contains conjugacy classes, the size of S is going to infinity with G.

There are still many open problems concerning the expansion and diameters of finite simple groups. Just as an illustration let us mention that it is not known if a random k-tuple of elements of $SL_2(p)$ has a logarithmic diameter. It is not even known whether the family $SL_2(p)$ has any system of generators with respect to which the diameter is not logarithmic. For S_n (or A_n) on the other hand, "non-logarithmic" generators exists (e.g. $\tau = (1,2)$ and $\sigma = (1,\ldots,n)$ for which $diam(Cay(S_n;\{\tau,\sigma\})) = O(n^2) > n\log n$) but it is not known what is the diameter of random generators. For more results and more questions see [9] and more recently [7].

5.5 Ramanujan groups

The bound in the definition of Ramanujan graphs is due to the fact that λ_0 for the Laplace operator on a k-regular tree is $1 - \frac{2\sqrt{k-1}}{k}$. This is an instance of a more general phenomenon, discussed in Section 4.6.

Theorem 5.29 ([79],[81]) *(i) Let X be an infinite connected graph and X_n a family of its quotients. Then*

$$\limsup_{|X_n|\to\infty} \lambda_1(X_n) \leq \lambda_0(X).$$

(ii) If Γ is a group generated by a finite set S and $\mathcal{L} = \{N_i\}$ a family of finite index subgroups then

$$\limsup_{n\to\infty} \lambda_1(Cay(\Gamma/N_i, S)) \leq \lambda_0(Cay(\Gamma, S)).$$

This leads to the following definition

Definition 5.30 *Let Γ be a group generated by a finite set S and $\mathcal{L} = \{N_i\}$ a family of finite index subgroups of the group Γ. The triple $(\Gamma, S, \mathcal{L})$ is called Ramanujan if for every i*

$$\lambda_1(Cay(\Gamma/N_i, S)) \geq \lambda_0(Cay(\Gamma, S)).$$

The group Γ will be said to be **residually Ramanujan** if there exists a Ramanujan triple $(\Gamma, S, \mathcal{L})$ with $\cap\{N_i | N_i \in \mathcal{L}\} = \{1\}$.

In [137] examples of residually Ramanujan and non-residually Ramanujan groups are constructed. Note that the definition depends on the choice of generators, though at this point it is not clear whether this is really the case. What is known is that the definition strongly depends on the algebraic structure of Γ and not only on the Cayley graph $Cay(\Gamma, S)$. For example there are two groups Γ_1 and $\Gamma_2 = F_{\frac{p+1}{2}} \times F_{\frac{q+1}{2}}$ both lattices in $SL_2(\mathbb{Q}_p) \times SL_2(\mathbb{Q}_q)$ acting simply transitively on the product of two trees $X_{p+1} \times X_{q+1}$ (hence with respect to suitable generators have the same Cayley graphs) but the first is residually Ramanujan while the second is non-residually Ramanujan. Indeed,

Proposition 5.31 ([137]) *Let Γ_1 and Γ_2 be two nonamenable, residually finite groups. Then $\Gamma_1 \times \Gamma_2$ is not residually Ramanujan.*

Proof Let S be a finite set of generators for $\Gamma_1 \times \Gamma_2$. In this proof all Cayley graphs we consider are with respect to the set S or its quotient. Suppose that $\Gamma_1 \times \Gamma_2$ is Ramanujan with respect to the infinite family of finite index normal subgroups N_i and S, i.e.

$$\lambda_1((\Gamma_1 \times \Gamma_2)/N_i) \geq \lambda_0(\Gamma_1 \times \Gamma_2).$$

One of the families $\{\Gamma_1 \cap N_i\}$ or $\{\Gamma_1 \cap N_i\}$ is infinite. Suppose that the first case holds. Then by Theorem 4.22 and Corollary 4.21

$$\limsup_{i\to\infty} \lambda_1(\Gamma_1/N_i \cap \Gamma_1) \leq \lambda_0(\Gamma_1) < \lambda_0(\Gamma_1 \times \Gamma_2) \leq \liminf_{i\to\infty} \lambda_1((\Gamma_1 \times \Gamma_2)/N_i).$$

The inequality in the middle is strict, because Γ_2 is non-amenable. But

$$\lambda_1(\Gamma_1/N_i \cap \Gamma_1) \geq \lambda_1((\Gamma_1 \times \Gamma_2)/N_i),$$

which gives a desired contradiction. □

Chapter 6

The product replacement algorithm

In this chapter we present, following [129] a quite unexpected application of property (τ) to analysis of the product replacement algorithm (PRA for short) which generates (pseudo) random elements in a finite group G.

The analysis suggests one old and one new problem on the possibility of a "non-commutative Selberg Theorem".

6.1 The algorithm

In this section we describe the product replacement algorithm which produces a pseudo random elements from a finite group given by a set of generators.

The algorithm, proposed by Leedham-Green and Soifer has shown outstanding performances in practice [39] and became very quickly popular in computational group theory. Its theoretical analysis is still not fully understood. It works as follows: Given a finite group G, let $\Omega_k(G)$ be the set of k-tuples $(g) = (g_1, \ldots, g_k)$ of elements of G such that $\langle g_1, \ldots, g_k \rangle = G$. We call the elements of $\Omega_k(G)$ the *generating k-tuples*. Given a generating k-tuple $(g_1, \ldots, g_k)$, define a *move* on it in the following way: Choose uniformly a random pair (i, j), such that $1 \leq i \neq j \leq k$, then apply one of the following four operations with equal probability :

$$R^{\pm}_{i,j} : (g_1, \ldots, g_i, \ldots, g_k) \to (g_1, \ldots, g_i \cdot g_j^{\pm 1}, \ldots, g_k)$$
$$L^{\pm}_{i,j} : (g_1, \ldots, g_i, \ldots, g_k) \to (g_1, \ldots, g_j^{\pm 1} \cdot g_i, \ldots, g_k)$$

Note that these moves map a generating k-tuple into a generating k-tuple. Now apply these moves t times and return a random component of the resulting generating k-tuple. This is the desired "random" element of the group G.

Another way to think about the algorithm, is to define on $\Omega_k(G)$ a structure of a graph induced by maps $R^{\pm}_{i,j}$ and $L^{\pm}_{i,j}$. This makes $\Omega_k(G)$ into a $4k(k-1)$-regular graph with no orientation on edges, but with loops when $k > d(G)$, where $d(G)$ is the minimal number of generators of G. Now the algorithm consists of running a nearest neighbor random walk on this graph (for t steps) and returning a random component. We refer to this random walk as *product replacement random walk*. By abuse of notation, we denote this graph $\Omega_k(G)$ as well.

A crucial ingredient in analyzing the performances of the algorithm (though not the only ingredient [11]) is to evaluate the mixing time of the random walk on $\Omega_k(G)$.

To present the connection with (τ), let us start with some general definitions and questions.

6.2 Congruence subgroups of $Aut(H)$

Let H be a finitely generated residually finite group. An old result of Baumslag [17] asserts that $\Gamma = Aut(H)$ is also residually finite i.e., Γ is Hausdorff with respect to the profinite topology. Let us now define on Γ a weaker topology in the following way:

For every finite index characteristic subgroup I of H, let

$$\Gamma(I) = Ker(Aut(H) \to Aut(H/I)).$$

Define the "congruence topology" on Γ to be the one for which the groups

$$\{\Gamma(I); I \text{ characteristic finite index in } H\}$$

serve as a fundamental system of neighborhoods of the identity. The inclusion $H \to \widehat{H}$ induces an inclusion $i : \Gamma = Aut(H) \to Aut(\widehat{H})$. Note, that as $\widehat{H}$ is a finitely generated profinite group, $Aut(\widehat{H})$ is a profinite group [57] and one can easily check that the closure $\Gamma^{\sim} = \overline{i(\Gamma)}$ is isomorphic to the completion of Γ with respect to the congruence topology. Moreover, as $\widehat{\Gamma}$ and $Aut(\widehat{H})$

are profinite groups containing Γ we have a map

$$\pi : \widehat{\Gamma} = \widehat{Aut(H)} \to Aut(\widehat{H})$$

where $Im(\pi) = \Gamma^{\sim}$.

All this is just a generalization of the classical case for $H = \mathbb{Z}^k$ when we get

$$\widehat{Aut(\mathbb{Z}^n)} = \widehat{GL_n(\mathbb{Z})} \to GL_n(\widehat{\mathbb{Z}}) = Aut(\widehat{\mathbb{Z}}^n). \tag{6.1}$$

Note that in (6.1), unlike with

$$\widehat{SL_k(\mathbb{Z})} \to SL_k(\widehat{\mathbb{Z}}) \tag{6.2}$$

the map is not surjective. The surjectivity of (6.2) is due to the strong approximation theorem, which is true for SL_n but not GL_n. It is also not true, in general, for our π, i.e. $\Gamma^{\sim}$ is usually a proper subgroup of $Aut(\widehat{\Gamma})$.

One may still formulate the **congruence subgroup problem** for Γ, i.e.

$$\text{Is } Ker(\pi) = 1?$$

There are two cases which are of special interest (see [95])

Problem 6.1 *Let H be the free group F_k on $k \geq 2$ generators, or $H = \pi_1(\mathcal{M}_g)$ the fundamental group of a surface of genus $g \geq 2$. Does $\Gamma = Aut(H)$ have the congruence subgroup property (CSP for short), i.e. is*

$$\widehat{Aut(H)} \to Aut(\widehat{H})$$

injective, or equivalently, does every finite index subgroup of Γ contain $\Gamma(I)$ for some I?

In [126] it is shown that for $H = F_k$, $k = 2, 3$ and for $\pi_1(\mathcal{M}_g)$, $g = 2$ the answer is negative - but in all other cases the problem is open. One may wonder whether these cases indicate the true answer in the general case or maybe, just as in the abelian case - where $GL_2(\mathbb{Z})$ does not have the CSP, but $GL_k(\mathbb{Z})$ has for $k \geq 3$, also in the non-abelian setting, we should expect positive results if k and g are large enough.

Let us call the attention that Mozes [156] has proved an affirmative answer to a different kind of a congruence subgroup problem for free groups, which are tree lattices.

Anyway once we define congruence subgroups we can also define:

Definition 6.2 *The group $\Gamma = Aut(H)$ is said to have the* **Selberg property** *if it has (τ) with respect to the family $\mathcal{L} = \{\Gamma(I)\}$ as above.*

Problem 6.3 *Let H be F_k or $\pi_1(\mathcal{M}_g)$ as in Problem 6.1. Does $\Gamma = Aut(H)$ have property (T)? Does it have property (τ)? Does it have the Selberg property?*

All these problems are widely open, except again if $k = 2$ or 3 or $g = 2$, when Γ does not have (T) nor (τ). It is not clear if it has the Selberg property [126].

We are now ready to relate the algorithm and the Selberg property.

6.3 The PRA and the Selberg property

Let F_k be the free group on k generators $x_1, \dots, x_k$. For every group G, the set $\Omega_k(G)$ introduced in Section 6.1, can be identified with $E = Epi(F_k, G)$ - the set of epimorphisms from F_k onto G. Now, $\Gamma = Aut(F_k)$ acts on E in the following way: If $\alpha \in \Gamma$ and $\phi \in E$, $\alpha(\phi) = \phi\alpha^{-1}$. Moreover, if $R_{ij}^{\pm}$ and $L_{ij}^{\pm}$ are the Nielsen automorphisms of F_k defined by

$$\begin{aligned} R_{ij}^{\pm}(x_i) &= x_i x_j^{\pm 1}, \text{ and } R_{ij}^{\pm}(x_l) = x_l \text{ if } l \neq i \\ L_{ij}^{\pm}(x_i) &= x_j^{\pm 1} x_i, \text{ and } L_{ij}^{\pm}(x_l) = x_l \text{ if } l \neq i \end{aligned}$$

then the action of them on $\Omega_k(G)$ is exactly the same as the $R_{ij}^{\pm}$ and $L_{ij}^{\pm}$ defined in Section 6.1. It is known that the subgroup Γ_1 of Γ generated by $R_{ij}^{\pm}$ and $L_{ij}^{\pm}$ is of index 2 in Γ (it is the preimage of $SL_k(\mathbb{Z})$ under the canonical map $Aut(F_k) \to GL_k(\mathbb{Z})$). Altogether, we can deduce that the product replacement algorithm graphs introduced in Section 6.1 are Schreier graphs of Γ_1. (More precisely: in general $\Omega_k(G)$ is not necessarily connected, but every connected component of it is such a Schreier graph). Moreover, given the finite group G, let

$$I = I(G) = \bigcap \{Ker(\phi); \phi \in Epi(F_k, G)\}.$$

Then it is easy to see that $\Gamma(I)$ acts trivially on $\Omega_k(G)$. So every connected component of the latter is a quotient graph of the Cayley graph of $\Gamma_1/\Gamma(I)$. One can therefore deduce

Theorem 6.4 (Lubotzky-Pak [129]) *If $Aut(F_k)$ has the Selberg property then*

1. *for every finite group G, every connected component of $\Omega_k(G)$ is an expander;*

2. *the mixing time $mix_{(g)}$ of the lazy random walk on every connected component of $\Omega_k(G)$ is bounded by $C(k)\ln|G|$ where $C(k)$ is a constant depending on k.*

By the lazy random walk we mean the random walk which with probability $\frac{1}{2}$ stays at the same vertex and with probability $\frac{1}{2}$ moves to a neighbor (this is just to avoid the complications in case the graph is bipartite). Note that 2 follows from 1 by considerations from Section 2.1.2.

As mention before in Problem 6.3 above, it is not known whether $Aut(F_k)$ has the Selberg property. Still, a variant of the above theorem can give some unconditional results: Let W be a characteristic subgroup of F_k. There is a natural homomorphism $\pi : Aut(F_k) \to Aut(F_k/W)$, whose image we will denote by $A(F_k/W)$. For general W the group $A(F_k/W)$ can have an infinite index in $Aut(F_k/W)$. Still one can talk about congruence subgroups of $A(F_k/W)$ as $A(F_k/W) \cap \Gamma(I)$ when I is a finite index characteristic subgroup of F_k/W and $\Gamma = Aut(F_k/W)$. This is just the topology induced on $A(F_k/W)$ from its embedding into $Aut(\widehat{F_k/W})$. Similarly we can talk about the Selberg property of $A(F_k/W)$, i.e., (τ) with respect to the congruence subgroups.

As before we deduce

Theorem 6.5 *If the group $A(F_k/W)$ has the Selberg property, then for every finite group G which is a quotient of F_k/W, every connected component of $\Omega_k(G)$ is an expander.*

For example, if W is the commutator subgroup of F_k, then $A(F_k/W)$ is $GL_k(\mathbb{Z})$. The latter has the Selberg property for every k and even (T) if $k \geq 3$. Moreover, the Nielsen moves are projected to the elementary matrices for which Shalom [183] and Kassabov [103] estimated the Kazhdan constant.

Theorem 6.6 ([103]) *The Kazhdan constants for $SL_n(\mathbb{Z})$ with respect to elementary matrices are bounded below by $(33\sqrt{n}+317)^{-1}$*

One can deduce:

Theorem 6.7 (Lubotzky-Pak [129]) *Let G be an abelian group, $(g) = (g_1, \ldots, g_k)$ be the initial generating k-tuple, and let $\Gamma' \subset \Gamma_k(G)$ be a connected component containing (g). Then for the mixing time of the lazy product replacement random walk starting at (g), we have $mix_{(g)} \leq C \cdot k^2 \cdot \log |G|$, where C is a universal constant.*

Proof We estimate $mix_{(g)}$ using Proposition 2.3

$$mix_{(g)} \;\leq\; \frac{1}{\lambda_1(\Gamma_k(G))}\,(\log|\Gamma_k(G)| + 1).$$

First of all we relate the spectral gap to Kazhdan constants using a result which takes into account the symmetries of the set of generators.

Proposition 6.8 ([158]) *Let Γ be a dicrete group generated by a finite set S which has property (T) with Kazhdan constant $\varepsilon(S)$. Assume that there is a finite group $H < Aut(\Gamma)$ such that $H(S) = S$ and the action of H on S has m orbits. Then for every finite index subgroup N of Γ*

$$\lambda_1(\Gamma/N, S) \geq \frac{\varepsilon^2(S)}{2m}.$$

In the case of the elmentary matrices, $S_n < SL_n(\mathbb{Z})$ as a subgroup of the group of automorphism of $SL_n(\mathbb{Z})$ acts transitively on them. Thus by Proposition 6.7 and Theorem 6.6 we get $\lambda_1 \geq C'k^2$. As $\log|\Gamma_k(G)| = k \log |G|$ we get the desired bound. $\square$

More generally

Theorem 6.9 *Let $W = \gamma_{i+1}(F_k)$, where $\gamma_{i+1}(F_k)$ is the $(i+1)$-th term of the lower central series of F_k. Then $A(F_k/W)$ has (T) for every $k \geq 3$. Hence for a fixed k and i and any nilpotent group G of class at most i we have $mix_{(g)} \leq C(k, i) \cdot \log |G|$.*

Proof The group theoretic structure of $Aut(F_k(i))$ was described by Andreadakis [5]: First, denote $J = Ker(Aut(F_k(i)) \to Aut(F_k(i-1)))$. Every $\alpha \in J$ is an automorphism which takes each of the free generators $x_1, \ldots, x_k$ of $F_k(i)$ to $x_1\zeta_1, \ldots, x_k\zeta_k$, where $\zeta_1, \ldots, \zeta_k \in \gamma_i(F_k)/\gamma_{i+1}(F_k)$. It is not difficult to check that $\alpha \to (\zeta_1, \ldots, \zeta_k)$ defines an isomorphism from J onto

$(\gamma_i(F_k)/\gamma_{i+1}(F_k))^k$. From Witt formula ([144], Theorem 5.11) and by induction we can now deduce that $Aut(F_k(i))$ is an extension:

$$(*) \quad 1 \to \widetilde{M}_k(i) \to Aut(F_k(i)) \to GL_k(\mathbb{Z}) \to 1\,,$$

where $\widetilde{M}_k(i)$ is the group of IA-automorphisms of $F_k(i)$, i.e. the group of automorphisms which act trivially on the commutator quotient. The group $\widetilde{M}_k(i)$ is a nilpotent group of class $(i-1)$ and of Hirsh rank m_i:

$$m_k(i) = k\left(\sum_{j=1}^{i-1} \frac{1}{j} \sum_{d\,|\,j} \mu(d)\, k^{j/d}\right),$$

where μ is a classical Möbius function. See [5] for details.

Let $U_k(i)$ be a free nilpotent group over $\mathbb{R}$ associated to $F_k(i)$. The group $Aut(U_k(i))$ has a similar structure:

$$(**) \quad 1 \to \widetilde{N}_k(i) \to Aut(U_k(i)) \to GL_k(\mathbb{R}) \to 1\,,$$

where $\widetilde{N}_k(i)$ is a simply connected nilpotent group of dimension $m_k(i)$.

Now, from the description it is clear that $Aut(F_k(i))$ is a discrete subgroup of $Aut(U_k(i))$. It is not a lattice there, but it is a lattice in the preimage of $SL_k^{\pm}(\mathbb{R})$ under π, where $SL_k^{\pm}(\mathbb{R})$ denotes the group of $k \times k$ matrices of determinant ± 1.

Unlike $(*)$, the sequence $(**)$ splits, and so

$$Aut(U_k(i)) = \widetilde{N}_k(i) \rtimes SL_k^{\pm}(\mathbb{R}).$$

Let us look now at $A_k(i)$, which is the image of $A^+(F_k)$ in $Aut(F_k(i))$, as a subgroup of $Aut(U_k(i))$. Let $M_k(i)$ be the intersection of $A_k(i)$ with $\widetilde{M}_k(i)$.

We have that $M_k(i)$ is a discrete subgroup of $\widetilde{N}_k(i)$; it is a subgroup of

$$\widetilde{M}_k(i) = Ker(Aut(F_k(i)) \to GL_k(\mathbb{Z})),$$

which is a lattice in $\widetilde{N}_k(i)$. Let $N_k(i)$ be the Zariski closure of $M_k(i)$ in $\widetilde{N}_k(i)$. One can prove by induction on the dimension of a nilpotent unipotent group that every subgroup of a lattice is a lattice in its Zariski closure. Hence $M_k(i)$ is a lattice in $N_k(i)$.

The image of $A_k(i)$ in $SL_k^{\pm}(\mathbb{R})$ is $SL_k(\mathbb{Z})$ which is Zariski dense in $SL_k(\mathbb{R})$, and since $A_k(i)$ normalizes $M_k(i)$, $SL_k(\mathbb{R})$ normalizes $N_k(i)$. We can summarize: The group $A_k(i)$ is a lattice in the Lie group $G_k(i) = N_k(i) \rtimes SL_k(\mathbb{R})$.

Now, for $k \geq 3$ and $i \geq 1$, the Lie group $G_k(i) = N_k(i) \rtimes SL_k(\mathbb{R})$ has property (T). Indeed it follows from [192] that a group like $G_k(i)$, i.e. a semidirect product of a nilpotent unipotent group and a connected non-compact semisimple Lie group with property (T), has Kazhdan's property (T) if and only if $G_k(i) = [G_k(i), G_k(i)]$. Now, since $A_k(i)$ is Zariski dense in $G_k(i)$ it therefore suffices to show that $[A_k(i), A_k(i)]$ is of finite index in $A_k(i)$. To see this it is enough to show that $[A, A]$ is of finite index in A, where $A = A^+(F_k)$. In fact $[A, A]$ is equal to A as can be seen from the explicit presentation for A given by Gersten [74]. $\square$

So, again the connected components are expanders (in fact, if $k > d(G)$, $\Omega_k(G)$ is connected) and the mixing time is linear. It is not known how $C(k, i)$ depends on k and i. Ideally (as predicted by Theorem 6.4) it should depend only on k, and hopefully polynomially in k. But this is still an open problem. See [129] for further discussion.

Another interesting problem is whether $Aut(F_k/\gamma_i(F_k))$ have property (T) for $k \geq 3$ and every i. The proof of Theorem 6.9 shows that this is the case for $A(F_k/\gamma_i(F_k))$, but in general the latter is of infinite index in the first. Kassabov [103] showed that $Aut(F_k/\gamma_i(F_k))$ has property (T) for $k \geq 3$ and $i \leq k(k-1)$. For lager values of i the problem is still open.

6.4 The PRA and dependence on generators

In Chapter 5 we have discussed the issue to what extend the expansion coefficient of (finite) groups depends on the choice of generators. We end the current chapter with two remarks concerning that topic and the PRA.

Let $\Omega_k(G)$ be the product replacement graph for G as above. The group $Aut(G)$ acts on it in the following way: $\beta \in Aut(G)$ and $(g_1, \ldots, g_k) \in \Omega_k(G)$ then

$$\beta(g_1, \ldots, g_k) = (\beta(g_1), \ldots, \beta(g_k)).$$

This action commutes with the action of $Aut(F_k)$ on $\Omega_k(G)$. Let $\widetilde{\Omega}_k(G) = \Omega_k(G)/Aut(G)$. The vertices of $\widetilde{\Omega}_k(G)$ are in one to one correspondence with the set $\{Ker\phi; \phi \in Epi(F_k, G)\}$. Indeed, $(g_1, \ldots, g_k) \in \Omega_k(G)$ gives rise to $\phi \in Epi(F_k, G)$ with $\phi(x_i) = g_i$, $i = 1, \ldots, k$. $Ker\phi$ depends only on the orbit of $Aut(G)$ in its action on $Epi(F_k, G)$ which, as said before, can be identified with $\Omega_k(G)$. The set $\widetilde{\Omega}_k(G)$ has been studied in the group theoretical literature under the name "T-systems". (This T and the T of property (T) has nothing to do with each other - except of the curious connection found in [129] and described above).

Anyway, Gilman [76] studied the action of $Aut(F_k)$ on $\widetilde{\Omega}_k(G)$ for $G = PSL_2(p)$ and showed that for $k \geq 4$, the action is transitive. Moreover, if $N = |\Omega_k(G)|$, $Aut(F_k)$ acts on $\widetilde{\Omega}_k(G)$ as $Sym(N)$ or $Alt(N)$. This implies, in particular, that for infinitely many values of n, $Sym(n)$ or $Alt(n)$ are quotients of $Aut(F_k)$. Moreover, they are quotients even through congruence subgroups of $Aut(F_k)$. This implies that if the answer to Problem 5.22 (ii) is negative, then $Aut(F_k)$ does not have the Selberg property and hence also not (τ) and (T).

A second related remark: In [70] Gamburd and Pak related the expansion coefficient of $\Omega_k(G)$ to the minimal expansion coefficient of a Cayley graph of G with k generators. Instead of bringing here their most general result, we mention a corollary: If the answer to Problem 5.20 is positive, i.e. the family $SL_2(p)$ is a uniform family of expanders, then for sufficiently large fixed k, the PRA graphs $\{\Omega_k(PSL_2(p)); p \text{ prime}\}$ are expanders.

Chapter 7

Hyperbolic manifolds

In this chapter we shall describe the most unexpected applications of property (τ). These applications are related to the following conjecture, which is usually attributed to Thurston, though probably goes back to Waldhausen, at least for $n = 3$.

Conjecture 7.1 *Let M be a finite volume n-dimensional hyperbolic manifold. Then M has a finite sheeted cover $M' \twoheadrightarrow M$ with $\beta_1(M') > 0$, where $\beta_1(M') = dim H_1(M', \mathbb{R})$.*

An equivalent formulation is: let Γ be a lattice in $SO(n,1)$. Then Γ has a finite index subgroup Δ with $\Delta \twoheadrightarrow \mathbb{Z}$ (i.e. $|\Delta/[\Delta,\Delta]| = \infty$).

Conjecture 7.1 is an important one especially for 3-manifolds. Dunfield and Thurston [62] have recently checked by computer 10,986 hyperbolic manifolds and showed that all of them satisfy the conjecture. The conjecture implies that every 3-dimensional hyperbolic manifold is virtually Haken, i.e. has a finite sheeted cover which is Haken. A 3-dimensional manifold M is called Haken (or sufficiently large) if it is irreducible and if it contains an incompressible surface, i.e. a properly embedded orientable surface S (other than S^2) such that $\pi_1(S)$ injects into $\pi_1(M)$.

A central conjecture in 3-manifolds theory is the following:

Conjecture 7.2 (Virtual Haken conjecture) *A compact orientable irreducible 3-manifold with infinite fundamental group is virtually Haken.*

Thurston showed that if a 3-manifold is Haken, then it satisfies the geometrization conjecture. So, if M is virtually Haken, then M is a manifold satisfying the geometrization conjecture modulo an action of a finite group. It then follows, though still highly non-trivial, that also M satisfies the geometrization conjecture. Thus proving the **Virtual Haken conjecture** would amount to finishing up half of the geometrization conjecture (the half dealing with manifolds with infinite fundamental groups - so it does not include the Poincaré conjecture).

Anyway, answering this problem just for hyperbolic 3-manifolds will be a great step forward. Let us mention that being Haken also has a purely group theoretical interpretation: a compact orientable, irreducible 3-manifold M is Haken if and only if its fundamental group is either an HNN extension or a free product with amalgamation in a non-trivial way. Now, a group is an HNN extension in a non-trivial way if and only if it is mapped onto $\mathbb{Z}$. So we see that the Thurston conjecture (for $n = 3$) implies the virtual Haken conjecture, for hyperbolic 3-manifolds.

In this chapter we show how property (τ) is related to all this in few different ways. We start in Sections 7.1 and 7.2 with a proof based on (τ) that arithmetic lattices in $SO(n, 1)$ (for $n \neq 3, 7$) satisfy the Thurston conjecture. We continue in Section 7.3 with a discussion of the Lubotzky-Sarnak conjecture, which asserts that no lattice in $SO(n, 1)$ has (τ). This conjecture is weaker than Thurston's but Lackenby showed that together with another very plausible conjecture in 3-manifolds, it would imply the virtual Haken conjecture for hyperbolic 3-manifolds. His work will be described in Section 7.4. In Section 7.5 we show how the theory of pro-p groups in general and a recent result of Zelmanov in particular, can be relevant to attack the above mentioned conjectures for hyperbolic 3-manifolds. We end in 7.6 with a discussion of lattices in other rank one Lie groups. Along the way we also describe the connections and applications these topics have towards Serre's conjecture on the congruence subgroup problem.

7.1 Thurston's conjecture for arithmetic lattices

Theorem 7.3 *Conjecture 7.1 is true if* Γ *is an arithmetic lattice in* $SO(n, 1)$ *and* $n \neq 3, 7$.

In fact also for $n = 3, 7$ it is true for many of the arithmetic lattices - see below.

Theorem 7.3 is the accumulation of the work of Millson [151], Labesse and Schwermer [112], Li [117], Raghunathan-Venkataramana [164] and Li-Millson [118] (see [125]). A unified proof was given in [125] using (τ), which we will sketch here.

The main idea is the following Sandwich Lemma.

Lemma 7.4 (The Sandwich Lemma) *Assume $G_1 \leq G_2 \leq G_3$ are three non-compact simple Lie groups. Assume for $i = 1, 2, 3$, Γ_i is an arithmetic lattice in G_i, with $\Gamma_2 = G_2 \cap \Gamma_3$ and $\Gamma_1 = G_1 \cap \Gamma_3$ $(= G_1 \cap \Gamma_2)$.*

(a) *If Γ_1 has the Selberg property and Γ_3 does not have property (τ), then Γ_2 does not have the congruence subgroup property.*

(b) *Assume Γ_1 has the Selberg property and Γ_3 has a congruence subgroup Φ which is mapped onto $\mathbb{Z}$. Then Γ_2 has a congruence subgroup which is mapped onto $\mathbb{Z}$.*

Proof (a) Follows easily from the Burger-Sarnak result from Section 4.2 (Proposition 4.5). Indeed, by Proposition 4.5 (i), Γ_2 has the Selberg property but Γ_2 cannot have (τ) as by Proposition 4.5 (ii) Γ_3 does not have (τ). So Γ_2 has the Selberg property and no (τ) which means that it has non-congruence subgroups.

(b) Let $\phi : \Phi \to \mathbb{Z}$ be the given epimorphism and $N = Ker(\phi)$. For every $0 < n \in \mathbb{Z}$, there is a unique subgroup Φ_n such that $N \subseteq \Phi_n \subseteq \Phi$ and $[\Phi : \Phi_n] = n$, so $\Phi_1 = \Phi$. For every $x \in C = \mathrm{Comm}(\Gamma_3)$ denote by Φ_n^x the group $x^{-1}\Phi_n x$ and ϕ^x will be the homomorphism $\phi^x : \Phi^x \to \mathbb{Z}$ given by $\phi^x(x^{-1}\gamma x) = \phi(\gamma)$, for $\gamma \in \Phi$.

Note that since Φ is a congruence subgroup of Γ_3, so are $\Phi^x \cap \Gamma_3$ in Γ_3 and $\Phi^x \cap \Gamma_2$ in Γ_2. Also observe that if π_m is the natural projection from $\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$, $0 < m \in \mathbb{Z}$, then $\Phi_m^x = Ker(\pi_m \circ \phi^x)$. Now, to prove the lemma it suffices to show that for some $x \in C$, $\phi^x|_{\Phi^x \cap \Gamma_2}$ is non-trivial.

To this end, note that Φ does not have property (τ) with respect to the family $\{\Phi_m\}_{0<m\in\mathbb{Z}}$. This means that the G_3-module $\bigoplus_m L_0^2(\Phi_m \setminus G_3)$ weakly contains the trivial representation. The same holds when we consider this as a G_2-module. By Proposition 4.6, it follows that the G_2-module $\bigoplus_{x\in C} \bigoplus_{m\in\mathbb{Z}} L_0^2(\Gamma_2 \cap \Phi_m^x \cap \Phi \setminus G_2)$ weakly contains that trivial representation. But Γ_2 has the Selberg property, since Γ_1 has it and we may apply Proposition 4.5 (i). Thus for some $m_0 \in \mathbb{Z}$ and some $x_0 \in C$, $\Gamma_2 \cap \Phi_{m_0}^{x_0} \cap \Phi$ is not a congruence subgroup

of Γ_2. Since $\Gamma_2 \cap \Phi$ is a congruence subgroup, this means that $\Gamma_2 \cap \Phi^{x_0}_{m_0}$ is not.

On the other hand, as said before $\Gamma_2 \cap \Phi^{x_0}$ is a congruence subgroup. This shows that $\Gamma_2 \cap \Phi^{x_0}_{m_0} \lneqq \Gamma_2 \cap \Phi^{x_0}$. This means that $\pi_m \circ \phi^{x_0}$ is non trivial when restricted to $\Gamma_2 \cap \Phi^{x_0}$. Hence, ϕ^{x_0} is non-trivial when restricted to $\Gamma_2 \cap \Phi^{x_0}$ and the lemma is proved. □

The second step in the proof of Theorem 7.3 is to put the arithmetic lattice Γ of $SO(n,1)$ in the middle of a Sandwich as in Lemma 7.4 (b). Note that while doing so our hands are very tied; G_2 is $SO(n,1)$ but G_3 should also be a group of $\mathbb{R}$-rank one, otherwise it would have property (T) and could not have a lattice with an infinite abelianization. Well, such an embedding is still possible for most arithmetic lattices, but one should go into the details of their structure. This will be done in the next section.

7.2 Arithmetic lattices in $SO(n,1)$

In $SO(n,1)$ there are arithmetic lattices of two kinds:

(a) The following lattices Γ_2 are called "the lattices of the simplest type" in [189]: Let K be a totally real number field and $f = \sum_{i=1}^{n+1} a_i x_i^2$ a quadratic form defined over K, i.e., $a_i \in K$. Let $\sigma_1, \dots, \sigma_\ell$ be the ℓ-different embeddings of K into $\mathbb{R}$ where $\ell = [K:\mathbb{Q}]$. Consider for $j = 1, \dots, \ell$, $f^{\sigma_j} = \sum_{i=1}^{n+1} \sigma_j(a_i) x_i^2$ as a real quadratic form. Assume f^{σ_1} is of signature $(n,1)$ while f^{σ_j} is either positive definite or negative definite for every $j = 2, \dots, \ell$. If $\mathcal{O}$ denotes the ring of integers in K and $G_2 = SO(f)$ the K-algebraic subgroup of GL_{n+1} preserving f, then $G_2(\mathcal{O})$ is embedded diagonally in $\prod_{i=1}^{\ell} SO(f^{\sigma_i}, \mathbb{R})$. For $i = 2, \dots, \ell$, $SO(f^{\sigma_i}, \mathbb{R}) \cong SO(n+1)$ is compact and the image Γ_2 of the projection of $G_2(\mathcal{O})$ to $SO(f^{\sigma_1}, \mathbb{R}) \cong SO(n,1)$ is a lattice in $SO(n,1)$.

Let W be a three dimensional subspace of K^{n+1} and f_0 be the form f restricted to W. Choose W in such a way that $f_0^{\sigma_1}$ is of signature $(2,1)$. Then $G_1 = SO(f_0)$ is a K-algebraic subgroup of G_2 and denote by Γ_1 the projection of $G_1(\mathcal{O})$ into $SO(f_0^{\sigma_1}, \mathbb{R}) \cong SO(2,1) \approx PSL_2(\mathbb{R})$. Γ_1, which is a lattice in $PSL_2(\mathbb{R})$, has the Selberg property (see Section 4.1).

Let $L = K(\sqrt{-1})$ and $\tilde{f}$ the Hermitian form on L^{n+1} given by $\tilde{f} = \sum_{i=1}^{n+1} a_i |x_i|^2$. Let G_3 be the K-algebraic group $SU(\tilde{f})$. Then $G_3(\mathcal{O})$ contains

$G_2(\mathcal{O})$, $G_3(K \otimes \mathbb{R}) = SU(\widetilde{f^{\sigma_1}}, \mathbb{R}) \times C$ where C is a compact group and the projection Γ_3 of $G_3(\mathcal{O})$ to $SU(\widetilde{f^{\sigma_1}}) \cong SU(n,1)$ is an arithmetic lattice there. Γ_3 has a finite index congruence subgroup Φ which is mapped onto $\mathbb{Z}$, by Kazhdan [105] if $n=2$, by Shimura [184] for general n if Γ_3 is a non-uniform lattice in $SU(n,1)$, and by Borel-Wallach [22] if Γ_3 is cocompact.

(b) If n is odd, $SO(n,1)$ has more arithmetic lattices. They are constructed as follows (see [189] and [118] for missing details): Let K be a totally real number field, $\mathcal{D}$ a quaternion algebra over K with the involution σ given by $\sigma(x) = tr(x) - x$, $x \in \mathcal{D}$. Let V be an m-dimensional $\mathcal{D}$-vector space and $h : V \times V \to \mathcal{D}$ a non-degenerate skew-Hermitian form (so that for $\lambda, \mu \in \mathcal{D}$ and $v, w \in V$, $h(\lambda v, \mu w) = \sigma(\lambda) h((v,w)\mu))$. Let $G_2 = SU(h)$ the special unitary group of this form h. Assume h was chosen in such a way that $G_2(K \underset{\mathbb{Q}}{\otimes} \mathbb{R}) \cong SO(n,1) \times C$ where C is a compact group and $n+1 = 2m$. If $\mathcal{O}$ is the ring of integers of K, then the projection Γ_2 of $G_2(\mathcal{O})$ to $SO(n,1)$ is a cocompact arithmetic lattice.

Let W be a two dimensional $\mathcal{D}$-subspace of V and h_0 be the form h restricted to W. Choose W in such a way that if $G_1 = SU(h_0)$, then $G_1(K \underset{\mathbb{Q}}{\otimes} \mathbb{R}) \cong SO(3,1) \times C$ where C is a compact group (such a choice indeed exists!). Then $G_1(\mathcal{O})$ gives rise to a lattice in $SO(3,1) \approx SL_2(\mathbb{C})$. Thus Γ_1 has the Selberg property (see Section 4.1).

Let now L/K be a quadratic extension which is totally imaginary and splits $\mathcal{D}$. Denote by $\ell \to \bar{\ell}$ the action of the non-trivial element of $Gal(L/K)$. Let $\mathcal{D}_L = \mathcal{D} \underset{K}{\otimes} L$ and $\tilde{\sigma} : \mathcal{D}_L \to \mathcal{D}_L$ be the involution $\tilde{\sigma}(\lambda \otimes \ell) = \sigma(\lambda) \otimes \bar{\ell}$, $\lambda \in \mathcal{D}$ and $\ell \in L$. Let $V_L = V \underset{K}{\otimes} L$ and extend h to $h_L : V_L \times V_L \to \mathcal{D}_L$ by $h_L(v_1 \otimes \ell_1, v_2 \otimes \ell_2) = h(v_1, v_2) \otimes \ell_1 \ell_2$ for all $v_1, v_2 \in V$ and $\ell_1, \ell_2 \in L$. Consider now the K-algebraic group $G_3 = SU(h_L)$. Then $G_3(K \underset{\mathbb{Q}}{\otimes} \mathbb{R}) = SU(n,1) \times C$ where C is a compact group. Thus the projection Γ_3 of $G_3(\mathcal{O})$ to $SU(n,1)$ is an arithmetic lattice in $SU(n,1)$ which contains Γ_2. Moreover, as L splits $\mathcal{D}$, $\mathcal{D}_L$ is isomorphic to $M_2(L)$ and thus G_3 can be presented as a unitary group of an Hermitian form in $n+1 = 2m$ variables defined via K and L. So as in (a), we can deduce from [105], [184] and [22] that Γ_3 has a finite index congruence subgroup Φ with a positive Betti number.

It is interesting to note that the arithmetic lattices of type (a) above contain lattices of $SO(2,1) \approx SL_2(\mathbb{R})$ while for type (b) we must use $SO(3,1) \approx SL_2(\mathbb{C})$. Fortunately for both $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$, the arithmetic lattices

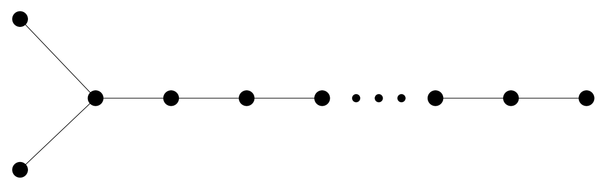

Figure 7.1: The Dinkin diagram of type $D_{\frac{n+1}{2}}$

have the additional interpretation as coming from units of quaternion algebras. Thus as explained in Section 4.1, the Jacquet-Langlands correspondence can be applied and they all have the Selberg property. Well, as we said in Chapter 4, all arithmetic lattices in characteristic zero have the Selberg property including therefore all lattices in $SO(n,1)$. But we showed explicitly how the Burger-Sarnak method enables one to deduce it directly from $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$.

One should also remark that for the lattices of type (a) we do not need to apply this. A much easier proof exists which even shows that they have a subgroup of finite index which is mapped onto a non-abelian free group [122]. It is not known whether this holds for the lattices of type (b).

If $n \neq 3,7$ the above methods give all possible arithmetic lattices in $SO(n,1)$. In $SO(3,1)$ there are more arithmetic lattices (all come from units of suitable quaternion algebras). The case $n = 3$ is different from the general n as $SO(3,1)$ is locally isomorphic to $SL_2(\mathbb{C})$ and hence has also a complex structure. The case $n = 7$ is also special; for every odd n , $SO(n,1)$ is a real form of $D_{\frac{n+1}{2}}$ whose diagram is as in Figure 7.1, where this diagram has $l = \frac{n+1}{2}$ vertices. If $l \neq 4$ this diagram has only two automorphisms while for $l = 4$ it has 6. These automorphisms give rise to more "forms" on D_4 and more arithmetic lattices. For these arithmetic lattices the method described here does not work and the Thurston conjecture is still open.

7.3 The Lubotzky-Sarnak conjecture

Conjecture 7.5 *Let M be an n-dimensional finite volume hyperbolic manifold. Then for every $\varepsilon > 0$, M has a finite cover M' with $\lambda_1(M') < \varepsilon$.*

By Proposition 2.9, this conjecture is equivalent to

Conjecture 7.5' *Let Γ be a lattice in $SO(n,1)$. Then it does not have property (τ).*

To put it into perspective, let us observe

$$\text{Thurston conjecture} \Rightarrow \text{Lubotzky-Sarnak conjecture} \Rightarrow \text{Serre conjecture}$$

By Serre's conjecture we mean here the assertion that arithmetic lattices in $SO(n,1)$ do not have the congruence subgroup property (see Section 1.4.2 Conjecture 1.23). To see these implications note that if the Thurston conjecture holds, Γ does not have FAb and hence does not have (τ) by Corollary 1.29. Now if an arithmetic lattice in $SO(n,1)$ does not have (τ) it follows that it has plenty of non-congruence subgroups since Γ is known to have the Selberg property. This shows that the Lubotzky-Sarnak conjecture implies the Serre conjecture.

The current state of knowledge is as follows: In dimension $n \geq 4$ the Thurston conjecture is known for all arithmetic lattices except from $n = 7$. For the exceptional lattices in dimension 7 (i.e. those coming from the triality effect of D_4) nothing is known (these lattices deserve a special study!). Now, for non-arithmetic lattices, Thurston's conjecture is known (see [124]) for all known non-arithmetic lattices (which are either generated by reflections or constructed by the interbreeding method of Gromov and Piatetski-Shapiro [85]) but not in general. Of course, Thurston's conjecture implies all other conjectures so in these cases they are also known, but when Thurston's conjecture is not known, nothing is known about the other conjectures.

For $n = 2$ and 3 the situation is different. For $n = 2$ all conjectures are known and easy. For $n = 3$ Thurston's conjecture is known for all non-uniform lattices. In fact, by passing to a finite index torsion free subgroup we get a group Γ with deficiency $def(\Gamma) = 1$, i.e. a group presented by one more generators then relations. So $\Gamma \to \mathbb{Z}$ and satisfies the Thurston conjecture. It is also known for the two types of arithmetic lattices and the two types of non-arithmetic lattices mentioned before, as well as for some more arithmetic lattices (see [44]). But the conjecture is still open and is considered to be one

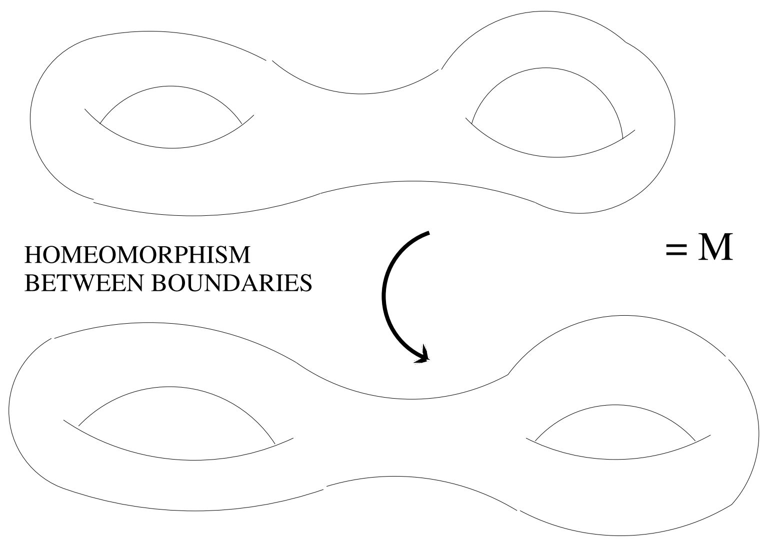

Figure 7.2: The Heegaard splitting

of the most outstanding problems in the geometry of 3-manifolds. On the other hand, Serre's conjecture is proved in full generality for $SO(3,1)$ (see Theorem 7.18 below). Thus the Lubotzky-Sarnak conjecture is of special interest in this case, as an intermediate step towards the Thurston conjecture. In fact, in the next section we will see that it may lead to the virtual Haken conjecture.

7.4 3-manifolds, Heegaard splittings and property (τ)

Throughout this section M is a connected, closed, orientable and irreducible (i.e. any S^2 bounds B^3) 3-manifold.

Heegaard splitting of M is a way to decompose M as a union of "simple" pieces. More precisely, this is a pair (V_1, V_2) where V_i is a handle-body $(i = 1, 2)$ such that $M = V_1 \cup V_2$ and $V_1 \cap V_2 = \partial V_1 = \partial V_2$ (see Figure 7.2). Every M has such a splitting [93].

Note that the boundary of a handle-body with n handles V is a closed surface of Euler characteristic $2 - 2n$, so V_1 and V_2 have the same number of handles - called the genus of the splitting. The minimal n for which M has a Heegaard splitting into two handle-bodies with n handles is called the **Heegaard genus** of M and denoted $C(M)$. A slightly better invariant for M is $\chi_-^h(M)$ which is $-\chi(F)$ where F is the boundary surface in a minimal Heegaard spliting and $\chi(F)$ is the Euler characteristic of F, so $\chi_-^h(M) = 2(C(M) - 1)$.

Given a Heegaard splitting one can construct a presentation for $\pi_1(M)$ with n generators and n relators, where n is the number of handles in V_1 (and V_2). This gives another proof of the fact used in Section 7.5 below, that M has a presentation with the same number of generators and relations. It also proves un upper bound on the size $d(\pi_1(M))$ of the smallest generating subset.

Proposition 7.6

$$d(\pi_1(M)) \leq C(M) = \frac{\chi_-^h(M)}{2} + 1.$$

Our interest is in the behavior of $\chi_-^h(M)$ when one passes to finite sheeted covers M'. One way is to lift the Heegaard splitting. This simply multiples the genus of the splitting surface by the degree of the cover. This shows that $\chi_-^h(M') \leq [M' : M]\chi_-^h(M)$. In fact, Lackenby even shows that if M' can be chosen to be of sufficiently large injectivity radius that one can arrange to ensure that $\chi_-^h(M') < [M' : M]\chi_-^h(M)$. This applies for example to hyperbolic manifolds.

Definition 7.7 *Let* $\Gamma = \pi_1(M)$, $\mathcal{L} = \{N_i\}$ *a family of finite index subgroups,* $\{M_i\}$ *the corresponding covers. Define*

$$\chi_{\mathcal{L}}^h(M) = \inf \left\{ \frac{\chi_-^h(M_i)}{[M_i : M]} | N_i \in \mathcal{L} \right\}.$$

$\chi_{\mathcal{L}}^h(M)$ *is the* **infimal Heegaard gradient** *of* M *with respect to* $\mathcal{L}$.

This number $\chi_{\mathcal{L}}^h(M)$ is non-negative provided that M is not covered by the 3-sphere. It can be zero. This happens for example if M is a hyperbolic manifold that fibers over the circle and one takes the family of cyclic covers dual to the fibers. In group theoretical terms this means that $\Gamma = \pi_1(M)$ has a finitely generated normal subgroup N (isomorphic to a surface group) such that $\Gamma/N \simeq \mathbb{Z}$ and we take $\mathcal{L} = \{N_i\}$ where N_i is the unique subgroup containing N with $\Gamma/N_i \simeq \mathbb{Z}/i\mathbb{Z}$.

Conjecture 7.8 (Heegaard gradient conjecture) [113] *M has a family of covers $\mathcal{L}$ as before with $\chi^h_{\mathcal{L}}(M) = 0$ if and only if M fibers over a circle.*

On the other hand, there is a connection - at least in one direction - between $\chi^h(M)$ and $h(M)$ the isoperimetric Cheeger constant of M.

Theorem 7.9 (Lackenby [113]) *Let M be a closed Riemannian 3-manifold and $K < 0$ its supremal sectional curvature ($K = -1$ if M is hyperbolic). Then*

$$h(M) \leq \frac{4\pi\chi^h_-(M)}{|K|vol(M)}.$$

It follows therefore that if $\pi_1(M)$ has (τ) with respect to $\mathcal{L} = \{N_i\}$ then $\chi^h_{\mathcal{L}}(M) > 0$. This gives an interesting topological information on the covers M_i.

For example

Corollary 7.10 *Let M be an arithmetic hyperbolic 3-manifold. Then there are positive constants c and C such that for every congruence cover $M_i \to M$*

$$cvol(M_i) \leq \chi^h_-(M_i) \leq Cvol(M_i).$$

Proof The upper bound follows from the general fact that the Heegaard genus of covers of M grows at most linearly. The lower bound follows from the previous theorem and the fact that arithmetic lattices in $SL_2(\mathbb{C})$ have the Selberg property (see Chapter 4). □

Combining with Theorem 3.1 above, Theorem 7.9 can give a potential way to attack the Thurston conjecture via Heegaard splittings.

Corollary 7.11 *Let M be a closed hyperbolic 3-manifold, $\Gamma = \pi_1(M)$ and $\mathcal{L} = \{N_i\}$ the family of all finite index normal subgroups of Γ and M_i the corresponding covers. If $\chi^h_-(M_i)/\sqrt{[M_i : M]}$ has zero infimum then there exists i such that $\beta_1(M_i) = dimH^1(M_i, \mathbb{R}) \neq 0$.*

Proof By Theorem 7.9 we have that $h(M_i)\sqrt{[M_i : M]}$ has zero infimum and so by Theorem 3.1 we get the positive virtual β_1. □

In fact, Lackenby's result is stronger: he defined an invariant $C_+(M)$ which depends on "generalized Heegaard splitting" of M and satisfies $C_+(M) \leq \chi_-^h(M)$. He then shows that the virtual positive β_1 for M is equivalent to $C_+(M_i)/\sqrt{[M_i : M]}$ has zero infimum (or to $C_+(M_i)$ has a bounded subsequence). Hence, these conditions are equivalent to all the conditions in Theorem 3.1. We refer the reader to the original paper and to [38] and [175].

Moreover Lackenby showed that the infimal Heegaard conjecture together with Lubotzky-Sarnak conjecture implies the virtual Haken conjecture. Indeed, he proved

Theorem 7.12 *Let M be a closed, orientable 3-manifold and $\mathcal{L} = \{M_i\}$ a family of finite regular covers of M. Suppose that*

1. *$\pi_1(M)$ fails to have (τ) with respect to $\{\pi(M_i)\}$,*

2. *the infimal Heegaard gradient $\chi_{\mathcal{L}}^h(M)$ is positive.*

Then M is virtually Haaken.

In fact, Lackenby showed that by the assumptions of Theorem 7.12, for infinitely many i, M_i has a "thin generalized Heegaard splitting" which is not a Heegaard splitting. Now by the work of Casson and Gordon [38], it is known that, if a 3-manifold has a generalized Heegaard splitting, which is not Heegaard splitting, then it has an incompressible surface, i.e. it is Haken.

Anyway we can deduce

Claim 7.13 *The Lubotzky-Sarnak conjecture (Conjecture 7.5) and the infimal Heegaard gradient conjecture (Conjecture 7.8) together imply the virtual Haken conjecture (Conjecture 7.2) for hyperbolic 3-manifolds.*

Proof Let M be a 3-dimensional hyperbolic manifold and $\Gamma = \pi_1(M)$. By Lubotzky-Sarnak conjecture, Γ does not have (τ) so there exists $\mathcal{L} = \{N_i\}$ of finite index normal subgroups with respect to which Γ does not have property (τ). Let $\chi_{\mathcal{L}}^h(M)$ be the infimal Heegaard gradient of M with respect to $\mathcal{L}$. If $\chi_{\mathcal{L}}^h(M) = 0$, then by the infimal Heegaard gradient conjecture, M virtually fibers over the circle and in particular is virtually Haken. If $\chi_{\mathcal{L}}^h(M) > 0$ then Theorem 7.12 implies that M is anyway virtually Haken. □

7.5 Pro-p groups and 3-manifolds

Let us suggest now some new ways to tackle the 3-manifold problems mentioned in the previous section using group theory in general and pro-p groups in particular.

We start with the following

Conjecture 7.14 *Let G be a group presented by d generators and d relations and such that for some prime p, $d_p(\Gamma) \geq 5$, where $d_p(\Gamma) = dim_{\mathbb{F}_p}(\Gamma/[\Gamma,\Gamma]\Gamma^p)$. Then*

$$\limsup \left(\log |N/[N,N]|/[G:N]\right)$$

is infinite when N runs over all finite index normal subgroups of Γ.

Note that Conjecture 7.14 implies Thurston's conjecture for compact hyperbolic 3-manifolds (for finite volume non compact it is known anyway). Indeed, if M is such a manifold, the $\pi = \pi_1(M)$ is a cocompact lattice in $SL_2(\mathbb{C})$.

Recall:

Proposition 7.15 ([121]) *If Γ_0 is a finitely generated not virtually solvable linear group, then for every prime p, and every positive integer k, Γ_0 has a finite index subgroup Γ, with $d_p(\Gamma) \geq k$.*

We can apply Proposition 7.15 to $\Gamma_0 = \pi$ to replace π by a finite index subgroup Γ with $d_p(\Gamma) \geq 5$ (note that we can choose p arbitrarily and we can in fact make $d_p(\Gamma)$ as large as we wish). Now, Γ being the fundamental group of a finite sheeted cover of M has also deficency zero (as does every closed 3-manifold group - see [63]), i.e. it is presented by d generators and d relators. So by Conjecture 7.14, the abelian quotients are not exponentially bounded. Theorem 3.9 now implies that there exists a finite index subgroup N of Γ with $N/[N,N]$ infinite, i.e. Thurston's conjecture holds.

Let us remark that Conjecture 7.14 is not true without the extra assumption that $d_p(\Gamma)$ is large. Indeed, the Higman group

$$H = \langle x_0, x_1, x_2, x_3 | x_{i+1}^{-1} x_i x_{i+1} = x_i^2, \text{ for } i \in \mathbb{Z}/4\mathbb{Z}\}$$

is a 4-generator 4-relator group with no finite index subgroup (see [178] page 18) and $H^{\star(n)} = H \star \ldots \star H$ is a $4n$-generated, $4n$-related group with the same property. Thus some assumption on $d_p(\Gamma)$ is necessary. The assumption that $d_p(\Gamma) \geq 5$ implies that the pro-p completion $G = \Gamma_{\hat{p}}$ of Γ is a Golod-Shafarevitch group (see below) and opens the door to try to apply pro-p methods for proving Conjecture 7.14. Various methods in pro-p theory have been applied to the study of the size of the abelian quotients of finite index subgroups (see [134], [97], [127]) but usually in the opposite direction, i.e., to get upper bound on $|N/[N,N]|$ in terms of $[\Gamma : N]$. This time we need lower bounds.

One may try to use pro-p methods to prove the Lubotzky-Sarnak conjecture. To explain this we should elaborate on the notion of Golod-Shafarevitch groups. After doing so we will show how these considerations can prove Serre's conjecture for lattices in $SL_2(\mathbb{C})$. As explained earlier (Section 7.3) the Lubotzky-Sarnak conjecture is stronger than Serre's conjecture, but there is some hope that pro-p methods can also be useful for it.

Consider the free group $F(d)$ on d generators and its free pro-p completion $F(d)_{\hat{p}}$ that is the free pro-p group on d generators.

Now let A be the free associative algebra over the field F_p, $|F_p| = p$, on the free generators $a_1, \ldots, a_d$. The algebra A is graded: $A = \sum_{i=0}^{\infty} A_i$, where A_i is spanned by all monomials of length i while $A_0 = F_p \cdot 1$. Consider the ideal $I = \sum_{i=1}^{\infty} A_i$ of A, $A = F_p \cdot 1 + I$. The ideals $I^m = \sum_{i \geq m}^{\infty} A_i$, $m = 1, 2, \ldots$ define a topology on A. Let $\hat{A}$ be the completion of A with respect to this topology. $\hat{A}$ is the algebra of infinite series over F_p on d non-commuting variables.

Consider the group of the invertible elements from $\hat{A}$ (we denote it by $\hat{A}^*$) with the induced topology. The closed subgroup of $\hat{A}^*$ generated by $1 + a_1, \ldots, 1 + a_d$ is the free pro-p group having $1 + a_i$, $1 \leq i \leq d$, as free generators.

Hence, an arbitrary element f of the free pro-p group $F(d)_{\hat{p}}$ on d generators can be viewed as an element of the algebra $\hat{A}$ upon substituting $x_i = 1 + a_i$, $1 \leq i \leq d$,

$$f(1 + a_1, \ldots, 1 + a_d) = 1 + f'(a_1, \ldots, a_d),$$

where $f'(a_1, \ldots, a_d)$ is an element of degree ≥ 1.

Consider a d-generated pro-p group G presented by generators and relations $G = \langle x_1, \ldots, x_d | f_1, f_2, \ldots \rangle$. Substituting $1 + a_i$ for x_i, $1 \leq i \leq d$, we get

$$f_j(1 + a_1, \ldots, 1 + a_d) = 1 + f_j'(a_1, \ldots, a_d), f_j' \in I.$$

The algebra

$$B = \langle a_1, \ldots, a_d | f'_j(a_1, \ldots, a_d) = 0, j = 1, 2, \ldots \rangle$$

is closely related to the group G in the following way.

For an open normal subgroup H of G, let $\omega(H)$ denote the ideal of the group algebra $F_p[G]$ generated by the elements $1-h$, $h \in H$. In other words, $\omega(H)$ is the kernel of the homomorphism $F_p[G] \to F_p[G/H]$ induced by the natural homomorphism $G \to G/H$. So $\omega(G)$ is the augmentation ideal of $\mathbb{F}_p[G]$.

The ideals $\omega(H)$ define a topology on the group algebra $F_p[G]$. The completion of $F_p[G]$ with respect to this topology is isomorphic to the algebra B [89]. Hence,

$$H_B(t) = 1 + \sum_{i=1}^{\infty} \dim(\omega(G)^i/\omega(G)^{i+1})t^i,$$

where $H_B(t)$ is the Hilbert series of the algebra B.

Suppose that all relators $f_1, f_2, \ldots$ have degrees ≥ 2. Suppose further that there are r_2 relators of degree 2, there are r_3 relators of degree 3 and so on. Denote

$$H_R(t) = r_2 t^2 + r_3 t^3 = \ldots.$$

Then the fundamental result of Golod and Shafarevitch is (see[78])

$$\frac{H_B(t)}{1-t}(1 - dt + H_R(t)) \geq \frac{1}{1-t} \tag{7.1}$$

where the inequality of the power series means term by term.

Definition 7.16 *We say that a set of relators of a pro-p group G is small in the sense of Golod-Shafarevitch if all relators have degrees ≥ 2 and there exists a number $0 < t_0 < 1$ such that $1 - dt_0 + H_R(t_0) < 0$. Groups which admit small presentations are called Golod-Shafarevitch groups.*

A group presented by a small set of relators is necessarily infinite. Indeed, the series $H_B(t_0)$ cannot converge as otherwise the left-hand side of (7.1) for $t = t_0$ would be negative, while the right hand side of (7.1) is positive.

Let $d(G)$ be the minimal number of generators for G and $r(G)$ the minimal number of relations in a pro-p presentation of G.

Examples of groups with small sets of relators are for instance given by the following.

Proposition 7.17 *(i) A pro-p group G which is presented by $d(G) \geq 4$ generators and $r(G) < \frac{d(G)^2}{4}$ relators is a Golod-Shafarevitch group.*

(ii) Let Γ be a discrete group generated by d generators and r relators. Assume $d_p(\Gamma) \geq 4$ and $r \leq d = \frac{d_p(\Gamma)^2}{4} - d_p(\Gamma)$. Then $G = \Gamma_{\hat{p}}$ is a Golod-Shafarevitch group.

In [120] it was observed that groups defined by small sets of relators are even not p-adic analytic (see also [57]). Let now Γ be an arithmetic lattice in $SL_2(\mathbb{C})$. As explained above, we may, after replacing Γ by a finite index subgroup, assume that $d_p(\Gamma)$ is as large as we want and thus Γ has a balanced presentation (i.e. a presentation with the same number of generators and relators). We conclude from Proposition 7.17 that $G = \Gamma_{\hat{p}}$ is a Golod-Shafarevitch group and hence not a p-adic analytic group.

On the other hand, it is not difficult to show that if Γ is an arithmetic group (in characteristic 0) satisfying the congruence subgroup property, then $\Gamma_{\hat{p}}$ must be p-adic analytic. We can deduce Serre's conjecture:

Theorem 7.18 (Lubotzky [120]) *If Γ is an arithmetic cocompact lattice in $SL_2(\mathbb{C})$, then it does not have the congruence subgroup property.*

Remark If Γ is a torsion-free arithmetic lattice in $SL_2(\mathbb{C})$ which is not cocompact, then it has a presentation with one more generator than relators. So $|\Gamma/[\Gamma,\Gamma]| = \infty$ which easily implies that Γ does not have the congruence subgroup property.

Let us observe now, that the arguments above give more. Let Γ be any cocompact lattice in $SL_2(\mathbb{C})$ and p an arbitrary prime. As before, by passing to a finite index subgroup we can assume

(a) Γ is torsion free and residually finite,

(b) $d_p(\Gamma)$ is as large as we need,

(c) $G = \Gamma_{\hat{p}}$ has a presentation with $d_p(\Gamma)$ generators and $d_p(\Gamma)$ relations, so G is a Golod-Shafarevitch group.

The following theorem shows that $\Gamma_{\hat{p}}$ is a very large pro-p group.

Theorem 7.19 (Zelmanov [196]) *Let G be a Golod-Shafarevich pro-p group. Then G contains a non-abelian free pro-p group F on two generators.*

Let us show right away a non-trivial application to hyperbolic 3-manifolds.

Proposition 7.20 *Let M be a 3-dimensional hyperbolic manifold and H an arbitrary finite p-group. Then M has two finite sheeted covers M' and M'' such that M'' is a Galois cover of M' with Galois group isomorphic to H.*

Proof Let $\Gamma = \pi_1(M)$. The proposition is equivalent to the assertion that Γ has two finite index subgroups $\Gamma'' \lhd \Gamma' \leq \Gamma$ such that $\Gamma'/\Gamma'' \simeq H$.

Now, by replacing Γ by a finite index subgroup as above, we can assume that $G = \Gamma_{\hat{p}}$ is a GS-group. So G contains F by Zelmanov's theorem. F has a free pro-p subgroups on d generators for any d, in particular it has a closed subgroup F_1 with an epimorphism $\pi : F_1 \to H$. Let $N = ker\pi$, and K an open normal subgroup of G with $K \cap F_1 \leq N$ (such K clearly exists as G is a profinite group).

Let now $G' = KF_1$ and $\Gamma' = G' \cap \Gamma$. So $G'/K = KF_1/K \simeq F_1/K \cap F_1$ is mapped onto $F_1/N \simeq H$. As Γ' is dense in G' it is mapped onto H and proposition is proved. □

We expect that Proposition 7.20 holds for every finite group. We do not know how to prove this proposition directly even for p-groups. It shows that the theory of pro-p groups has the potential to be useful for studying 3-manifolds.

Zelmanov's theorem implies that given a cocompact lattice Γ in $SL_2(\mathbb{C})$ and a finite p-group H, Γ has a finite index subgroup Λ with an epimorphism $\Lambda \to H$. Unfortunately it does not (and neither its proof) give any information on the index of Λ. Such an information could be very useful in trying to prove the Lubotzky-Sarnak conjecture.

The following conjecture suggests itself:

Conjecture 7.21 *(i) Let $\{G_i\}_{i\in I}$ be a family of finite groups such that every finite group H is a subquotient of G_i for some i (i.e. for some i, G_i has two subgroups $A \lhd B$ with $B/A \simeq H$). Then the family $\{G_i\}_{i\in I}$ is not an expanding family (i.e. cannot be made into a family of expanders with respect to a bounded number of generators).*

(ii) The same conjecture with the relaxed condition that H is an arbitrary finite p-group (for a fixed prime p).

So Conjecture 7.21 (ii) (with Zelmanov's theorem) would imply the Lubotzky-Sarnak conjecture. It would also answer in the negative Problems 5.22 and 5.23 from Chapter 5.

7.6 Lattices in other rank one groups

In the previous sections we have looked at lattices in $SO(n,1)$. There are more simple Lie groups of rank one: $SU(n,1)$, $Sp(n,1)$ and $\mathbb{F}_4^{(-20)}$. These are the isometry groups of the complex hyperbolic spaces, the quaternionic hyperbolic spaces and the hyperbolic plane over the Cayley numbers, receptively. The lattices in these groups suggest some interesting problems and challenges that we briefly review here.

Let us start with $SU(n,1)$. The arithmetic lattices of $SU(n,1)$ are constructed in the following way: Let E be a CM-field (this means that E is a totally imaginary quadratic extension of a totally real field K, e.g. $E = \mathbb{Q}(\sqrt{-d})$, $d > 0$). Let $D = D_b$ be a central algebra over E of rank b with an involution $d \to d'$ of the second kind (i.e., this is antiautomorphism of D which induces on the center E the unique nontrivial element of $Gal(E/K)$). The involution can be extended to $M_a(D_b)$ and we can define the unitary group $G = \{g \in GL_a(D_b); gg' = 1\}$, which is a K-algebraic group. It follows that if $\mathcal{O}$ is the ring of integers in K, then $G(\mathcal{O})$ is a lattice in $G(K \otimes \mathbb{R})$. If $n+1 = ab$, then under various arithmetical conditions on E, K and D, it may happen that $G(K \otimes \mathbb{R})$ is equal to $SU(n,1) \times \mathcal{K}$ when $\mathcal{K}$ is a compact group. Hence $G(\mathcal{O})$ will give us an arithmetic lattice in $SU(n,1)$. All arithmetic lattices in $SU(n,1)$ are obtained in this way.

Now, if $b = 1$, $D_b = E$ and G is just the group preserving an Hermitian form of E over K. In this way we get the *lattices of the simple type*. These are the lattices Γ into which the arithmetic lattices of $SO(n,1)$ were embedded (see Section 7.2 above). At the other end of the spectrum, we have the lattices for which $a = 1$. These are obtained from G, the group of units of the division algebra D.

All the arithmetic lattices in $SU(n,1)$ have property $\tau(\mathcal{L})$ when $\mathcal{L}$ is the family of congruence subgroups, i.e. they all have the Selberg property. But apparently there is a big difference concerning property $FAb(\mathcal{L})$. Kazhdan [105], Shimura [184] and Borel-Wallach [22] showed that if Γ is a lattice of the simple type it has congruence subgroups with infinite abelianizations (and we have made a crucial use of this in Section 7.2 to deduce such a result for the arithmetic lattices on $SO(n,1)$). On the other hand Rapoport-Zink [165], Rogawski [168] and Clozel [42] showed that for the lattices obtained from units of division algebras, $H^1(\Delta, \mathbb{R})$ always vanishes for congruence subgroup Δ, i.e. they have property $FAb(\mathcal{L})$.

This shows (see Proposition 2.15) that for the lattices of the simple type

$H^1(\Gamma, L^2(\widehat{\Gamma}_{\mathcal{L}})) \neq 0$ while for division algebra units $H^1(\Gamma, L^2(\widehat{\Gamma}_{\mathcal{L}})) = 0$.

This mysterious phenomenon deserves some attention: It goes against the common believe that "lattices in the same group behave similarly".

Let us mention that if $n+1$ is not a prime then beside the two cases described above there are also "intermediate cases" when both a and b are greater than 1. For these nothing has been proven, but we have learned from Rogawski that we should expect they behave like lattices of the simple type, namely, they also satisfy the Thurston conjecture (i.e. do not have $FAb(\mathcal{L})$).

We should mention however that FAb is not known for the lattices coming from units of division algebras, only $FAb(\mathcal{L})$ with respect to $\mathcal{L} = \{$congruence subgroups$\}$. The congruence subgroup problem is completely open for them.

Regarding non-arithmetic lattices even less is known: It is known that $SU(n,1)$ for $n=2$ and 3 have some non-arithmetic lattices. Examples were constructed by Mostow [155], Deligne-Mostow [51] and Livne. For one of the examples constructed by Livne (see [51]) it is known that it is mapped onto a non-abelian free group, so it does not have FAb or (τ). It is not known whether $SU(n,1)$ for other n's have non-arithmetic lattices.

The story of the lattices in $Sp(n,1)$ and $\mathbb{F}_4^{(-20)}$ is completely different. As mentioned in Theorem 1.6 these groups and their lattices have property (T). So the analogue of Thurston's and Lubotzky-Sarnak's conjectures do not hold here. Moreover as lattices in these groups satisfy super-rigidity and are all arithmetic one may suspect that the lattices in $Sp(n,1)$ and $\mathbb{F}_4^{(-20)}$ satisfy the congruence subgroup property (see the discussion in Section 4.4). We mention in passing that solving affirmatively the congruence subgroup problem even for one such lattice would solve a well known problem on hyperbolic groups:

Claim 7.22 *If there exists a cocompact arithmetic lattice Γ in $Sp(n,1)$ or $\mathbb{F}_4^{(-20)}$ satisfying the congruence subgroup property, then there exists a non-residually finite hyperbolic group.*

Proof Recall that Γ is a hyperbolic group (which is of course residually finite as Γ is linear). Now, in a hyperbolic group Γ, for most sufficiently large elements $\gamma \in \Gamma$, the normal closure N of γ in Γ is infinite and the quotient group Γ/N is a hyperbolic group Δ (see [82] and [75]). We claim that such Δ is not residually finite. Indeed, if Δ has infinitely many finite index normal subgroups $\{\Delta_i\}_{i\in I}$ then their preimages $\{\widetilde{\Delta}_i\}_{i\in I}$ in Γ give an infinite family of finite index normal subgroups in Γ. They are all congruence subgroups of

Γ by our assumption. Now, in an arithmetic subgroup of a simple algebraic group, the intersection of infinitely many normal congruence subgroups must be finite and central. This is a contradiction since all the $\{\widetilde{\Delta}_i\}_{i\in I}$ contain N, which is, by our construction, an infinite normal subgroup. This shows that Δ has only finitely many finite index normal subgroups and hence it is not residually finite. □

Remark The proof actually shows that by replacing Δ by the intersection of its finite index subgroups, we get even a hyperbolic group with no finite index subgroup. Indeed, it was proved in [102] that if a non-residually finite hyperbolic group exists then there exists also a hyperbolic group without any finite index subgroup.

Chapter 8

Uniqueness of invariant measures

In this chapter we apply (τ) to the Ruziewicz problem: The uniqueness of the Haar measure on some compact groups. Some counter intuitive examples will be given in Section 8.2.

8.1 Open compact subgroups of local adelic groups

Let G be a compact group. It has a uniquely defined Haar measure, which is G invariant (under the left multiplication, in fact also under right) and countably additive. If one relaxes the condition from "countably additive" to "finitely additive" the uniqueness is not automatic any more. In fact, S^1-the circle has uncountably many different finitely additive invariant measures. The study of finitely additive measures has its origin in the Banach-Tarski paradox and in the theory of amenable groups - see [122] and the references there. It turns out that this topic is very much related to our discussion and this is due to what we have seen in Section 2.3.1 Proposition 2.10: The Haar measure is the only finitely additive invariant measure of G if and only if G as a discrete group acting on $L^2_0(G,\mu)$ does not weakly contain the trivial representation. This actually amounts to saying that μ is unique if and only if G has a finitely generated dense subgroup Δ whose representation on $L^2_0(G,\mu)$ does not weakly contain the trivial representation. This is exactly what brought us to Corollary 2.11 that Γ has $\tau(\mathcal{L})$ if and only if μ is the unique finitely additive measure on $\widehat{\Gamma}_{\mathcal{L}}$.

As we have shown various examples of $\tau(\mathcal{L})$, we can now deduce:

Theorem 8.1 *Let G be a connected semi-simple algebraic group defined over a global field k with the ring of integers $\mathcal{O}$. Then for every profinite open subgroup K of $G(\mathbb{A}_k^f)$ or $G(k_v)$, for some valuation v of k, the Haar measure of K is the only finitely additive invariant measure of K ($\mathbb{A}_k^f$ denotes the finite adèles).*

Proof Let us first consider the local case: $K \subset G(k_v)$. As we saw in Chapter 4 Theorem 4.4 for some valuation $v_0 \notin S_\infty \cup \{v\}$, $\Gamma = G(\mathcal{O}_S)$ has the Selberg property when $S = S_\infty \cup \{v_0\}$. The closure of Γ in $G(k_v)$ is commensurable with K and the induced topology on Γ is weaker than the congruence topology of Γ. This shows that the Haar measure is unique on the closure of Γ and this implies the uniqueness also for K.

Let now K be open in $G(\mathbb{A}_k^f)$. Let v_0 be a valuation of k provided by Theorem 4.4. Write $G(\mathbb{A}_k^f) = G(k_{v_0}) \times H$. Replacing K by a finite index subgroup (which is allowed) we can assume that $K = K_1 \times K_2$ where K_1 is open compact in $G(k_v)$ and K_2 in H. $G(k)$ is also a subgroup of H. Look at $\Gamma = G(k) \cap K_2$. This is an S arithmetic group for $S = S_\infty \cup \{v_o\}$ which is dense in K_2. By Theorem 4.4, Γ has the Selberg property, so the Haar measure is the unique invariant measure on K_2. The uniqueness for K_1 and for K_2 implies the existence of suitable finitely generated subgroups and ensures the uniqueness for K. □

8.2 Uniqueness and non-uniqueness with respect to dense subgroups

In [139] the question whether the property of being expanders depends on the generators or not was discussed. As a way to construct examples which depend on generators, it was suggested to look for a profinite group G with two dense finitely generated subgroups Γ_1 and Γ_2 such that Γ_1 is amenable and Γ_2 has (T). By Theorem 4.12 and Proposition 2.5 the finite quotients of G are expanders with respect to the generators of Γ_2 and are not with respect to the generators of Γ_1. But, at the same time, it was suggested that maybe such a triple (G, Γ_1, Γ_2) is impossible unless $G = \Gamma_1 = \Gamma_2$ is finite. This is still an open problem, but let us show that the solution to "dependence problem" in [2] as presented here in Theorem 5.10 above gives us the following interesting situation:

Theorem 8.2 *There exists a profinite group G with two finitely generated dense subgroups Γ_1 and Γ_2 such that the representation of Γ_1 on $L_0^2(G)$ weakly contains the trivial representation and the representation of Γ_2 on $L_0^2(G)$ does not. Or equivalently, the Haar measure is the only finitely additive Γ_2 invariant measure on G, but there are different finitely additive Γ_1 invariant measures.*

Proof Let $\Gamma = SL_d(\mathbb{Z})$ and $\Gamma(q^n) = Ker(SL_d(\mathbb{Z}) \to SL_d(\mathbb{Z}/q^n\mathbb{Z}))$. We saw in Example 5.6 that the family $\{\Gamma(q)/\Gamma(q^n); n \in \mathbb{N}\}$ is a family of finite q-groups which are expanders with respect to a finite fixed set S_1 of generators for $\Gamma(q)$.

Fix two different primes p and q. Let $H_n = \Gamma(q)/\Gamma(q^n)$, $G_n = \mathbb{F}_q[H_n] \rtimes H_n$, $H = \lim_{\leftarrow} H_n$ and $G = \lim_{\leftarrow} G_n$ all with respect to the natural epimorphisms. Then $G = \mathbb{F}_p\left[[H]\right] \rtimes H$ where $\mathbb{F}_p\left[[H]\right] = \lim_{\leftarrow} \mathbb{F}_p[H_n]$ is the completed group algebra of H. Now, Example 5.15 and the proof of Theorem 5.13 show that the subgroup Γ_1 of G generated by S_1 and the element $1 \cdot e$ in $\mathbb{F}_p\left[[H]\right]$ is dense in G. The quotients are not expanders, so the representation of Γ_1 on $L_0^2(G)$ weakly contains the trivial representation, so the Haar measure is not the unique finitely additive Γ_1 invariant measure of G. On the other hand, the remark after Theorem 5.10 shows that for a suitable constant t, with positive probability (with respect to the Haar measure of $\mathbb{F}_p\left[[H]\right]$ which, let us recall, is different from the restriction to $\mathbb{F}_p\left[[H]\right]$ of the Haar measure of G) a choice of a random set T of t elements in $\mathbb{F}_q\left[[H]\right]$, the set $T \cup S_1$ generates a finitely generated dense subgroup Γ_2, and its projection to $\mathbb{F}_p\left[[H_n]\right] \rtimes H_n$ form a family of expanders. Thus the representation of Γ_2 on $L_0^2(G)$ does not weakly contain the trivial representation and μ is the only finitely additive Γ_2 invariant measure on G. □

Chapter 9

Property (τ) and C^* algebras

Property (T) found its way into the theory of operators and C^*-algebras in few different ways (see for example the works of Connes [46] and of Voiculescu [190]). Sometimes property (τ) suffices for the applications and in some cases it is even essential. For example, in [98] Junge and Pisier showed that $B(\mathcal{H}) \otimes B(\mathcal{H})$ has two different norms (when $B(\mathcal{H})$ is the C^* algebra of the bounded linear operators of an infinite dimensional separable Hilbert space $\mathcal{H}$). Valette obtained some optimal bounds regarding the ratio between these norms (see [186]) using the Ramanujan graphs. What was really used there is the fact that the free group on $q+1$ generators, q being a prime power, has an infinite series of finite dimensional representations with a special property expressed by the bound on the eigenvalues of the graphs, i.e. they have (τ) with respect to a special class of normal subgroups and with an explicit optimal constant.

In another direction the work of Benveniste and Szarek [21] uses (τ) to get explicit examples of $n \times n$ matrices of norm 1 which cannot be approximated within an explicit constant by any matrix which decomposes as an orthogonal direct sum of smaller matrices.

Still, in the above mentioned applications what plays the crucial role is either property (T) or the explicit constant which comes with (τ).

We choose to present here in more details an application of (τ) to the question of separating the C^*-algebra of a group Γ by the finite dimensional representations of Γ. This application is based essentially on the difference between (T) and (τ). Then we will take a broader perspective and will consider the "location" of the finite representations $\widetilde{\Gamma}_f$ of Γ within $\widetilde{\Gamma}$-the unitary dual of Γ. Property (τ) says that the non-trivial finite representations are bounded away from the trivial representation. We will consider a very different property: the finite dimensional representations are dense in $\widetilde{\Gamma}$. This, in particular, implies that the C^* algebra of Γ is seperated by the finite (dimensional) representations. Free groups and surface groups have this property while lattices in higher rank simple groups tend not to have it.

9.1 Separating $C^*(\Gamma)$ by finite dimensional representations: negative results

Let Γ be a finitely generated group and $C^*(\Gamma)$ the full C^*-algebra of Γ. This is the completion of the group algebra $\mathbb{C}[\Gamma]$ under the norm

$$\left\| \sum_\gamma c_\gamma \gamma \right\| = \sup \left\{ \left\| \sum_\gamma c_\gamma \pi(\gamma) \right\| ; \pi \in \widetilde{\Gamma} \right\}$$

where $\widetilde{\Gamma}$ is the unitary dual of Γ, i.e. the set of all (equivalent classes of) unitary representations of Γ.

The following proposition follows easily from the fact that every finitely generated linear group is residually finite.

Proposition 9.1 *Let Γ be a finitely generated group. The following conditions are equivalent:*

1. *the group Γ is residually finite;*
2. *the finite dimensional unitary representations of Γ separate the points of Γ;*
3. *the finite dimensional unitary representations of Γ separate the points of $\mathbb{C}[\Gamma]$.*

Condition 3 means that for every $\alpha \in \mathbb{C}[\Gamma]$ there exists a finite dimensional unitary representation ρ of Γ such that $\widetilde{\rho}(\alpha) \neq 0$, when $\widetilde{\rho}$ is the representation induced on $\mathbb{C}[\Gamma]$ by Γ.

A much more difficult question is when the finite dimensional unitary representations of Γ separate the points of $C^*(\Gamma)$, equivalently, when is it possible to embed $C^*(\Gamma)$ in the C^* direct sum of matrix algebras?

The following fact follows from the basic definition and enables us to change the language to a question about density in the unitary dual.

Proposition 9.2 *Finite dimensional unitary representations of Γ separate the points of $C^*(\Gamma)$ if and only if the set of finite dimensional unitary representations of Γ is dense in the unitary dual of Γ with respect to the Fell topology.*

Proof See [56] Chapter 18. □

Let us say that a discrete group Γ has the **finite dimensional density property** (FDD for short) if the set of finite dimensional unitary representations is dense in $\hat{\Gamma}$. We say that it has the **finitary density property** (FD for short) if the finite representations are dense in $\hat{\Gamma}$. Clearly, FD implies FDD.

Proposition 9.3 *Let Γ be a discrete group and Λ a subgroup of Γ . If Γ has FDD (respectively FD) the same holds for Λ.*

Proof Let π be a representation of Λ. We have to prove that π is the limit of finite dimensional representations. Let $\rho = Ind_\Lambda^\Gamma(\pi)$, so by assumption ρ is a limit of such representations $\{\rho_i\}_{i\in I}$ of Γ. Thus $Res_\Lambda^\Gamma(\rho)$ is a limit of $\{Res_\Lambda^\Gamma(\rho_i)\}_{i\in I}$ and the later are clearly also finite dimensional. We only need to observe now that as Γ is discrete π is a subrepresentation of $Res_\Lambda^\Gamma(\rho) = Res_\Lambda^\Gamma(Ind_\Lambda^\Gamma(\pi))$. Indeed, if V is the representation space of π then the space of functions

$$\{f_v : \Gamma \to V | v \in V, f(\delta) = \pi(\delta)v \text{ for } \delta \in \Lambda \text{ and } f(\gamma) = 0 \text{ for } \gamma \notin \Gamma \setminus \Lambda\}$$

is a subspace of $Ind_\Lambda^\Gamma(\pi)$ which is Λ-invariant and isomorphic to (V, π). □

The next proposition gives exemples of residually finie groups without FDD.

Proposition 9.4 *Let Δ be a finitely generated group such that:*

1. *Δ has (τ) but not (T),*
2. *every finite dimensional representation of Δ factors through a finite quotient.*

Then the finite dimensional representations of Δ do not separate the points of $C^*(\Delta)$.

Proof Indeed the finite dimensional representations of Δ are bounded away from the trivial, but the unitary dual is not. So they are not dense there. □

Proposition 9.4 applies for example to $SL_2(\mathbb{Z}[\frac{1}{p}])$ or $SL_2(\mathbb{Z}[\sqrt{p}])$, p a prime.

But also for many groups with property (T), the finite dimensional representations are not dense.

Theorem 9.5 (Bekka [19]) *Let K be a number field, $\mathcal{O}$ its ring of integers, S a finite set of primes of K including all archimedean ones, $\mathcal{O}_S = \{x \in K | v(x) \geq 0 \quad \text{for all } v \notin S\}$ and let $\Gamma = G(\mathcal{O}_S)$, when G is an absolutely simple, simply connected K-algebraic group. Assume*

1. *K-rank $(G) \geq 1$,*
2. *S-rank $(G)(= \sum_{v \in S} K_v -$ rank $(G)) \geq 2$.*

Then the finite dimensional representations of Γ do not separate the points of $C^*(\Gamma)$, i.e. Γ does not have FDD.

Corollary 9.6 *For $\Gamma = SL_d(\mathbb{Z})$, $d \geq 3$, the finite dimensional unitary representations do not separate the points of $C^*(\Gamma)$ while for $\Gamma = SL_2(\mathbb{Z})$ they do.*

The fact that $SL_2(\mathbb{Z})$ has FDD and even FD will be shown in the next section. Here we prove Theorem 9.5 from which the first half of Corollary 9.6 follows.

Proof of Theorem 9.5 Under the assumptions of theorem, the group Γ satisfies:

(a) Every normal subgroup of it is either finite or of finite index (this is Margulis normal subgroup theorem [149]).

(b) Γ has a non-trivial unipotent element γ since $K - rank(G) \geq 1$. This element is a u-element by [128]. Namely the length of γ^n with respect to a fixed finite set of generators of Γ is $O(\log n)$.

(c) Γ satisfies the congruence subgroup property (see [161] and [162]).

Properties (a), (b) and (c) imply that every finite dimensional representation of Γ factors through a finite representation (in fact a congruence one). Indeed, if ρ is a finite dimensional representation of Γ then by [128] the image of a u-element γ must be a virtual unipotent element, i.e. $\rho(\gamma^k)$ is unipotent for some $k \in \mathbb{N}$. But the unitary group $U_d(\mathbb{C})$ has no non-trivial unipotent elements, so $\rho(\gamma^k) = 1$. This shows that $Ker(\rho)$ is infinite and hence by (a) of finite index and by (c) a congruence subgroup.

On the other hand, as K-rank $(G) \geq 1$, there is an algebraic homomorphism with finite kernel from SL_2 into G. This shows that $\Lambda = SL_2(\mathbb{Z})$ (or $\Lambda = PSL_2(\mathbb{Z})$) is embedded in Γ, up to commensurability, in such a way that the congruence topology of Γ induces the congruence topology of $SL_2(\mathbb{Z})$.

Now if the finite representations of Γ separate the points of $C^*(\Gamma)$, their restrictions to Λ separate the points of $C^*(\Lambda)$. But in our case, the finite representations of Γ restricted to Λ give only the congruence quotients representations of $SL_2(\mathbb{Z})$. By the Selberg Theorem (see Section 4.1) the latter are bounded away from the trivial representation, while Λ does not have (T) and so the finite congruence representations cannot be dense in the unitary dual of Λ. This contradiction finishes the proof. □

Remarks 1) Theorem 9.5 holds also in $char(K) = p > 0$. The proof is similar, but $\Lambda = SL_2(\mathbb{Z})$ should be replaced by $\Lambda_p = SL_2(\mathbb{F}_p[t])$. The latter is not finitely generated, but this is not crucial in the proof. The group Λ_p has the Selberg property in the following sense: the representations of it factoring through the congruence quotients are bounded away from the trivial representation. This highly non-trivial fact is a corollary of a theorem of Drinfeld who proved the Ramanujan conjecture for Λ_p (see [60]).

2) The proof can still work for many such groups of $K-rank$ 0 (i.e. when Γ is a cocompact lattice in the suitable semi-simple group), but not for all. The proof needs that the congruence subgroup property holds for Γ. This is known in many cases and conjecturally in all cases. We also need to show that a K-rank zero group H with S-rank one can be embedded into G and hence a subgroup Λ into Γ. This is a problematic point: in some cases Γ is a minimal semi-simple group (e.g., if G is the norm one elements of a division algebra of prime degree over $\mathbb{Q}$) and contains no proper K-simple group and in particular not such H. If such H exists, the rest of the proof will work as the Selberg property is now known for all arithmetic lattices, provided Λ does not have (T). This is the case in most S-rank one groups. The exceptional cases being $Sp(n,1)$ and $\mathbb{F}_4^{(-20)}$ over $\mathbb{R}$. Over non-archemedian local fields, the lattices in rank one groups never have (T).

It will be an interesting challenge to determine for all lattices in Lie groups whether their finite dimensional representations separate their C^*-algebras.

9.2 Seperating $C^*(\Gamma)$ by finite dimensional representations: positive results

In this section we show that some interesting groups have FDD. In fact we will even show that they have FD. This is not just a minor improvement, in fact, the FD property will be essential. It turns out that in some cases one can deduce FD for a larger group Γ from FD of a subgroup Λ.

Easy examples of groups with FD are

Proposition 9.7 *Let Γ be an amenable residually finite group. Then Γ has property FD.*

Proof For every residually finite group Γ, $l^2(\Gamma)$ is in the closure of the finite representations (see Theorem 4.13). Now, if Γ is amenable then it is well known that the closure of $l^2(\Gamma)$ is the whole $\widetilde{\Gamma}$ and the proposition follows. □

We now turn to FD for free groups. In [41] Choi proved FDD for the free groups. The proof of FD given here is from [137] who attribute the main idea to Tim Steeger.

Theorem 9.8 *Let $\Gamma = F_r$ be the free group on r generators $2 \leq r \leq \aleph_0$. Then Γ has property FD, i.e., the finite representations of Γ are dense in the unitary dual of Γ.*

Let us first recall a general useful lemma.

Lemma 9.9 ([197]) *If Γ is a discrete group and π a unitary Γ-representation. Then there exists a probability measure preserving Γ-action on a standard Lebesgue space (X, μ) such that π is a subrepresentation of the unitary Γ representation on $L^2(X, \mu)$. Moreover X can be chosen to be a compact metric space on which Γ acts continuously.*

Here is the main lemma.

Lemma 9.10 *Assume $\Gamma = F_r$ acts continuously on a compact metric space X, preserving a probability measure μ. Then the induced unitary representation of Γ on $L^2(X, \mu)$ is in the closure of finite representations of Γ.*

Proof The idea of the proof is to show that there are "arbitrarily small perturbations" of any given action, which factor through a finite quotient. First however, observe that we may assume μ has no atoms since for any $\alpha > 0$ there can be only finitely many atoms of weight α (μ has total mass 1), hence they form Γ-invariant subset on which the Γ-action factors through a finite quotient. By removing all the atoms of μ we are reduced to the non-atomic case. Let $S \subseteq \Gamma$ be a free generating subset. Abusing notation, we shall denote both the Γ-action on X and on $L^2(X)$ by ρ, i.e., for $f \in L^2(X)$ we have: $[\rho(g)f](x) = f(\rho(g^{-1})x)$. Our strategy now goes as follows: Because the subspace of continuous functions on X is dense in $L^2(X)$, it is enough, given any finite number of continuous functions $f_1, \cdots, f_n$ on X, and any $\epsilon > 0$, to construct a new (μ-)measure preserving Γ-action ρ' on X, which factors through a finite quotient of Γ, such that for every $g \in S \cup S^{-1}$ (as well as $g = e$) one has for the L^2 norm: $||\rho(g)f_i - \rho'(g)f_i|| < \epsilon$ for all $1 \leq i \leq n$.

Let then ϵ and $f_1, \cdots, f_n$ be given. Choose $\delta > 0$ so that $d(x, y) < 2\delta$ implies $|f_i(x) - f_i(y)| < \epsilon$ for all i and $x, y \in X$. Now, again using continuity, choose some $0 < \theta < \delta$ such that $d(x, y) < \theta$ implies $d(\rho(g)x, \rho(g)y) < \delta$ for $g \in S \cup S^{-1}$. Next observe that X can be divided into a disjoint union of subsets $B_1, \cdots, B_k$ which have equal measure (of size $1/k$) and diameter less than θ. We shall leave the straightforward, yet rather tedious verification of this fact to the reader, hinting only that it is easier to find first such a subdivision where the measure of every B_i is rational (and then subdivide arbitrarily all B_i's to get pieces of equal measure), which of course uses the fact that μ is non-atomic (it seems most convenient here to homemorphically embed X in $[0, 1]^N$ and work with the latter).

We can now define the new Γ-action ρ' on X. By freeness, it is enough to define it for any $g \in S$ and we fix such an element. Consider the two disjoint subdivisions of X: $X = \cup B_i = \cup C_i$, where $C_i = g(B_i)$. Let us call a couple B_i and C_j "matched" if $\mu(B_i \cap C_j) > 0$. Because $\mu(B_i) = \mu(C_j) = 1/k$ for all i, j, it is easy to see that for every family of m B_i's, the family of C_j's matched to at least one of its members has least m elements (and vice versa), that is, Hall's marriage theorem applies. We can therefore find a permutation $\sigma_g \in S_k$ with the property that $g(B_i) \cap B_{\sigma_g(i)}$ is non empty for all i. Notice that defining $\sigma_{g^{-1}} = (\sigma_g)^{-1}$, the latter holds also when g is replaced by g^{-1}.

By freeness the permutation action of S on the set of B_i's extends to a permutation action of Γ, factoring through a finite quotient F ($\subseteq S_k$). In order to define a measure preserving Γ-action on X (and not only on the collection of subsets B_i), which factors through an action of F and induces the previous permutation action on the B_i's, we choose as a "model space" any non-atomic standard Lebesgue space Y on which S_k acts by permuting a disjoint subdivision of it to k subsets Y_i. Identifying measure preservingly each B_i and Y_i, induces an F- (hence also $\Gamma-$)action on X, which in turn induces the previously defined permutations $\{\sigma_g\}$ of the sets B_i. This is the promised ρ'.

Finally, given any $1 \leq i \leq n$ we show that $||\rho(g)f_i - \rho'(g)f_i|| < \epsilon$ for any $g \in S \cup S^{-1}$. In fact, we show that for all $x \in X$: $|\rho(g)f_i(x) - \rho'(g)f_i(x)| < \epsilon$. Let $1 \leq j \leq k$ be such that $x \in B_j$. Then $\rho(g^{-1})x \in \rho(g^{-1})B_j$, and $\rho'(g^{-1}x) \in B_{\sigma_{g^{-1}}(j)}$. By the construction of σ the two B_i's on the right hand sides intersect, and by the choice of θ and δ they both have diameter $< \delta$. By triangle inequality it follows that $d(\rho(g^{-1})x, \rho'(g^{-1})x) < 2\delta$, which by the choice of δ implies $|\rho(g)f_i(x) - \rho'(g)f_i(x)| = |f_i(\rho(g^{-1})x) - f_i(\rho'(g^{-1})x)| < \epsilon$. As this holds for every $x \in X$ and μ is a probability measure, this establishes the lemma. $\square$

Theorem 9.8 follows now immediately from the last two lemmas.

Let us now show, following [137] how knowing property FD for a subgroup Λ can sometimes enable one to deduce FD for a bigger group Γ.

Proposition 9.11 *Let Λ be a normal subgroup of Γ such that:*
1) Λ is a finitely generated
2) $Z(\hat{\Lambda}) = 1$ where $\hat{\Lambda}$ is the profinite completion of Λ.
3) Γ/Λ is a amenable group.
4) Γ/Λ is residually finite.
If Λ has FD then so does Γ.

Proof We first show that conditions (1) and (2) imply that the profinite topology of Γ induces on Λ the profinite topology of Λ. (Note, the profinite topology of a group always induces a profinite topology on a subgroup, but in many cases this is a weaker topology than the profinite topology of the subgroup). This is equivalent to the statement that the map $\hat{i}$ from $\hat{\Lambda}$ to $\hat{\Gamma}$, continuing the inclusion $i : \Lambda \to \Gamma$ is injective. To see this is really the case, note that the conjugating the action of Γ on Λ, induces a map from $\Gamma \to Aut(\Lambda) \to Aut(\hat{\Lambda})$. Now, $Aut(\hat{\Lambda})$ is a profinite group since $\hat{\Lambda}$ is a finitely generated profinite group [57], hence we have a map $\pi : \hat{\Gamma} \to Aut(\hat{\Lambda})$. The latter is injective (since $Z(\hat{\Lambda}) = 1$), hence $\pi \circ \hat{i}$ and $\hat{i}$ are injective.

We can now prove the proposition: if ρ is a representation of Γ then $\pi = Res^{\Gamma}_{\Lambda}(\rho)$ is a representation of Λ and hence a limit of finite representations

$\{\pi_i\}_{i\in I}$ of Λ, each one with kernel say N_i of finite index in Λ. By the argument above, there exists $N_i' \triangleleft \Gamma$ of finite index such that $M_i = N_i \cap \Lambda$ is contained in N_i. Now, $\pi_i \prec l^2(\Lambda/N_i) \prec l^2(\Lambda/M_i)$. Inducing up to Γ we get that $ind_\Lambda^\Gamma(\pi)$ is a limit of $Ind_\Lambda^\Gamma(l^2(\Lambda/M_i)) = l^2(\Gamma/M_i)$.

As Γ/Λ is amenable, the trivial representation $I \prec l^2(\Gamma/\Lambda)$ hence $\rho \prec \rho \otimes l^2(\Gamma/\Lambda) = Ind_\Lambda^\Gamma(Res_\Lambda^\Gamma(\rho)) = Ind_\Lambda^\Gamma(\pi)$. So ρ is a limit of the $l^2(\Gamma/M_i)$'s. But each one of the Γ/M_i is residually finite (by condition 4 and the choice of M_i) - so $l^2(\Gamma/M_i)$ is a limit of finite representations and so is ρ. The proposition is now proven. $\square$

We can now deduce

Theorem 9.12 (Lubotzky-Shalom [137]) *(i) The surface groups*

$$T_g = \langle a_1, \ldots, a_g, b_1, \ldots, b_g | \prod [a_i, b_i] = 1 \rangle$$

have property FD.

(ii) $SL_2(\mathbb{Z}[\sqrt{-1}])$ *and* $SL_2(\mathbb{Z}[\sqrt{-3}])$ *have FD.*

Proof We start with (ii). It is known that if Γ_0 is as in (ii) then it is commensurable with a group Γ when Γ has a finitely generated normal free (non-abelian) subgroup N and Γ/N is abelian []. Note also that for non abelian free groups F, $Z(\hat{F}) = 1$ [138]. So Proposition 9.11 gives the desired result.

As far as (i) is concerned the surface groups can be embedded into the groups of part (ii) of the theorem. In fact Maclachlan [143] showed that for every d, $PSL(2, \mathbb{Z}[\sqrt{-d}])$ contains a surface group. Moreover, it follows from [47] that this holds for every non-uniform lattice in $PSL(2, \mathbb{C})$.

Note that if $\Gamma_1 = PSL(2, \mathbb{Z}[\sqrt{-d}])$ contains T_g then $\Gamma_2 = SL(2, \mathbb{Z}(\sqrt{-d})$ contains $T_{g'}$ for some $g' \geq g$ (since Γ_1 and Γ_2 are commensurable, i.e. have isomorphic finite index subgroups and a finite index subgroup of T_g is isomorphic to $T_{g'}$ for some $g' \geq g$). Now, if a finite index subgroup has FD, so does the full group. $\square$

We end this topic with some remarks. The results of this chapter suggest to see property FD as a strong negation of the congruence subgroup property, i.e., there are many finite representations, enough to be dense in $\hat{\Gamma}$. We do not know how to connect the two properties directly. But the results here strongly suggest that an aritmetic group Γ has the congruence subgroup property if and only if it does not have FD. Combined with Conjecture 4.9

the two conjectures imply that there is no arithmetic group with (τ) and FD. One is tempted to believe that there is no group with (τ) and FD but we have no idea how to start to prove such a speculation.

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List of Notations

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