
1 Introduction

The allocation or assignment problem is one of the more important problems of management science. It may be stated very briefly: We are given a system with a number of vacant positions and an equal number of available parts. We know how well each part performs in each position; we wish to assign the parts to the positions so that system performance is optimized.

Applications range far and wide, from employment to aircraft assignment to naval overhaul programs. The computational aspects of the problem have been solved, under the assumption that a numerical value is associated with each assignment and that the value of the system is given by the sum of the values of the individual assignments [1,2]. The crux of the problem, therefore, becomes the finding of values to use for the individual assignments.

Most of the prior literature has either ignored this problem or failed to reach a definitive conclusion. In certain applications it may be possible to determine values in a unique natural way; the transportation problem is an example [1]. But in most cases the problem is much more complex, and there is no natural utility (like dollar cost in the transportation case) available. The employment problem is a case in point. Here we have a number of vacancies and a number of candidates, and we wish to assign the candidates to the vacancies. The problem is of the utmost importance; all organizations but the smallest are plagued by it. It has been suggested that, as a first approximation, we classify each candidate as either "suitable" or "unsuitable" for a job and solve the problem by using the values 1 and 0, respectively. This approach is manifestly unrealistic; it is probably worse than letting the personnel office struggle along as best it can, using nonnumerical subjective evaluation of the candidates and the jobs.

The technique presented in this paper was originally developed for the problem of allocating major electronic equipment to Naval Ships [3,4]. For concreteness, we will refer to this application throughout the sequel, except in the last section, where we will return to the general case.

The technique is constituted so as to take advantage of any special internal relationships that might exist between the positions or the parts. Thus it will vary somewhat from application to application; and even for a given application, several variations may be possible. In essence, though, it is completely general; the same basic ideas apply to all allocation

problems in which the assignment value is not easily measurable or not measurable at all.

The naval electronics problem is extremely complex. In order not to obscure the main outlines of the technique, we have drastically simplified its statement. A more realistic and detailed treatment of that particular problem, of great importance in itself, will be given in a subsequent paper [5]. Aspects of procurement and allocation have been covered previously in the Quarterly [4].

The basic idea contained in this paper is that of Jack W. Smith; the authors are responsible only for mathematical formulation, elaboration, and technique and, even for these, only partly. Much credit is also due John P. Mayberry, Norman Shapiro, and George Suzuki.

The strictly mathematical part of this paper is contained in Sections 4, 5, and 6; Sections 2, 3 and 7 contain more explanatory material than strict mathematical analysis.

2 Objectives of the Technique

Most problems of operations research are handled in the following way: We are given some physical system and a physical utility (dollars, expected kill, etc.) which we would like to optimize, subject to the restraints of the system. We set up a mathematical model of the system that allows us to express the physical utility (objective function) and the restraints mathematically, and then we do our best to find that set of parameters that optimize the physical utility.

This method, although it works very well with many OR problems, fails completely with ours. To start with, there is no measurable physical utility that we are trying to optimize. We say we are trying to optimize military effectiveness, but this is obviously a subjective idea rather than an objective, physical, measurable utility. One can always count his dollars to see how much profit he's made; but there's no way of measuring how much "military effectiveness" is contributed by some radio transmitter. And even if we had a physical utility, the operating conditions of the electronic sets are too complex and involve too many unmeasurable and unpredictable factors to be describable at all usefully in mathematical terms.

The navy handles this problem in the following way: It appoints someone familiar with navy objectives and experienced with the equipment and the ships involved, it helps him out with a few directives, and it tells him to make up an allocation plan. Although a quantitative statement of the objective function and the restraints is out of the question, the responsible person can qualitatively weigh all the factors and come up

with a “reasonable” or “acceptable” plan. In fact, this is how most decisions in the military and business world are reached—the responsible person comes up with his decision on the basis of qualitative considerations and shrewd guesswork rather than by the use of mathematical optimizations. This process is a perfectly accepted and valid one.

This process works very well when the choice the responsible person must make is a clear-cut one with more or less clear-cut implications. Thus if the allocation problems under discussion involved but a few sets and ships, there would be no point in trying to improve current methods. Actually, these allocation problems involve hundreds of electronic sets and ships; and at this point, the responsible person is no longer able to exercise his judgment soundly, because he is overwhelmed by the combinatorial difficulties inherent in problems of this size.

The job of solving large allocation problems really is two jobs: the combinatorial or mathematical job and the naval judgment job. Under current methods these two are hopelessly mixed together, with the result that probably neither one is being done as well as it could be. *Our objective is to separate the two:* the mathematical or combinatorial problems should be solved by mathematicians and computing machines, and decisions involving naval judgment should be made by naval personnel. As a result of this process, *both* jobs will be handled more efficiently.

Note that we do *not* want to eliminate or replace qualitative naval judgment. In fact, it forms the basis of our technique. We are simply out to find a more effective way of using it.

3 A Brief Description of the Technique

Roughly speaking, the technique works as follows: The Navy appoints an officer or board of officers (henceforth called “the board”) to be responsible for making allocation decisions. These decisions will come in response to hypothetical allocation problems of a very small size, usually involving no more than two sets of electronic equipment and two possible ships to which they could be assigned. Because of the small size of the problems, the board will not get entangled in combinatorial difficulties and will be able to reach a decision based solely on its judgment, experience, and knowledge, taking into account all factors that affect the problem.

These qualitative decisions of the officers are used in the following way: We assumed in Section 1 that each alternative in each of the choices made by the board had a numerical value, given by the sum of the values of the individual assignments in it. The choices made by the officers therefore yield a set of inequalities involving the coefficients we

are seeking. These inequalities, together with some additional results on the algebraic make-up of the individual coefficients, enable us to estimate the value of the coefficients fairly precisely.

It is important to note that:

- a. The value of an assignment is not a function of the assignment only, but also of the board that makes the decisions.
- b. The board is *never* asked for a numerical estimate, *only* for qualitative yes-or-no decisions. Quantitative values are mathematically deduced from these qualitative decisions, assuming only the *existence* and linearity of the value function. There is nothing surprising in this; the process of deducing important numerical information from the mere existence of a quantity, without any *a priori* knowledge of its value, is a commonplace in mathematics.

We would like to stress the subjectivity of the allocation plan resulting from this technique. Our results will be based entirely on the judgment, knowledge, and experience of the board; all we have done is to find a technique for obtaining an allocation plan from their judgment. If we substitute a different board, we will probably get somewhat different allocation plans. There is nothing anomalous in this. All executive organizations, whether business, governmental, or military, are based on a chain of command, and it is to be expected that different people within this chain would make different decisions; this does not negate the validity of the command system or the applicability of a decision once it is made.

4 The Mathematical Model—Basic Theorems

The two classes of objects which we are pairing off are *electronic sets*, and *positions*, i.e., places on the ships where the sets will be put. We wish to find the value of all possible assignments of a set to a position.

The electronic sets are classified into *models*; two sets of the same model are essentially indistinguishable. We define the *void* model to consist of no set at all. The positions are classified into *priority groups*, depending on how important they are, and into *state groups*, depending on the model already installed which the new set will replace. The *void* position is no position at all (conceptually the warehouse). A *priority-state group* is the intersection of a priority group and a state group.

If a is a set and b a position, we will denote by (a, b) the assignment of a to b . Let X be a class of electronic sets and Y a class of positions. An allocation plan T defined on (X, Y) is a class of individual assignments (a, b) , where $a \in X$ and $b \in Y$, and each a and b occurs at most once in T .

X and Y will not be fixed in the sequel, but will vary in accordance with the context. (a, b) itself may also be considered an allocation plan (defined on $(\{a\}, \{b\})$).

ASSUMPTION 1 With each allocation plan T and board B , there is associated a numerical value $v(T; B)$.

Intuitively, $v(T; B)$ can be interpreted as the military effectiveness of T in the opinion of B .

ASSUMPTION 2 $v(T; B) = \sum_{(a,b) \in T} v((a, b); B)$.

From the strictly logical viewpoint, assumption 2 could equally well have been stated as a definition. It involves important conceptual assumptions, however, and is used not only in theoretical work but in the deduction of numerical values from the board's decisions. It therefore seems more honest to state it as an assumption.

Denote by $f(T)$ the function that associates with each pair consisting of a model M and a priority group P the number of sets of M in positions of P after the plan T is carried out; in other words, $f(T)$ specifies the cardinality of the new priority-state groups. Of course, we are here counting only those positions on which T is defined.

ASSUMPTION 3 If $f(T_1) = f(T_2)$, then $v(T_1; B) = v(T_2; B)$.

This assumption tells us that the only quantity the Board considers significant can be deduced from the number of models of each kind that end up in positions of each kind. Which set replaced which set is of no interest to the board. Of course T_1 and T_2 are here assumed to be defined on the same set of positions.

THEOREM 4.1 If a is of the same model as the set installed in b , then

$$v((a, b); B) = 0.$$

Proof Let T be a null plan, i.e., a plan containing no assignments at all. Then, by the hypothesis,

$$f(T) = f(a, b)$$

Hence, by assumption 3,

$$v((a, b); B) = v(T; B). \quad (*)$$

But since T is null, the sum on the right side of assumption 2 is empty, whence

$$v(T; B) = 0. \quad (**)$$

Combining $(*)$ and $(**)$, we obtain our theorem.

THEOREM 4.2 If a_1 and a_2 are of the same model and b_1 and b_2 of the same priority-state group, then

$$v((a_1, b_1); B) = v((a_2, b_2); B).$$

Proof For $i = 1$ and 2 , let T_i be defined on $(\{a_1, a_2\}, \{b_1, b_2\})$ and $T_i = \{(a_i, b_i)\}$. Then $f(T_1) = f(T_2)$, and hence it follows from assumption 3 that $v(T_1; B) = v(T_2; B)$. Since T_i consists of (a_i, b_i) only, our theorem follows.

In consequence of 4.2, we can speak of values $v(M, A; B)$ for triples consisting of a model M , a priority-state group A , and a board B . Formally, we have:

DEFINITION 4.3 If $a \in M$ and $b \in A$, then

$$v(M, A; B) = v((a, b); B).$$

(The definition is unique because of 4.2.)

For a given model M , let $S(M)$ denote the set of those positions that have sets of model M already installed (before the implementation of the contemplated allocation plan). $S(M)$ is a state group; if P is a priority group, then $P \cap S(M)$ is a priority-state group. Define

$$v(M_1, P, M_2; B) = v(M_1, P \cap S(M_2); B). \quad (4.4)$$

$$\text{THEOREM 4.5 } v(M, P, M; B) = 0.$$

Proof Follows at once from theorems 4.1 and 4.2.

Denote the void model by 0.

$$\text{THEOREM 4.6 } v(M_1, P, M_2; B) = v(M_1, P, 0; B) - v(M_2, P, 0; B).$$

Proof If $M_1 = M_2$, then 4.6 follows at once from 4.5. Assume $M_1 \neq M_2$. Let b_1 and b_2 be positions in priority group P . Suppose that b_1 has a set of model M_2 currently installed, and that b_2 has a void currently installed. Let a_1 and a_2 be sets of model M_1 and M_2 , respectively. Let T_1 be the plan that assigns a_1 to b_1 and a_2 to b_2 . Let T_2 be the plan that assigns a_1 to b_2 and a_2 to b_1 . (Both are defined on $(\{a_1, a_2\}, \{b_1, b_2\})$.) Then

$$f(T_1)(M_1, P) = 1 = f(T_2)(M_1, P),$$

$$f(T_1)(M_2, P) = 1 = f(T_2)(M_2, P),$$

and $f(T_1) = 0 = f(T_2)$ for all other pairs. Thus $f(T_1) = f(T_2)$, and it follows from assumption 3 that

$$v(T_1; B) = v(T_2; B). \quad (*)$$

Now

$$\begin{aligned}
 & v(M_1, P, M_2; B) + v(M_2, P, 0; B) \\
 &= v(M_1, P \cap S(M_2); B) + v(M_2, P \cap S(0); B) \quad (\text{by 4.4}) \\
 &= v((a_1, b_1); B) + v((a_2, b_2); B) \quad (\text{by 4.3}) \\
 &= v(T_1; B) \quad (\text{by assumption 2}) \\
 &= v(T_2; B) \quad (\text{by } (*)) \\
 &= v((a_1, b_2); B) + v((a_2, b_1); B) \quad (\text{by assumption 2}) \\
 &= v(M_1, P \cap S(0); B) + v(M_2, P \cap S(M_2); B) \quad (\text{by 4.3}) \\
 &= v(M_1, P, 0; B) + v(M_2, P, M_2; B) \quad (\text{by 4.4}) \\
 &= v(M_1, P, 0; B) \quad (\text{by 4.5})
 \end{aligned}$$

Subtracting $v(M_2, P, 0; B)$ from both sides, we obtain (4.6).

Intuitively, the expression $v(M, P, 0; B)$ gives the value of *having* a set of model M in a position of priority P , rather than the value of a *replacement*. When P is a priority group, we will use the expression $v(M, P; B)$ instead of $v(M, P, 0; B)$. These values will be called *basic* values. Theorem 4.6 tells us that, in order to find the values, it is sufficient to find the basic values.

THEOREM 4.7 If b is the void position, then

$$v((a, b); B) = 0.$$

Proof If T is a null assignment, then

$$f(T) = f(a, b).$$

The result now follows from assumptions 2 and 3. (Actually, (a, b) is a null assignment.)

Finally, in order to make use of the decisions of the board, we need:

ASSUMPTION 4 If of two plans, T_1 and T_2 , a board B prefers T_1 to T_2 , then

$$v(T_1; B) > v(T_2; B).$$

If B is indifferent as to which plan is used, then

$$v(T_1; B) \approx v(T_2; B),$$

where \approx denotes approximate equality.

Those readers who are unhappy about the use of \approx in a formal assumption may replace it by an equality sign. To us, the \approx seems more reasonable.

We remark that it would have been possible to state and prove the axioms and theorems of this section by using only the terminology of set theory (sets, functions, transformations, etc.). Such a procedure would have detracted from the intuitive appeal and added nothing to the mathematical rigour.

Throughout this section, we have used the notation $v(T; B)$ to stress the dependence of the numerical value on the board. Since this notation is somewhat clumsy, we will henceforth write simply $v(T)$, the dependence on B being understood. B will remain fixed throughout the discussion. In the new notation, (4.6) becomes

$$v(M_1, P, M_2) = v(M_1, P) - v(M_2, P). \quad (4.8)$$

There is one additional assumption, which will be stated in the following section.

5 Determination of the Relative Values within a Priority Group

Let us fix attention on a certain priority group P .

Let b be a position in priority group P that has nothing currently installed. By asking the board which model it would most like to install in b , which model would be its second choice, and so on, we can obtain a list of all the $v(M, P)$ in decreasing order. (Here P is fixed and M ranges over all models.) This is called the *goodness order* for P . Unfortunately, the mere linear order on the functional values $v(M, P)$ does not give us significant numerical information about them; in other words, it does not determine their scaling. There is, however, a theorem that under certain conditions enables us to obtain a function numerically—to within an additive and a multiplicative constant—from a linear order on the *differences* between the functional values. In our case, the additive constant is fixed by the requirement that installing a void in b have value 0 (Theorem 4.1). The multiplicative constant will be fixed if we specify the value of the highest $v(M, P)$ in the goodness order. This will in general depend on P ; we will denote it by $q(P)$. Define

$$r(M, P) = v(M, P)/q(P). \quad (5.1)$$

The $r(M, P)$ are called *relative basic values*, or just relative values. They are used only as a convenience, so that the top member of the goodness order should be 1.

THEOREM 5.2¹ Let g be a strictly decreasing continuous function on a closed interval X . Then g is completely determined by the set of all inequalities of the form

$$g(x_1) - g(x_2) \left\{ \begin{array}{l} \geqslant \\ < \end{array} \right\} g(x_3) - g(x_4),$$

where $x_1, x_2, x_3, x_4 \in X$. In other words, if g' is a strictly decreasing continuous function on X distinct from g , then there is a quadruple (x_1, x_2, x_3, x_4) for which

$$g(x_1) - g(x_2) > g(x_3) - g(x_4)$$

and

$$g'(x_1) - g'(x_2) < g'(x_3) - g'(x_4).$$

In other words, a *linear order* on the differences between the functional values determines them *numerically*. The principle of Theorem 5.2 provides the key to solve our problem. It suggests that, in order to obtain the basic values quantitatively, we *qualitatively* compare the differences between them; these will be called *improvement* values. But because we do not yet know the value of $q(P)$, we can at most hope to obtain *relative* values by this method. Furthermore, since our problem is discrete (the variable M has a finite domain), whereas theorem 5.2 deals with a continuous situation, it is not very surprising that we will be able to obtain these relative values only approximately from the qualitative comparisons.

Let M_1, \dots, M_{n+1} be all the models, arranged so that

$$r(M_1, P), \dots, r(M_{n+1}, P)$$

is in decreasing order (M_{n+1} is the void model). This can be done because we already know the goodness order. In order to obtain a linear order on the relative improvement values

$$d_{ij} = r(M_i, P) - r(M_j, P), \tag{5.3}$$

we must ask the Board to consider all the improvements that can be made, and list in order the one it considers most important, the one it considers next most important, and so on. To be specific, we ask the board to consider assigning model M_i to a position in P that has model M_j already installed. By (4.4), (4.8), (5.1), and (5.3), this assignment has

1. Theorem 5.2 is not new, but we have not found it stated or proved in the literature. A proof will be given in a subsequent paper. (Added in proof: It was proved in [7].)

value $q(P)d_{ij}$. By ranging over all i and j and using assumption 4, we get a linear order on the $q(P)d_{ij}$, which yields a linear order on the d_{ij} when $q(P)$ is divided out.

By means of the goodness order, we can tell at once when a d_{ij} is positive and when negative. The order on the *positive* improvement values is called the *improvement order*. Let us denote the largest relative improvement value by d_1 , the next largest by d_2 , and so on down to d_k . The improvement order can be written in the form of $k - 1$ inequalities

$$\begin{aligned} d_1 &> d_2 \\ d_2 &> d_3 \\ &\dots \\ d_{k-1} &> d_k. \end{aligned} \tag{5.4}$$

It turns out that the inequalities (5.4) are by no means independent and that in fact the entire improvement order *and* goodness order can usually be characterized by a fairly small number of inequalities, usually few more than n of them [5].

We remarked above that one would need a continuum of models in order for the improvement order to determine the relative values precisely. As it is, each of the linear inequalities constituting the improvement order can be thought of as restricting the relative value vector to lie in some portion of $(n + 1)$ -dimensional space bounded by a hyperplane. The entire improvement order thus restricts the relative value vector to lie within a certain polyhedron. The smaller this polyhedron is, the better the improvement order determines the relative value vector. In practice, the centroid of the polyhedron has been used for the relative value vector, but there is really no a priori reason to prefer this point to other points within the polyhedron.

The only information used to determine the relative values is the improvement order. Thus, if for two priority groups the improvement order turns out to be the same, then the relative value vector for these two priority groups will also be the same. In actual fact, it turns out that the improvement order is independent of the priority groups. (This also makes sense intuitively, because the relative value is in a sense a measure of the goodness or suitability of a piece of equipment to a particular mission and should not depend on the importance of the position to which it is assigned.) We have been led to:

ASSUMPTION 5 $r(M, P)$ is independent of P .

As a consequence of Assumption 5, we may write

$$r(M, P) = g(M),$$

(g standing for “goodness”) and conclude from (5.1) and (4.8) that

$$v(M, P) = q(P)g(M) \quad (5.5)$$

and

$$v(M_1, P, M_2) = q(P)(g(M_1) - g(M_2)). \quad (5.6)$$

$q(P)$ is called the *priority rating* of P , $g(M)$ the *goodness rating* of M . We have already determined the goodness rating of M . It coincides with the relative value of (M, P) for an arbitrary P . It remains only to determine the priority ratings.

6 Determination of Priority Ratings

Let P^i and P^j be two priority groups. Let b^i and b^j be positions in P^i and P^j , respectively, having sets of model M^i and M^j , respectively, currently installed. Let M be an arbitrary model, and let a be a set of model M . By (5.6) we have

$$\begin{aligned} v(a, b^i) &= q(P^i)(g(M) - g(M^i)) \\ v(a, b^j) &= q(P^j)(g(M) - g(M^j)). \end{aligned} \quad (6.1)$$

We may now confront the board with an allocation problem in which there are only 2 positions, namely b^i and b^j , and only one available set, namely a , and ask it which of the two plans (a, b^i) and (a, b^j) it would prefer. According to its answer, we may deduce from (6.1) and Assumption 4 which of

$$q(P^i) - (g(M) - g(M^i)) \left\{ \begin{array}{c} > \\ \approx \\ < \end{array} \right\} q(P^j) - (g(M) - g(M^j)) \quad (6.2)$$

holds, whence we may deduce which of

$$q(P^i)/q(P^j) \left\{ \begin{array}{c} > \\ \approx \\ < \end{array} \right\} (g(M) - g(M^j))/(g(M) - g(M^i)) \quad (6.3)$$

holds. We now draw up a list in size order of all the ratios

$$(g(M) - g(M^j))/(g(M) - g(M^i)),$$

where M , M^i and M^j range over all models for which both sides of (6.2) are positive.

By asking questions of the above type, and using (6.3), we may find between which two items of this list, or near which item, the ratio

$q(P^i)/q(P^j)$ lies. Often the list is sufficiently dense so that fairly sensitive results can be obtained for any given set of goodness ratings as obtained in Section 5. Of course, the “uncertainty” interval surrounding the ratings considered in this section depends on the size of the polyhedron considered.

Once we have obtained all the ratios $q(P^i)/q(P^j)$, we may obtain the $q(P^i)$ and $q(P^j)$ themselves up to a multiplicative constant (which may of course be ignored). In doing this, it is well to fix arbitrarily one of the P^j , set $q(P^j) = 1$, and then use $q(P^i)/q(P^j)$ as the value of $q(P^i)$, in order to avoid the “snowballing” of uncertainties that would result if we combine $q(P^i)/q(P^k)$ and $q(P^k)/q(P^j)$ to obtain $q(P^i)/q(P^j)$. Also, in making up the list on which our questions are based, it is well to try to stay away from ratios involving small denominators, as uncertainties in a small denominator are tremendously blown up in the quotient.

7 Reservations and Conclusions

1. Use of the technique presented here should be confined to problems which, like the naval electronics problem, involve too many unpredictable or unmeasurable factors or are too vaguely defined or complex to be amenable to treatment by the conventional methods of Operations Research, i.e., by methods involving physical, objective constraints, and utilities. When objective methods are available, they are naturally preferable to subjective methods that depend on the opinions of an individual or group, no matter how well informed it may be. It is only when we are forced to use subjective methods anyway that the technique described here is applicable.
2. The application of our technique to a particular kind of allocation problem (employment, naval electronics, aircraft assignment, etc.) involves the assumption that the model is meaningful and valid for that kind of problem. In the final analysis, the validity of the model for a given kind of problem must be judged by the military or management authorities themselves. There are, however, mathematical tests which we can use to see if the method is at all applicable. The most important of these involves the consistency of the numerical results obtained. Before the technique is adopted for use with a particular kind of problem, a pilot problem should be run in which several sets of questions are asked (of the same board, of course). If widely divergent results are obtained, the model is not applicable to that kind of problem. If fairly consistent results are obtained, and if the model is acceptable from other viewpoints as well, then it is reasonable to assume that the model is applicable. It is then no longer necessary to test for consistency every time a particular

allocation is determined, because consistency is a logical consequence of the model. Nevertheless, it may be wise to throw in one or two more questions than are strictly necessary to obtain the value, just to see if the answers check. In the case of naval electronics, limited tests have yielded consistent results. It has been pointed out that if linearity (assumption 2) holds, then the answers to the questions should be independent of what other assignments have already been made. This also can and should be tested.

It should be noted that no assumptions, implicit or explicit, are made about the "nature of psychological judgment," except insofar as the routine use of any priority list comes under this heading. Psychological or not, all the assumptions are explicitly stated, and the only ways in which we can check applicability of the model are by examining for acceptability both the assumptions themselves and the results obtained by using them.

3. There are only three basic assumptions necessary for the application of the kind of technique presented here. These are assumptions 1, 2, and 4. Assumptions 1 and 4 tell us that associated with each allocation plan and board there is a *numerical* military value that satisfies the intuitive notion of the word "value." Assumption 2 gives us a way of going from small allocation problems to big ones, which is of course the crux of our technique. It is important to note, however, that the particular form of assumption 2—linearity—could equally well be replaced by some other formula, and the technique would still be applicable, though possibly more complicated computationally.

Assumption 1 and something like assumption 2 are necessary not only for finding the coefficients but also for the application of *any* method that arrives at allocation plans by linear or nonlinear programming. As we remarked in Section 1, standard computational techniques for solving the allocation problem require assumptions 1 and 2. As for assumption 4, this tells us that we may rely on the decision of the board in arriving at a plan; this assumption is certainly justified in the problem we are considering here, because, as stated early in this section, we have to rely on subjective decisions anyway. We may therefore conclude that if in a given kind of allocation problem linear or nonlinear programming is at all applicable, then the kind of technique presented here also should be.

Assumptions 3 and 5 are not basic; they are merely ways of formalizing the structure that the real-life problem in fact has. Most real-life allocation problems probably present opportunities for simplifications of this kind. Incidentally, inconsistencies in numerical results may sometimes be traceable to ignoring the "fine structure" of a problem in the formulation of a "type 3" assumption. If that occurs, the assumption has to be revised

accordingly. On the other hand, we are sure that there are some allocation problems in which linear or nonlinear programming are not applicable at all.

4. Even within the framework of assumptions 1 to 5, the computational procedures outlined in Sections 5 and 6 are by no means the only ones. We are not even sure that they are the most efficient, in the sense that they yield the values as precisely and with as few questions as possible. There is another consideration to be taken into account when asking questions. This is the need to phrase the questions as realistically as possible, i.e., to present actual situations which the Board can readily picture, so as to make it as easy as possible for the Board to consider all relevant factors in arriving at an answer. From this point of view, also, it is possible that the procedure in Sections 5 and 6 could be improved. On the other hand, the methods of Sections 5 and 6 are easily grasped and lead in a rather straightforward manner to numerical ratings.

In alternative procedures, we might want to determine goodness and priority ratings simultaneously, for instance, by exclusive use of questions of the type discussed in Section 6. Or else, we might use a completely new type of question; for instance, we could attach realistic prices to the models, and use procurement-allocation questions, i.e., questions which ask: Given these positions, this money, and these models available for procurement, what would you buy and how would you allocate it? The resulting inequalities would yield valuable numerical information which could be converted into approximate numerical ratings.

Some of these various computational systems should be used in checking the applicability of the model to the problem, as outlined in paragraph 2 of this section.

5. Another cause for concern is possible variation of the value within the restraints set by the decisions we have obtained. Let us consider two sets of assignment values, both consistent with the answers we have received. In accordance with assumptions 1 and 2, they will each induce value functions, which we will call v_1 and v_2 . By maximizing $v_1(T)$ and $v_2(T)$ over all allocation plans T , we obtain allocation plans T_1 and T_2 that are optimal for v_1 and v_2 , respectively. T_1 and T_2 may differ considerably without causing undue alarm; this is because it is not the plan itself that is important but its military effectiveness. Thus if

$$\begin{aligned} v_1(T_1) &\approx v_1(T_2), \\ v_2(T_1) &\approx v_2(T_2), \end{aligned} \tag{7.1}$$

then we probably need not worry. It is only when there is a pair of value functions not satisfying (7.1) that we must try to remedy the situation. In

that case we must ask more questions, or different kinds of questions, in order to reduce the range of permissible values. In general, the more information we have, the less uncertainty there is in the numerical ratings (but we must be careful not to ask so many questions that the board ceases to think about the answers).

If these methods do not help, the usefulness of the technique is seriously impaired; but even then it is well to remember that if for two plans T_1 and T_2 , we have

$$v(T_1) > v(T_2)$$

for *all* value functions v consistent with the answers received, then T_1 is probably preferable to T_2 .

On the other hand, it may be possible to obtain plans that are essentially unique (in the sense of (7.1)), with numerical information of only the most general and cursory kind. Only experience with a particular kind of problem can determine which kind and how many questions to ask.

6. Subjective problems of the kind considered here can have no unique “correct” solutions. The technique outlined in this paper is just *a* way to go about finding acceptable allocation plans, by no means *the* way. In a particular problem, it may be difficult to evaluate this technique; but since the allocation involved usually *must* be performed in one way or another, the technique presented here should, practically speaking, be evaluated in comparison with the possible alternatives rather than on its own merits alone.

Morse [6] has stated that in operations research, the scientist supplies the executive with a quantitative basis for the qualitative decision the executive will make. Actually, this process is typical of operations research problems on or near the operating level, where it is possible to use meaningful physical utilities and restraints to reach conclusions that will be used in making decisions on a higher level. The problems with which we are concerned are usually several stages removed from the operating level. The solutions must therefore make use of the qualitative, subjective decisions made on the lower levels. On the other hand, our problems are complicated from the combinatorial-mathematical viewpoint and cannot be solved efficiently by the executive without mathematical help. As a result, the process described by Morse is here reversed; the executive supplies the scientist and the computing machine with qualitative decisions which the scientist and the computing machine use to arrive at quantitative conclusions. It remains to be seen whether the same general type of thinking can be applied as well to other problems

(i.e., not necessarily allocation problems) in which quantitative decisions on a high level must be based at least in part on subjective, qualitative decisions on lower levels.

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