40 Introduction to "Some Thoughts on the Theory of Cooperative Games"

The author of this paper, Gerd Jentzsch, died while still a young man on March 26, 1959. This is apparently his only publication. Judging from its originality and all-around brilliance, his death was a loss of the first magnitude to game theory.

The von Neumann-Morgenstern (N-M) theory of n-person games [5] is concerned with cooperative games in which side payments are permitted and utility is "unrestrictedly transferable"—that is, each player's utility for money is linear in money.[1] Jentzsch's investigations grew out of an attempt to generalize the N-M theory either by eliminating the requirement that the utility functions[2] be linear, or more generally, by eliminating side payments altogether. He notices at the outset that the notion of "effectiveness"—which is crucial in the N-M theory—does not generalize in a straightforward manner. In the classical theory, a coalition K is effective for a payoff vector f if, roughly speaking, the coalition can assure itself of getting at least f. An equivalent definition of effectiveness is that the *opposition*—the complement of K—cannot prevent K from obtaining at least f. But when utilities are nonlinear in money or side payments are forbidden, these two definitions of effectiveness are in general no longer equivalent—in Jentzsch's terminology, the game need not be "clear" (Example 4). Jentzsch addresses himself to the task of broadening the class of games considered by von Neumann and Morgenstern, while still retaining the clearness property.

The chief result is Theorem 21. Rather than stating it here in its most general form, we will describe its application to games with side payments ("money games" for short) in which the utility functions need not be linear. The problem that Jentzsch considers is, what kinds of utility functions of the players will always lead to clear games (as linear utility functions do)? More precisely, what conditions, when placed on the utility functions of the players, will ensure that all money games in which these players participate are clear? The answer is that each coalition must have a kind of "social utility function" for money. For example, this involves the demand that \$50 be indifferent—from the point of view of the coalition as a whole—to some probability combination of 0 dollars and \$100 (though not necessarily the 1/2-1/2 combination). The sums of money involved (\$50, \$0, \$100) are not given to the individual players,

This chapter is an introduction to [4]. It originally appeared in *Advances in Game Theory*, Annals of Mathematics Studies 52, edited by M. Dresher, L. S. Shapley, and A. W. Tucker, pp. 407–409, Princeton University Press, Princeton, 1964. Reprinted with permission.

1. See R. D. Luce and H. Raiffa, *Games and Decisions*, p. 168.

2. By the phrase "utility function" we shall henceforth *always* mean "utility of money as a function of money."

but to the coalition as a whole for distribution among its members. "Indifferent" has a very precise meaning here: The two sets of (utility) payoff vectors that can result from the two possibilities must coincide.

The existence of such a "social utility function" is a considerable restriction. Jentzsch remarks without proof that Bernoullian (i.e., logarithmic) individual utility functions lead to a social utility function (Examples 11, 16) and that other individual utility functions that lead to a social utility function can be obtained as solutions of a third-order differential equation with one parameter (which he does not specify). These questions must certainly be investigated further. But on the whole, Jentzsch's result shows that clearness is the exception rather than the rule—that games with nonlinear utility functions or without side payments cannot be "expected" to be clear.

The difference between the two kinds of effectiveness was appreciated by others, working independently of Jentzsch, as far back as 1957—which is probably the approximate date of Jentzsch's investigations.[3] It was explicitly mentioned by Aumann and Peleg [2], who used the names α- and β-effectiveness for the two kinds. A survey of the whole field of cooperative games without side payments is given in [1], which has a bibliography of 51 items; but of this work, Jentzsch knew only of the pioneering investigation of Shapley and Shubik [6]. This is another example of the known phenomenon of the intrinsic "ripeness" of a scientific concept—leading to simultaneous independent discovery by widely separated investigators. It should be emphasized, though, that it is only in the basic recognition of the difference between α- and β-effectiveness that Jentzsch's work overlaps that of others; the main result of this paper has not been found by anybody else, and appears here for the first time. Indeed other workers have approached the subject from a somewhat different viewpoint—they have tried to "live with" the difference, whereas Jentzsch characterized the conditions under which it could be eliminated (see [1]).

An attempt has been made to keep editorial comment separate from Jentzsch's original text. All the footnotes are the editor's, as are the two "Editor's Notes." The long formal proofs given by Jentzsch for Theorems 10 and 13 have been replaced by short intuitive sketches. There has been some rearranging of the material, and the more straightforward proofs have been left out. Those are all the changes.

3. In fact, the idea is related to Blackwell's approachability-excludability theory [3] which appeared already in 1956.

Since Jentzsch is interested only in the question of effectiveness, he fixes once and for all a coalition K, and considers only the joint strategies of the coalition, the joint strategies of the opposition, and the payoff to the coalition. The resulting formal object is called a "K-game," and this is the object of investigation throughout.[4] The individual strategies of members of the coalition and of the opposition, and the payoff to the opposition, are of no interest in this context, and are therefore suppressed in the formal model.

References

[1] Aumann, R. J., "Cooperative Games without Side Payments," in *Essays in Mathematical Economics in Honor of Oskar Morgenstern*, M. Shubik (ed.), Princeton University Press, Princeton, 1967, pp. 3–27 [Chapter 41].

[2] Aumann, R. J. and B. Peleg, "Von Neumann–Morgenstern Solutions to Cooperative Games without Side Payments," *Bull. Amer. Math. Soc.* 66, 1960, pp. 173–179 [Chapter 38].

[3] Blackwell, D., "An Analog of the Minimax Theorem for Vector Payoffs," *Pac. J. Math.* 6, 1956, pp. 1–8.

[4] Jentzsch, G., "Some Thoughts on the Theory of Cooperative Games," in *Advances in Game Theory*, Annals of Mathematics Studies 52, M. Dresher, L. S. Shapley, and A. W. Tucker (eds.), Princeton University Press, Princeton, 1964, pp. 407–442.

[5] Von Neumann, J. and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, 1944.

[6] Shapley, L. S. and M. Shubik, "Solutions of n-Person Games with Ordinal Utilities" (abstract), *Econometrica* 21, 1953, p. 348.

4. It is formally identical to Blackwell's "game with vector payoffs" [3].