49 A Note on Gale's Example
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David Gale (1974) constructs an example of a pure exchange economy with 3 traders in which two of the traders, by exchanging goods among themselves only, can affect prices in the entire economy in such a way so that *both* will benefit (and so necessarily, the third trader will lose). Gale's example involves indifference curves with sharp corners, and he raises the question as to whether an example of this kind can be found with smooth preferences.

In this note we discuss a related phenomenon which at first glance is even more striking but which is even simpler to prove. In a 2-trader, 2-commodity market, it is possible for a trader simply to discard some of his initial bundle and to gain from this act – at the expense of the other trader, of course. This example may be modified to yield Gale's phenomenon – instead of discarding, he donates to a third trader. Moreover, the preferences in our example are smooth, thus settling Gale's question as to smoothness.

The idea is quite simple, and is similar to that of Gale's example. Each trader initially holds a 'corner' on one of the two commodities, i.e., the initial bundles are of the form $(\alpha, 0)$ and $(0, \beta)$. If trader 1 throws away some of his initial bundle, the price of commodity 1 goes up; as in Gale's example, this rise in the *price* of the commodity he holds is more than enough to compensate for the drop in the amount.

It seems to us that this is what acreage restrictions and similar tactics are all about.

In the example, there are 2 goods; traders 1 and 2 initially hold (2, 0) and (0, 1), respectively. Both traders have the same preferences, which are homothetic; at (1, 1) the indifference curve has slope -1, and at (2, 1) it has slope $-\frac{1}{8}$ (see fig. 1). There is of course no difficulty at all in constructing arbitrarily smooth preferences that obey these conditions (in addition to quasi-concavity, strict monotonicity, and practically whatever one wishes). A numerical example is given below.

Since the preferences are homothetic, the budget line is tangent to the indifference curve at the total initial bundle; i.e., it has slope $-\frac{1}{8}$ (see fig. 1).

This chapter originally appeared in *Journal of Mathematical Economics* 1 (1974): 209–211. Reprinted with permission.

Again because of the homotheticity, the competitive bundle of trader 1 lies on the line $x = 2y$, and hence the competitive bundle is $(\frac{2}{5}, \frac{1}{5})$. If trader 1 discards one unit of good 1 before trading starts, the total initial bundle is (1, 1), and hence the prices stand in the ratio 1 : 1 (see fig. 1). The competitive bundle of trader 1 then lies on the line $x = y$, and hence it is $(\frac{1}{2}, \frac{1}{2})$ – yielding more of *each* commodity than he got before he discarded anything.

To get an example of Gale's kind from this, add another trader (trader 3), and give him the initial bundle (1, 0) and the utility $2x+y$. If trader 1 gives trader 3 one unit of good 1, then trader 1 gains, and so does trader 3. Of course trader 2 loses.

To obtain appropriate preferences (for traders 1 and 2), consider the utility function defined to be 0 at the origin and $[(x+\alpha y)^{-3}+(\alpha x+y)^{-3}]^{-1}$ elsewhere

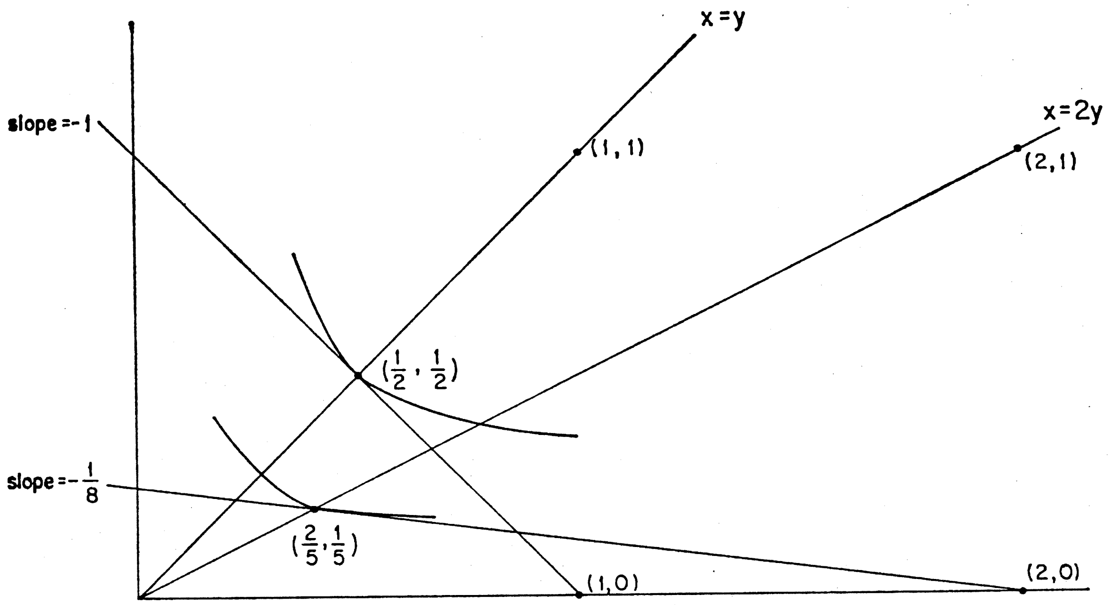

Fig. 1

in the non-negative orthant Ω, where α is a parameter that may take values in the half-open interval (0, 1]. This is continuous, quasi-concave, strictly monotonic, and induces homothetic preferences. The indifference curve has slope -1 at (1, 1) for all values of α. When α is near 0 the indifference curve at (2, 1) has slope near $-\frac{1}{16}$, and when $\alpha = 1$ it has slope -1. From continuity considerations it then follows that for an appropriate α, the indifference curve has slope $-\frac{1}{8}$ at (2, 1). The utility function is of class C^∞ in the entire non-negative orthant. The gradient is strictly positive everywhere in Ω, except at the origin, where it vanishes. There is perhaps a certain lack of smoothness at the origin, since the normalized gradient cannot be continuously extended to the origin. But this is unavoidable when the preferences are homothetic (unless the utility function is linear). In our case, if one wants to avoid this, one can re-define the preferences in an essentially arbitrary way in a neighborhood of the origin. The homotheticity will be destroyed, but if the neighborhood is sufficiently small, the example will not be affected in any essential way.

Reference

Gale, D., Exchange equilibrium and coalitions: An example, Journal of Mathematical Economics 1, 63–66.