1. Introduction. This paper is one of a series of studies (cf. [1], [5], [6]) in which orderability and various continuity notions for set functions are investigated and related to each other. Throughout we assume familiarity with the concepts summarized in § 2 of [6]. Our main result (§ 5) concerns the absolute continuity of set functions (see [1, § 5] or § 2 of this paper). In [1, Prop. 12.8] it was shown that every absolutely continuous set function is orderable; here (§ 5) we construct an example to show that the converse is false. The example is a function of two nonatomic measures, and is in a sense "simplest possible": In § 4 we show that for functions of a single nonatomic measure, orderability and absolute continuity are equivalent.

2. Notations and definitions. We refer the reader to § 2 of [6] for a summary of some notations and definitions from [1] and [5] that will be used in this paper. Familiarity with the above section will be assumed throughout our discussions.

For $x$ in the Euclidean space $E^n$, $\|x\|$ will always mean the summing norm, i.e., $\|x\| = \sum_{i=1}^{n} |x_i|$. If $x, y \in E^n$, write $x \leq y$ if $x_i \leq y_i$ for all $i$. If $\mu$ is a vector measure $(\mu_1, \ldots, \mu_n)$, then $\sum \mu$ will denote $\sum_{i=1}^{n} \mu_i$.

We next summarize some definitions and conventions from [1] which were not used in [6] and will be needed in this paper. The norm on $BV$ is the variation norm, defined by

$$\|v\| = \inf \{ u(I) + w(I) \mid u - w = v, \text{where } u \text{ and } w \text{ are monotonic} \}.$$ 

A chain is a nondecreasing sequence of sets of the form $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = I$. A link of this chain is a pair of successive elements. A subchain is a set of links. A chain will be identified with the subchain consisting of all links. If $v$ is a set function and $\Lambda$ is a subchain of a chain, then the variation of $v$ over $\Lambda$ is defined by $\|V\|_{\Lambda} = \sum |v(S_i) - v(S_{i-1})|$, where the sum ranges over $\{i \mid (S_{i-1}, S_i) \in \Lambda \}$. For a fixed $\Lambda$, $\| \cdot \|_{\Lambda}$ is a pseudonorm on $BV$, i.e., it enjoys all the properties of a norm except $\|v\|_{\Lambda} = 0 \Rightarrow v = 0$. It is known (see [1, Prop. 4.1]) that for every $v \in BV$, $\|v\| = \sup \|v\|_{\Lambda}$, where the supremum is taken over all subchains $\Lambda$. It is also known that

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the linear subspaces \( M, NA, WC \) and ORD are closed subspaces of \( BV \) [5, Prop. 4.2 and 4.3].

A set function \( v \) is said to be absolutely continuous with respect to a set function \( w \) (written \( v \ll w \) [1, p. 35]) if for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that for every chain \( \Omega \) and every subchain \( \Lambda \) of \( \Omega \), \( \|w\|_\Lambda \leq \delta \) implies \( \|v\|_\Lambda \leq \varepsilon \). Note that this relation is transitive, and that if \( v \) and \( w \) are measures, it coincides with the usual notion of absolute continuity. A set function is absolutely continuous if there is a measure \( \mu \in NA^+ \) such that \( v \ll \mu \). The set of all absolutely continuous set functions forms a closed linear subspace of \( BV \) [1, Prop. 5.2], denoted AC. Finally, \( pNA \) denotes the closed subspace of \( BV \) spanned by all powers of nonatomic measures.

3. Weak continuity and absolute continuity. A real-valued function on a subset of \( E^n \) is said to be monotonically absolutely continuous if for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that if \( x_1 \preceq y_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq y_n \), then

\[
\sum_{i=1}^{n} \|y_i - x_i\| \leq \delta \Rightarrow \sum_{i=1}^{n} |f(y_i) - f(x_i)| \leq \varepsilon.
\]

If the domain of \( f \) is one-dimensional, then monotonic absolute continuity coincides with the usual absolute continuity.

**Proposition 1.** Let \( \mu \) be an \( n \)-dimensional \( \sigma \)-additive measure whose components are in \( NA^+ \) and are mutually singular. Let \( f \) be a real-valued function on the range of \( \mu \) with \( f(0) = 0 \). Let \( v = f \circ \mu \). Then \( v \ll \sum \mu \Leftrightarrow f \) is monotonically absolutely continuous.

**Proof.** The direction \( \Leftarrow \) is obvious. To prove the direction \( \Rightarrow \), recall Lyapunov's theorem [4], according to which the range of a nonatomic \( \sigma \)-additive vector measure is convex and compact. From this and the mutual singularity it follows that if \( x_1 \preceq y_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq y_n \), then there exist \( S_1, T_1, \cdots, S_n, T_n \) in \( \mathcal{C} \) such that \( \mu(S_i) = x_i, \mu(T_i) = y_i \), and \( S_1 \subseteq T_1 \subseteq \cdots \subseteq S_n \supseteq T_n \), completing the proof of Proposition 1.

**Proposition 2.** Let \( v \in BV \) and \( \mu, \xi \in M^+ \). If \( v \ll \xi \), then \( v \ll \mu \) if and only if \( v \ll \mu \).

**Proof.** Sufficiency of the condition is obvious. To see the necessity, let \( \xi = \xi^{ac} + \xi^\perp \) be the Lebesgue decomposition of \( \xi \) with respect to \( \mu \), i.e., \( \xi^\perp \) and \( \xi^{ac} \) are nonnegative measures such that \( \xi^{ac} \ll \mu \) and \( \xi^\perp \perp \mu \) [3, Thm. C, p. 134]. Let \( A \in \mathcal{G} \) be such that \( \xi^\perp(A) = 0 \) and \( \mu(I \setminus A) = 0 \).

We shall show that \( v \ll \xi^{ac} \), and since \( \xi^{ac} \ll \mu \) it will follow that \( v \ll \mu \). Let \( \delta > 0 \) correspond to a given \( \varepsilon \) in accordance with the absolute continuity \( v \ll \xi \); i.e.,

\[
(3.1) \quad \|\xi\|_\Lambda \leq \delta \Rightarrow \|v\|_\Lambda \leq \varepsilon.
\]

We shall prove that \( v \ll \xi^{ac} \) by showing that

\[
(3.2) \quad \|\xi^{ac}\|_\Lambda \leq \delta \Rightarrow \|v\|_\Lambda \leq \varepsilon.
\]

If we intersect each set in each link of \( \Lambda \) with \( A \) then we get a subchain \( \Lambda^* \) such that \( \|\xi\|_{\Lambda^*} = \|\xi^{ac}\|_\Lambda \leq \delta \), and therefore by (3.1), \( \|v\|_\Lambda \leq \varepsilon \). But because \( v \ll \mu \) and \( \mu(I \setminus A) = 0 \), it follows that \( \|v\|_\Lambda = \|v\|_{\Lambda^*} \leq \varepsilon \). This proves (3.2).
COROLLARY 1. Let \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \) be an \( n \)-dimensional vector of measures in \( \text{NA}^+ \). Let \( f \) be a real-valued function on the range of \( \mu \), such that \( v = f \circ \mu \in \text{BV} \). Then \( v \in \text{AC} \) if and only if \( v \ll \sum \mu \).

Proof. Sufficiency of the condition is obvious. To verify the necessity note that \( f \circ \mu \ll \sum \mu \) and use Proposition 2.

COROLLARY 2.\(^1\) Let \( v = f \circ \mu \), where \( \mu \in \text{NA}^+ \); then \( v \in \text{pNA} \) if and only if \( v \in \text{AC} \).

Proof. The fact that \( \text{pNA} \subseteq \text{AC} \) has been proved in [1, Cor. 5.3]. Now let \( f \circ \mu \in \text{AC} \). Then by Corollary 1, \( f \circ \mu \ll \mu \), and hence by Proposition 1 and Theorem C in [1], \( f \circ \mu \in \text{pNA} \).

COROLLARY 3. The inclusions \( \text{BV} \supseteq \text{WC} \supseteq \text{AC} \) are strict.

Proof. The unanimity game \( v \) defined by

\[
v(S) = \begin{cases} 1, & S = I, \\ 0, & \text{otherwise}, \end{cases}
\]

shows that \( \text{BV} \neq \text{WC} \). Next, let \( \lambda \) be Lebesgue measure, and let \( g \) be the Cantor function, which is not absolutely continuous; then \( g \circ \lambda \in \text{WC} \), and by Propositions 1 and 2, \( g \circ \lambda \notin \text{AC} \).

4. Ordered absolute continuity. Let \( R \) be a measurable order. A chain \( \emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_m = I \) is called an \( R \)-chain if all the \( S_i \) are \( R \)-initial segments. Note that an \( R \)-chain is defined by a finite sequence of elements in \( I \), \( s_m \subseteq s_{m-1} \subseteq \cdots \subseteq s_1 \subseteq s_0 = -\infty \), such that \( I(s_i, R) = S_i \).

If \( v \) and \( w \) are in \( \text{BV} \), then \( v \) is said to be ordered absolutely continuous with respect to \( w \) (written \( v \ll w \)), if for every measurable order \( R \) and \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for every \( R \)-chain \( \Omega \) and every subchain \( \Lambda \) of \( \Omega \), \( \| w \|_{\Lambda} \leq \delta \) implies \( \| v \|_{\Lambda} \leq \varepsilon \). Note that the relation is transitive.

PROPOSITION 3. Let \( v \in \text{BV} \), \( \mu \in M^+ \).\(^2\) Then \( v \) is ordered absolutely continuous with respect to \( \mu \) if and only if \( v \in \text{ORD} \) and \( v \ll \mu \).

Proof. First assume that \( v \) is ordered absolutely continuous. It is easily verified that this implies \( v \ll \mu \). Using the argument of the proof of Proposition 12.8 of [1] we obtain that \( v \in \text{ORD} \). This completes the proof of one direction.

To prove the second direction, let us assume \( v \ll \mu \) and \( v \in \text{ORD} \). By [5, Thm. 3.2], we know that \( v \ll \mu \) implies that \( \varphi_R v \ll \mu \) for all measurable orders \( R \). Recall that weak continuity and absolute continuity between members of \( M \) coincide [2, § III. 4.3, p. 131]; hence \( \varphi_R v \ll \mu \) for all measurable orders \( R \). But then it follows that \( v \ll \mu \).

A set is said to be ordered absolutely continuous if there is a measure \( \mu \in \text{NA}^+ \) such that \( v \) is ordered absolutely continuous with respect to \( \mu \). The set of all ordered absolutely continuous functions in \( \text{BV} \) is denoted \( \text{OAC} \).

\(^1\) Cf. [1, Thm. C].

\(^2\) One may extend this theorem and require only \( \mu \in M \), and not \( \mu \in M^+ \). This would slightly complicate the proof.

\(^3\) In Proposition 12.8 of [1] one assumes \( v \ll \mu \) and obtains in addition to \( v \in \text{ORD} \), also that \( \varphi_R v \ll \mu \) uniformly in \( R \). Here we assume only \( v \ll \mu \), and can also obtain \( \varphi_R v \ll \mu \), but not uniformly.
Corollary 4. ORD ∩ WC = OAC

Corollary 5. OAC is a closed linear subspace of BV.

Remark. One may conjecture that if \( v \in \text{ORD} \) and every point in \( I \) is \( v \)-null then there exists a measure \( \mu \in \text{NA}^+ \) such that \( v \preccurlyeq \mu \). If this is true then clearly it should yield that OAC equals the set of all set functions in ORD for which every point is null.

Proposition 4. Let \( v = f \circ \mu \), where \( \mu \in \text{NA}^+ \); then \( v \ll \mu \) if and only if \( v \preccurlyeq \mu \).

Proof. If \( v \ll \mu \), then trivially \( v \ll \mu \). Assume now that \( v \triangleleft \mu \). By Proposition 3, \( v \in \text{ORD} \) and \( v \preccurlyeq \mu \). Let \( R \) be an arbitrary fixed measurable order, then by [5, Thm. 3.2], \( \varphi^R v \preccurlyeq \mu \). Since weak continuity and absolute continuity between totally finite measures coincide, it follows that \( \varphi^R v \ll \mu \). For a given \( \epsilon \), let \( \delta \) be given in accordance with the absolute continuity \( \varphi^R v \ll \mu \); i.e., for every subchain \( \Lambda \),

\[
\|\mu\|_{\Lambda} \leq \delta \Rightarrow \|\varphi^R v\|_{\Lambda} \leq \epsilon.
\]

We shall show that \( v \ll \mu \) by showing that for every subchain \( \Lambda \),

\[
\|\mu\|_{\Lambda} \leq \delta \Rightarrow \|v\|_{\Lambda} \leq \epsilon.
\]

Let \( \Lambda \) be a subchain satisfying \( \|\mu\|_{\Lambda} \leq \delta \) whose links are \( \{S_j, T_j\}_{1 \leq j \leq m} \), where \( \emptyset \subseteq S_1 \subseteq T_1 \subseteq S_2 \subseteq \cdots \subseteq S_m \subseteq T_m \subseteq I \). Let

\[
\bar{S}_j = \cap \{I(s, R) \mid s \in I, \mu((s, R)) \geq \mu(S_j)\},
\]

\[
\bar{T}_j = \cap \{I(s, R) \mid s \in I, \mu((s, R)) \geq \mu(T_j)\}.
\]

By [1, Lem. 12.15] it follows that for \( 1 \leq j \leq m \), \( \bar{S}_j \), \( \bar{T}_j \) are measurable and that

\[
\mu(\bar{S}_j) = \mu(S_j) \quad \text{and} \quad \mu(\bar{T}_j) = \mu(T_j).
\]

Note also that \( \bar{S}_j \) and \( \bar{T}_j \) are \( R \)-initial sets; hence, by [6, Lem. 2], it follows that for \( 1 \leq j \leq m \),

\[
(\varphi^R v)(\bar{T}_j) = v(\bar{T}_j) \quad \text{and} \quad (\varphi^R v)(\bar{S}_j) = v(\bar{S}_j).
\]

Let \( \Omega \) be the chain \( \emptyset \subseteq \bar{S}_1 \subseteq \bar{T}_1 \subseteq \bar{S}_2 \subseteq \cdots \subseteq \bar{S}_m \subseteq \bar{T}_m \subseteq I \) and let \( \Lambda \) be a subchain of \( \Omega \) whose links are \( \{\bar{S}_j, \bar{T}_j\}, 1 \leq j \leq m \). Note that (4.3) implies that \( \|\mu\|_{\Lambda} = \|\mu\|_{\bar{\Lambda}} \leq \delta \). Hence, by (4.1), \( \|\varphi^R v\|_{\bar{\Lambda}} \leq \epsilon \), and therefore (4.4) and (4.3) imply that

\[
\epsilon \geq \|\varphi^R v\|_{\bar{\Lambda}} = \|v\|_{\bar{\Lambda}} = \sum_{j=1}^{m} |f(\mu(T_j)) - f(\mu(S_j))|,
\]

\[
= \sum_{j=1}^{m} |f(\mu(T_j)) - f(\mu(S_j))| = \|v\|_{\Lambda}.
\]

We have established (4.2), thus completing the proof of Proposition 4.

Corollary 6. Let \( v = f \circ \mu \) where \( \mu \in \text{NA}^+ \); then

\[ v \in \text{AC} \iff v \ll \mu \iff v \in \text{OAC} \iff v \preccurlyeq \mu \iff v \in \text{ORD} \iff v \in \text{pNA}. \]

Proof. The above follows from Proposition 2, Corollary 2, Proposition 3 and Proposition 4.
Remark. It clearly follows from Corollary 6 that if we wish to construct an example of the form $v = f \circ \mu$ that is in ORD \ AC, then $\mu$ has to be at least two-dimensional.

5. ORD includes AC strictly. It was proved in [1, Prop. 12.8] that ORD $\supseteq$ AC. We are now going to construct an example of a set function in ORD that is not in AC. The example that we are going to describe appears, in a different context, at the beginning of § 9 in [1]. For each $k \geq 2$ let $A_k \subset [0, 1]^2$ be the parallelogram whose vertices are: $(2^{-k}, 0)$, $(2^{-k} + 4^{-k}, 0)$, $(2^{-k+1} + 4^{-k}, 1)$ and $(2^{-k+1}, 1)$ (see Fig. 1). Define a nondecreasing continuous function $f$ on the square such that for $x \in A_k$,

$$f(x) = f(x_1, x_2) = 2^k x_1 + 2^{-k+1} - 1;$$

for $x$ between $A_k$ and $A_{k-1}$,

$$f(x) = 2^{-k+1} + x_2 + \frac{(x_1 - 2 \cdot 4^{-k})}{(1 - 2^{-k} + x_2)};$$

for $x$ to the right of $A_2$ let $f$ be defined by the same formula that defines $f$ on $A_2$, i.e., $f(x) = 4x_1 - 1/2$; and finally for $x_1 = 0$ let $f(x) = x_2$. Let $\mu$ be any 2-dimensional vector measure on $(I, \mathcal{E})$, whose range is $[0, 1]^2$. We shall show that $v = f \circ \mu \in$ ORD \ AC.

![Diagram](image)

To show that $v \not\in$ AC, let

$$(2^{-k}, 0) = x_1^k \leq x_2^k \leq \cdots \leq x_n^k = (2^{-k+1}, 1)$$

be a “staircase” sequence of points in $A_n$, i.e., each point differs from the preceding one in one coordinate only (see Fig. 1). On the vertical segments of this sequence, $f$ does not change; all the change is concentrated on the horizontal segments. But the total length of the horizontal segments goes to 0, whereas the total change in $f$ is 1. Therefore $f$ is not monotonically absolutely continuous.

---

4 One can easily see that by “smoothing” our example one can get a set function in MIX [1, § 13] that is not in AC.
therefore $f \circ \mu$ is not absolutely continuous with respect to $\Sigma \mu$ (Proposition 1), and therefore $f \circ \mu \not\in \text{AC}$ (Corollary 1).

Let us now prove that $v \in \text{ORD}$. Set $\mu = \mu_1 + \mu_2$. We shall show that $v$ is ordered absolutely continuous with respect to $\mu$, and then use Proposition 3. Let $\mathcal{R}$ be a fixed measurable order. For a given $\varepsilon > 0$ we may choose a $1 > \delta_1 > 0$ such that

\begin{equation}
\|x - y\| \leq \delta_1 \Rightarrow |f(x) - f(y)| \leq \varepsilon / 2.
\end{equation}

This is possible because of the uniform continuity of $f$ in $[0, 1]^2$.

Let $J_1$ denote the intersection of all $\mathcal{R}$-initial segments of $\mu_1$-measure $> 0$. By [1, Lem. i2.15] it follows that $J_1$ is measurable and $\mu_1(J_1) = 0$. Let $J$ denote the intersection of all $\mathcal{R}$-initial segments of $\mu$-measure $> \mu(J_1) + \delta_1$. By the same lemma we mentioned before, it follows that $J$ is measurable and $\mu(J) = \mu(J_1) + \delta_1$, therefore $J \supseteq J_1$. Finally, observe that $\|\mu(J) - \mu(J_1)\| = \delta_1$; hence by (5.1) it follows that $|v(J) - v(J_1)| \leq \varepsilon / 2$.

Now let $p$ be an integer $\geq 2$ such that $2^{p-1} \geq 1 / \mu_1(J)$. Note that $p$ depends only on $\mathcal{R}$ and $\varepsilon$. One can easily verify that $f$ fulfills a Lipschitz condition on $\{x \in [0, 1]^2 \mid x_1 \geq \mu_1(J)\}$ with constant $2^p$, i.e., $|f(y) - f(x)| \leq 2^p \|x - y\|$; this implies that if $S, T \in \mathcal{E}$ and $J \subseteq S \subseteq T$, then $|v(T) - v(S)| \leq 2^p(|\mu(T) - \mu(S)|$. Define $\delta = \min \{\delta_1, (2^p + 1)^{-1} \varepsilon / 2\}$ and note that $\delta$ depends only on $\mathcal{R}$ and $\varepsilon$.

Let $\Lambda$ be a subchain of an $\mathcal{R}$-chain $\Omega$, with links $\{S_i, T_i\}$ ($1 \leq i \leq n$), where $\emptyset \subseteq S_1 \subseteq T_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq T_n \subseteq I$. By definition of $\mathcal{R}$-chain, $S_i$ and $T_i$ are $\mathcal{R}$-initial segments ($1 \leq i \leq n$). We shall show that $\|\mu\|_\Lambda \leq \delta$ implies $\|v\|_\Lambda \leq \varepsilon$, which implies that $v$ is ordered absolutely continuous with respect to $\mu$, and hence by Proposition 4 that $v \in \text{ORD}$. Let $\|\mu\|_\Lambda \leq \delta$, i.e., $\|\mu\|_\Lambda = \sum_{i=1}^{\delta} \|\mu(T_i) - \mu(S_i)\| \leq \delta$. Without loss of generality we may assume that if $T_i \supseteq J$, then $S_i \supseteq J$; otherwise split $\{S_i, T_i\}$ into two links $\{S_i, J\}$ and $\{J, T_i\}$. Similarly we may assume that if $S_i \subseteq J$, then $T_i \subseteq J$. Note that since $\mu$ and $v$ are monotonic, $\|\mu\|_\Lambda$ and $\|v\|_\Lambda$ remain unchanged. Let

$I_1 = \{1 \leq i \leq n \mid T_i \subseteq J\}$,

$I_2 = \{1 \leq i \leq n \mid J \subseteq S_i\}$,

$I_3 = \{1 \leq i \leq n \mid J \subseteq S_i \subseteq T_i \subseteq J\}$.

$I_1$, $I_2$ and $I_3$ are disjoint, and by our previous assumption $I_1 \cup I_2 \cup I_3 = \{1, 2, \cdots, n\}$. Now

$\|v\|_\Lambda = \sum_{i=1}^{n} |v(T_i) - v(S_i)| = \sum_{i=1}^{3} \sum_{i \in I_1} |v(T_i) - v(S_i)|$

$\leq \sum_{i \in I_1} \|\mu_2(T_i) - \mu_2(S_i)\| + \sum_{i \in I_2} \|2^p(\mu(T_i) - \mu(S_i)) + (v(T_i) - v(S_i)) - v(J) - v(J_1)\|

\leq \delta + 2^p \cdot \delta + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

\[\text{The lemma must be modified to apply to measures } \mu \text{ in } \text{NA}^+ \text{ for which } u(I) \neq 1. \text{ Note that } \mu(J_1) + \delta_1 < \mu(I).\]
This completes the proof that $v \in \text{ORD} \setminus \text{AC}$. Hence we have shown

\begin{equation}
\text{ORD includes AC strictly.}
\end{equation}

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