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ON BALANCED GAMES WITH INFINITELY MANY PLAYERS

by

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Let S be an arbitrary set, and Σ a field of subsets of S.* Let v be a bounded real function defined on Σ with non-negative values, such that $v(\Phi) = 0$ and $v(S) > 0$. We shall call the triple $[S,\Sigma,v]$ a <u>game</u>; S is the set of <u>players</u>, Σ the set of <u>coalitions</u> and v is the <u>payoff** function</u>. An <u>outcome</u> of the game is a bounded additive real function λ defined on Σ, for which $\lambda(S) = v(S)$. If an outcome λ fulfils $\lambda(A) \geq v(A)$ for each $A \in \Sigma$, it belongs, by definition, to the <u>core</u> of the game. For $A \in \Sigma$, let χ_A be the characteristic function of A, i.e. $\chi_A(s) = 1$ if $s \in A$ and $\chi_A(s) = 0$ if $s \in S \setminus A$, for all $s \in S$. A game is <u>balanced</u> if

$$\sup \Sigma_i a_i v(A_i) \leq v(S) ,$$

when the sup is taken over all finite sequences of a_i and A_i, where the a_i are non-negative numbers, the A_i are in Σ, and $\Sigma_i a_i \chi_{A_i} \leq \chi_S$.***

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* Σ fulfils: (a) $\Phi \in \Sigma$. (b) $A \in \Sigma \Rightarrow S \setminus A \in \Sigma$.
(c) $A \in \Sigma,\ B \in \Sigma \Rightarrow A \cup B \in \Sigma$.

** v is often called the "characteristic function" of the game. We refrain from that terminology because the same term is used with a different meaning in this paper.

*** It is easy to verify that this sup does not change even if it is constrained by $\Sigma_i a_i \chi_{A_i} = \chi_S$ (instead of the inequality); also, for balanced games, the sup equals $v(S)$.

This concept is due to L. S. Shapley [2], who proved that a finite game (S is finite) has a non-empty core if and only if it is balanced. In this paper we extend this result to an arbitrary set S.

U. Liberman [3] dealt with the case when (S,Σ,μ) is a finite, separable, non-atomic measure space (μ is a measure). He required, in addition to the balancedness condition, that the payoff function v be continuous on (Σ,ρ) where ρ is the metric induced on Σ by μ. Then, using Shapley's theorem for the finite case, he proved the existence of a measure in the core, absolutely continuous w.r.t. μ. We prove such a result with weaker assumptions.

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<u>The Main Theorem</u> A game has a non-empty core iff it is balanced.

<u>Proof</u> We shall show that a balanced game has a non-empty core. The other side of the implication is easily verified.

Let X denote the linear space of all finite real combinations of characteristic functions of sets in Σ. (The completion of X in the sup metric is denoted in Dunford and Schwartz [1] by $B(S,\Sigma)$.) For non-negative vectors x in X, (i.e. $x(s) \geq 0$ for $s \in S$) we define:

$$p(x) = \sup \Sigma_i a_i v(A_i) \;,$$

where the sup is taken over all finite sequences of a_i and A_i, where the a_i are non-negative real numbers, the A_i are in Σ, and $\Sigma_i a_i \chi_{A_i} \leq \chi_S$.

Let X^+ denote the positive cone of X, i.e.

$$X^+ = \{x \in X \mid x \geq 0\} .$$

Then p is a super-additive positive-homogeneous**** functional on X^+. We shall prove the existence of a linear functional F on X for which $F(x) \geq p(x)$ when $x \in X^+$, and $F(\chi_S) = v(S)$. Naturally, the set function induced by F on Σ is in the core of the game. The idea behind the proof is closely related to the Hahn-Banach theorem. (See, for instance, [1]).

Let Y be a subspace of X containing χ_S, and let $Y^+ = Y \cap X^+$. Assume that a linear functional F is defined on Y for which $F(x) \geq p(x)$ when $x \in Y^+$, and $F(\chi_S) = v(S)$. If $Y \neq X$, there is a set A in Σ such that $\chi_A \notin Y$. Define $Z = \operatorname{span}\{Y \cup \{\chi_A\}\}$. Every vector in Z has a unique representation in the form $y + a\chi_A$ with $y \in Y$. For any real c the function G defined on Z by the equation $G(y + a\chi_A) = F(y) + ac$ is a proper extension of F. It remains to show that c may be chosen so that $G(y + a\chi_A) \geq p(y + a\chi_A)$ when $y + a\chi_A \geq 0$. Because of the positive-homogeneity of p, it is sufficient to prove the

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**** A functional p is called super-additive if $p(x+y) \geq p(x) + p(y)$. It is called positive-homogeneous if $ap(x) = p(ax)$ when $a \geq 0$.

existence of c, such that for any y, z in Y, $y + \chi_A \geq 0$ and $z - \chi_A \geq 0$ imply $F(y) + c \geq p(y + \chi_A)$ and $F(z) - c \geq p(z - \chi_A)$. The last two inequalities are equivalent to

$$F(z) - p(z - \chi_A) \geq c \geq p(y + \chi_A) - F(y).$$

So it is sufficient to prove:

$$F(z) - p(z - \chi_A) \geq p(y + \chi_A) - F(y)$$

or,

$$F(z) + F(y) \geq p(z - \chi_A) + p(y + \chi_A) .$$

But $y + \chi_A \geq 0$ and $z - \chi_A \geq 0$ imply $y + z \in Y^+$, so

$$p(z - \chi_A) + p(y + \chi_A) \leq p(z + y) \leq F(z + y) = F(z) + F(y)$$

and the desired inequality is proved. The proof is completed by a standard use of Zorn's lemma. Q.E.D.

Next we deal with the problem of existence of a σ-additive outcome in the core, assuming that Σ is a σ-field.

A necessary and sufficient condition for an additive set function λ to be σ-additive is that $\lambda(A_i) \to \lambda(S)$ for any monotone increasing sequence $\{A_i\}_{i=1}^{\infty}$ in Σ with $\cup_{i=1}^{\infty} A_i = S$. If for every such sequence $v(A_i) \to v(S)$ and λ is in the core, i.e. $\lambda(A) \geq v(A)$ $A \in \Sigma$, we can easily conclude the desired condition $\lambda(A_i) \to v(S) = \lambda(S)$. So we have proved:

Lemma A If $v(A_i) \to v(S)$ for any monotone increasing sequence $\{A_i\}_{i=1}^{\infty}$ in Σ, the union of which is S, then every outcome in the core is σ-additive.

Indeed we know a little bit more. If λ belongs to the core, then $\lambda(A) \geq p(A)$, $A \in \Sigma$ and we get a somewhat stronger result:

Lemma B If $p(A_i) \to p(S)$ for any monotone increasing sequence $\{A_i\}_{i=1}^{\infty}$ in Σ, the union of which is S, then every outcome in the core is σ-additive.

Of course the second condition is not necessary for the existence of a G-additive outcome in the core. For example let:

$$v(A) = \begin{cases} 1 & A = S \\ 0 & \text{otherwise} \end{cases} .$$

So two open questions may be asked: Is the condition of lemma B necessary that every outcome in the core should be σ-additive? and what is a necessary and sufficient condition for the existence of a σ-additive outcome in the core? (Assuming the core is non-empty).

The treatment of another problem was found to be more successful. Assume a game $[S, \Sigma, v]$ and an additive function μ on Σ. What is the "continuity" condition on v with respect to μ, such that every outcome in the core will be "continuous" with respect to μ?

If $v(S \setminus A) = v(S)$ then for each λ in the core $\lambda(A) = 0$; otherwise

$$v(S) = v(S \setminus A) \leq \lambda(S \setminus A) = \lambda(S) - \lambda(A) < \lambda(S) = v(S) ,$$

a contradiction. On the other hand, if λ is in the core and $\lambda(A) = 0$, then $v(A) = 0$. We can state this result as follows:

<u>Lemma C</u> (i) If $v(S \setminus A) = v(S)$ for every μ-null set A, then any outcome in the core vanishes on the μ-null sets.

(ii) If an outcome in the core vanishes on μ-null sets, then v does the same.

Another similar simple result is given below:

<u>Lemma D</u> If v fulfils the conditions of lemma B and lemma C (i), then any outcome in the core is absolutely continuous w.r.t. μ.

REFERENCES

[1] N. Dunford and J.T. Schwartz, "Linear Operators", part I, Interscience Publishers, New York 1966.

[2] L.S. Shapley, "On balanced sets and cores", RAND RM-4601-PR, June 1965.

[3] U. Liberman, Master's thesis, Department of Mathematics, Hebrew University, 1966.