SHAPLEY VALUE[1]

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Abstract*:* The *Shapley value* is an a priori evaluation of the prospects of a player in a multi-person game. Introduced by Lloyd S. Shapley in 1953, it has become a central solution concept in cooperative game theory. The Shapley value has been applied to economic, political, and other models.

The *value* of an uncertain outcome (a 'lottery') is an a priori measure, in the participant's utility scale, of what he expects to obtain (this is the subject of 'utility theory'). The question is, how would one evaluate the prospects of a *player* in a multi-person interaction, that is, in a *game*?

This question was originally addressed by Lloyd S. Shapley (1953a). The framework was that of *n-person games in coalitional form with side-payments*, which are given by a set N of 'players', say 1, 2, …, n, together with a 'coalitional function' v that associates to every subset S of N ('coalition') a real number $v(S)$, the maximal total payoff the members of S can obtain (the 'worth' of S). An underlying assumption of this model is that there exists a medium of exchange ('money') that is freely transferable in unlimited amounts between the players, and moreover every player's utility is additive with respect to it (that is, a transfer of x units from one player to another decreases the first one's utility by x units and increases the second one's utility by x units; the total payoff of a coalition can thus be meaningfully defined as the sum of the payoffs of its members). This requirement is known as existence of 'side payments' or 'transferable utility'. In addition, the game is assumed to be adequately described by its coalitional function (that is, the worth $v(S)$ of each coalition S is well defined, and the abstraction from the extensive structure of the game to its coalitional function leads to no essential

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loss; such a game is called a ‘*c*-game’). These assumptions may be interpreted in a broader and more abstract sense. For example, in a voting situation, a ‘winning coalition’ is assigned worth 1, and a ‘losing’ coalition, worth 0. The essential feature is that the prospects of each coalition may be summarized by one number.

The *Shapley value* associates to each player in each such game a unique payoff – his ‘value’. The value is required to satisfy the following four axioms. (EFF) *Efficiency* or *Pareto optimality:* The sum of the values of all players equals $v(N)$, the worth of the grand coalition of all players (in a superadditive game $v(N)$ is the maximal amount that the players can jointly get); this axiom combines feasibility and efficiency. (SYM) *Symmetry* or *equal treatment:* If two players in a game are substitutes (that is, the worth of no coalition changes when replacing one of the two players by the other one), then their values are equal. (NUL) *Null* or *dummy player:* If a player in a game is such that the worth of every coalition remains the same when he joins it, then his value is zero. (ADD) *Additivity:* The value of the sum of two games is the sum of the values of the two games (equivalently, the value of a probabilistic combination of two games is the same as the probabilistic combination of the values of the two games; this is analogous to ‘expected utility’). The surprising result of Shapley is that these four axioms *uniquely determine* the values in *all* games.

Remarkably, the Shapley value of a player in a game turns out to be exactly his *expected marginal contribution to a random coalition.* The marginal contribution of a player i to a coalition S (that does not contain i) is the change in the worth when i joins S, that is, $v(S \cup \{i\}) - v(S)$. To obtain a random coalition S not containing i, arrange the n players in a line (for example, 1, 2, …, n) and put in S all those that precede i in that order; all $n!$ orders are assumed to be equally likely. The formula for the Shapley value is striking, first, since it is a consequence of very simple and basic axioms and, second, since the idea of marginal contribution is so fundamental in much of economic analysis.

It should be emphasized that the value of a game is an *a priori measure*, that is, an evaluation before the game is actually played. Unlike other solution concepts (for example, core, von Neumann–Morgenstern solution, bargaining set), it need not yield a ‘stable’ outcome (the probable final result when the game is actually played). These final stable outcomes are in general not well determined; the value – which is uniquely specified – may be thought of as their *expectation* or average. Another interpretation of the value axioms regards them as rules for 'fair' division, guiding an impartial 'referee' or 'arbitrator'. Also, as suggested above, the Shapley value may be understood as the utility of playing the game (Shapley, 1953a; Roth, 1977).

In view of both its strong intuitive appeal and its mathematical tractability, the Shapley value has been the focus of much research and many applications. We can only briefly mention some of these here (together with just a few representative references). The reader is referred to the survey of Aumann (1978) and, for more extensive coverage, to the *Handbook of Game Theory* (Aumann and Hart, vol 1: 1992 [HGT1], vol 2: 1994 [HGT2], vol 3: 2002 [HGT3]), especially Chapters 53–58, as well as parts of Chapters 32–34 and 37.

Variations

Following Shapley's pioneering approach, the concept of *value* has been extended, modified, and generalized.

Weighted values

Assume that the players are of unequal 'size' (for example, a player may represent a 'group', a 'department', and so on), and this is expressed by given (relative) weights. This setup leads to 'weighted Shapley values' (Shapley, 1953b); in unanimity games, for example, the values of the players are no longer equal but, rather, proportional to their weights [HGT3, Ch. 54].

Semi-values

Abandoning the efficiency axiom (EFF) yields the class of 'semi-values' (Dubey, Neyman and Weber, 1981). An interesting semi-value is the *Banzhaf index* (Penrose, 1946; Banzhaf, 1965; Dubey and Shapley, 1979), originally proposed as a measure of power in voting games. Like the Shapley value, it is also an expected marginal contribution, but here all coalitions not containing player i are equally likely [HGT3, Ch. 54].

Other axiomatizations

There are alternative axiomatic systems that characterize the Shapley value. For instance, one may replace the additivity axiom (ADD) with a *marginality axiom* that requires the value of a player to depend only on his marginal contributions (Young, 1985). Another approach is based on the existence of a *potential* function together with efficiency (EFF) (Hart and Mas-Colell, 1989) [HGT3, Ch. 53].

Consistency

Given a solution concept which associates payoffs to games, assume that a group of players in a game have already agreed to it, are paid off accordingly, and leave the game; consider the 'reduced game' among the remaining players. If the solution of the reduced game is the same as that of the original game, then the solution is said to be *consistent*. It turns out that consistency, together with some elementary requirements for two-player games, characterizes the Shapley value (Hart and Mas-Colell, 1989) [HGT3, Ch. 53], [HGT1, Ch. 18].

Large games

Assume that the number of players increases and individuals become negligible. Such models are important in applications (such as competitive economies and voting), and there is a vast body of work on values of large games that has led to beautiful and important insights (for example, Aumann and Shapley, 1974) [HGT3, Ch. 56].

NTU games

These are games 'without side payments', or 'with non-transferable utility' (that is, the existence of a medium of utility exchange is no longer assumed). The simplest such games, two-person pure bargaining problems, were originally studied by Nash (1950). Values for general NTU games, which coincide with the Shapley value in the side payments case, and with the Nash bargaining solution in the two-person case, have been introduced by Harsanyi (1963), Shapley (1969), Maschler and Owen (1992) [HGT3, Ch. 55].

Non-cooperative foundations

Bargaining procedures whose non-cooperative equilibrium outcome is the Shapley value have been proposed by Gul (1989) (see Hart and Levy, 1999, and Gul, 1999) and Winter (1994) for strictly convex games, and by Hart and Mas-Colell (1996) for general games [HGT3, Ch. 53].

Other extensions

This includes games with communication graphs (Myerson, 1977), coalition structures (Aumann and Drèze, 1974; Owen, 1977; Hart and Kurz, 1983), and others [HGT2, Ch. 37], [HGT3, Ch. 53].

Economic applications

Perfect competition

In the classical economic model of perfect competition, the commodity prices are determined by the requirement that total demand equals total supply; this yields a *competitive* (or *Walrasian*) *equilibrium*. A different approach in such setups looks at the cooperative 'market game' where the members of each coalition can freely exchange among themselves the commodities they own. A striking phenomenon occurs: various game-theoretic solutions of the market games yield precisely the competitive equilibria. In particular, in perfectly competitive economies every Shapley value allocation is competitive and, if the utilities are smooth, then every competitive allocation is also a value allocation. This result, called the *value equivalence principle*, is remarkable since it joins together two very different approaches: competitive prices arising from supply and demand on the one hand, and marginal contributions to trading coalitions on the other. The value equivalence principle has been studied in a wide range of models (for example, Shapley, 1964; Aumann, 1975). While it is undisputed in the TU case, its extension to the general NTU case seems less clear (it holds for the Shapley NTU value, but not necessarily for other NTU values) [HGT3, Ch. 57].

Cost allocation

Consider the problem of allocating joint costs in a 'fair' manner. Think of the various 'tasks' (or 'projects', 'departments', and so on) as players, and let $v(S)$ be the total cost of carrying out the set S of tasks (Shubik, 1962). It turns out that the axioms determining the Shapley value are easily translated into postulates appropriate for solving cost allocation problems (for example, the efficiency axiom becomes 'total-cost-sharing'). Two notable applications are airport landing fees (a task here is an aircraft landing; Littlechild and Owen, 1973) and telephone billing (each time unit of a phone call is a player; the resulting cost allocation scheme was put into actual use at Cornell University; Billera, Heath and Raanan, 1978) [HGT2, Ch. 34].

Other applications

The value has been applied to various economic models; for example, models of taxation where a political power structure is given in addition to the economic data (Aumann and Kurz, 1977). Further references to economic applications can be found in Aumann (1985) [HGT3, Ch. 58], [HGT2, Ch. 33].

Political applications

What is the 'power' of an individual or a group in a voting situation? A trivial observation – though not always remembered in practice – is that the political power need not be proportional to the number of votes (see Shapley, 1981, for some interesting examples). It is therefore important to find an objective method of measuring power in such situations. The Shapley value (known in this setup as the *Shapley–Shubik index*; Shapley and Shubik, 1954) is, by its very nature, a most appropriate candidate. Indeed, consider a simple political game, described by specifying whether each coalition is 'winning' or 'losing'. The Shapley value of a player i turns out to be the probability that i is the 'pivot' or 'key' player, namely, that in a random order of all players those preceding i are losing, whereas together with i they are winning. For example, in a 100-seat parliament with simple majority (that is, 51 votes are needed to win), assume there is one large party having 33 seats and the rest are divided among many small parties; the value of the large party is then close to 50%, considerably more than its voting weight (that is, its 33% share of the seats). In contrast, when there are two large parties each having 33 seats and a large number of small parties, the value of each large party is close to 25% – much less than its voting weight of 33%. To understand this, think of the competition between the two large parties to attract the small parties to form a winning coalition; in contrast, when there is only one large party, the competition is between the small parties (to join the large party).

The Shapley value has also been used in more complex models, where 'ideologies' and 'issues' are taken into account (thus, not all arrangements of the voters are equally likely; an 'extremist' party, for example, is less likely to be the pivot than a 'middle-of-the-road' one; Owen, 1971; Shapley, 1977).

References to political applications of the Shapley value may be found in Shapley (1981); these include various parliaments (USA, France, Israel), the United Nations Security Council, and others [HGT2, Ch. 32].

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See also: game theory, cooperative game theory

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