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Values of Large Market Games[1])

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Abstract: Three aspects of the application of the game theoretic concept of "value" to non-atomic economies – such as markets or production – are studied: first, the relation between value and equilibria; second, the problems of existence and non-existence of value; and third, a new way of defining value for these games, in order to guarantee its existence, which leads to interesting economic interpretations.

1. Introduction

The relations between game theoretic and economic concepts have been studied for a long time, trying to get a better insight into the laws governing the behavior of economic agents.

Much interest has been devoted to "large" economies[3]), where the individual is "negligible". Such situations are called "perfectly competitive", and the appropriate economic concept is that of "competitive equilibrium".

The first game theoretic solution studied in this context is the *core*, the main result being:

Core Equivalence Theorem: In a perfectly competitive economy, the core and the set of competitive allocations coincide [cf. *Debreu/Scarf; Aumann* [1964]*; Vind; Hildenbrand*, and others].

The next most used concept is the (*Shapley*) *value* – in particular, since it captures traditional economic ideas of "marginal contribution" (or, "worth"). The corresponding result is the following:

Value Theorem: In a perfectly competitive economy, every value allocation is competitive, and the two sets of allocations coincide if the economy is "sufficiently differentiable".

There are two main ways to model perfect competition. One is a limit approach, where sequences of finite economies, increasing in size, are considered (e.g., replicas). The other is using a non-atomic continuum as the space of agents.

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[3]) By "economy" we mean a market, or a production economy – as in Section 2.

Also, the two kinds of economic models are studied: Walrasian exchange (markets *without* transferable utility), and "monetary" markets (*with* transferable utility). As it will be pointed out in Section 2, the latter also represents production economies.

The following table summarizes the research done on the Value Theorem:

	Limit of Finite Economies	Non-atomic Economies	
Monetary (with transferable utility)	*Shapley* [1964]	*Aumann/Shapley* [1974]	Differentiable
	Champsaur [1975]	*Hart* [1977b]	Non-differentiable
Walrasian (without transferable utility)	*Mas-Colell* [1977]	*Aumann* [1975]	Differentiable
	Champsaur [1975].	*Hart* [1977b]	Non-differentiable

Tab. 1

In this paper we deal with non-atomic economies. After presenting the basic models and defining the generalized asymptotic values in Section 2, we divide the results in three parts. The first one (Section 3) is devoted to the Value Theorem; the second one (Section 4), to the existence (and non-existence) of the asymptotic value; and in the last one (Section 5), we try to overcome these problems by defining a new measure-based value.

2. Preliminaries

This section includes the basic models of non-atomic economies, the definitions of (generalized) asymptotic values, and some preliminary results.

We start by describing a *non-atomic economy* (*market*) – as in *Aumann/Shapley* [1974], *Aumann* [1975], and *Hart* [1977b].

The *trader space* is a measurable space (I, C), which we assume to be standard[4]. A non-negative, σ-additive and non-atomic measure μ on C is given, called the *population measure*. To simplify our notations, we will sometimes write[5] $\int_S \underset{\sim}{f} d\mu$ and $\int_S \underset{\sim}{f}$ to mean $\int_S \underset{\sim}{f}(t)\, d\mu(t)$.

The *commodity space* is Ω, the non-negative orthant of the l-dimensional Euclidean space $\mathbf{R}^l$, where l is the number of commodities. For x in $\mathbf{R}^l$, x^j will denote its j-th coordinate.

The *initial allocation* $\underset{\sim}{a}$ is an integrable function from I to Ω. We assume that every commodity is actually present in the market, i.e.,

[4]) I.e., isomorphic to the unit interval with the Borel σ-field. This assumption is not too restrictive, since any uncountable Borel subset of any Euclidean space, and indeed of any complete separable metric space, with the corresponding Borel σ-field, is standard – cf. Proposition (1.1) in *Aumann/Shapley* [1974].

[5]) Letters with 'wiggle' underneath will denote function defined on I.

$$\int_{I} \underset{\sim}{a}^j > 0, \text{ for all } 1 \leqslant j \leqslant l \tag{2.1}$$

(commodities with no initial supply can be obviously ingnored).

An *allocation* is an integrable function $\underset{\sim}{x}$ from I to Ω, such that $\int_I \underset{\sim}{x} = \int_I \underset{\sim}{a}$.

Here we distinguish between the two kinds of economies: monetary and Walrasian.

In the *transferable utility case* (monetary markets), to each t in I there corresponds a real-valued function u_t defined on Ω, called the *utility function of t*. All these functions are normalized by $u_t(0) = 0$, and they further satisfy:

(2.2) $x \geqslant y$ implies[6]) $u_t(x) \geqslant u_t(y)$ (*weak monotonicity*),

(2.3) u_t is a continuous function (*continuity*),

(2.4) $u_t(x)$, as a function of the pair (t, x), is measurable in the product field $\mathcal{C} \times \mathcal{B}^l$, where $\mathcal{B}^l$ denotes the Borel σ-field on Ω (*measurability*), and

(2.5) $u_t(x) = o(\| x \|)$ as $\| x \| \to \infty$, integrably in t (i.e., for every $\epsilon > 0$ there is an integrable function $\underset{\sim}{\eta}$ defined on I, such that $| u_t(x) | \leqslant \epsilon \| x \|$ whenever $\| x \| \geqslant \underset{\sim}{\eta}(t)$).

Given the above economy, we will define the corresponding *market game* v by

$$\mathrm{v}(S) = \max \{\int_S u_t(\underset{\sim}{x}(t))\, d\mu(t) \mid \int_S \underset{\sim}{x} = \int_S \underset{\sim}{a} \text{ and } \underset{\sim}{x}(t) \in \Omega \text{ for all } t \in S\}, \tag{2.6}$$

for all $S \in \mathcal{C}$, the maximum being attained by the main theorem in *Aumann/Perles* [1965]. Then v is a non-atomic game with side payments, and further it belongs to the space H'_+, as defined in *Hart* [1977a, Section 2]; namely, it is the limit in the supremum norm of polynomials of non-atomic measures, it is superadditive, monotone, and homogeneous of degree one [see Proposition (3.4) in *Hart*, 1977b].

In terms of our exchange market, the interpretation of v (S) is as follows: there is an additional commodity, called "money", such that each trader's utility increases by one unit for each one unit of money. The maximum utility the coalition S can get, by reallocating its own initial resources between its members, is then exactly v (S).

The second interpretation of this model is that of a production economy. There are l "inputs" (i.e., raw goods), and one "output" (i.e., a finished good). Each participant t can produce out of a vector x (in Ω) of inputs, an amount $u_t(x)$ of the output good[7]). The initial allocation of raw goods being $\underset{\sim}{a}$, v (S) is then the maximal quantity of the finished good that S can produce, again by using its own resources alone.

A *transferable utility competitive equilibrium* (t.u.c.e.) is a pair $(\underset{\sim}{x}, p)$, where $\underset{\sim}{x}$ is an allocation and p is a *price* vector in Ω, such that

$$u_t(x) - p \cdot (x - \underset{\sim}{a}(t)) \leqslant u_t(\underset{\sim}{x}(t)) - p \cdot (\underset{\sim}{x}(t) - \underset{\sim}{a}(t)) \tag{2.7}$$

[6]) For x and y in Ω, $x \geqslant y$ means $x^j \geqslant y^j$ for all $1 \leqslant j \leqslant l$.

[7]) A more precise interpretation will be that a producer dt gets $u_t(x)\, \mu(dt)$ out of $x\, \mu(dt)$ – cf. Section 30 in *Aumann/Shapley* [1974].

for all x in Ω and (almost) all t in I. The corresponding *competitive payoff distribution* is the measure[8]) ν_p defined by

$$\nu_p(S) = \int_S [u_t(\underset{\sim}{x}(t)) - p \cdot (\underset{\sim}{x}(t) - \underset{\sim}{a}(t))]\, d\mu(t), \tag{2.8}$$

for all $S \in \mathcal{C}$.

The t.u.c.e. is actually a usual (Walrasian) competitive equilibrium, the price of "money" being normalized to 1 – see Section 32 in *Aumann/Shapley* [1974] for a more detailed discussion.

From now on, P will always denote *the set of all equilibrium prices*, i.e., the set of all p in Ω such that $(\underset{\sim}{x}, p)$ is a t.u.c.e. for some allocation $\underset{\sim}{x}$.

Proposition 2.9:

(i) P is a non-empty, convex and compact subset of Ω.

(ii) $\underset{\sim}{x}$ is a t.u.c.e. allocation if and only if v (I) is attained[9]) at $\underset{\sim}{x}$. Moreover, $(\underset{\sim}{x}, p)$ is then a t.u.c.e. for *all* p in P, and the corresponding competitive payoff distribution does *not* depend on $\underset{\sim}{x}$ (i.e., is the same for all such $\underset{\sim}{x}$[10]).

(iii) The core of v and the set of competitive payoff distributions coincide.

Proof: Propositions 32.2 and 32.5 in *Aumann/Shapley* [1974], Propositions (2.10) and (2.20) in *Hart* [1977b], and Theorem 23.4 in *Rockafellar* [1970].

The second model is that of a *Walrasian non-atomic market*. Unlike the previous case, no (cardinal) utility functions are given. Instead, for each t in I there is an (ordinal) *preference relation* $\succ_t$ on Ω, satisfying:

(2.10) $x \geqslant y$ and $x \neq y$ imply $x \succ_t y$ (*desirability*),

(2.11) for each x in Ω, the sets $\{y \mid y \succ_t x\}$ and $\{y \mid x \succ_t y\}$ are open relative to Ω (*continuity*), and

(2.12) for any two measurable functions $\underset{\sim}{x}$ and $\underset{\sim}{y}$ from I to Ω, the set $\{t \mid \underset{\sim}{x}(t) \succ_t \underset{\sim}{y}(t)\}$ belongs to $\mathcal{C}$ (*measurability*).

A *competitive equilibrium* is a pair $(\underset{\sim}{x}, p)$, where $\underset{\sim}{x}$ is an allocation and $p \neq 0$ is a *price* vector in Ω, such that $\underset{\sim}{x}(t)$ is maximal with respect to $\succ_t$ in the budget set of trader t

$$B_p(t) = \{x \in \Omega \mid p \cdot x \leqslant p \cdot \underset{\sim}{a}(t)\},$$

for (almost) all t in I.

We come now to the definition of value – first in the transferable utility case – using the asymptotic approach, due to *Kannai* [1966].

[8]) Because of Proposition 2.9, we can use the notation ν_p (instead of $\nu_{(\underset{\sim}{x},p)}$).

[9]) I.e., v $(I) = \int_I u_t(\underset{\sim}{x}(t))$ and $\int_I \underset{\sim}{x} = \int_I \underset{\sim}{a}$.

[10]) But it *does* depend on p.

Let v be a non-atomic game on the measurable space $(I, \mathcal{C})$ (i.e., v is a real function on $\mathcal{C}$ with $v(\emptyset) = 0$).

A *partition* Π of $(I, \mathcal{C})$ is a finite family of subsets of I, which are measurable (i.e., belong to $\mathcal{C}$) and disjoint, and whose union is I. A sequence $\mathcal{P} = \{\Pi_m\}_{m=1}^{\infty}$ of partitions is called *admissible* if it satisfies

(2.13) it is *decreasing*, i.e. for all m, each member of Π_m is a union of members of Π_{m+1}; and

(2.14) it is *separating*, i.e. for each s, t in I, $s \neq t$, there is m such that s and t are in different members of Π_m.

For a given partition Π, let v_Π denote the finite game derived from v, whose players are the members of Π, namely

$$v_\Pi(\Lambda) \equiv v\left(\bigcup_{B \in \Lambda} B\right)$$

for all $\Lambda \subset \Pi$.

Let $T \in \mathcal{C}$ and let $\mathcal{P} = \{\Pi_m\}$ be an admissible sequence of partitions whose first term is $\Pi_1 = \{T, I \setminus T\}$. For each m, let $T_m = T_{\Pi_m}$ be the coalition corresponding to T in v_{Π_m}, namely $T_m = \{B \in \Pi_m \mid B \subset T\}$. Let ϕv_{Π_m} denote the Shapley value of the finite game v_{Π_m}. If the numbers $(\phi v_{\Pi_m})(T_m)$ approach a limit as $m \to \infty$, and this limit is independent of the sequence $\mathcal{P}$, then it will be denoted by $(\phi v)(T)$. It $(\phi v)(T)$ exists for all $T \in \mathcal{C}$, then the function ϕv will be called the *asymptotic value* of v.

In view of the non-existence of the asymptotic value in some cases of interest (see Section 4), we will also consider *generalized asymptotic values*, where the limit of $(\phi v_{\Pi_m})(T_m)$ should exist and be the same *for all* $\mathcal{P}$ *in a given class of admissible sequences.* Examples of such values are the measure based values, to be defined in Section 5. The reason for this definition is that the Value Theorem holds for *any* generalized asymptotic value (and not only for the usual one), as shown in Section 3. A last immediate remark is that the asymptotic value exists if and only if all generalized asymptotic values are identical.

In what regards the non-transferable utility case, we will use the following Nash-Harsanyi-Shapley procedure [cf. *Harsanyi; Shapley* 1969; *Aumann*, 1975].

Let $U = \{u_t\}_{t \in I}$ be a family of utility functions, representing the given preferences $\{\succ_t\}_{t \in I}$, namely,

$$u_t(x) > u_t(y) \text{ if and only if } x \succ_t y, \tag{2.15}$$

for all t in I, x and y in Ω. If U also satisfies (2.5), a transferable utility market $v \equiv v_U$ can be defined by (2.6) (note that (2.2), (2.3) and (2.4) are implied by (2.10), (2.11) and (2.12), respectively).

An allocation $\underset{\sim}{x}$ is called a *(generalized) asymptotic value allocation* if there exists a family U of utilities, satisfying (2.5) and (2.15), such that v_U has a (generalized) asymptotic value ϕv_U, and

$$(\phi v_U)(S) = \int_S u_t(\underset{\sim}{x}(t))\, d\mu(t),$$

for all S in C.

For discussions of this approach, the reader is referred to the above noted papers of *Shapley* [1969] and *Aumann* [1975].

3. The Value Theorem

The results in this section are the same as those in *Hart* [1977b], extended here to the generalized asymptotic values (using essentially the same proofs).

We start with the monetary markets.

Theorem 3.1: Let v be a market game arising from a non-atomic economy with transferable utility, satisfying (2.1) – (2.5). Let ϕv be a generalized asymptotic value. Then ϕv is a competitive payoff distribution.

Proof: By Proposition (3.4) in *Hart* [1977b], v belongs to H'_+. Applying now Proposition (7.1) and (5.4) in *Hart* [1977a], we get

$$(\phi v)(T) = \lim_{m \to \infty} (\phi v_{\Pi_m})(T_m) \geqslant \partial v^*(\chi_I; \chi_T) \geqslant v(T),$$

from which it follows that ϕv belongs to the core of v, hence by Proposition 2.9 (iii) it is a competitive payoff distribution.

The second part of the Value Theorem assumes differentiability. The following theorem is actually stronger than the asymptotic part of the results of *Aumann/Shapley* [1974] (see Table 1).

Theorem 3.2: Let v be a market game arising from a non-atomic economy with transferable utility, satisfying (2.2) – (2.5) and

(3.3) for every t in I and $1 \leqslant j \leqslant l$, $\partial u_t(x) / \partial x^j$ exists at every x in Ω with $x^j > 0$.

Then the set of competitive payoff distributions and the set of (generalized) asymptotic values of v concide, and consist of one element only.

Proof: Theorem D in *Hart* [1977b].

We come now to the non-transferable utility case.

Theorem 3.4: In a non-atomic Walrasian market satisfying (2.1) and (2.10) – (2.12), every generalized asymptotic value allocation is competitive.

Proof: The same proof as that of Theorem E in *Hart* [1977b], using Theorem 3.1 above instead of Theorem A there.

4. Existence of Asymptotic Value

Since the value for the Walrasian markets, by its definition, depends on the existence of value in the transferable utility case, we will deal in the next two sections with the latter only.

We start with some "positive" results.

Theorem 4.1: Let v be a market game arising from a non-atomic economy with transferable utility, satisfying (2.1) – (2.5). If there is a unique competitive payoff distribution, then v has an asymptotic value.

Proof: Theorem C in *Hart* [1977a], Proposition (3.4) in *Hart* [1977b], and Proposition 2.9 (iii).

The next theorem is a "generic existence theorem".

Theorem 4.2: Given utility functions $U = \{u_t\}_{t \in I}$ satisfying (2.2) – (2.5), let $A \equiv A_U$ be the set of all vectors a in Ω such that there is a transferable utility non-atomic market $(\underset{\sim}{a}, U)$ with $\int_I \underset{\sim}{a} = a$, for which the asymptotic value does not exist. Then A is a set of Lebesgue measure zero in Ω.

Proof: See the proof of Theorem C in *Hart* [1977b].

Remark 4.3: Actually, a stronger result is proved as Theorem C in *Hart* [1977b], namely, that the set of competitive payoff distributions coincides with the asymptotic value "almost everywhere" (which is defined in the same way as in the above Theorem 4.2).

However, in general, the asymptotic value need not exist. A necessary condition is given in the next theorem. For a subset X of a linear space, x_0 is a *center of symmetry* of X if, for every x in X, its symmetric image with respect to x_0, $2x_0 - x$, belongs also to X.

To avoid inessential complications, we will assume that the excess demand in equilibrium has full dimension, namely, that given a t.u.c.e. allocation $\underset{\sim}{x}$, the linear (affine) subspace $L(\underset{\sim}{x} - \underset{\sim}{a})$ of $\mathbf{R}^l$ spanned by all vectors $\int_S (\underset{\sim}{x} - \underset{\sim}{a})\, d\mu$, for all $S \in C$, has full dimension (i.e., dimension l). In case this is not satisfied, P (the set of equilibrium prices) should be replaced by its projection on $L(\underset{\sim}{x} - \underset{\sim}{a})$.

Theorem 4.4: Let v be a market game arising from a non-atomic economy with transferable utility satisfying (2.1) – (2.5). If v has an asymptotic value, then the set of competitive payoff distributions and the set P of equilibrium prices each have a center of symmetry.

Proof: Theorem B in *Hart* [1977b] (see also Added in Proof (2) there).

As an example where this condition is not satisfied (hence, there is no asymptotic value) – one can consider the "three-handed glove market" [cf. *Aumann/Shapley*, p. 203]. It should be also noted that the above condition is necessary but *not sufficient* [cf. example 8.1 in *Hart*, 1977a].

5. Measure-Based Values

In order to get a value for all market games, we will define in this section a specific generalized asymptotic value [see *Hart*, 1978].

First, let us consider the reasons for the non-existence of the asymptotic value. The definition requires that for *all* admissible sequences, the limit of the Shapley values of the corresponding finite games should exist, and be *independent* of the particular sequence chosen. In all the cases studied in this context, admissible sequences with different limits have been constructed. A deeper look reveals that all those partitions consisted of one (small) set which was "much bigger" than all the others. For example, consider the partition of $[0, 1]$ into one interval of length $1/n$ and $n(n-1)$ intervals of length $1/n^2$, and let $n \to \infty$ (to ensure that the partitions get "finer", one can take $n = 2^m$ and $m \to \infty$).

In case the only "data" is a game v, there is nothing that can make the above sequence of partitions "inadmissible". However, when one is considering an economy, and v is the derived "market game", additional data is given – namely, an underlying "population measure" μ. E.g., assume $[0, 1]$ to be the set of traders, and μ the Lebesque measure. Then the sequence of partitions described above does not seem to be a good approximation of the traders' space (some of them being always given much more weight than others!).

This indicates the way to proceed in order to guarantee the existence of the value. It was first used in *Aumann/Kurz* [1977], by restricting the admissible partitions to have all their elements equal in μ-measure. Here we adopt a slightly more "liberal" approach, requiring the measure of the elements of the partitions to get "close" one to another as $m \to \infty$.

Formally, given a measure μ on (I, C), a sequence $P = \{\Pi_m\}_{m=1}^{\infty}$ of partitions is called *μ-admissible* if it is admissible (i.e., satisfies (2.13) and (2.14)), and furthermore

$$\lim_{m \to \infty} \frac{\max_{B \in \Pi_m} \mu(B)}{\min_{B \in \Pi_m} \mu(B)} = 1. \tag{5.1}$$

The generalized asymptotic value corresponding to the class of all μ-admissible sequences is called *μ-based value,* or *μ-value.*

In order to guarantee existence of the μ-value for a market game, we need one further assumption, which can be interpreted as an added "competitiveness" requirement: that the variance of the excess demand, in equilibrium, should be finite. Intuitively, this implies that no arbitrarily "small" group of traders can have an arbitrarily "large" excess demand (recall that the total – hence, average – excess demand in equilibrium is zero). Usually, all allocations will be bounded, therefore this assumption will be surely satisfied.

We can now state our main result. As in Section 4, we will make the simplifying assumption that $L(\underset{\sim}{x} - \underset{\sim}{a})$ has full dimension (see Theorem 4.4; in the degenerate case, replace P by proj $_{L(\underset{\sim}{x}-\underset{\sim}{a})}P$ and $\mathbf{R}^l$ by $L(\underset{\sim}{x} - \underset{\sim}{a})$).

Theorem 5.2: Let v be a market game arising from a non-atomic economy with transferable utility, satisfying (2.1) – (2.5). Let P be the set of all equilibrium prices, and let $\underset{\sim}{x}$ be a t.u.c.e. allocation, such that

$$\int_I (\underset{\sim}{z}^j)^2 \, d\mu < \infty, \text{ for all } 1 \leqslant j \leqslant l, \qquad (5.3)$$

and $L(\underset{\sim}{z})$ has full dimension, where $\underset{\sim}{z} \equiv \underset{\sim}{x} - \underset{\sim}{a}$.

Then v has a μ-value ϕv, which coincides with a competitive payoff distribution ν_{p^*}. The price vector p^* in P is given by

$$p^* = \int_{\mathbf{R}^l} p(z) \, dN(z), \qquad (5.4)$$

where

(5.5) $p(z)$ maximizes $p \cdot z$ over all $p \in P$, for all z in $\mathbf{R}^l$, and N is the normal probability measure on $\mathbf{R}^l$ with the same first and second moments as $\underset{\sim}{z}$, namely, with expectation vector $0 = \int_I \underset{\sim}{z} d\mu$, and covariance matrix

$$\Sigma = (\int_I \underset{\sim}{z}^j \cdot \underset{\sim}{z}^k d\mu)_{j,k=1}^l. \qquad (5.6)$$

Proof: *Hart* [1978].

For a more detailed discussion of this theorem and its conditions, the reader is referred to *Hart* [1978, Section 3]; it also includes a set of assumptions on $\underset{\sim}{a}$ and $\{u_t\}_{t \in I}$ implying (5.3).

The theorem also raises the following interesting question: what is the equilibrium price p^* that corresponds to the μ-value (p^* is called: *value price*)?

The first observation is that in this model, the set P of all competitive prices is determined by *macro-economic considerations only*. Indeed, one needs to know *only* the aggregated utility function[11] u_I and the *aggregated* (initial) supply $\int_I \underset{\sim}{a}$, the competitive prices being then exactly the set of super-gradients (i.e., supporting hyperplanes) of u_I at $\int_I \underset{\sim}{a}$ [see *Hart*, 1978, Corollary 6.19]. All this data is not only "agent

[11]) For a in Ω, $u_I(a) = \max \{\int_I u_t(\underset{\sim}{x}(t)) \, d\mu(t) \mid \int_I \underset{\sim}{x} = a \text{ and } \underset{\sim}{x}(t) \in \Omega \text{ for all } t \in I\}$ [cf. *Aumann/Shapley*, p. 213].

free", but also "distribution free" – i.e., the utility function and the income (initial endowment) of any one trader are unspecified; furthermore, not even the distribution of those characteristics in the population need be given. In the case the competitive price is uniquely determined, no problem arises. But which price should be chosen when P is a "large" set? The value considerations point out one such price p^* – as the customary interpretation of this concept indicates, on grounds of "fairness" and "equity". Obviously, additional data – at the *microeconomic* level – will be needed for this purpose.

As in the statement of the theorem, let P be the set of all equilibrium prices, $\underset{\sim}{x}$ a (fixed) competitive allocation, and $\underset{\sim}{z} = \underset{\sim}{x} - \underset{\sim}{a}$.

Let S be a large random sample (coalition), drawn from the set of traders I. If the total supply of S, $\int_S \underset{\sim}{a}$, equals its total demand $\int_S \underset{\sim}{x}$, then every price vector p in P is also an "S-price", namely an equilibrium price for the economy formed by S. In general, however, $\int_S (\underset{\sim}{x} - \underset{\sim}{a}) = \int_S \underset{\sim}{z}$ will be small (by the Law of Large Numbers, since $\int_I \underset{\sim}{z} = 0$), but will *not* vanish. Therefore, the S-prices will be close to P; in fact, they will be close to those p in P maximizing $p \cdot \int_S \underset{\sim}{z}$, i.e., following our notation (5.5), to $p\,(\int_S \underset{\sim}{z})$. Mathematically, this is an easy consequence of the properties of super-gradients [cf. *Rockafellar*, Theorem 24.6]. Economically, it corresponds to a high price for commodities with a high excess demand, and a low price for those with a low excess demand. It can be also thought of as some kind of an auctioneer's rule in a tâtonnement process [cf. *Arrow/Hahn*, Chapter 11].

Let Z be the distribution of the excess demand $\underset{\sim}{z}$ in the population (i.e., Z is the probability measure on $\mathbf{R}^l$ defined by $Z = \mu \circ \underset{\sim}{z}^{-1}$). Then, if we choose *traders* at random, the distribution of their excess demand will be Z. However, if we choose *samples* (*coalitions*) at random, their aggregated excess demand will no longer be Z-distributed. By the Central Limit Theorem, we will get instead (with the adequate normalization) the normal distribution with the same first and second moments as those of Z – namely, N.

Combining these two results, and noting that the normalization does not matter, since $p\,(z) = p\,(\alpha z)$ for all $\alpha > 0$, we finally get: p^*, as defined by (5.4), *is the expected equilibrium price vector of the economy formed by a random subset* (*or, random sample*) *of the agents.*

A close look reveals that this interpretation actually follows from the value considerations. Indeed, let dt be a small trader, then his value, $(\phi v)\,(dt)$, is the expected incremental worth ("contribution") of dt to a large sample (coalition), picked at random from the population. Let S be such a sample, then the contribution of dt equals the utility of his allocation, minus its net cost, namely

$$[u_t\,(\underset{\sim}{x}^S\,(t)) - p^S \cdot (\underset{\sim}{x}^S\,(t) - \underset{\sim}{a}\,(t))]\,\mu\,(dt), \tag{5.7}$$

where $(p^S, \underset{\sim}{x}^S)$ is a competitive equilibrium in the economy formed by S. By the Law of Large Numbers, S is a "good representation" of the traders space I, hence $\underset{\sim}{x}^S$ will be close to $\underset{\sim}{x}$ (our fixed competitive allocation for the whole economy[12]). Taking expectation in (5.7) we get (in the limit)

$$(\phi v)(dt) = [u_t(\underset{\sim}{x}(t)) - E(p^S) \cdot \underset{\sim}{z}(t)]\,\mu(dt), \tag{5.8}$$

therefore the value payoff distribution is competitive, and the corresponding price p^* is precisely $E(p^S)$, i.e., the expected equilibrium price for a random sample (coalition).

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[12]) The non-uniqueness of $\underset{\sim}{x}$ is "inessential": by (2.7), $u_t(\underset{\sim}{x}(t)) - p \cdot (\underset{\sim}{x}(t) - \underset{\sim}{a}(t))$ does not depend on the particular t.u.c.e. allocation $\underset{\sim}{x}$ chosen. Therefore, only the non-uniqueness of the equilibrium price vectors matters in getting (5.8) from (5.7).