Chapter IX

Value Equivalence Theorems: The TU and NTU Cases

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1 Introduction

The study of large economic models, where agents are individually insignificant, has been a most interesting and fruitful subject. In particular, the application of various game theoretic solution concepts to these models has yielded important understandings. Most notably are the two well-known "principles": the *Core Equivalence* and the *Value Equivalence*. They show that Walrasian equilibria arise from completely new foundations.

The *Core Equivalence Theorem* states that in a perfectly competitive market, the core coincides with the set of competitive allocations.

We will not prove but only use this theorem here.

The *Value Equivalence Theorem* states that in a sufficiently differentiable perfectly competitive market, the set of value allocations coincides with the set of competitive allocations.

Originally defined for games with transferable utilities (TU) in coalitional (or characteristic) function form, the Shapley value, Shapley [1953], assigns a payoff to each player in each game. It turns out to equal the expected marginal contribution of the player to a randomly formed coalition. Using an asymptotic approach, Shapley [1964] showed that the set of value allocations of a TU market game coincides with the set of core allocations, provided that the coalitional function was assumed to be differentiable. Going directly to the case of a continuum of players, Aumann-Shapley [1974] showed the same result. Champsaur [1975] showed that in the asymptotic case every value allocation is in the core, without assuming differentiability; Hart [1977] proved the same result for the case of a continuum.

Now, quite clearly, the TU case is not appropriate when we consider economies or market games since it implies the possibility of interpersonal comparisons of utility. A solution concept, similar to the Shapley value but for the case of non-transferable utility (NTU) games, was provided by Harsanyi [1959, 1963]. The Harsanyi solution turned out to be quite complex and Shapley [1969] proposed an alternative solution concept, the Shapley

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NTU value (or λ-transfer value) which was easier to work with.

Assuming differentiability and a continuum of traders, Aumann [1975] showed that the set of Shapley *NTU* value allocations coincides with the set of core allocations and hence with the set of Walrasian allocations. Mas-Colell [1977] proved the same result in the asymptotic case. Once again, Champsaur [1975] and Hart [1977] proved that without differentiability the inclusion obtained in the *TU* case (every value allocation is also a core allocation) continues to hold. Note that all the results require the consideration of perfectly competitive economies, i.e., a large number of agents.

Table 1 summarizes these results.

Table 1

	ASYMP	*CONTINUUM*	
	Shapley [1964]	Aumann and Shapley [1974]	Differentiable
TU	Champsaur [1975]	Hart [1977]	Non-diff.
	Mas-Colell [1977]	Aumann [1975]	Differentiable
NTU	Champsaur [1975]	Hart [1977]	Non-diff.

Recently, the possibility of obtaining equivalence results for the Harsanyi solution has been examined by Hart and Mas-Colell [1991a,b,c]. Using their potential function approach (introduced in Hart and Mas-Colell [1989]), they are able to provide conditions under which the Harsanyi solutions of large market games lie in the core (the case of "tight" Harsanyi solutions). However, it turns out that in general the Harsanyi solution and the core may yield completely different outcomes. For a discussion on this "non-equivalence", see Section 5 and Hart and Mas-Colell [1991c].

The rest of this chapter is organized as follows: In Section 2 we present our model and basic assumptions. Section 3 is devoted to the case of *TU* games; Section 4 to the Shapley *NTU* value; and Section 5 to the Harsanyi *NTU* value. It should be stressed that the "proofs" and arguments brought here are informal, and meant only to suggest some of the basic ideas. For a precise treatment see the referred papers.

2 The Model

As we saw in the Introduction, there are various ways to model "large markets". One is the asymptotic approach — sequences of finite games with increasing number of players. The other is to consider the limit game with a continuum of players. It turns out that the most convenient model is that of a continuum game with finitely many types of players; some of the more complex technical points are easily handled in this model, and the results are the same.

The basic model consists of a *non-transferable utility, NTU*, economy with a *continuum* of agents. There are finitely many types of agents. It will suffice to describe the resulting *(market) game.*

Each *coalition* is characterized by its composition, namely how many players of each type it contains. Let n be the number of types. The *profile* of a coalition is a vector $x = (x_1, ..., x_n)$ in $\mathbb{R}^n_+$, where x_i is the measure (or, mass) of players of type i in the coalition.

The basic data consists of specifying the sets of feasible payoff vectors for all coalitions. We consider only *type-symmetric* imputations, where all players of the same type get the same payoff. For every profile $x \in \mathbb{R}^n_+$, let thus $V(x) \subset \mathbb{R}^n$ be the set of *feasible per-capita payoff vectors* for a (coalition with) profile x. That is, $a = (a_1, ..., a_n) \in V(x)$ whenever it is feasible that each one of the x_i players of type i will get a payoff a_i (simultaneously for all i). We assume for convenience that the coordinates of a that correspond to types which are not present in x are arbitrary; more precisely, if $x_i = 0, a \in V(x)$ and $a'_j = a_j$ for all $j \neq i$, then $a' \in V(x)$ too. This (set-valued) function V is called the *coalitional* (or *characteristic*, or *worth*) *function* of the game.

A *non-transferable utility (NTU) game* $(\overline{x}, V)$ is obtained by specifying, in addition to its coalitional function V, also the *grand coalitional profile* $\overline{x} \in \mathbb{R}^n_+$. We will assume without loss of generality that $\overline{x}_i > 0$ for all $i = 1, ..., n$ (i.e. $\overline{x} \in \mathbb{R}^n_{++}$), since types whose total mass is zero may obviously be dropped.

We will find it useful to consider also *per-type* payoffs. For z and $w \in \mathbb{R}^n$, let $z * w \in \mathbb{R}^n$ denote the vector whose i-th coordinate is $z_i w_i$. We then define for every $x \in \mathbb{R}^n_+$ the set of *feasible total per-type payoffs* by $\hat{V}(x) := \{x * a \mid a \in V(x)\}$. Thus $b \in \hat{V}(x)$ whenever, for each i with $x_i > 0$, the total payoff of all x_i players of type i is b_i; i.e., each one gets b_i/x_i (note that $b_i = 0$ when $x_i = 0$).

The special case where utilities are actually transferable – the game is then called a *transferable utility (TU)* game – corresponds to $V(x) = \{a \mid \sum_i x_i a_i \leq v(x)\}$ and $\hat{V} = \{b \mid \sum_i b_i \leq v(x)\}$, where $v : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ is the *TU* coalitional function.

We make the following assumptions on the game $(\overline{x}, V)$:

A1. Basic: For every $x \in \mathbb{R}^n_+$, the set $V(x)$ is a strict subset of $\mathbb{R}^n$, it contains 0, it is closed, convex and comprehensive (the last one means that if $a \in V(x)$ and $a' \leq a$, then $a' \in V(x)$ too).

A2. Super-additivity: $\hat{V}(x+y) \supset \hat{V}(x) + \hat{V}(y)$ for every $x, y \in \mathbb{R}^n_+$. (Recall that $\hat{V}(x)$ is the set of total per-type payoffs).

A3. Non-levelness: The feasible set $V(\overline{x})$ of the grand coalition is non-level, i.e. $a, a' \in \text{bd } V(\overline{x})$ and $a \leq a'$ implies $a = a'$.

A4. Homogeneity: V is positively homogeneous of degree 0; i.e., $V(tx) = V(x)$ for every $x \in \mathbb{R}^n_+$ and $t > 0$. (Recall that $V(x)$ is the set of feasible *per-capita* payoff vectors; in terms of *total* payoffs, $\hat{V}(tx) = t\hat{V}(x)$).

Assumptions A1 and A3 are standard (and mainly technical; see Hart & Mas-Colell [1991a] for an extensive discussion). Super-additivity A2, and in particular the homogeneity A4, are the essential properties of market games. It is easy to check that a pure exchange economy where, for each type i, the utility function u_i is concave and non-decreasing, with slope (i.e., any super-gradient) everywhere bounded away from 0 and infinity, and $u_i(\omega_i) = 0$ (an irrelevant normalization; ω_i is the initial endowment), will yield a market game such that A1-A4 will be satisfied.

Note that some of the results discussed below may require further hypotheses.

We end this section with a well-known definition. The **core** of the game $(\overline{x}, V)$consists of all payoff vectors $a \in \mathbb{R}^n$ such that:

$$a \in bd\ V(x); \quad \text{and} \tag{1}$$

$$a \notin int\ V(x) \quad \text{for all } x \leq \overline{x}. \tag{2}$$

That is, a is feasible and efficient for the grand coalition, and it cannot be improved upon by any coalition.

3 The *TU* Case

In this section we are going to prove the *Value Equivalence Theorem* in the *TU* case. As defined above, $(\overline{x}, v)$ is a *TU* game with a continuum of players of n types. Conditions A2 and A4 become

(*) v is super-additive; i.e., $v(x+y) \geq v(x) + v(y)$ for every $x, y \in \mathbb{R}^n_+$.

(*) v is positively homogeneous of degree 1; i.e. $v(tx) = tv(x)$ for every $x \in \mathbb{R}^n_+$ and $t \geq 0$.

Therefore v is concave. The graph of $v(\cdot)$ is composed of rays from the origin of $\mathbb{R}^{n+1}$ which lie in the non-negative orthant and are "patched" together so as to form a concave surface.

Let $\overline{a}$ $(\in \mathbb{R}^n) \in core(\overline{x}, v)$, then by efficiency $\sum_{i=1}^n \overline{x}_i \overline{a}_i = v(\overline{x})$. Moreover $\sum_{i=1}^n x_i \overline{a}_i \geq v(x)$ for every $x \leq \overline{x}$. Thus an element of the core of $(\overline{x}, v)$ corresponds to a supporting hyperplane to $v(.)$ at $\overline{x}$. If v is differentiable near $\overline{x}$, then $|\, core \,| = 1$ and $core = \{\nabla v(\overline{x})\}$.

We now turn to the Shapley value for the *TU* game. As indicated in the introduction, the value can be interpreted as giving to each player the expected marginal contribution that the player makes to randomly formed coalitions. With a continuum of players, a randomly chosen coalition Q will, with high probability, look like the grand coalition rescaled by the size of Q. Also, if $v(x)$ is differentiable in a neighborhood of $\bar{x}$, then the marginal contribution of a player will be given by the derivative of the coalitional function which, by positive linear homogeneity, is invariant to positive rescaling. It follows that $\nabla v(\bar{x})$ is precisely the Aumann-Shapley [1974] *TU* value, i.e.,

$$\varphi(\bar{x}, v) = \nabla v(\bar{x}).$$

So we have our equivalence result in the differentiable case (recall that, by the Core Equivalence Theorem, the core coincides with the set of Walrasian allocations).

Even if the coalitional function is not differentiable at $\bar{x}$, the set of core allocations is still given by the convex set of supporting normals to the graph of $v(\cdot)$ at $\overline{x}$. For the value, one computes the marginal contributions based on the appropriate hyperplane (corresponding to the random Q) and then takes the average. But since each of the points that we are averaging is in the core (since each corresponds to a normal to a plane that supports the graph of the function $v(\cdot)$ and therefore corresponds to a payoff vector in the core) the average is also in the core (the set being convex). Therefore, we have shown that in the non-differentiable case every value allocation is a core allocation.

Back to the differentiable case, we now provide an alternative proof of the *TU* value equivalence theorem using the potential function approach of Hart and Mas-Colell [1989]. A potential for the game $(\bar{x}, v)$ is a differentiable function $P : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ such that for any x we have $x \cdot \nabla P(x) = v(x)$ and $P(0) = 0$. It is shown that these conditions define a unique potential for each game and that the gradient of the potential corresponds to the Aumann-Shapley value. Since v is homogeneous of degree one, we have $x \cdot \nabla v(x) = v(x)$

for all x (from Euler's theorem) so that our condition for a function to be the potential for the game (x, v) — namely, $x \cdot \bigtriangledown P(x) = v(x)$ — implies that $P(x) = v(x)$ for all x, so that $\bigtriangledown P(x) = \bigtriangledown v(x)$ and we have the result that every value allocation is a core allocation.

4 The *NTU* Case: Shapley Value

Let us consider the Shapley [1969] *NTU* value. It is defined as follows.

Given a vector of weights, $\lambda \in \mathbb{R}^n_{++}$, define a *TU* game by:

$$v_\lambda(x) := \max\{\Sigma\lambda_i x_i a_i \mid a \in V(x)\}.$$

Find the Shapley *TU*-value of the game $(\bar{x}, v_\lambda)$. If the resulting payoff vector is on the boundary of $V(\bar{x})$ rescaled by λ, then we have obtained a Shapley *NTU*-value. That is, a payoff vector $\bar{a} \in bd\ V(\bar{x})$ is a Shapley *NTU*-value if there exists a vector of weights λ such that $\lambda_i \bar{a}_i = Sh_i(\bar{x}, v_\lambda)$ for all $i = 1, \ldots, n$. The weights λ are precisely the marginal rates of efficient substitution between the payoff of the various (types of) players, at the solution point $\bar{a}$.

To show that every value allocation is a competitive allocation, let $\overline{a}$ be a value allocation. Without loss of generality, we assume $\lambda = (1, ...1)$; otherwise we may just rescale all payoff vectors by λ. Thus, $\overline{a} \in bdV(\overline{x})$. We know that $\overline{a}$ is the Shapley value of the *TU* game $(\overline{x}, v)$, where $v(x) = \max\{\sum_i x_i a_i \mid a \in V(x)\}$. Hence by the proof in the *TU* case, we have $\overline{a} = \varphi(\overline{x}, v) \in core(\overline{x}, v) = \bigtriangledown^* v(\overline{x})$ (where $\bigtriangledown^*$ is the Clarke [1983] generalized gradient). Therefore $x \cdot \overline{a} \geq v(x)$ for all $x \in \mathbb{R}^n_+$, implying that $\overline{a} \notin intV(x)$, for all $x \in \mathbb{R}^n_+$. Therefore $\overline{a} \in core(\overline{x}, V)$, which implies that $\overline{a}$ is competitive.

5 *NTU* Case: Harsanyi Value

A Harsanyi value of a (finite) game is defined as a payoff vector which is simultaneously "egalitarian" and "utilitarian". The former is a generalization to many players of the "equal-split" allocation for two players; the latter means that the sum of the utilities is maximized. In order for these to be both satisfied, one may (independently) rescale the utility scales of the various players.

We define now these notions in our setup. First, to define an egalitarian allocation, we use the notion of a potential function, introduced in Hart & Mas-Colell [1989] and studied extensively for games with a continuum of players in Hart & Mas-Colell [1991a]. A real-valued function $P \equiv PV : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ is the **(smooth) potential function** of V if

$$\text{P(0)=0;} \tag{3}$$

$$P \text{ is continuously differentiable } (C^1) \text{ on } \mathbb{R}^n_+ \setminus \{0\} \text{ and continuous at } 0; \tag{4}$$

$$\bigtriangledown P(x) \in bd\ V(x) \quad \text{for every } x \in \mathbb{R}^n_+ \setminus \{0\}. \tag{5}$$

The payoff vector $a \in \mathbb{R}^n$ is **egalitarian** for the game $(\overline{x}, V)$ if

$$\text{the potential } P \text{ of } V \text{ exists and } a = \bigtriangledown P(\overline{x}). \tag{6}$$

As we see in (5), an egalitarian solution is obtained from a solution to the *partial differential equation* (*PDE*) $\nabla P(x) \in bd\ V(x)$, which is usually nonlinear (the linear case corresponds to a transferable utility game, and the theory there is indeed relatively simple and coincides with Aumann-Shapley [1974]).

Unfortunately, even for V extremely well behaved, the *PDE* equation may have no solution. One is therefore led to consider **generalized (Lipschitz) solutions**, i.e., functions P that are Lipschitz (thus differentiable almost everywhere — by Rademacher's Theorem) and satisfy the *PDE* almost everywhere:

$$\nabla P(x) \in bd\ V(x) \qquad \text{for almost every } x \in \mathbb{R}^n_{++}. \tag{7}$$

As we will see below, generalized solutions do indeed exist. However, there may be many such solutions, and we would like to be able to select the "right one".

To do so, one needs to recall that these continuous games and economies are actually an idealization of situations with finitely many participants, each one individually insignificant. Therefore, the question of which generalized *PDE* solution is the right one can only be settled by considering sequences of finite approximations.

We come now to the informal statement of our main results. For ease of exposition we skip the statement of the assumptions (see Hart and Mas-Colell [1991a]).

For every $x \in \mathbb{R}^n_{++}$ and $p \in \mathbb{R}^n$, define the support function $v(x,p) := \sup\{p \cdot a \mid a \in V(x)\}$. Consider the *Variational Problem* (where a.c. stands for absolutely continuous):

$$P(x) := \inf\left\{\int_0^1 v(\mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt \;\middle|\; \mathbf{x}\colon [0,1] \to \mathbb{R}^n_{++} \cup \{0\} \text{ a.c., } \mathbf{x}(0) = 0 \text{ and } \mathbf{x}(1) = x\right\} \tag{8}$$

Theorem 1 *The function P in (8) is well defined. Moreover*

1. *P is a solution of (7);*
2. *$P \geq Q$ for any solution Q of (7) with $Q(0) = 0$;*
3. *If (7) admits a differentiable (C^1) solution, then it must coincide with P; and*
4. *P fails to be differentiable at x if and only if problem (8) has more than one solution.*

We will thus call the function P (given by (8)), the **variational potential**. It is the maximal potential, and, in those cases where a differentiable potential exists, P is that potential (hence, there may be at most one differentiable solution to (7)). We are also able to show that for hyperplane games (i.e., games where $v(x,p) < \infty$ for a single p, which may vary with x) a differentiable solution exists with great generality. As a matter of interpretation this means that the lack of differentiability of P, which is a robust phenomenon, is essentially related to the *NTU* character of the game.

We come now to the convergence results, that show that the variational potential is indeed the right one. For every positive integer r, we define V_r (the *r-approximation* of V) by replacing each mass of $1/r$ of each type with a single player. Let P_r denote the potential for the finite game V_r and let P be the variational potential for V. The results are:

Theorem 2 $\frac{1}{r} P_r(x) \underset{r \to \infty}{\longrightarrow} P(x)$ *for every* $x \in \mathbb{R}^n_{++}$.

Theorem 3 *If P is differentiable at $\overline{x}$, then $DP_r \longrightarrow \nabla P(\overline{x})$ as $r \longrightarrow \infty$; otherwise $DP_r \longrightarrow (\nabla^* P(x) + \mathbb{R}^n_+) \cap bd\ V(\overline{x})$. Here DP_r stands for the vector of normalized finite (discrete) differences of P_r, and $\nabla^* P(x) :=$ convex hull $\{ \lim_{m\to\infty} \nabla P(x_m) : P$ is differentiable at x_m and $x_m \underset{m\to\infty}{\longrightarrow} x \}$ is the Clarke [1983] generalized gradient.*

Thus, when P is not differentiable, we obtain in the limit points that are $\geq$ some generalized gradient of P at $\overline{x}$ (and, of course, lie on the boundary $bd\ V(x)$).

We will therefore define the *egalitarian solution* of $(\overline{x}, V)$ as

$$Eg(\overline{x}, V) := (\nabla^* P(\overline{x}) + \mathbb{R}^n_+) \cap bd\ V(\overline{x}),$$

where P is the potential function. Thus, if P is differentiable at $\overline{x}$, then there will be just one point in $Eg(\overline{x}, V)$, namely $\nabla P(\overline{x})$.

Next, a payoff vector $a \in \mathbb{R}^n$ is **utilitarian** for the game $(\overline{x}, V)$ if it maximizes the sum of the utilities over the feasible set $V(\overline{x})$ of the grand coalition; i.e.

$$a \in V(\overline{x}); \quad \text{and} \tag{9}$$

$$\sum_{i=1}^{n} \overline{x}_i a_i \geq \sum_{i=1}^{n} \overline{x}_i a'_i \quad \text{for all } a' \in V(\overline{x}). \tag{10}$$

(recall that a_i is the payoff of each player of type i, and there are $\overline{x}_i$ many of them).

Now let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_{++}$ be a **vector of weights**. We denote by V_λ the λ-rescaling of V; i.e., $V_\lambda(x) := \{\lambda * a \mid a \in V(x)\}$. A payoff vector $a \in \mathbb{R}^n$ is **λ-egalitarian** (respectively, **λ-utilitarian**) for the game $(\overline{x}, V)$ if it is egalitarian (respectively, utilitarian) for the game $(\overline{x}, V_\lambda)$. This can be easily translated back to the original game, as follows: $P_\lambda \equiv PV_\lambda : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ is the λ-potential function of V if it satisfies:

$$P_\lambda(0) = 0; \tag{11}$$

$$\nabla P_\lambda(x) \in bd\ (\lambda * V(x)) \equiv \lambda * bd\ V(x) \quad \text{for almost every } x \in \mathbb{R}^n_+. \tag{12}$$

The payoff vector $a \in \mathbb{R}^n$ is then **λ-egalitarian** for the game $(\overline{x}, V)$ if

$$\lambda * a = \nabla P_\lambda(\overline{x}). \tag{13}$$

The payoff vector $a \in \mathbb{R}^n$ is **λ-utilitarian** if it satisfies: (9) $a \in V(\overline{x})$; and

$$\sum_{i=1}^{n} \lambda_i \overline{x}_i a_i \geq \sum_{i=1}^{n} \lambda_i \overline{x}_i a'_i \quad \text{for all } a' \in V(\overline{x}). \tag{14}$$

Finally, a payoff vector $a \in \mathbb{R}^n$ is a **Harsanyi value** for the game $(\overline{x}, V)$ if there exists a weight vector $\lambda \in \mathbb{R}^n_{++}$ such that a is both λ-egalitarian and λ-utilitarian for $(\overline{x}, V)$; we will refer to the λ as the weight vector associated with a.

We will say that a Harsanyi value a is *tight* if the associated potential function is differentiable at $\overline{x}$ (and thus $a = \nabla P_\lambda(\overline{x})$).

Theorem 4 *Let $(\overline{x}, V)$ be a market game. Every tight Harsanyi value belongs to the core.*

However, this result does not hold in general. In Hart and Mas-Colell [1991c] we provide an example of a market game where there is a unique element in the core, which is not the Harsanyi solution (of course, we are in a non-"tight" case). Thus the *Value Equivalence Principle does not apply to the Harsanyi NTU-solution.* For a comprehensive discussion on these issues and their interpretations, see Hart and Mas-Colell [1991c].

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