The Query Complexity of Correlated Equilibria

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Paper

Paper

- Sergiu Hart and Noam Nisan
  The Query Complexity of Correlated Equilibria
  - Center for Rationality 2013
  - Revised September 2015

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Correlated Equilibria

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$\Longrightarrow$ There is an algorithm for computing **CORRELATED EQUILIBRIA** with **COMPLEXITY** $=$ **POLY**$(2^n) =$ **EXP**(n)

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BUT:

- **Regret-based** dynamics yield ϵ-**CORRELATED EQUILIBRIA** with high probability (Hart & Mas-Colell 2000, 2001; ...)

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- QUERY COMPLEXITY (QC) $:=$ maximal number of pure payoff QUERIES (out of $n \cdot 2^n$)

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- QUERY COMPLEXITY (QC) $:=$ maximal number of pure payoff QUERIES (out of $n \cdot 2^n$)

$\Rightarrow$ There are randomized algorithms for computing ϵ-CORRELATED EQUILIBRIA with QC $=$ POLY(n)

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Surprise ? ***... perhaps not so much ...***

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  - use Lipton and Young 1994
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NOTE: Regret-based dynamics converge ***as fast as possible*** (up to a constant factor)

Correlated Equilibria (recall)

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- ***exact*** **CORRELATED EQUILIBRIA** **?**

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- ***exact*** **CORRELATED EQUILIBRIA ?**

- ***deterministic*** algorithms **?**

Query Complexity of CE

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	Algorithm	
	Randomized	Deterministic
ε-CE		
exact CE		

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	Randomized	***Deterministic***
ε-***CE***	**POLY**(n) [1]	
exact CE		

$[1]$ = regret-based dynamics

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	Algorithm	
	Randomized	*Deterministic*
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exact CE		EXP(n) [2]

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[2] = Babichenko and Barman 2013

Query Complexity of CE

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[3] = this paper

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- **Theorem A.** *Every* **DETERMINISTIC** *algorithm that finds a* $1/2$*–approximate correlated equilibrium in every* n*-person bi-strategy games with payoffs in* $\{0, 1\}$ *requires* $2^{\Omega(n)}$ **QUERIES** *in the worst case.*

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- **Theorem B.** *Every algorithm (randomized or deterministic) that finds an* **EXACT** *correlated equilibrium in every n-person bi-strategy games with payoffs specified as b-bit integers with $b = \Omega(n)$ requires $2^{\Omega(n)}$ expected* **COST** *in the worst case.*

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COST = # of **QUERIES** + size of support of output

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Query Complexity of COARSE CE

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When every player has 2 strategies:

COARSE CORRELATED EQUILIBRIUM =

CORRELATED EQUILIBRIUM

Idea of Proof of Theorem A

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- Each edge is labelled with the regret of the player whose strategy changes
- A query at node v provides the n regrets of all edges adjacent to v
- If the number of queries is $2^{\Omega(n)}$ then we can make the sum of the queried regrets high so that no $1/2$-approximate correlated equilibrium is found within the queried nodes (use the edge iso-perimetric inequality)

Idea of Proof of Theorem B

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- Take a random path in the hypercube and define the regrets so that in order to get an exact correlated equilibrium one must find the endpoint of the path
- To find the endpoint one must essentially follow the path (because every $n \log(n)$ steps there is "full mixing"), which requires $2^{\Omega(n)}$ queries

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Complexity of CE

- **VERIFICATION** of **CORRELATED EQUILIBRIUM** with support of size **POLY**(n) is **POLY**(n)
- **QUERIES** of ***mixed*** strategies: only **POLY**(n) are needed
  - Papadimitriou and Roughgarden 2008
  - Jing and Leyton-Brown 2011

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  randomized algorithms for ***approximate*** solutions

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  - ***algorithm***:
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- **QUESTION**:
  Why does this help **ONLY** for ***randomized*** algorithms for ***approximate*** **CE** ?

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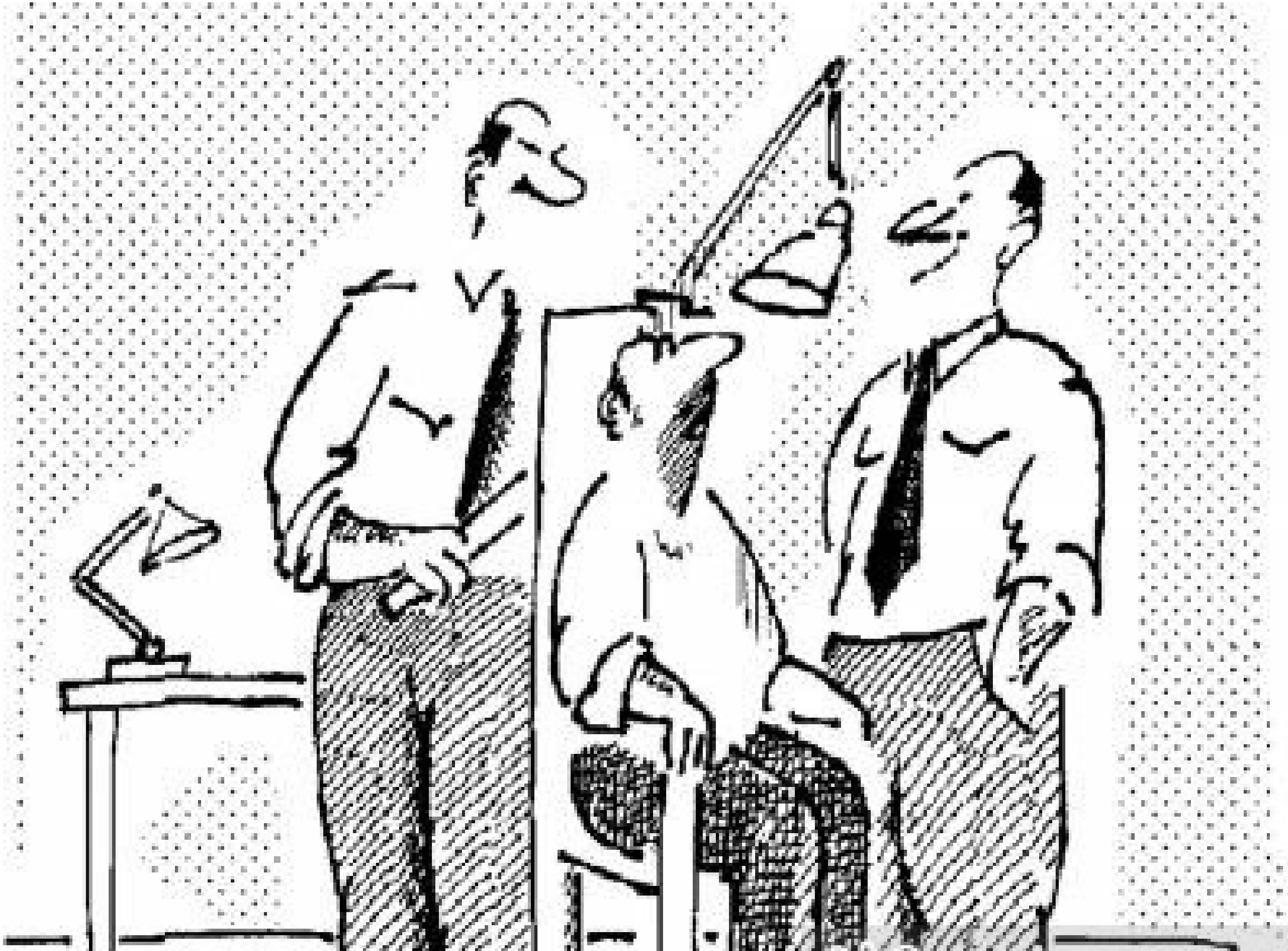
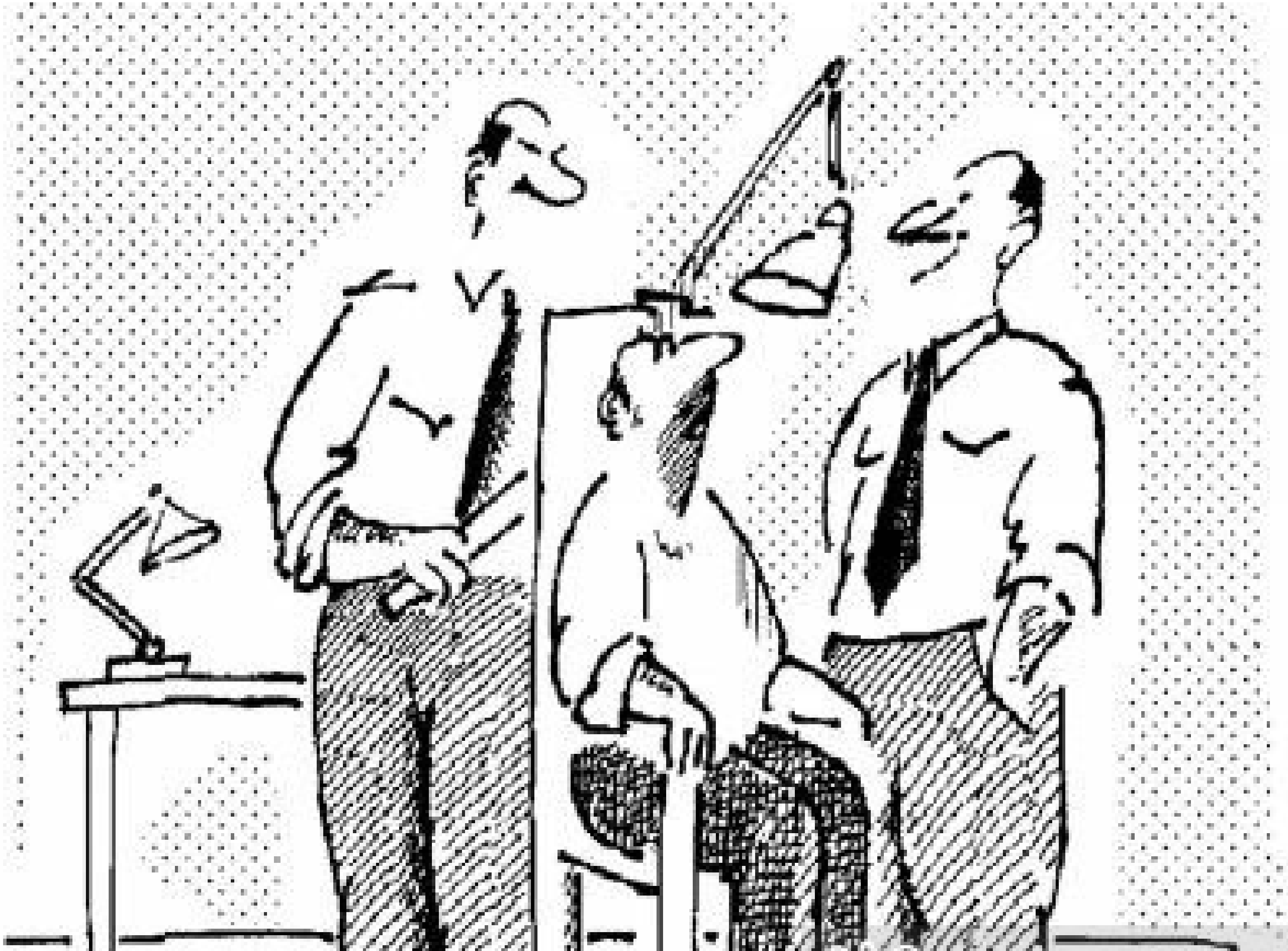
- **QUESTION**:
  Complexity of approximate ***Nash Equilibria*** ?





"Police brutality is a thing of the past, mate, these days we apply structured query language!" © BART