The Number of Commodities Required to Represent a Market Game*

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An n -person game with side payments that can arise from an economic market is called a market game. It is proved here that any n -person market game can be obtained from a market with at most $n - 1$ commodities. Moreover, no smaller number will suffice in some cases (e.g., the unanimity game). This result is then applied to settle a question regarding market games with a continuum of traders. *Journal of Economic Literature* Classification Numbers: 021, 022.

The following problem is studied here: What is the minimal number of commodities required to represent a totally balanced n -player game (with side payments)?¹ Shapley and Shubik [4] showed that every such game is a market game—i.e., it can be derived from a “market”; their construction (of the “direct market”) requires n commodities. It is proved here that $n - 1$ suffice (Proposition 2); moreover, this bound is exact—there are n -player market games for which $n - 1$ goods are necessary (Proposition 1). This last result is then applied to non-atomic games (where the original question—settled here as Proposition 3—was raised).

The n -player unanimity game is defined as follows: $v(S) = 0$ for all $S \subsetneq N$ and $v(N) = 1$, where $N = \{1, 2, \dots, n\}$. It is clearly a totally balanced game; a market which generates it, is for example,² $(N, \mathbb{R}_+^{n-1}, A, U)$, where, for $i \neq n$,

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¹ For the non-side-payments case, where the problems are not yet completely settled, see [6], which includes an overview and references.

² The notations of Shapley and Shubik [4] are followed here. N is the set of agents, $\mathbb{R}_+^m$ the commodity space, $A = \{a^i\}_{i \in N}$ the initial endowments and $U = \{u^i\}_{i \in N}$ continuous concave utility functions (not necessarily non-decreasing). The market game v is defined by $v(S) = \max\{\sum_{i \in S} u^i(x^i) \mid \sum_{i \in S} x^i = \sum_{i \in S} a^i, \text{ all } x^i \in \mathbb{R}_+^m\}$ for all $S \subset N$.

a^i is³ the i th unit vector in $\mathbb{R}^{n-1}$ and $u^i \equiv 0$; for $i = n$, $a^n = 0$ and $u^n(x_1, x_2, \dots, x_{n-1}) = \min\{x_1, x_2, \dots, x_{n-1}\}$.

PROPOSITION 1. *Let $(N, \mathbb{R}_+^m, A, U)$ be a market generating the unanimity game v on N . Then $m \geq n - 1$.*

Proof. Since $v(\{i\}) = 0$, we have $u^i(a^i) = 0$; since $v(N \setminus \{i\}) = 0$, we have

$$0 = \max \left\{ \sum_{j \neq i} u^j(x^j) \mid \sum_{j \neq i} x^j = \sum_{j \neq i} a^j, \text{ all } x^j \in \mathbb{R}_+^m \right\},$$

hence the maximum is attained at $x^j = a^j$ for all $j \neq i$ ($\sum_{j \neq i} u^j(a^j) = \sum_{j \neq i} 0 = 0$).

Now $v(N) = 1$, hence there is an allocation $(y^i)_{i \in N}$ such that

$$\sum_{i \in N} y^i = \sum_{i \in N} a^i, \quad \sum_{i \in N} u^i(y^i) = 1.$$

Let us define now new concave functions $\hat{u}^i: \mathbb{R}_+^m \rightarrow [-\infty, +\infty)$ as follows:

$$\begin{aligned} \hat{u}^i(x) &= tu^i(a^i) + (1-t)u^i(y^i), & \text{for } x &= ta^i + (1-t)y^i, \\ & & \text{with } 0 &\leq t \leq 1, \\ &= -\infty, & \text{otherwise.} \end{aligned}$$

Then $\hat{u}^i$ are polyhedral concave functions (cf. Rockafellar [3, pp. 172–173]), and $\hat{u}^i \leq u^i$. Let $\hat{v}$ be the market game generated by $(N, \mathbb{R}_+^m, A, \hat{U})$, then $\hat{v} \leq v$; however, since $\hat{u}^i(a^i) = u^i(a^i)$ and $\hat{u}^i(y^i) = u^i(y^i)$, we actually have $\hat{v} = v$.

Now consider the functions $\hat{u}_S$, defined for $S \subset N$, as follows:

$$\hat{u}_S(x) = \sup \left\{ \sum_{i \in S} \hat{u}^i(x^i) \mid \sum_{i \in S} x^i = x, \text{ all } x^i \in \mathbb{R}_+^m \right\}.$$

Then $\hat{u}_S$ are polyhedral concave functions (cf. Rockafellar [3, Corollary 19.3.4]). In particular, for each $i \in N$, the function $\hat{u}_{N \setminus \{i\}}$ is subdifferentiable at $\sum_{j \in N \setminus \{i\}} a^j$ (cf. Rockafellar [3, Theorem 23.10]); i.e., there is $p^i \in \mathbb{R}^m$ such that

$$p^i \in \partial \hat{u}_{N \setminus \{i\}} \left(\sum_{j \in N \setminus \{i\}} a^j \right).$$

³ Subscripts will be used for coordinates, and superscripts for players.

But this clearly implies $p^i \in \partial \hat{u}^k(a^k)$ for all $k \neq i$; indeed,

$$\begin{aligned}\hat{u}^k(x) &= \sum_{j \neq i, k} \hat{u}^j(a^j) + \hat{u}^k(x) \leq \hat{u}_{N \setminus \{i\}} \left(\sum_{j \neq i, k} a^j + x \right) \\ &\leq \hat{u}_{N \setminus \{i\}} \left(\sum_{j \neq i} a^j \right) + p^i \cdot \left(\sum_{j \neq i, k} a^j + x - \sum_{j \neq i} a^j \right) \\ &= \hat{u}^k(a^k) + p^i \cdot (x - a^k).\end{aligned}$$

Hence $\{\partial \hat{u}^k(a^k)\}_{k=1}^n$ is a collection of n convex sets in $\mathbb{R}^m$, such that the intersection of every $n-1$ of them is non-empty; if $m \leq n-2$, Helly's theorem (Rockafellar [3, Theorem 21.6]) would imply the existence of a vector p with $p \in \partial \hat{u}^k(a^k)$ for all $k \in N$. Then

$$1 = \sum_{k=1}^n \hat{u}^k(y^k) \leq \sum_{k=1}^n [\hat{u}^k(a^k) + p \cdot (y^k - a^k)] = 0,$$

since $\hat{u}^k(a^k) = 0$ and $\sum_k y^k = \sum_k a^k$. This contradiction shows that $m \geq n-1$.

Remark. Note that each u^i could be just a concave function with values in $[-\infty, +\infty)$ (i.e., not necessarily finite on all $\mathbb{R}_+^m$).

PROPOSITION 2. *Let (N, v) be a totally balanced game. Then v can be derived from a market with $n-1$ goods. Moreover, the set of competitive payoff vectors of this market and the core of v coincide.*

Remark. In Shapley and Shubik [5], it is shown that the competitive vectors of the direct market—with n goods—enjoy the same property.

Proof. Consider the restriction of v to $N' = \{1, 2, \dots, n-1\}$. It is a totally balanced game of $n-1$ players; let $(N', \mathbb{R}^{n-1}, A', U')$ be the corresponding direct market. Define $a^n = 0$ and $u^n: \mathbb{R}_+^{n-1} \rightarrow \mathbb{R}$ by

$$u^n(x) = \max \left\{ \sum_{S \subset N'} \gamma_S v(S) \mid \sum_{S \subset N'} \gamma_S e^S = (x, 1), \text{ all } \gamma_S \geq 0 \right\},$$

where $(x, 1) = (x_1, x_2, \dots, x_{n-1}, 1) \in \mathbb{R}^n$, and $e^S \in \mathbb{R}_+^n$ is the characteristic vector of S (i.e., $e_i^S = 1$ or 0 according to $i \in S$ or $i \notin S$). It is easy to check that u^n is a continuous concave function. Denote by $\hat{v}$ the market game generated by $(N, \mathbb{R}_+^{n-1}, A, U)$.

We show now that $\hat{v} = v$. For $S \subset N'$, it is clear that $\hat{v}(S) = v(S)$. When $n \in S$, we have

$$\hat{v}(S) \geq u^n \left(\sum_{i \in S} a^i \right) + \sum_{i \in S'} u^i(0),$$

where $S' = S \setminus \{n\}$. Now $u^i(0) = 0$ and $(\sum_{i \in S} a^i, 1) = e^S$; therefore, choosing $\gamma_S = 1$ and $\gamma_T = 0$ for $T \neq S$ in the definition of $u^n(\sum_{i \in S} a^i)$, we obtain $\hat{v}(S) \geq v(S)$. Conversely, let $(x^i)_{i \in S}$ be an S -allocation (i.e., $\sum_{i \in S} x^i = \sum_{i \in S} a^i$). Then, for each $i \in S'$, we obtain $\gamma_T^i \geq 0$ for all $T \subset N'$ such that

$$u^i(x^i) = \sum_{T \subset N'} \gamma_T^i v(T),$$

$$\sum_{T \subset N'} \gamma_T^i e^T = (x^i, 0)$$

(see Shapley and Shubik [4, Eqs. (4-2) and (4-3)]). For $i = n$, we have $\gamma_T^n \geq 0$ for all $T \subset N$ such that

$$u^n(x^n) = \sum_{T \subset N} \gamma_T^n v(T),$$

$$\sum_{T \subset N} \gamma_T^n e^T = (x^n, 1).$$

Now let $\gamma_T = \sum_{i \in S} \gamma_T^i$ for $T \subset N'$, and $\gamma_T = \gamma_T^n$ for $T \ni n$; then

$$\sum_{T \subset N} \gamma_T e^T = \left(\sum_{i \in S} x^i, 1 \right) = \left(\sum_{i \in S} a^i, 1 \right) = e^S,$$

which implies

$$\sum_{T \subset N} \gamma_T v(T) \leq v(S),$$

since v is totally balanced. Hence

$$\sum_{i \in S} u^i(x^i) = \sum_{T \subset N} \gamma_T v(T) \leq v(S),$$

from which it follows that $\hat{v}(S) \leq v(S)$. This⁴ completes the proof that the above market indeed generates v .

We come now to the second part. We have to show that if $\beta = (\beta^1, \beta^2, \dots, \beta^n)$ belongs to the core of v , then it is a competitive payoff vector of our market (the other direction is well known). Denote $\pi = (\beta^1, \beta^2, \dots, \beta^{n-1}) \in \mathbb{R}^{n-1}$, $x^i = (0, 0, \dots, 0) \in \mathbb{R}^{n-1}$ for $i \neq n$ and $x^n = (1, 1, \dots, 1) \in \mathbb{R}^{n-1}$. We will show that (π, x) is a competitive equilibrium, and that the corresponding payoff vector is precisely β .

⁴ The reason the above construction need not work if we do not use the direct market for N' , is that the vectors $\sum_{i \in S} a^i$ for $S \subset N'$ may be linearly dependent and the function u^n cannot be defined.

Indeed, $\sum_{i=1}^{n-1} \pi_i = \sum_{i \in N'} \beta^i \geq v(N')$; hence, for all $i \in N'$, we have $u^i(x) \leq \pi \cdot x$ for all $x \in \mathbb{R}_+^{n-1}$ (the same proof as Shapley and Shubik [5, Eq. (16)]). Therefore

$$\beta^i = u^i(0) - \pi \cdot (0 - a^i) \geq u^i(x) - \pi \cdot (x - a^i).$$

As for $i = n$, note that

$$u^n(x^n) = v(N)$$

(see the proof above that $v(S) = \hat{v}(S)$, for $S = N$); hence

$$u^n(x^n) - \pi \cdot (x^n - a^n) = v(N) - \sum_{i \in N'} \beta^i = \beta^n.$$

Let $x \in \mathbb{R}_+^{n-1}$, then

$$\sum_{S \subset N} \gamma_S e^S = (x, 1)$$

implies, since $\beta \in \text{core } v$, that

$$\sum_{S \subset N} \gamma_S v(S) \leq \sum_{S \subset N} \gamma_S (e^S \cdot \beta) = (x, 1) \cdot \beta = \pi \cdot x + \beta^n;$$

therefore

$$u^n(x) - \pi \cdot (x - a^n) \leq \beta^n = u^n(x^n) - \pi \cdot (x^n - a^n),$$

completing the proof.

Remark. The idea behind the above representation of v is that the n th player does not “bring” to the market any goods—he contributes, instead, his “good” utility function (or, production function—depending on the interpretation). This is possible because the utilities are transferable.

An n -player game v can be regarded as a (real) function defined on the vertices of the unit n -cube (by identifying $S \subset N$ with $e^S \in \{0, 1\}^n$). Then v is totally balanced if and only if it can be (uniquely) extended to a continuous, concave, and positively homogeneous of degree one function on the unit n -cube⁵ $[0, 1]^n$ (or even, on all $\mathbb{R}_+^n$); see Shapley and Shubik ([4, Sect. 4; 5, Footnote 5]).

This suggests a natural candidate for the space of *non-atomic* market games. With the additional requirement of differentiable utilities (rather than just continuous), Aumann and Shapley [1, p. 240] consider the space H of all games in pNA (the space of “differentiable” non-atomic games) that are

⁵ This is the so-called set of “fuzzy” or “ideal” subsets of N .

concave⁶ and positively homogeneous of degree one. H_+ is then the subset of all monotone games in H .

It is known that H_+ contains the differentiable non-atomic market games—see Aumann and Shapley [1, proof of Proposition 40.26]). We show here that the converse is false.

PROPOSITION 3. *There are non-atomic games in H_+ that are not market games (i.e., cannot be generated by markets).*

We prove first a preliminary result which is a direct consequence of Proposition 1. Let $(I, \mathcal{C})$ be a measure space, assumed to be isomorphic to the unit interval $[0, 1]$ with its Borel σ -field. Let $\mu_1, \mu_2, \dots, \mu_n$ be n mutually singular non-atomic probability measures on $(I, \mathcal{C})$, and define

$$v = \sqrt[n]{\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n}.$$

Then v is a market game. For example, it can be generated using n commodities as follows: let $T_1, T_2, \dots, T_n$ be a partition of I such that T_i is a support of μ_i ; the initial endowment $\mathbf{a}(t)$ of each $t \in T_i$ is the i th unit vector $e^i \in \mathbb{R}_+^n$; the utility function of each $t \in I$ is $u(x) = n \cdot \sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n}$, and the population measure is $\mu = (\mu_1 + \mu_2 + \dots + \mu_n)/n$. Actually, $n - 1$ commodities suffice, as in Proposition 2.

PROPOSITION 4. *A non-atomic market generating the above v has at least $n - 1$ commodities.*

Proof. Let $((I, \mathcal{C}, v), \mathbb{R}_+^m, \{\mathbf{a}(t)\}_{t \in I}, \{u_t\}_{t \in I})$ be a market generating v . Let $T_1, T_2, \dots, T_n$ be as above, and define, for $i = 1, 2, \dots, n$,

$$a_i = \int_{T_i} \mathbf{a} \, dv,$$

$$u^i(x) = \sup \left\{ \int_{T_i} u_t(\mathbf{x}(t)) \, dv(t) \mid \int_{T_i} \mathbf{x} \, dv = x, \mathbf{x}(t) \in \mathbb{R}_+^m \text{ for all } t \right\}.$$

Then u^i is a concave function (with possible values $-\infty$). Note that, if we replace each T_i by a player i , we obtain the n -player unanimity game; moreover, $(N, \mathbb{R}_+^m, \{a^i\}_{i \in N}, \{u^i\}_{i \in N})$ is an n -player market generating it. By Proposition 1 (see the remark following its proof), $m \geq n - 1$.

Proof⁷ of Proposition 3. Let $\{\mu_{ni}\}_{n=1,2,\dots; i=1,2,\dots,n}$ be a (countably) infinite triangular array of mutually singular non-atomic probability

⁶ Their definition actually requires superadditivity instead of concavity; however, in the presence of homogeneity, the two concepts are equivalent.

⁷ The idea of this proof is due to R. J. Aumann.

measures on $(I, \mathcal{C})$ (for example, take the normalized restrictions of a non-atomic measure to a countable partition of I into non-null sets). Define

$$v = \sum_{n=1}^{\infty} 2^{-n} \sqrt[n]{\mu_{n1} \cdot \mu_{n2} \cdot \dots \cdot \mu_{nn}},$$

and let v_N be the partial sum to N . Then $v_N \in pNA$ (by Theorem C in Aumann and Shapley [1, p. 25], $\sqrt[n]{\mu_{ni}} \in pNA$, which is an algebra), v_N is positively homogeneous of degree 1 (see [1, Eq. (22.18)]), it is superadditive and monotone. Hence $v_N \in H_+$. Moreover, $v - v_N$ is also a monotone game, hence

$$\|v - v_N\| = (v - v_N)(I) = \sum_{n=N+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^N} \xrightarrow{N \rightarrow \infty} 0,$$

which, since H_+ is closed, implies $v \in H_+$.

If v is generated by a market, then, by considering the restriction to the support of $\{\mu_{ni}\}_{i=1}^n$, it must have at least $n - 1$ goods (Proposition 3). This is true for all n , so no finite number of commodities suffice.

The importance of this last result lies in showing that, in order to study (differentiable) market games, one need not consider the whole space H_+ ; the finite dimensional "character" of the markets may be a useful property. A similar situation occurs in the non-differentiable case—cf. Hart [2, Open Problem A]. See also the forthcoming paper of Dubey and Neyman.

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