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VALUES OF NON-DIFFERENTIABLE MARKETS WITH A CONTINUUM OF TRADERS*

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The main result is that in perfectly competitive markets, every value allocation is competitive. The model used is that of a non-atomic continuum of traders, both in a Walrasian and in a transferable utility (monetary) market. No differentiability assumptions are made. The problems of existence of the value and of the converse to the above result (i.e., that every competitive allocation is a value allocation) are also studied.

1. Introduction

The *Value Equivalence Principle* states that in a sufficiently differentiable perfectly competitive market, the set of value allocations coincides with the set of competitive allocations.

By a 'perfectly competitive' market it is meant one in which every single trader is negligible. A 'differentiable' market is one where the preference relations of all the traders are representable by differentiable (or smooth) utility functions.

The purpose of this paper is to investigate the above principle in general markets – i.e., not necessarily differentiable.

We represent the 'perfectly competitive' market by a non-atomic continuum of traders [cf. Aumann (1964)], and we use the asymptotic approach for the concept of value [cf. Aumann and Shapley (1974), Aumann (1975)]. The two kinds of markets are studied – Walrasian (without transferable utility) and monetary (with transferable utility).

The main result is that one direction of the Value Principle is always true, namely, that *in a perfectly competitive market, every value allocation is competitive*. This can be interpreted to mean that the allocation of *marginal contributions*, if it is feasible, is also competitive. The reader is referred to Shapley

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(1967) and Aumann (1975, §1 and §6), for detailed intuitive discussions and interpretations of these concepts.

Next, it is shown that the second direction of the Principle – i.e., that every competitive allocation is a value allocation – is no longer true for all markets. However, it is proved that, at least in the transferable utility case, the Value Equivalence Principle holds for *almost* every market.

A parallel development should be noted here. It is based on a limit approach – finite markets with a fixed number of types of traders, in which the number of traders in each type goes to infinity (replica economies).

To place this work in its 'context' of research done on the relation between competitive and value allocations, table 1 is useful.

Table 1

	LR	NA	
M	Shapley (1964)**	Aumann and Shapley (1974)**	D
	Champsaur (1975)* – S	Hart (this paper)*	ND
W		Aumann (1975)**	D
	Champsaur (1975)* – S	Hart (this paper)*	ND

M = Monetary (with transferable utility)
W = Walrasian (without transferable utility)
LR = Limit of replicas
NA = Non-atomic
D = Differentiable
ND = Non-differentiable
S = Symmetric value allocations only
* = {value allocations} $\subset$ {competitive allocations}
** = {value allocations} = {competitive allocations} (Equivalence)

The results of this paper were made possible by the development of the theory of asymptotic value for a class of non-atomic games which are, in some sense, non-differentiable [Hart (1977)].

The paper is organized as follows: the main results are included in section 3 (for monetary markets) and in section 4 (for Walrasian markets), whereas section 2 is devoted to the model and some preliminary results.

The book of Aumann and Shapley (1974) being a used reference, it will be abbreviated by A&S.

2. Preliminaries

We begin by recalling the basic mathematical model. The measurable space $(I, \mathscr{C})$ is the *trader space* and μ is a non-atomic non-negative σ-additive measure defined on $\mathscr{C}$, satisfying, without loss of generality, $\mu(I) = 1$. We also assume $(I, \mathscr{C})$ to be a standard measurable space [i.e., isomorphic to the unit interval with the Borel sets – cf. A&S (2.1)]. Since all integrals will be with respect to μ, we will usually drop the symbol $d\mu$: when the integral is over all I, we will write $\int$ instead of $\int_I$. The *commodity space* is Ω, the non-negative orthant of the l-dimensional Euclidian space. For x in Ω, x^j will denote its jth coordinate. The *initial allocation*[1] $\boldsymbol{a}$ is a measurable function from I to Ω. We assume every commodity to be actually present in the market, i.e.,[2]

(2.1) $\int \boldsymbol{a} \gg 0.$

To each trader t in I corresponds a real-valued function u_t defined on Ω, called the *utility function of t*. All utility functions will be normalized by $u_t(0) = 0$. The following assumptions are made:

(2.2) *Weak monotonicity.*[2] $x \geqq y$ implies $u_t(x) \geqq u_t(y)$ for all t in I,

(2.3) *Continuity.* u_t is a continuous function, for all $t \in I$,

(2.4) *Measurability.* $u_t(x)$, as a function of (x, t), is measurable in the product field $\mathscr{C} \times \mathscr{B}^l$, where $\mathscr{B}^l$ denotes the Borel σ-field on Ω,

(2.5) $u_t(x) = o(\|x\|)$ as $\|x\| \to \infty$, integrably in t.

The first three assumptions are standard. As for the last one, this is the Aumann–Perles (1965) condition.

The non-atomic game v is defined by

(2.6) $v(S) = \max\{\int_S u_t(\boldsymbol{x}(t))\, d\mu(t) \mid \int_S \boldsymbol{a} = \int_S \boldsymbol{x}$ and $\boldsymbol{x}(t) \in \Omega$ for all $t \in S\}$,

for all $S \in \mathscr{C}$. The maximum is attained [this follows from (2.4) and (2.5) by the main theorem in Aumann–Perles (1965)]. For a discussion on the economic meaning of v see A&S (§30). The usual interpretation is that there is money in the market, and each trader's utility increases by one unit for each unit of money added. Then $v(S)$ is the maximum utility the coalition S can get by using its own initial resources alone, with unrestricted side payments between its members. This model is called: a *market with transferable utility*, or a *monetary market*. From now on, we will identify v with the market it arose from.

[1]Boldface letters will denote functions on I.

[2]For x, y in Ω, $x \gg y$ means $x^j > y^j$ for all $j = 1, \ldots, l$, and $x \geqq y$ means $x^j \geqq y^j$ for all $j = 1, \ldots, l$.

An *allocation* is an integrable function $\boldsymbol{x}$ from I to Ω such that

$$\int \boldsymbol{x} = \int \boldsymbol{a}.$$

A *transferable utility competitive equilibrium* (t.u.c.e.) is a pair $(\boldsymbol{x}, p)$, where $\boldsymbol{x}$ is an allocation and $p \in \Omega$, such that for almost all $t \in I$,

$$(2.7) \quad u_t(x) - p \cdot (x - \boldsymbol{a}(t)) \leqq u_t(\boldsymbol{x}(t)) - p \cdot (\boldsymbol{x}(t) - \boldsymbol{a}(t)),$$

for all $x \in \Omega$. The measure v defined by

$$(2.8) \quad v(S) = \int_S [u_t(\boldsymbol{x}(t)) - p \cdot (\boldsymbol{x}(t) - \boldsymbol{a}(t))] \,\mathrm{d}\mu(t),$$

for all $S \in \mathscr{C}$, is called the *competitive payoff distribution*, with respect to the t.u.c.e. $(\boldsymbol{x}, p)$. Again, see A&S (§32) for an intuitive discussion of this concept.

(2.9) *Proposition. The core of v is not empty, and it coincides with the set of competitive payoff distributions.*

Proof. Propositions 32.5 and 32.2 in A&S.

(2.10) *Proposition. Let z be such that $v(I)$ is attained at z. Let $(\boldsymbol{x}, p)$ be a t.u.c.e. Then (z, p) is a t.u.c.e., and their competitive payoff distributions coincide.*

Proof. Apply (2.7) to $x = z(t)$ to get

$$(2.11) \quad u_t(z(t)) - p \cdot (z(t) - \boldsymbol{a}(t)) \leqq u_t(\boldsymbol{x}(t)) - p \cdot (\boldsymbol{x}(t) - \boldsymbol{a}(t)),$$

for almost all $t \in I$. Since $\int z = \int \boldsymbol{x} = \int \boldsymbol{a}$, the integration over I gives

$$(2.12) \quad \int u_t(z(t)) \leqq \int u_t(\boldsymbol{x}(t)).$$

But $v(I)$ is attained at z, therefore we must have equality in (2.12), hence also in (2.11), for almost all $t \in I$, which proves the proposition.

In the following, let z be a fixed allocation at which $v(I)$ is attained, and define

$$(2.13) \quad \zeta^0(S) = \int_S u_t(z(t)) \,\mathrm{d}\mu(t),$$

$$(2.14) \quad \zeta^j(S) = \int_S (a^j - z^j), \quad \text{for} \quad j = 1, \ldots, l,$$

for all $S \in \mathscr{C}$. Then ζ^0 is a non-atomic measure, and $\zeta = (\zeta^1, \ldots, \zeta^l)$ is a vector of non-atomic measures.

The set of all competitive prices, i.e., the set of all p in Ω such that $(\boldsymbol{x}, p)$ is a t.u.c.e. for some allocation $\boldsymbol{x}$, will be denoted by P. For every p in Ω, let

(2.15) $v_p = \zeta^0 + p \cdot \zeta,$

then we get:

(2.16) *Corollary. The core of v is the set of all v_p for p in P.*

Proof. Propositions 2.9 and 2.10.

Two characterizations of P are given in the next propositions.

Let f be a real-valued function on Ω, and let $z \in \Omega$. The vector $p \in \Omega$ is called a *super-gradient* of f at z if

$$f(x) \leqq f(z) + p \cdot (x - z),$$

for all $x \in \Omega$, i.e., if the hyper-plane with 'slope' p through $f(z)$ is always 'above' f. Let $\Delta f(z)$ denote the set of supergradients of f at z. If f is concave, then $\Delta f(z)$ is non-empty for every $z \gg 0$; if, moreover, f is differentiable, then the only point in $\Delta f(z)$ is the gradient of f at z [see Rockafellar (1970, §23, §25 and p. 308)].

Let A_t be a set for all t in I. The *essential intersection* of A_t for all t in I is defined to be the set of all points belonging to *almost*[3] all A_t. We will denote this set by $\text{ess.}\cap_{t \in I} A_t$.

(2.17) *Proposition.*[4] *Let z and P be as above. Then*

$$P = \text{ess.}\cap_{t \in I} \Delta u_t(z(t)).$$

Proof. Immediate from the definition of t.u.c.e. and Proposition 2.10.

This means that a vector P in Ω is a competitive price if and only if it is a super-gradient of u_t at $z(t)$ for almost all t in I.

(2.18) *Example.* Let $l = 1$, and assume half the traders have utility function u_1, and the other half utility u_2 (see fig. 2.19). Then the *only* competitive price is p.

[3]With respect to μ.

[4]Robert J. Aumann, private communication.

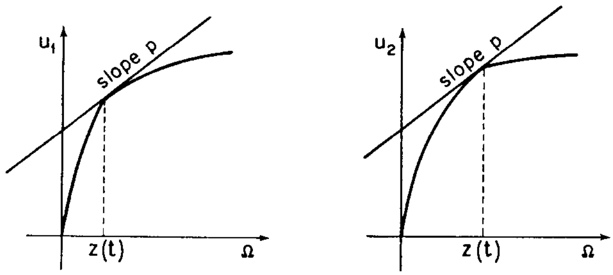

Fig. 2.19

As in A&S (§36) we will define the function u_I on Ω by

$$u_I(a) = \max\left\{ \int u_t(\boldsymbol{x}(t))\,\mathrm{d}\mu(t) \mid \int \boldsymbol{x} = a, \quad \boldsymbol{x}(t) \in \Omega \quad \text{for all} \quad t \in I \right\},$$

for all $a \in \Omega$. Note that the maximum is attained, and

$$v(I) = u_I\left(\int \boldsymbol{a}\right) = \int u_t(z(t))\,\mathrm{d}\mu(t),$$

where z is the allocation defined above.

(2.20) *Proposition.*[5] *Let P and u_I be as above, then*

$$P = \Delta u_I\left(\int \boldsymbol{a}\right).$$

Proof. First, assume $p \in P$, and let $a \in \Omega$. Then $u_I(a)$ is attained at some $\boldsymbol{x}$. Using Proposition 2.17, we get

$$u_t(\boldsymbol{x}(t)) \leqq u_t(z(t)) + p \cdot (\boldsymbol{x}(t) - z(t)),$$

for almost every t in I. By integrating over I, it follows that

$$u_I(a) \leqq u_I\left(\int \boldsymbol{a}\right) + p \cdot \left(a - \int \boldsymbol{a}\right),$$

hence

$$p \in \Delta u_I\left(\int \boldsymbol{a}\right).$$

Second, assume $p \notin P$. Again, by Proposition 2.17, there is a set $S \in \mathscr{C}$ of positive measure and a vector $\boldsymbol{x}(t) \in \Omega$ for every t in S, such that

[5]Robert J. Aumann, private communication.

(2.21) $u_t(\boldsymbol{x}(t)) > u_t(\boldsymbol{z}(t))+p\cdot(\boldsymbol{x}(t)-\boldsymbol{z}(t))$,

for all $t\in S$. Define $\boldsymbol{x}(t)=\boldsymbol{z}(t)$ for $t\notin S$, then we get equality in (2.21) for those t.

Integrating over all I (recall that S has positive measure) we get

$$\int u_t(\boldsymbol{x}(t))\,\mathrm{d}\mu(t) > u_I(\textstyle\int\boldsymbol{a})+p\cdot(\int\boldsymbol{x}-\int\boldsymbol{a}).$$

Denote $a=\int\boldsymbol{x}$, then the definition of u_I implies that

$$u_I(a) \geqq \int u_t(\boldsymbol{x}(t)\,\mathrm{d}\mu(t)),$$

hence $p\notin \Delta u_I(\int\boldsymbol{a})$.

3. Monetary markets

This section includes the main results for markets with transferable utility (i.e., monetary markets).

First, we will recall some notions and theorems from Hart (1977).

The space H'_+ consists of all non-atomic games in pNA' (i.e., limits in the supremum norm of all polynoms in non-atomic measures), which are homogeneous of degree one, superadditive and monotone. For a subset X of a linear space, x_0 is a *center of symmetry of* X if for every x in X, its symmetrical image with respect to x_0, i.e., $2x_0-x$, also belongs to X.

The next three theorems are proved there (as Theorems A, B and C, respectively):

(3.1) *Theorem. Let $v\in H'_+$. If v has an asymptotic value ϕv, then ϕv is a member of the core of v.*

(3.2) *Theorem. Let $v\in H_+$. If v has an asymptotic value ϕv, then ϕv is the center of symmetry of the core of v.*

(3.3) *Theorem. Let $v\in H'_+$. If the core of v contains only one member v_0, then v has an asymptotic value $\phi v=v_0$.*

We return now to the monetary markets. To use the above results, we have to prove first the following:

(3.4) *Proposition. Let v be a transferable utility non-atomic market satisfying (2.2)–(2.5). Then $v\in H'_+$.*

The proof will be given at the end of this section.

We come now to the Value Equivalence Principle. One should note that in the case of transferable utility markets, the set of value allocations is either empty, or consists of one member (the asymptotic value of the corresponding game).

The first theorem is the one part of the Value Principle which is always true.

Theorem A. Let v be a transferable utility non-atomic market satisfying (2.1)–(2.5). *If v has an asymptotic value ϕv, then ϕv is a competitive payoff distribution.*

Proof. Follows immediately from Theorem 3.1 and Propositions 2.9 and 3.4.

The next theorem gives a necessary condition for the existence of the value:

Theorem B. Let v be a transferable utility non-atomic market satisfying (2.1)–(2.5). *If v has an asymptotic value, then the set of competitive payoff distributions and the set P of competitive prices each have a center of symmetry.*

Proof. It is easy to see that the assertions for the two sets are equivalent, since

$$v_{2p_0-p} = 2v_{p_0} - v_p,$$

by (2.15), and since v_p is a competitive payoff distribution if and only if $p \in P$ (Corollary 2.16). The theorem follows now from Theorem 3.2 and Propositions 2.9 and 3.4.

As a consequence of this theorem, we will show now that the Value Equivalence Principle is not always true. Indeed, take any market for which P has no center of symmetry [e.g., whenever P is the convex hull of more than two price vectors – cf. Hart (1977); see also Example 4.6]. The set of value allocations is then empty, whereas the set of competitive allocations is not (Proposition 2.9).

However, the next theorem will show that *in general*, this is *not* the case:

Theorem C. Let $U = \{u_t\}_{t \in I}$ be utility functions satisfying (2.2)–(2.5). *Let $A = A_U$ denote the set of all vectors $\boldsymbol{a}$ in Ω such that there is a transferable utility non-atomic market*[6] *$(\boldsymbol{a}, U)$ with*[7] *$\int \boldsymbol{a} = a$, for which the Value Equivalence Principle does not hold. Then A is a set of (Lebesque) measure zero in Ω.*

Proof. By Proposition 2.19, the set P of competitive prices is the set of supergradients of u_I at $\boldsymbol{a}$. The function u_I is concave on Ω [cf. A&S (Proposition

[6] I.e., the market with initial allocation $\boldsymbol{a}$ and utility functions U.

[7] The vector $\boldsymbol{a}$ is the 'total initial allocation'.

36.3)], hence the set of points where it is not differentiable has measure zero in Ω [cf. Rockafellar (1970, Theorem 25.5)]. Therefore P consists of a unique point [the gradient – cf. Rockafellar (1970), Theorem 25.1)] outside a set of measure zero in Ω. Applying now Theorem 3.3 and recalling Corollary 2.16 Propositions 2.9 and 3.4, the theorem is proved.

Remarks. (i) The proof of Theorem C reveals the following interesting fact: although the utility functions are not differentiable, one gets *almost always unique* competitive price and payoff distribution (recall also Example 2.18).

(ii) Assumption 2.1 is necessary only to ensure that the core and the set of competitive payoff distributions coincide (Proposition 2.9). In case it is not satisfied, both Theorem A and Theorem B must be formulated in terms of the core. For the same reason, Theorem D does not depend on this assumption.

Theorem D. Let v be a transferable utility non-atomic market satisfying (2.2)–(2.5). *Assume that*

(3.5) *for every t, $\partial u_t(x)/\partial x^j$ exists at every x in Ω for which $x^j > 0$, for all $1 \leqq j \leqq l$.*

Then v has an asymptotic value ϕv, and the core of v consists of the single measure ϕv.

Proof. Assume first that (2.1) is also satisfied. We will prove that (3.5) implies that the set P of competitive prices consists of a single point. Let $p \in P$, then Proposition 2.17 implies that p is a super-gradient of u_t at $z(t)$ for almost all t. Let $1 \leqq j \leqq l$, then $\int a^j > 0$ by (2.1), therefore $z^j(t) > 0$ for a set of positive measure in I. Let e_j denote the jth unit vector in $\mathbf{R}^l$ (i.e., whose jth coordinate is 1 and all others are 0), and let t be such that $z^j(t) > 0$ and $p \in \Delta u_t(z(t))$. Then, by definition of super-gradient, we get for $\lambda > 0$ small enough

$$[u_t(z(t)+\lambda e_j) - u_t(z(t))]/\lambda \leqq [p \cdot \lambda e_j]/\lambda = p^j,$$

and

$$[u_t(z(t)-\lambda e_j) - u_t(z(t))]/(-\lambda) \geqq [p \cdot (-\lambda e_j)]/(-\lambda) = p^j.$$

By (3.5), the limit as $\lambda \to 0$ of the left-hand side exists, and equals $\partial u(z(t))/\partial x^j$, hence p^j is uniquely determined, which proves that there is only one p in P.

We have proved that (3.5) implies that the core of v consists of only one point, assuming (2.1). But v does not change whenever commodities with zero as total initial allocation are added, hence the assumption (2.1) is not needed. The theorem now follows from Theorem 3.3 and Propositions 2.9 and 3.4.

Remark. Theorem D includes the asymptotic part of Proposition 31.7 in A&S. In fact, we assume less (weak monotonicity instead of strict one,[8] and existence of the partial derivatives only, without their continuity), and the proof is much simpler – no need for the complicate approximations in A&S (see especially §40).

To end this section, we have to prove Proposition 3.4. We will follow quite closely the results of Chapter VI in A&S.

Let U denote the set of utility functions $\{u_t\}_{t\in I}$. U is of *finite type* if it is a finite set.

(3.6) *Lemma. Let U be of finite type. Then $v \in H'_+$.*

Proof. As in A&S (§39) we get a continuous, non-decreasing and concave function g, and a vector of NA measures ν, such that $v = g \circ \nu$ [(39.7), Lemma 39.9 and (39.18) in A&S]. By Lyapounov's (1940) theorem, the range of ν is compact, hence there is a sequence of polynomials $\{h_n\}$ converging to g in the supremum norm on the range of ν (Weierstrass' approximation theorem). Since $h_n \circ \nu$ are polynomials in NA measures, we get $h_n \circ \nu \in pNA'$ and $(h_n \circ \nu)^* = h_n \circ \nu^*$ by Proposition 22.16 in A&S. Hence $h_n \circ \nu$ are homogeneous of degree one, and $h_n \circ \nu \to g \circ \nu = v$ in the supremum norm. This implies that v is in pNA' and is homogeneous of degree one. From the definition of v follows $v \geqq 0$ and its superadditivity, which proves the lemma.

(3.7) *Lemma. For every $\varepsilon > 0$ there is $\hat{U} = \{\hat{u}_t\}_{t\in I}$ of finite type such that*

$$| v(S) - \hat{v}(S) | < \varepsilon,$$

for all $S \in \mathscr{C}$, where $\hat{v}$ denotes the set function obtained by replacing U with $\hat{U}$.

Proof. The same arguments prove that Propositions 35.5 and 35.6 in A&S are true for $\mathscr{F}_0$ and $\mathscr{U}_0$, respectively; i.e., for every $\delta > 0$ there is a δ-approximation to U of finite type. Let δ correspond to

$$\varepsilon \bigg/ \left(1 + \sum_{j=1}^{l} \int a^j \right)$$

[8]In what concerns the asymptotic approach, this solves positively Open Problem B in A&S (§ 41).

by Proposition 37.11 in A&S, and let $\hat{U}$ be the above δ-approximation, then

$$| v(S)-\hat{v}(S) | < \varepsilon\bigg/\left(1+\sum_{j=1}^{l} \int a^j\right)\cdot\left(1+\sum_{j=1}^{l} \int_S a^j\right) \leqq \varepsilon,$$

which proves the lemma.

Proof of Proposition 3.4. For every $\varepsilon > 0$ there is $\hat{U}$ of finite type such that $\| v-\hat{v} \|' < \varepsilon$, where $\| \, \|'$ denotes the supremum norm (Lemma 3.7). But $\hat{v}$ is in H'_+ (Lemma 3.6), which is closed in the supremum norm, therefore $v \in H'_+$.

4. Walrasian markets

In this section we will prove our theorem for non-atomic markets without transferable utility (i.e., Walrasian markets).

As in section 2, we are given a trader space $(I, \mathscr{C})$, a positive non-atomic measure μ on $\mathscr{C}$ and an initial allocation $\boldsymbol{a}$ satisfying (2.1). Unlike the transferable utility case, no (cardinal) utility functions are given. Instead, we have for each t in I, an *ordinal preference relation* $\succ_t$ on Ω, which satisfies:

(4.1) *Desirability.* $x \geqq y$ and $x \neq y$ imply $x \succ_t y$.

(4.2) *Continuity.* For each x in Ω, the sets $\{y \mid y \succ_t x\}$ and $\{y \mid x \succ_t y\}|$ are open relative to Ω.

(4.3) *Measurability.* For any two measurable functions $\boldsymbol{x}$ and $\boldsymbol{y}$ from I to Ω, the set $\{t \mid \boldsymbol{x}(t) \succ_t \boldsymbol{y}(t)\}$ is in $\mathscr{C}$.

An *allocation* is a measurable function $\boldsymbol{x}$ from I to Ω such that $\int \boldsymbol{x} = \int \boldsymbol{a}$. It is called *competitive* if there exists a vector $p \neq 0$ in Ω such that $(\boldsymbol{x}, p)$ is a Walras competitive equilibrium, i.e., for almost all t in I, $\boldsymbol{x}(t)$ is maximal with respect to $\succ_t$ in tth budget set,

$$\boldsymbol{B}_p(t) = \{x \in \Omega \mid p \cdot x \leqq p \cdot \boldsymbol{a}(t)\}.$$

Let $U = \{u_t\}_{t \in I}$ be a family of utility functions, representing the given preferences $\{\succ_t\}_{t \in I}$; i.e., for every t in I,

$$u_t(x) > u_t(y) \quad \text{if and only if} \quad x \succ_t y.$$

If U satisfies also[9] (2.5), a transferable utility market $v = v_U$ can be defined by (2.6).

[9]Note that (4.2) and (4.3) imply (2.3) and (2.4), respectively. As for (4.1), it implies strict monotonicity [i.e., $x \geqq y$ and $x \neq y$ imply $u_t(x) > u_t(y)$], which is stronger than (2.2).

The allocation $\boldsymbol{x}$ is called a *value allocation* if there exists a family U of utilities, representing the preferences and satisfying (2.5), such that v_U has an asymptotic value ϕv_U, and

$$\phi v_U(S) = \int_S u_t(\boldsymbol{x}(t))\,\mathrm{d}\mu(t),$$

for all S in $\mathscr{C}$.

This definition is much more natural than Aumann's (1975), since almost any family U is 'admissible' here, and not only bounded differentiable ones [the only requirement is (2.5), which ensures that $v_U(S)$ is attained for every S in $\mathscr{C}$].

The definition of value in the non-transferable utility case is due to Harsanyi (1959) and Shapley (1967). A good intuitive discussion of the non-transferable utility value in an economic context can be found in Aumann (1975). The reasoning is as follows: First, one artificially assumes that *utility transfers are permissible*. If the game v_U which results has a value, and this value can in fact be achieved by an allocation $\boldsymbol{x}$ *without* any utility transfers, then $\boldsymbol{x}$ is a value allocation.

Theorem E. In a non-atomic Walrasian market satisfying (2.1) *and* (4.1)–(4.3), *every value allocation is competitive.*

Proof. The proof is similar to that of Proposition 8.1 in Aumann (1975). Let $\boldsymbol{x}$ be a value allocation corresponding to U, i.e.,

$$\phi v_U(S) = \int_S u_t(\boldsymbol{x}(t))\,\mathrm{d}\mu(t), \tag{4.4}$$

for every S in $\mathscr{C}$. Then $\phi v_U(I) = v_U(I)$, hence $v_U(I)$ is attained at $\boldsymbol{x}$. By Theorem A, there is a vector p in P such that ϕv_U is the competitive payoff distribution with respect to the t.u.c.e. $(\boldsymbol{x}, p)$, i.e.,

$$\phi v_U(S) = \int_S [u_t(\boldsymbol{x}(t)) - p\cdot(\boldsymbol{x}(t)-\boldsymbol{a}(t))]\,\mathrm{d}\mu(t). \tag{4.5}$$

Combining (4.4) and (4.5), we get that

$$p\cdot(\boldsymbol{x}(t)-\boldsymbol{a}(t)) = 0,$$

for almost all t in I. A standard argument now implies that $(\boldsymbol{x}, p)$ is a competitive equilibrium [e.g., Lemma 13.3 in Aumann (1975)].

In the uniformly smooth case, the Value Equivalence Principle holds, as proved by Aumann (1975). In the general case, this is no longer true. The following example shows it:

(4.6) *Example.* Let $[0, 1]$ be the set of traders and let $l = 3$ be the number of commodities. The preference relations $\succ_t$ are the same for all t, and they are defined by the utility function

$$u(x) = 0.7 \cdot \min_{1 \leqq i \leqq 3} x^i + 0.1 \cdot \sum_{i=1}^{3} x^i.$$

This is the so-called 'three-handed' glove market, cf. A&S (p. 203). The initial allocation is $\boldsymbol{a}(t) \equiv (1, 1, 1)$.

Let P denote the convex hull of the 3 vectors,

$$p_1 = (0.8, 0.1, 0.1),$$

$$p_2 = (0.1, 0.8, 0.1),$$

$$p_3 = (0.1, 0.1, 0.8).$$

Then $u(x) = \min_{1 \leqq i \leqq 3} p_i \cdot x = \min_{p \in P} p \cdot x$, and P is the set of super-gradients of u at $(1, 1, 1)$.

Let w be any utility function representing the same preference relation as u. We claim that $\Delta w(1, 1, 1)$, the set of super-gradients of w at $(1, 1, 1)$, must be equal to

$$(4.7) \qquad \bigcup_{\alpha \in A} \alpha P,$$

where A is some interval of the positive real line, possibly empty. The proof is as follows: First, there must be a continuous and strictly increasing function h such that $w = h \circ u$. Second, let A be the set of super-gradients of h at $1 = u(1, 1, 1)$; then $\alpha p \in \Delta w(1, 1, 1)$ for every $\alpha \in A$ and $p \in P$. Third, by considering the indifference surface through $(1, 1, 1)$, we show that every q in $\Delta w(1, 1, 1)$ must be proportional to some p in P. And fourth, using the fact that $u(y, y, y) = y$ for every $y > 0$, we prove that whenever $\alpha p \in \Delta w(1, 1, 1)$ for some p in P, α must belong to $\Delta h(1)$.

Returning to the market, it is easy to check that the only competitive allocation is $\boldsymbol{x} = \boldsymbol{a}$ (this follows from the quasi-concavity of the preference relations).

Suppose $\boldsymbol{x}$ were also a value allocation, with respect to a family $W = \{w_t\}$ of utilities. Since $\Delta w_t(\boldsymbol{x}(t))$ is of the form (4.7) for every t, the set Q of transferable utility competitive prices at $\boldsymbol{x}$ must be of the same form (by Proposition 2.17). Hence Q is either empty or is a truncated triangular pyramid, which has no center of symmetry, and this implies by Theorems A and B, respectively, that v_W has no asymptotic value.

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(1) Added in proof

A. Mas-Colell (1977), Competitive and Value Allocations of Large Exchange Economies, Journal of Economic Theory, vol. 14, pp. 419–438, proves the Value Equivalence Principle for sequences of finite Walrasian differentiable markets (converging to a regular limit), thus filling in the empty box W-D-LR in table 1.

(2) Added in proof

Theorem B (p. 110) is correct as it stands only if the range $R(\zeta)$ of the vector measure ζ (see p. 107) has full dimension (i.e., dimension l). In the general case, the second sentence of Theorem B must be amended to as read as follows: *If v has an asymptotic value, then the set of competitive payoff distributions has a center of symmetry, and so does the projection of the set P of competitive prices on the linear space of $R(\zeta)$.* (The linear space of a set is the smallest linear space containing that set.)

Unfortunately, in Example 4.6, we by bad luck chose precisely the most degenerate case – far from having full dimension, $R(\zeta)$ in this case has dimension 0. To make the example correct, we should choose $\boldsymbol{a}(t)$ to be (3,0,0,),(0,3,0,), and (0,0,3,) when t is in [0,1/3), [1/3, 2/3),and [2/3,1] respectively. Similarly, on line 7 from the bottom of p. 115, $\boldsymbol{x} = \boldsymbol{a}$ should be replaced by $\boldsymbol{x} = (1,1,1,)$.