Commentary: Nash equilibrium and dynamics ☆

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ABSTRACT

John F. Nash, Jr., submitted his Ph.D. Dissertation entitled *Non-cooperative games* to Princeton University in 1950. Read it 58 years later, and you will find the germs of various later developments in game theory. Some of these are presented below, followed by a discussion concerning dynamic aspects of equilibrium.

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Nash equilibrium

What is a *Nash equilibrium*[1]? John Nash defines an *equilibrium point* as

"an *n*-tuple *s* such that each player's mixed strategy maximizes his pay-off if the strategies of the others are held fixed. Thus each player's strategy is optimal against those of the others." (page 3)

Non-cooperative games

Nash emphasizes the *non-cooperative* aspect of his theory, in contrast to the "cooperative" approach of von Neumann and Morgenstern (1944) that was prevalent at the time:

"Our theory, in contradistinction, is based on the *absence* of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others. [...] The basic requirement for a non-cooperative game is that there should be no pre-play communication among the players [...]. Thus, by implication, there are no coalitions and no side-payments." (pages 1, 21)

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[1] I have seen "nash equilibrium" in print, as if "nash" were an English word. Having one's name spelled with a lower-case initial letter is surely a sign of lasting fame!

The "Nash program"

In the last section of his thesis, "Applications," Nash introduces what has come to be known as the *Nash program*, namely:

"a 'dynamical' approach to the study of cooperative games based upon reduction to non-cooperative form. One proceeds by constructing a model of the pre-play negotiation so that the steps of negotiation become moves in a larger non-cooperative game [...] describing the total situation. This larger game is then treated in terms of the theory of this paper [...]. Thus the problem of analyzing a cooperative game becomes the problem of obtaining a suitable, and convincing, non-cooperative model for the negotiation." (pages 25–26)

Nash himself used this approach for the two-person bargaining problem in his 1953 paper. Since then, the Nash program has been applied to a large number of models (see, e.g., the surveys of Mas-Colell, 1997 and Reny, 1997). The whole area of *implementation*—discussed in this panel by Eric Maskin—may also be regarded as a successful offshoot of the Nash program.

The "mass-action" interpretation

The next-to-last section of the dissertation is entitled "Motivation and Interpretation." Two interpretations of the concept of Nash equilibrium are provided. The first, the "mass-action" interpretation, assumes that

"there is a population [...] of participants for each position in the game." (page 21)

This framework brings us to a significant area of research known as *evolutionary game theory*, with important connections to biology; it is discussed in this panel by Peyton Young.

The "rational" interpretation

The second interpretation of Nash equilibrium provided in the dissertation refers to

"a 'rational' prediction of the behavior to be expected of rational playing of the game [...]. In this interpretation we need to assume the players know the full structure of the game [...]. It is quite strongly a rationalistic and idealizing interpretation." (page 23)

With the development of the area known as *interactive epistemology*, which deals formally with the issues of knowledge and rationality in multi-player situations, one can now state precise conditions for Nash equilibria. For instance, to obtain Nash equilibria in pure strategies, it suffices for each player to be rational and to know his own payoff function as well as the choices of the pure strategies of the other players (see Aumann and Brandenburger, 1995; for mixed strategies the conditions are more complicated).

Dynamics

Nash equilibrium is by definition a *static* concept. So what happens in dynamic setups where the players adjust their play over time? Since Nash equilibria are essentially "rest points," it is reasonable to expect that appropriate dynamical systems should lead to them.

However, after more than half a century of research, it turns out that

> *There are no general, natural dynamics leading to Nash equilibria.* $\qquad(*)$

Let us clarify the statement $(*)$. First, "general" means "for all games," rather than "for specific classes of games only."[2] Second, "leading to Nash equilibria" means that the process reaches Nash equilibria (or is close to them) from some time on.[3] Finally, what is a "natural dynamic"? While it is not easy to define the term precisely, there are certain clear requirements for a dynamic to be called "natural": it should be in some sense adaptive—i.e., the players should react to what happens, and move in generally improving directions (this rules out deterministic and stochastic variants of "exhaustive search" where the players blindly search through all the possibilities); and it should be simple and efficient—in terms of how much information the players possess, how complex their computations are at each step, and how long it takes to reach equilibrium. What $(*)$ says is that, despite much effort, no such dynamics have yet been found.

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[2] Indeed, there are classes of games—in fact, no more than a handful—for which such dynamics have been found, e.g., two-player games, and potential games.

[3] Dynamics that are close to Nash equilibria *most of the time* have been constructed in particular by Foster and Young (2003) and recently by Young (2009). These dynamics will reach a neighborhood of equilibria and stay there for a long time, but always leave this neighborhood eventually and the process then restarts.

A natural informational restriction is for each player to know initially only his own payoff (or utility) function, but not those of the other players; this is called *uncoupledness* (Hart and Mas-Colell, 2003) and is a usual condition in many setups. However, it leads to various impossibility results on the existence of "natural" uncoupled dynamics leading to Nash equilibria (Hart and Mas-Colell, 2003, 2006; Hart and Mansour, 2010). Thus (∗) turns out to be not just a statement about the current state of research in game theory, but in fact a result:

There cannot be *general, natural dynamics leading to Nash equilibria.*

Correlated equilibrium

A generalization of the concept of Nash equilibrium is the notion of *correlated equilibrium* (Aumann, 1974): it is a Nash equilibrium of the extended game where each player may receive certain information (call it a "signal") before playing the game; these signals do not affect the game directly. Nevertheless, the players may well take these signals into account when deciding which strategy to play. When the players' signals are independent, this reduces to nothing more than Nash equilibria; when the signals are public, this yields a weighted average of Nash equilibria; and when the signals are correlated (but not fully), new equilibria obtain.

Interestingly, there *are* general, natural dynamics leading to correlated equilibria (such as "regret matching": Hart and Mas-Colell, 2000; Hart, 2005).

Summary

The dissertation that John Nash submitted in 1950 is relatively short (certainly by today's standards). But not in content! Besides the definition of the concept of non-cooperative equilibrium and a proof of its existence, one can find there intriguing and stimulating ideas that predate a lot of modern game theory.

HAPPY BIRTHDAY, JOHN!

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