A Colorful Urn

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An urn has n balls, each one of a different color. Two balls are drawn at random from the urn, and the color of the second ball is changed to be the same as the color of the first ball; the two balls are then returned to the urn, and the process repeats. Let T denote the number of rounds until all balls are of the same color.

Question 1. What is $\mathbf{E}[T]$?

Question 2. Let T_0 and T_1 be the number of rounds up to T that the two chosen balls are of the same color, and let $T_1 = T - T_0$ be the number of rounds that they are of different colors. What are $\mathbf{E}[T_0]$ and $\mathbf{E}[T_1]$?

Theorem 1

$$\mathbf{E}[T] = (n-1)^2,$$

Theorem 2

$$\mathbf{E}[T_0] = \frac{(n-1)(n-2)}{2} \quad \text{and} \quad \mathbf{E}[T_1] = \frac{n(n-1)}{2}.$$

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1 Proof of Theorem 1

Number the colors $1, 2, ..., n$. A *composition* of the urn is $K \equiv (k^1, ..., k^n)$ where k^i is the number of balls of color i (so $\sum_{i=1}^n k^i = n$). Let K_t be the composition of the urn at time t, so $K_0 = (1, 1, ..., 1)$ and, at the stopping time T when all the balls are of the same color, K_T is a permutation of $(n, 0, ..., 0)$.

For each color i, the sequence k_t^i is a random walk on $\{0, 1, , ..., n\}$ with absorbing states 0 and n: letting

$$p(k) := \frac{k(n-k)}{n(n-1)},$$

we have $k_{t+1}^i = k_t^i + 1$ with probability $p(k_t^i)$ (the first ball is of color i and the second of a different color $j \neq i$), $k_{t+1}^i = k_t^i - 1$ with the same probability $p(k_t^i)$ (the second ball is of color i and the first of a different color), and $k_{t+1}^i = k_t^i$ with the remaining probability $1 - 2p(k_t^i)$. The random walk is "symmetric"—the probabilities of increasing and decreasing by one unit are equal—and thus k_t^i is a martingale.

Let A^i be the event that one ends with all balls of color i, i.e., $k_T^i = n$, and put $T^i := T \cdot \mathbf{1}_{A^i}$. Clearly the distribution of T^i depends only on the number of balls of color i (the partition into the other colors does not matter). For each $0 \leq k \leq n$ let

$$\phi(k) := \mathbf{E}\left[T^i \mid k_0^i = k\right]$$

be the expectation of T^i starting with k balls of color i (and $n-k$ of other colors).[1] We have $\phi(0) = \phi(n) = 0$, and for $1 \leq k \leq n$,

$$\phi(k) = (1 - 2p(k))\phi(k) + p(k)\phi(k+1) + p(k)\phi(k-1) + 1 \cdot \mathbf{P}\left[A^i \mid k\right] \quad (1)$$

Lemma 3 *For each* $0 \leq k \leq n$

$$\mathbf{P}\left[A^i \mid k\right] = \frac{k}{n}.$$

[1] ϕ does not have a superscript i since it is the same for any color i.

Proof. Let U be the time that the bounded martingale k_t^i reaches either 0 or n, then $k_0^i = \mathbf{E}\left[k_U^i\right] = 0 \cdot \mathbf{P}\left[k_U^i = 0\right] + n \cdot \mathbf{P}\left[k_U^i = n\right]$, and so $\mathbf{P}\left[k_U^i = n\right] = k_0^i/n$. □

Proposition 4 *For each*[2] $0 \le k \le n$

$$\phi(k) = (n-1)(n-k)\sum_{r=0}^{k-1}\frac{1}{n-r}.$$

Proof. Rearranging equation (1) and using Lemma 3 yields

$$\phi(k) = \frac{1}{2}\phi(k+1) + \frac{1}{2}\phi(k-1) + \frac{1}{2}\frac{n-1}{n-k}.$$

Letting $\delta_k := \phi(k) - \phi(k-1)$ and $a_k := (n-1)/(n-k)$, this can be rewritten as

$$\delta_{k+1} = \delta_k - a_k.$$

Therefore $\delta_k = \delta_1 - \sum_{r=1}^{k-1} a_r$ for $1 \le k \le n$ and so $\phi(k) = \sum_{r=1}^{k}\delta_r = k\delta_1 - \sum_{r=1}^{k-1}(k-r)a_r$ (recall that $\phi(0) = 0$). Since $\phi(n) = 0$ we get $n\delta_1 = \sum_{r=1}^{n-1}(n-r)a_r = \sum_{r=1}^{n-1}(n-1) = (n-1)^2$, or $\delta_1 = (n-1)^2/n$. Therefore

$$\begin{aligned}\phi(k) &= k\delta_1 - \sum_{r=1}^{k-1}(k-r)a_r = \frac{k(n-1)^2}{n} - \sum_{r=1}^{k-1}\frac{(k-r)(n-1)}{n-r}\\ &= (n-1)\left(k - \frac{k}{n} - \sum_{r=1}^{k-1}\left(1 - \frac{n-k}{n-r}\right)\right) = (n-1)\sum_{r=0}^{k-1}\frac{n-k}{n-r}.\end{aligned}$$

□

Proof of Theorem 1. We have $\mathbf{P}\left[\cup_{i=1}^n A^i\right] = 1$ and $\phi(1) = (n-1)^2/n$ (by Proposition 4), hence

$$\mathbf{E}\left[T\right] = \sum_{i=1}^{n}\mathbf{E}\left[T \cdot \mathbf{1}_{A^i}\right] = \sum_{i=1}^{n}\phi(1) = (n-1)^2.$$

□

[2] For $k = 0$ the empty sum is 0.

Remark. For each urn composition K put

$$\Phi(K) \equiv \Phi(k^1, ..., k^n) := \sum_{i=1}^{n} \phi(k^i).$$

Since $\phi(k^i) = \mathbf{E}[T \cdot \mathbf{1}_{A^i} \mid k_0^i = k^i]$ we have

$$\begin{aligned} \Phi(K) &= \sum_{i=1}^{n} \mathbf{E}\left[T \cdot \mathbf{1}_{A^i} \mid k_0^i = k^i\right] = \sum_{i=1}^{n} \mathbf{E}[T \cdot \mathbf{1}_{A^i} \mid K_0 = K] \\ &= \mathbf{E}[T \mid K_0 = K]. \end{aligned}$$

This implies in particular that, if K_t is not a permutation of $(n, 0, ..., 0)$ (i.e., $t < T$), then

$$\mathbf{E}[\Phi(K_{t+1}) \mid K_t] = \Phi(K_t) - 1$$

(since from t to $t+1$ one period has passed and so the remaining time to T has decreased by one).[3]

2 Proof of Theorem 2

Consider T_1; one may just as well assume that if the two chosen balls are of the same color then they are returned to the urn, and the round does not count. Denote the resulting process $\tilde{K}_t$; thus $\tilde{K}_{t+1} \neq \tilde{K}_t$ as long as $\tilde{K}_t$ is not a permutation of $(n, 0, ..., 0)$, and T_1 is the resulting stopping time. Define:

$$\Psi(K) \equiv \Psi(k^1, ..., k^n) := \sum_{i=1}^{n} (k^i)^2.$$

Proposition 5 *If $\tilde{K}_t$ is not a permutation of $(n, 0, ..., 0)$, then*

$$\mathbf{E}\left[\Psi(\tilde{K}_{t+1}) \mid \tilde{K}_t\right] = \Psi(\tilde{K}_t) + 2.$$

[3] This can also be shown directly by using either (1) or the formula of Proposition 4.

Proof. If the color of the first ball is i and the color of the second ball is j (with $j \neq i$), then

$$\Psi(\tilde{K}_{t+1}) = \Psi(\tilde{K}_t) + 2\tilde{k}_t^i + 1 - 2\tilde{k}_t^j + 1;$$

if the color of the first ball is j and that of the second is i, then

$$\Psi(\tilde{K}_{t+1}) = \Psi(\tilde{K}_t) - 2\tilde{k}_t^i + 1 + 2\tilde{k}_t^j + 1.$$

These two events are equally probable, and so

$$\mathbf{E}\left[\Psi(\tilde{K}_{t+1}) \,|\, \tilde{K}_t, C_{t+1}^{\{i,j\}}\right] = \Psi(\tilde{K}_t) + 2,$$

where $C_{t+1}^{\{i,j\}}$ denotes the event that the colors of the two balls chosen at time $t+1$ are $\{i, j\}$ (without specifiying which one is first and which one is second). Now this holds for every $i \neq j$, yielding the result. □

Proof of Theorem 2. Proposition 5 implies that $\Psi(\tilde{K}_t) - 2t$ is a martingale, for $t < T_1$. Therefore[4] $\Psi(\tilde{K}_0) = \mathbf{E}\left[\Psi(\tilde{K}_{T_1}) - 2T_1\right]$ and so $2\mathbf{E}[T_1] = \mathbf{E}\left[\Psi(\tilde{K}_{T_1})\right] - \Psi(\tilde{K}_0) = n^2 - n \cdot 1^2 = n(n-1)$. Finally, $\mathbf{E}[T_0] = \mathbf{E}[T] - \mathbf{E}[T_1] = (n-1)^2 - n(n-1)/2 = (n-1)(n-2)/2$. □

[4]For each integer m we have $\mathbf{E}\left[\Psi(\tilde{K}_{T_1 \wedge m}) + T_1 \wedge m\right] = \Psi(\tilde{K}_0)$. As $m \to \infty$ the sequence $\Psi(\tilde{K}_{T_1 \wedge m})$ is bounded and $T_1 \wedge m$ is monotonic, so one can take the limit.