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EXISTENCE OF CORRELATED EQUILIBRIA*†

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An elementary proof, based on linear duality, is provided for the existence of correlated equilibria in finite games. The existence result is then extended to infinite games, including some that possess no Nash equilibria.

1. Introduction. The standard proof of existence of *correlated equilibria* (defined by Aumann 1974) in finite games consists of showing, first, that Nash equilibria are correlated, and second, that every game has at least one Nash equilibrium. The first argument is trivial; the second requires however the use of a fixed-point theorem.[1]

Since the correlated equilibria form a convex set, defined by a set of explicit linear inequalities, it is reasonable to expect a simpler existence proof. We provide here (in §2) such an elementary proof, based on the standard Linear Duality Theorem. We actually chose to use an equivalent result, which seems more appropriate in this game theoretic setup: the Minimax Theorem.[2]

We next consider infinite games, where the set of players and the sets of (pure) strategies are arbitrary. General results on the existence of correlated equilibria are obtained. Moreover, it is proved that a *countably additive* correlated equilibrium exists when the payoff functions are continuous and the strategy spaces are compact Hausdorff. All proofs are as "elementary" as possible: we use the result of the existence of correlated equilibria for finite games, together with a standard product compactness argument (e.g., Tychonoff's Theorem); this is equivalent to an infinite-dimensional separation theorem, but weaker than a fixed-point theorem (which is needed to show the existence of Nash equilibria). We study first, in §3, the simpler case where the sets of (pure) strategies are all finite; §4 then deals with the general case. We also provide several examples to illustrate the various difficulties.

2. Finite games. A *finite game* (an "n-person game in normal or strategic form") is given as follows: Let $N = \{1, 2, \ldots, n\}$ be a finite set of *players*. For each $i \in N$, let S^i be a finite set of (pure) *strategies* of i. Let S be the set of n-tuples of strategies: $S = S^1 \times S^2 \times \cdots \times S^n$; an element of S is $s = (s^i)_{i \in N}$. For each $i \in N$ and $s \in S$ let

$$s^{-i} = \left(s^1, \ldots, s^{i-1}, s^{i+1}, \ldots, s^n\right)$$

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[1]Another proof of existence of correlated equilibria has been obtained by Reinhard Selten (private communication); it applies however only to the case where each player has two pure strategies.

[2]Alternatively, we could have used the Theorem of the Alternative, or any other equivalent result.

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denote the strategies played by everyone but i; thus $s^{-i} \in S^{-i} = \prod_{j \neq i} S^j$ and $s = (s^{-i}, s^i)$. Finally, for each $i \in N$, let[3] $h^i: S \to \mathbb{R}$ be the *payoff function* of player i; $h^i(s)$ is the payoff to i when the n-tuple of strategies s is played.

A *correlated equilibrium* (Aumann 1974, 1987) consists of a probability vector[4] $p = (p(s))_{s \in S}$ on S such that the following is satisfied for all $i \in N$ and all $r^i, t^i \in S^i$:

$$(1) \qquad \sum_{s^{-i} \in S^{-i}} p(s^{-i}, r^i)\left[h^i(s^{-i}, r^i) - h^i(s^{-i}, t^i)\right] \geq 0.$$

The interpretation is as follows: An n-tuple of strategies $r \in S$ is chosen at random (by a referee, say), according to the distribution p. Each player i is then told (only) his own coordinate r^i of r, and the original game is played. A correlated equilibrium results if the n-tuple of strategies in which each player i always plays the "recommended" r^i is a Nash equilibrium in this extended game (note that all players are assumed to know the distribution p). Condition (1) says that, whenever player i is told r^i, he will have no incentive to play t^i instead. (Note that one should have in (1) the conditional probability[5] $p(s^{-i}|r^i)$ instead of the joint probability $p(s^{-i}, r^i)$; multiplying by the marginal probability $p(r^i)$ then yields (1), which also holds trivialy when $p(r^i) = 0$.)

THEOREM 1. *Every finite game has a correlated equilibrium.*

PROOF. Consider the following auxiliary two-person zero-sum game: Player I (the maximizer) chooses an n-tuple of strategies $s = (s^1, \ldots, s^n) \in S$; player II (the minimizer) chooses a triple (i, r^i, t^i), where $i \in N$ and $r^i, t^i \in S^i$. The payoff (from II to I) is: $h^i(s^{-i}, r^i) - h^i(s^{-i}, t^i)$ if $s^i = r^i$ and 0 otherwise. A correlated equilibrium in the original game is now easily seen to correspond to a strategy[6] of player I that guarantees a nonnegative payoff in this zero-sum game. By the Minimax Theorem, such a strategy exists if, for every *given* strategy of player II, there exists a strategy of player I yielding a nonnegative payoff. Let thus $y = (y^i(r^i, t^i))_{i \in N;\, r^i, t^i \in S^i}$ be a strategy of player II. We now need the following:

LEMMA. *Let $(a_{jk})_{j,k=1,\ldots,m}$ be nonnegative numbers. Then there exists a probability vector $x = (x_j)_{j=1,\ldots,m}$ such that, for any vector $u = (u_j)_{j=1,\ldots,m}$,*

$$\sum_{j=1}^{m} x_j \sum_{k=1}^{m} a_{jk}(u_j - u_k) = 0.$$

PROOF. Denote the expression above by $\Phi(x, u)$. Since $\Phi(x, -u) = -\Phi(x, u)$, we need only to show that $\Phi(x, u) \geq 0$ for all u. Next, note that it suffices to consider only probability vectors u, since one may add an arbitrary constant to all the coordinates of u (to make them nonnegative) and then multiply u by a positive scalar (to normalize it), without changing (the sign of) $\Phi(x, u)$. The function Φ is bilinear in x and u; therefore the Minimax Theorem applies to it: $\operatorname{Max}_x \min_u \Phi(x, u) = \min_u \operatorname{Max}_x \Phi(x, u)$, where x and u are both probability vectors. Given u, let j be such

[3] $\mathbb{R}$ denotes the real line.

[4] A probability vector is a vector whose coordinates are all nonnegative and sum up to 1.

[5] We regard p as a probability distribution on the product space S; marginal and conditional probabilities are thus well defined.

[6] By "strategy" we mean, of course, "mixed strategy".

that $u_j = \text{Max}_k u_k$; for x the jth unit vector, we then have $\Phi(x, u) \geq 0$. This shows that the min Max is nonnegative, hence so is the Max min, implying the existence of a vector x as claimed.[7] ■

PROOF OF THEOREM 1 (CONTINUED). For each i, apply the Lemma to $(y^i(r^i, t^i))_{r^i, t^i \in S^i}$ (as (a_{jk})), to obtain a probability vector $(x^i(r^i))_{r^i \in S^i}$ such that, in particular,

$$\sum_{r^i} x^i(r^i) \sum_{t^i} y^i(r^i, t^i)\left[h^i(s^{-i}, r^i) - h^i(s^{-i}, t^i)\right] = 0 \tag{2}$$

for every $s^{-i} \in S^{-i}$ (here, $u = (h^i(s^{-i}, r^i))_{r^i \in S^i}$). Define $x(r) = \prod_{j \in N} x^j(r^j)$ for each $r \in S$; x is clearly a strategy of player I in the auxiliary game. The payoff corresponding to the pair of strategies (x, y) (recall that y is the given strategy of player II) is

$$\sum_r \left[\prod_j x^j(r^j)\right]\left\{\sum_i \sum_{t^i} y^i(r^i, t^i)\left[h^i(r^{-i}, r^i) - h^i(r^{-i}, t^i)\right]\right\}.$$

This may be rewritten as

$$\sum_i \sum_{r^{-i}} \left[\prod_{j \neq i} x^j(r^j)\right]\left\{\sum_{r^i} x^i(r^i) \sum_{t^i} y^i(r^i, t^i)\left[h^i(r^{-i}, r^i) - h^i(r^{-i}, t^i)\right]\right\}$$

which equals zero by (2). This completes the proof. ■

REMARK. An explicit formula for x in the Lemma is as follows: For each j in $M = \{1, \ldots, m\}$, let G_j be the set of all functions g from M into itself, such that, for every $k \neq j$, there is a positive integer r (depending on j) with $g^{(r)}(k) = j$ (we write $g^{(r)}$ for the composition of g with itself r times). Then

$$x_j = \sum_{g \in G_j} \prod_{k \neq j} a_{k, g(k)}$$

for all $j \in M$.

3. Infinitely many players. Consider now the case where the set of players may be infinite. We will assume in this section that the (pure) strategies sets of all players are finite, since the definitions, the results, the proofs—and the difficulties—are simpler and more transparent than in the general case (which is studied in §4).

A *game* (in normal or strategic form) consists of:[8] (i) a nonempty set of players N; (ii) for each $i \in N$, a nonempty set of (pure) streategies S^i; and (iii) for each $i \in N$, a bounded payoff function h^i from $S = \prod_{i \in N} S^i$ to $\mathbb{R}$.

In this section we assume that all the sets S^i are finite.

For each $i \in N$, denote $S^{-i} = \prod_{j \neq i} S^j$; thus, every $s \in S$ can be written as $s = (s^i, s^{-i})$ with $s^i \in S^i$ and $s^{-i} \in S^{-i}$. The sets S^i are endowed with the discrete topology,[9] and their product S with the resulting product topology (note that S is a

[7] See the Remark following the proof of Theorem 1, for an explicit formula for x.

[8] The difference between a "finite game" (as defined in §2) and a "game" is that the sets N and S^i are finite in the former and arbitrary in the latter. (Note that the functions h^i are always bounded in a finite game.)

[9] A standard reference for the concepts and results of topology, measure theory, linear spaces, etc., that are used in §§3 and 4 is Dunford and Schwartz (1958).

compact Hausdorff space). Let Σ_0 denote the product σ-algebra on S: it is generated by the cylinders of the form $S^{-i} \times \{s^i\}$ for $i \in N$ and $s^i \in S^i$.

Let Σ be an algebra on S such that $\Sigma \supset \Sigma_0$ and all the payoff functions h^i are Σ-measurable. A *correlated equilibrium* (*with respect to* Σ) is a (finitely additive) probability measure p on the measurable space (S, Σ), such that the following[10] is satisfied for all $i \in N$ and all $r^i, t^i \in S^i$:

$$\int_{S^{-i} \times \{r^i\}} \left[h^i(s^{-i}, r^i) - h^i(s^{-i}, t^i) \right] dp(s) \geq 0. \tag{3}$$

(Note that, as in the finite case, we have multiplied by $p(r^i)$.)

THEOREM 2. *Assume that, for each $i \in N$, the set S^i is finite.*

(i) *Let $\Sigma \supset \Sigma_0$ be an algebra on S and assume that, for each $i \in N$, the function h^i is bounded and Σ-measurable. Then there exists a correlated equilibrium with respect to Σ.*

(ii) *Assume that, for each $i \in N$, the function h^i is continuous. Then there exists a countably additive correlated equilibrium with respect to Σ_0.*

REMARKS. 1. One may of course apply (i) to $\Sigma = 2^S$, in which case every function is Σ-measurable. Thus, if the payoff functions are bounded, then there always exists a (finitely additive) correlated equilibrium with respect to 2^S.

2. If h^i is a continuous function, then it is bounded (since S is compact) and Σ_0-measurable.

3. A real function on the product space S is continuous if and only if it is "almost finitely determined"; i.e., it is the uniform limit of functions depending only on finitely many coordinates (cf. Lemma 4.3 in Peleg 1969).

In the case that the payoff functions are continuous, Peleg (1969) has proved that there exists a Nash equilibrium, which is clearly also a countably additive correlated equilibrium with respect to Σ_0. However, his proof uses a fixed-point theorem, whereas the proof below is essentially an infinite-dimensional separation argument (or, equivalently, finite-dimensional separation together with compactness of an infinite product —which is the way our proof is presented).

Before proceeding to the proof of Theorem 2, it is instructive to consider two examples. The first one, due to Peleg (1969), is of a game that possesses no Nash equilibria (the payoff functions are not continuous); we exhibit a countably additive correlated equilibrium there (whose existence is *not* guaranteed by Theorem 2), and also a noncountably additive one, which shows that these equilibria (although they always exist) may well be quite "unreasonable". The second example, which is a slight modification of the first, possesses only noncountable additive correlated equilibria.

EXAMPLE 1. Let N be the set of positive integers; for each $i \in N$ let $S^i = \{0, 1\}$ and $h^i(s) = s^i$ if $\sum_{i \in N} s^i < \infty$ and $h^i(s) = -s^i$ if $\sum_{i \in N} s^i = \infty$. Call the first case (of the convergent series, which happens whenever there are only finitely many players that chose $s^i = 1$) *Case* 1 (since all players would like to chose $s^i = 1$ there), and the second case *Case* 0 (all players would prefer to choose $s^i = 0$ here). This game has no Nash equilibria: Let σ^i be a mixed strategy of player i, and denote by ξ^i the probability (under σ^i) that $s^i = 1$. By the Zero-One Law (mixed strategies are independent), either Case 1 happens almost surely, or Case 0 happens almost surely. In the former case, players strictly prefer their strategy $\xi^i = 1$ (i.e., $s^i = 1$ for sure); in the latter, $\xi^i = 0$ (i.e., $s^i = 0$).

[10] The above conditions on Σ and the functions h^i guarantee that (3) is well defined.

A countably additive correlated equilibrium (with respect to Σ_0) is obtained in this game as follows: For each $j \in N$, let z_j be that element of S whose first j coordinates are 1 and all the rest are 0; i.e., $z_j = (z_j^i)_{i \in N}$ with $z_j^i = 1$ if $i \leq j$ and $z_j^i = 0$ if $i > j$. Let p_1 be the probability measure on S with support $\{z_j : j \in N\}$ and $p_1(z_j) = 1/j - 1/(j+1)$ for all $j \in N$. Note that $p_1[s \in S : s^i = 1] = p_1[z_j : j \geq i] = 1/i$ for all $i \in N$ and that $p_1[\text{Case } 1] = 1$. Next, let p_0 be the product probability measure on S with marginals $p_0[s \in S : s^i = 1] = 1/i$ for all $i \in N$; note that $p_0[\text{Case } 0] = 1$ (by the Zero-One Law, since $\Sigma 1/i = \infty$). Finally, put $p_2 = (p_1 + p_0)/2$. It is now easily checked that p_2 is indeed a correlated equilibrium: For every $i \in N$ and $s^i = 0, 1$, one has $p_2[\text{Case } 1|s^i] = p_2[\text{Case } 0|s^i] = 1/2$, therefore (3) is always satisfied (as an equality).

A noncountably additive correlated equilibrium (with respect to $\Sigma = 2^S$) is as follows (this is essentially an equilibrium of the type constructed in our proof of Theorem 2(i) below): Let $A \subset S$; if A contains only finitely many of the points $\{z_j\}_{j \in N}$ (defined in the previous paragraph), put $p_3(A) = 0$; if A contains all but finitely many of these points, put $p_3(A) = 1$. Now extend p_3 to a finitely additive probability measure on $(S, 2^S)$ (apply the Hahn-Banach Theorem[11]). It can now be easily checked that p_3 satisfies the following two properties: $p_3[s^i = 1] = p_3[z_j : j \geq i] = 1$ for each i; and $p_3[\text{Case } 1] = p_3[\Sigma_{i \in N} s^i < \infty] = p_3[z_j : j \in N] = 1$. It is now clear that p_3 is indeed a correlated equilibrium. ■

The equilibria in this example illustrate two points: First, that a (countably additive) correlated equilibrium may exist even when there is no Nash equilibrium. And second, that noncountably additive correlated equilibria may well be unintuitive (indeed, p_3-almost surely every player chooses 1, and also p_3-almost surely there are only finitely many 1's!). However, if the payoff functions are not continuous, countably additive correlated equilibria need not exist at all, as the next example will show.

EXAMPLE 2. Identical to Example 1, except that in Case 1 (when there are finitely many 1's) we define $h^i(s) = s^i/i^2$ (instead of $h^i(s) = s^i$). Applying the same arguments used in Example 1 shows that there exists no Nash equilibrium, and also that p_3 is a (noncountably additive) correlated equilibrium.

Next, we prove that there exists no countably additive correlated equilibrium. Indeed, assume p is such an equilibrium. First, note that both Case 0 and Case 1 must have positive probability (otherwise, apply the same argument used to show that there is no Nash equilibrium). Now condition (3) for $r^i = 1$ and $t^i = 0$ is

$$p\left[\text{Case } 1 \; \& \; r^i = 1\right]/i^2 \geq p\left[\text{Case } 0 \; \& \; r^i = 1\right].$$

This implies that $p[\text{Case } 0 \; \& \; r^i = 1] \leq 1/i^2$, hence $p[\text{Case } 0 \; \& \; r^i = 1$ infinitely often$] = 0$ (by the Borel-Cantelli Lemma, since the series $\Sigma 1/i^2$ converges; here is the only use of the countably additivity of p). Thus $p[\text{Case } 0] = 0$, a contradiction. ■

PROOF OF THEOREM 2. (i) Consider finite sets $T = \prod_{i \in N} T^i \subset S$ such that $T^i \subset S^i$ for all i, and T^i is a singleton for all but finitely many players $i \in N$; we will call such a set T an "f-set". The game Γ_T obtained by replacing S^i with T^i for each $i \in N$ is equivalent to a finite game (only those i for which T^i is not a singleton are "real" players). By Theorem 1, every finite game has a correlated equilibrium; one therefore obtains a correlated equilibrium in Γ_T, which will be denoted by q_T. Clearly, we may regard q_T as a probability measure on S, with (finite) support T.

[11] The Axiom of Choice is used here.

Consider the net $\{q_T : T \text{ an } f\text{-set}\}$, ordered by inclusion of the sets T. It belongs to the unit ball of ba(S, Σ), the space of finitely additive measures on (S, Σ). By the Banach-Alaoglu Theorem (which is an easy consequence of Tychonoff's Theorem on the compactness of a product of compact spaces; see Dunford and Schwartz 1958, Theorem V.4.2), the unit ball is compact in the weak*-topology, which in this case is induced by $B(S, \Sigma)$, the space of bounded and measurable functions on (S, Σ) (cf. Dunford & Schwartz 1958, Theorem IV.5.1). Let p thus be a cluster point of $\{q_T\}$. We will show that p is a correlated equilibrium.

Indeed, fix $i \in N$ and $r^i, t^i \in S^i$. Define a real function f on S by: $f(s) = h^i(s) - h^i(s^{-i}, t^i)$ for $s \in S^{-i} \times \{r^i\}$ (i.e., when $s^i = r^i$), and $f(s) = 0$ otherwise. The assumptions on h^i imply that f too is bounded and Σ-measurable. Given $\varepsilon > 0$ and i, there exists therefore an f-set T with $T^i \supset \{r^i, t^i\}$, such that $|\int f\,dp - \int f\,dq_T| < \varepsilon$. It is easily checked that $\int f\,dq_T \geq 0$ (since q_T satisfies (1), hence (3), for all "real" players in Γ_T, in particular i, and $r^i, t^i \in T^i$). Therefore the inequality above implies $\int f\,dp > -\varepsilon$, hence ≥ 0, which is exactly (3) for p.

(ii) Replace ba(S, Σ) in the proof of (i) with rca(S, Σ_0), the space of regular countably additive measures on (S, Σ_0), which is the dual of $C(S)$, the space of continuous real functions on S (by Riesz's Representation Theorem; see Dunford and Schwartz 1958, Theorem IV.6.3). ∎

4. Infinitely many players and strategies. In this section we deal with the general case, where the set of players as well as the strategy sets may be infinite. A "game" was defined in §3; here it is not longer assumed that the (pure) strategies sets are finite. When defining a correlated equilibrium, it turns out that condition (3) is no longer appropriate, since $p(r^i)$ many well be zero for all $r^i \in S^i$ (for example, when S^i is a continuum). A first attempt is therefore to require (3) also for subsets $R^i \subset S^i$ and not only for singletons $\{r^i\}$. This is however not yet satisfactory, since t^i there need not be fixed—it may well depend on r^i. To obtain the correct conditions, we therefore consider again the *extended game* described in §2: An element $r \in S$ is chosen, each player i is informed only of the ith coordinate r^i of r, and then the original game is played. A strategy of player i in the extended game is thus a function ζ^i from S^i into itself, that associates to each "recommendation" $r^i \in S^i$ a choice of action $\zeta^i(r^i) \in S^i$ in the original game. A correlated equilibrium is obtained if, when each player i uses the strategy ζ^i = identity, a Nash equilibrium results (in the extended game).

Formally, let Σ^i be an algebra on S^i, let Σ_0 be the σ-algebra on S generated by the product of the Σ^i's, and let $\Sigma \supset \Sigma_0$ be any algebra on S. We assume that, for each $i \in N$, the payoff function h^i is bounded and Σ-measurable. A *correlated equilibrium* (*with respect to* $\{\Sigma^i\}_{i \in N}$ *and* Σ) is a probability measure p on (S, Σ), such that, for all $i \in N$ and all Σ^i-measurable[12] functions $\zeta^i : S^i \to S^i$, the following inequality holds:

$$(4) \qquad \int_S \left[h^i(s^{-i}, s^i) - h^i(s^{-i}, \zeta^i(s^i)) \right] dp(s) \geq 0.$$

The left-hand side in (4) is the difference between the payoff of player i (in the extended game) when he always follows the "recommendation", and his payoff when he plays the strategy ζ^i instead. It is easy to see that, when S^i is a finite set, conditions (4) and (3) are equivalent: (3) is obtained by taking in (4) $\zeta^i(s^i) = s^i$ for all $s^i \neq r^i$ and $\zeta^i(r^i) = t^i$; vice versa, given a function ζ^i, sum the inequalities (3) over all r^i with $t^i = \zeta^i(r^i)$, to yield (4).

[12] I.e., $\{s^i \in S^i : \zeta^i(s^i) \in A\} \in \Sigma^i$ for every $A \in \Sigma^i$.

As we saw in the previous section, noncountably additive correlated equilibria may be quite "unreasonable". We will therefore deal here only with the existence of countably additive correlated equilibria (however, a result parallel to Theorem 2(i) may be obtained here too).

THEOREM 3. *Assume that, for each $i \in N$, the space S^i is compact Hausdorff and the function h^i is continuous (where S is endowed with the product topology). Let Σ^i be the Borel σ-algebra on S^i, and let Σ be the Borel σ-algebra on*[13] *S. Then there exists a countably additive correlated equilibrium with respect to $\{\Sigma^i\}_{i \in N}$ and Σ.*

PROOF. The construction is similar to that used in the proof of Theorem 2. A set $T = \prod_{i \in N} T^i$ will be called an "f-set" if T^i is a finite subset of S^i for all $i \in N$, and moreover T^i is a singleton for all but finitely many i's. To each f-set T there corresponds an (essentially) finite game Γ_T (obtained by replacing S^i with T^i for all i); let q_T be a correlated equilibrium of Γ_T (whose existence follows from Theorem 1), regarded as a probability measure on S (with finite support T). The net $\{q_T : T$ an f-set$\}$ (ordered by inclusion of the sets T) belongs to the unit ball of rca(S) (the space of regular countably additive measures on S; recall that each S^i is Hausdorff, therefore S is Hausdorff, hence measures with finite support are regular). Again, the Banach-Alaoglu Theorem implies the existence of a cluster point p with respect to the topology induced by $C(S)$ (the space of continuous functions on S). We will show that p is indeed a correlated equilibrium.

Let $i \in N$ and let $\zeta^i : S^i \to S^i$ be a measurable function. If ζ^i were a continuous function, inequality (4) would easily be proved by an argument similar to that used in the Proof of Theorem 2 (here, p is a cluster point relative to continuous functions); since however ζ^i need not be continous, we need more elaborate arguments. We will deal first with a special case, from which the general case will then easily follow.

A special case. There exist $t^i \in S^i$ and $R^i \in \Sigma^i$ such that $\zeta^i(s^i) = t^i$ for all $t^i \in R^i$ and $\zeta^i(s^i) = s^i$ otherwise. Fix $\varepsilon > 0$. The measure p is regular; therefore there exist sets $F, G \subset S$ such that: F is closed, G is open, $F \subset R^i \times S^{-i} \subset G$ and[14] $p(G \setminus F) < \varepsilon$. Define[15] $F^i = \text{proj}_i F$ and $G^i = S^i \setminus \text{proj}_i(S \setminus G)$, then F^i is closed and G^i is open (since S is compact), $F^i \subset R^i \subset G^i$ and[16] $p((G^i \setminus F^i) \times S^{-i}) \le p(G \setminus F) < \varepsilon$. Next, apply Urisohn's Lemma (see Dunford and Schwartz 1958, Theorems I.5.2 and I.5.9), to obtain a continuous function $\varphi : S^i \to [0,1]$ such that $\varphi(s^i) = 1$ for all $s^i \in F^i$ and $\varphi(s^i) = 0$ for all $s^i \notin G^i$.

Define[17] $f(s) = \varphi(s^i)[h^i(s) - h^i(s^{-i}, t^i)]$; since $f : S \to \mathbb{R}$ is a continuous function, there exists an f-set T with $T^i \supset \{t^i\}$ such that (all the integrals are over S)

$$\left| \int f\, dp - \int f\, dq_T \right| < \varepsilon.$$

Next, let M be a bound on h^i; then

$$\left| \int f\, dp - \int \left[h^i(s) - h^i(s^{-i}, \zeta^i(s^i)) \right] dp \right| < 4M\varepsilon,$$

[13] Note that $\Sigma \supset \Sigma_0$ (= the product of the Σ^i's). A. S. Nowak has pointed out that the inclusion may be strict in the nonmetrizable case.

[14] The symbol $\setminus$ denotes set-theoretic subtraction.

[15] The projection from S onto S^i is denoted proj_i.

[16] What we have showed here is that in a compact space, the marginal of a regular measure is also regular.

[17] One may regard φ as a mixed strategy (more precisely, a behavioral strategy) of player i in the extended game: when the recommendation (to i) is s^i, he follows it (i.e., he plays s^i) with probability $1 - \varphi(s^i)$ and he plays t^i with probability $\varphi(s^i)$.

since the integrands may differ only on $(G^i \setminus F^i) \times S^{-i}$. Finally,

$$\int f\,dq_T = \sum_{s^i \in T^i} \varphi(s^i) \sum_{s^{-i} \in T^{-i}} \left[h^i(s) - h^i(s^{-i}, t^i)\right] q_T(s) \geq 0,$$

since (1) holds for q_T and $\varphi \geq 0$. The last three displayed inequalities together imply that the left-hand side of (4) is $> -(4M + 1)\varepsilon$, hence, ε being arbitrary, that (4) holds.

The general case. Let $\varepsilon > 0$ be given. Since both S^i and S^{-i} are compact and the function h^i is continuous, it is straightforward to obtain a finite partition of S^i into disjoint measurable sets $\{A_k\}_{k=1,\dots,K}$ such that $|h^i(s^{-i}, s^i) - h^i(s^{-i}, t^i)| < \varepsilon$ for any s^i, t^i belonging to the same A_k and every $s^{-i} \in S^{-i}$. For each k, fix some $t_k^i \in A_k$; let $R_k^i = \{s^i \in S^i : \zeta^i(s^i) \in A_k\}$ and define a Σ^i-measurable function ζ_k^i as follows: $\zeta_k^i(s^i) = t_k^i$ for $s^i \in R_k^i$ and ζ_k^i is the identity outside R_k^i. Inequality (4) holds for ζ_k^i, since the special case applies to it. Summing up these inequalities for all k, we obtain (4) for the admissible function η^i defined by $\eta^i(s^i) = t_k^i$ if $s^i \in R_k^i$ for some k. The construction above implies that $|h^i(s^{-i}, \zeta^i(s^i)) - h^i(s^{-i}, \eta^i(s^i))| < \varepsilon$ for all s, therefore the left-hand side of (4) for ζ^i is $> -\varepsilon$, hence ≥ 0. ■

REMARK. Under the assumptions of Theorem 3, one may prove—using a fixed-point theorem—the existence of a Nash equilibrium (e.g., see Fan 1952 [apply his Theorem 2] or Glicksberg 1952 [his proof in §2 may be easily applied to an arbitrary set of players]).

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References

Aumann, R. J. (1974). Subjectivity and Correlation in Randomized Strategies. *J. Math. Economics* **1** 67–95.

______. (1987). Correlated Equilibrium as an Expression of Bayesian Rationality. *Econometrica* **55** 1–18.

Dunford, N. and Schwartz, J. T. (1958). *Linear Operators*. Vol. I. Interscience Publishers, New York.

Fan, K. (1952). Fixed-Point Theorems in Locally Convex Topological Spaces. *Proc. Nat. Acad. Sci. USA* **38** 121–126.

Glicksberg, I. L. (1952). A Further Generalization of the Kakutani Fixed Point Theorem, with Application to Nash Equilibrium Points. *Proc. Amer. Math. Soc.* **3** 170–174.

Peleg, B. (1969). Equilibrium Points for Games with Infinitely Many Players. *J. London Math. Soc.* **44** 292–294.

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