Forecast-Hedging and Calibration

Sergiu Hart

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Joint work with

Dean P. Foster

University of Pennsylvania &
Amazon

Paper

Paper

- Dean P. Foster and Sergiu Hart
  “Forecast Hedging and Calibration”
  - *Journal of Political Economy* 2021

www.ma.huji.ac.il/hart/publ.html#calib-int

Other Papers

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- Sergiu Hart
  "Calibration: The Minimax Proof", 1995 [2021]
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- Dean P. Foster and Sergiu Hart
  “Smooth Calibration, Leaky Forecasts, Finite Recall, and Nash Dynamics”
  Games and Economic Behavior 2018
  www.ma.huji.ac.il/hart/publ.html#calib-eq

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  www.ma.huji.ac.il/hart/publ.html#calib-eq
- Dean P. Foster and Sergiu Hart
  " 'Calibeating': Beating Forecasters at Their Own Game", 2021
  www.ma.huji.ac.il/hart/publ.html#calib-beat

Calibrated Forecasts

Calibration

Calibration

- Forecaster says:
  "There is a 70% chance of rain tomorrow"

Calibration

- Forecaster says:
  "There is a p chance of rain tomorrow"

Calibration

- Forecaster says:
"There is a p chance of rain tomorrow"
- Forecaster is **CALIBRATED** if
  - For every p:
The proportion of rainy days among those days when the forecast was p equals p
(or: is close to p in the long run)

Calibration

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Theorem:

CALIBRATION ***can be guaranteed***

(no matter what the weather will be)

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  (e.g.: with probability $\mathbf{1/2}$, forecast $= \mathbf{25\%}$
  with probability $\mathbf{1/2}$, forecast $= \mathbf{60\%}$)

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The MINIMAX Theorem

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THEOREM (von Neumann 1928)

IF

$X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ are compact convex sets,
and $f : X \times Y \longrightarrow \mathbb{R}$ is a continuous function
that is convex-concave,
i.e., $f(\cdot, y) : X \longrightarrow \mathbb{R}$ is convex for fixed y,
and $f(x, \cdot) : Y \longrightarrow \mathbb{R}$ is concave for fixed x,

THEN

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y).$$

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  (for every strategy of the opponent)
- finite game; probabilistic (mixed) strategies

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  $\Rightarrow$ FINITE 2-person 0-sum game

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- Foster and Hart 2016 [**2021**]: even simpler

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Oakes 1985

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Foster and Kakade (2004, 2006)
Foster and Hart (2018, **2021**)

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A Simple Illustration

At time t:

- q = forecast (in $[0,1]$)
- a = rain/no-rain (in $\{0,1\}$)
- Change in $S = (G(q)+a-q)^2 - G(q)^2$
  - First-order approximation = 2Δ, where

$$\Delta := G(q)\cdot(a-q)$$

- $G(p) := r(p) - n(p)\cdot p$ = gap at p
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  $$\mathbb{E}_q[\Delta] \le 0 \ \ \textbf{for all } a$$

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  $\Rightarrow$ How ?

Forecast-Hedging (FH)

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Forecast-Hedging (FH)

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- **DETERMINISTIC FORECAST-HEDGING**: Guarantee that

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- **STOCHASTIC FORECAST-HEDGING**: Guarantee that

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Stochastic Forecast-Hedging

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Stochastic Forecast-Hedging

- **STOCHASTIC FORECAST-HEDGING**:

  $\mathbb{E}_q[\Delta] = \mathbb{E}_q[G(q) \cdot (a - q)] \leq 0$ **for all** a

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- $p_1 G(q_1) + p_2 G(q_2) = 0$
- $q_1 - q_2$ is small

Stochastic FH

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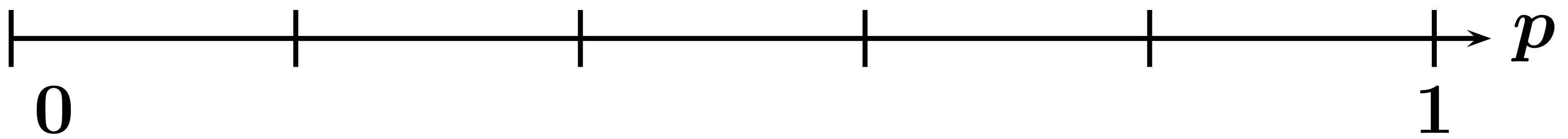

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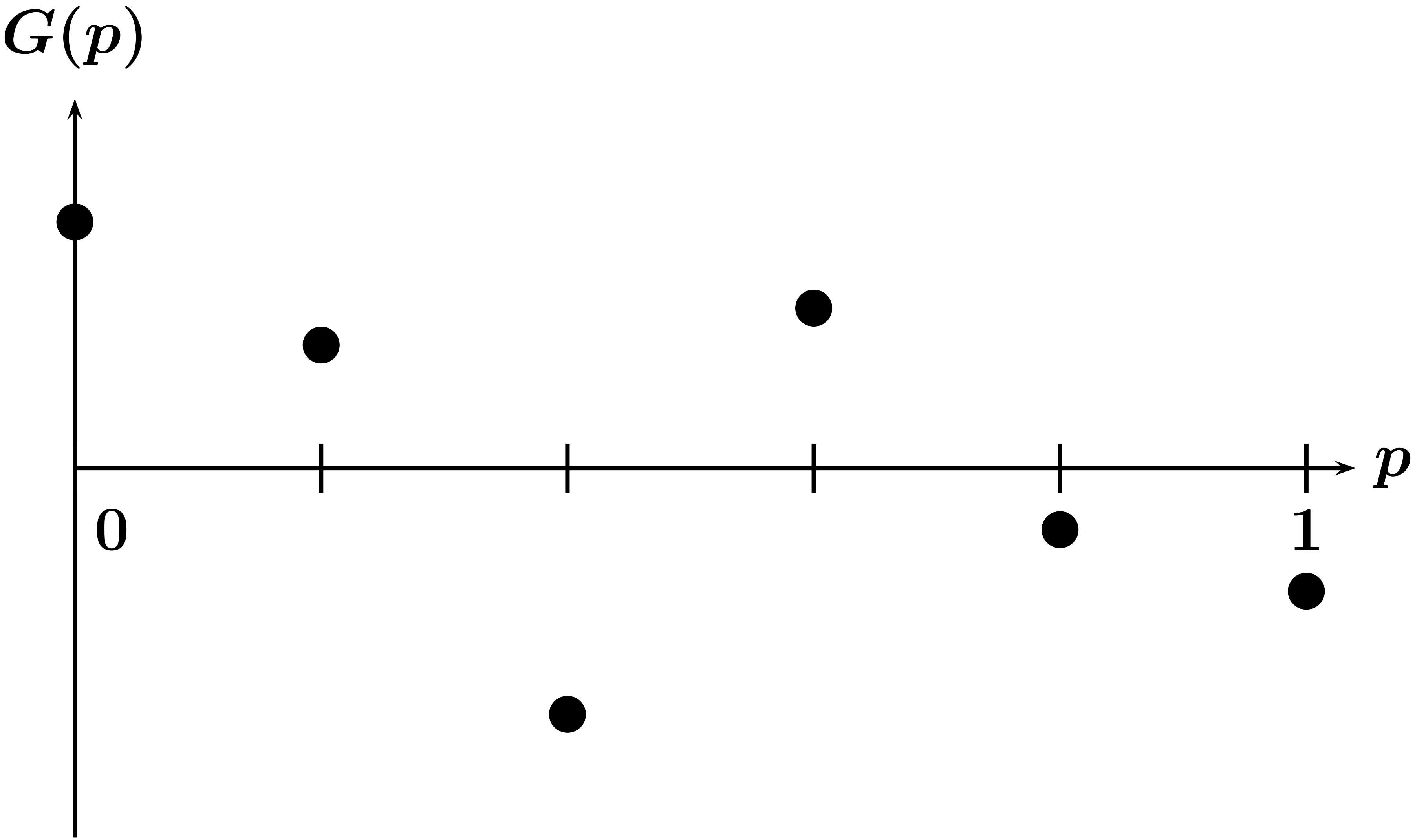

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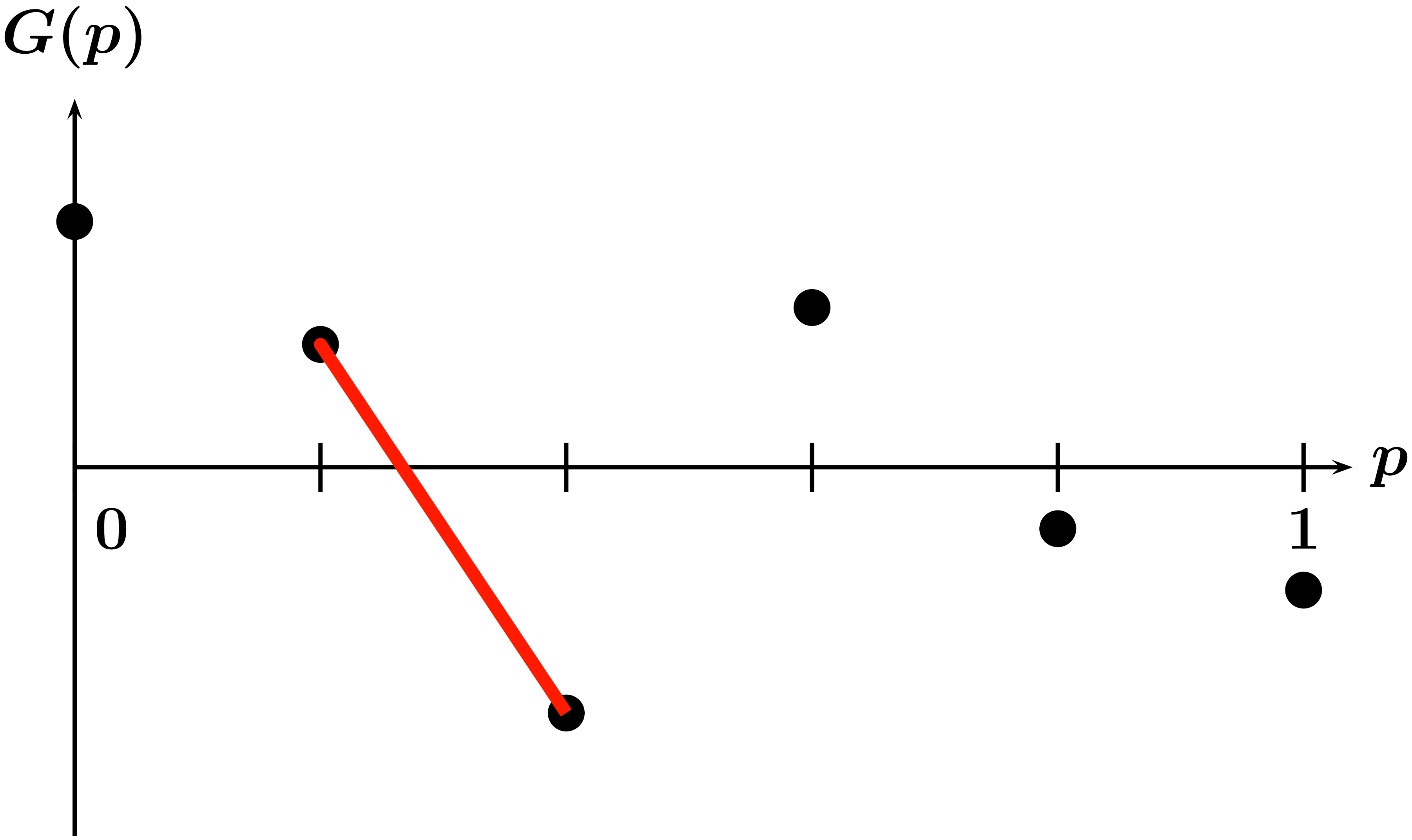

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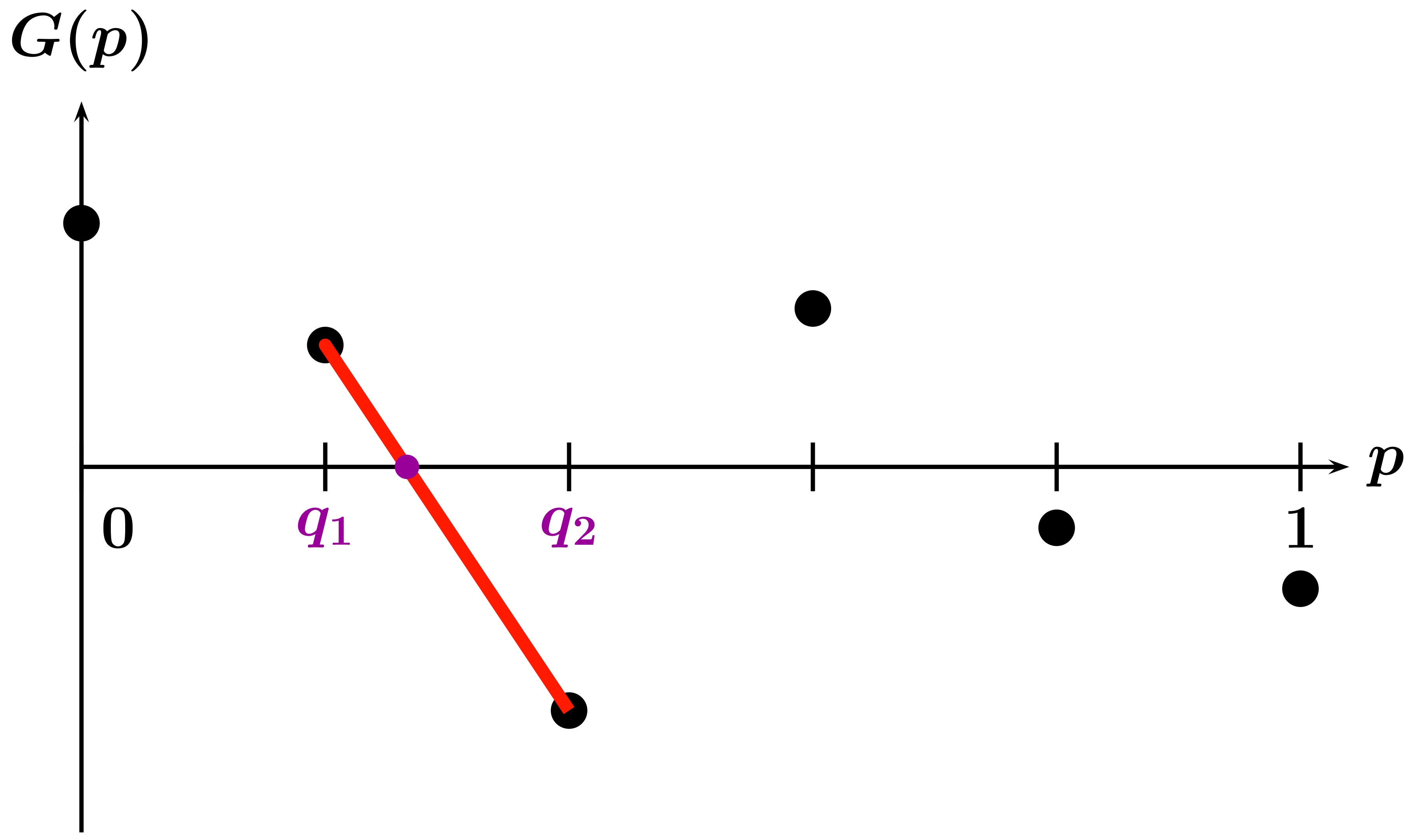

Deterministic FH

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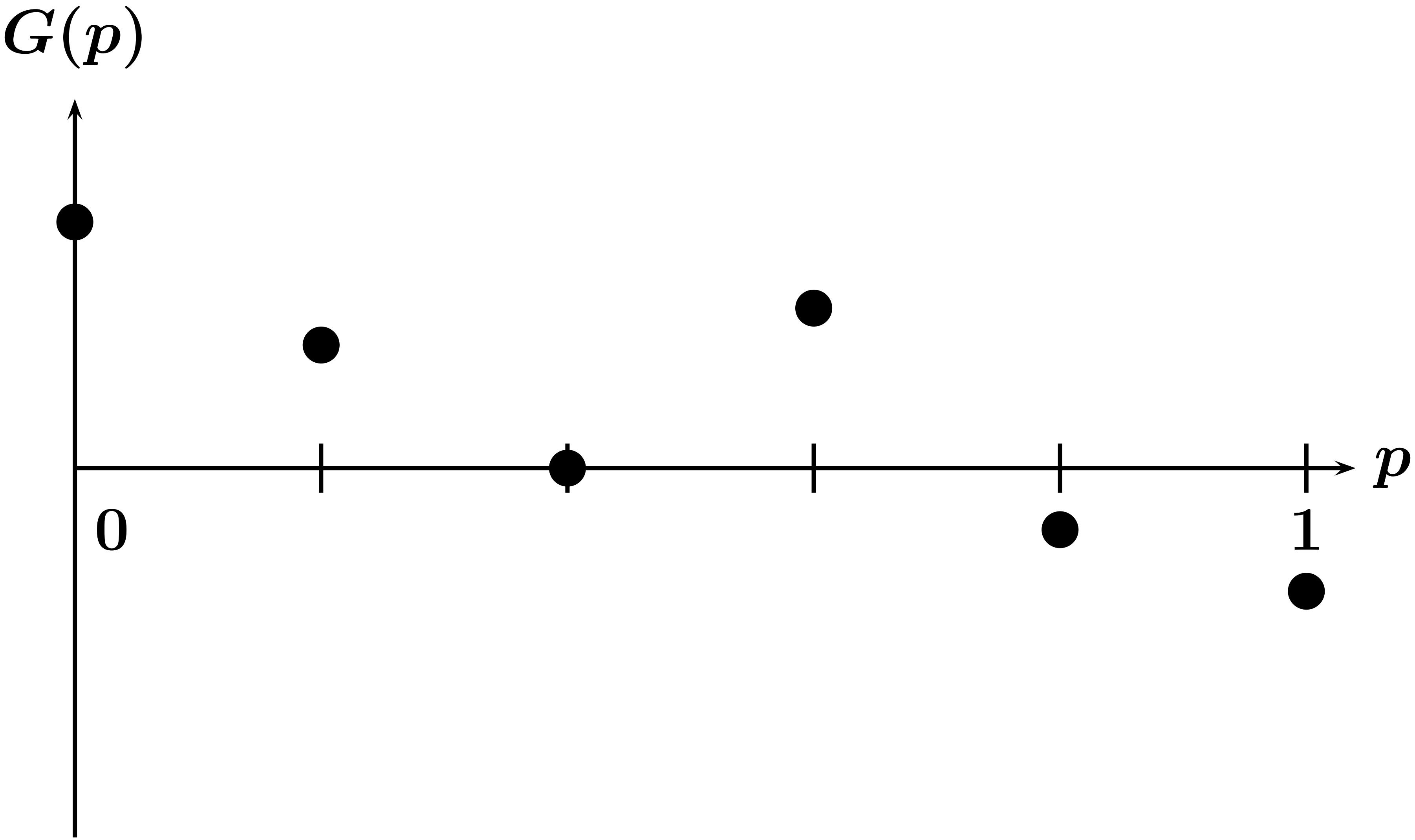

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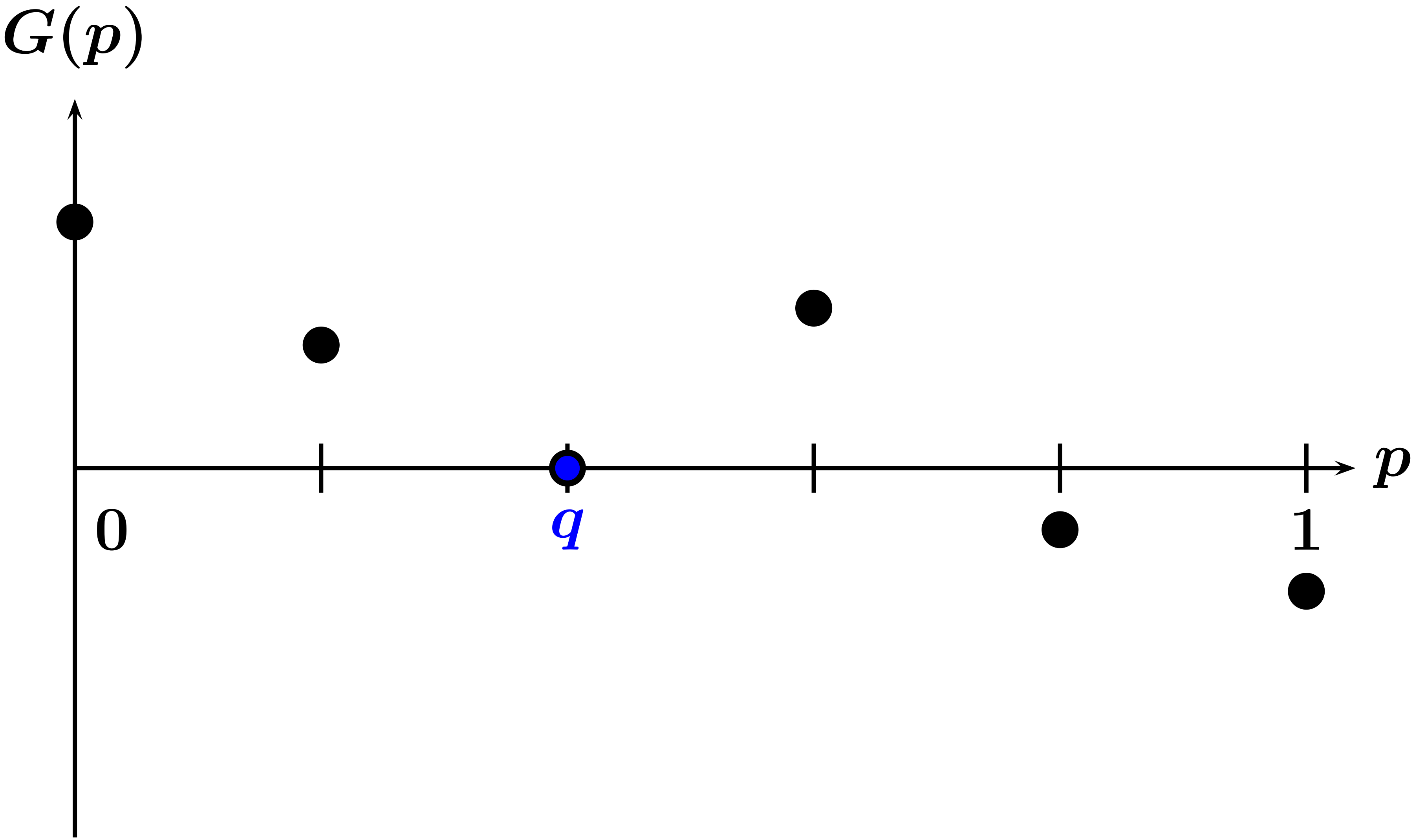

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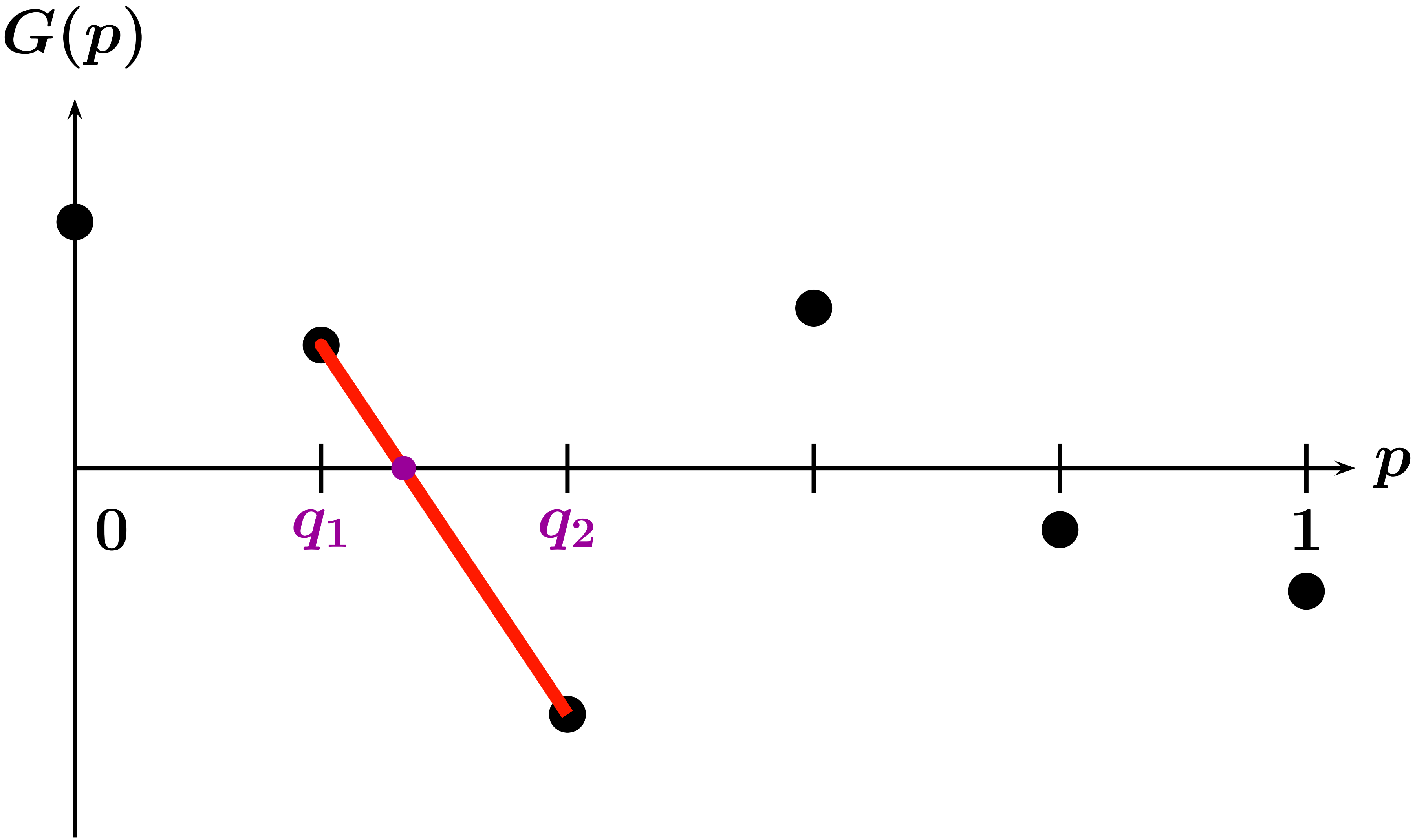

Forecast-Hedging

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Forecast-Hedging

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- forecasting ?

Forecast-Hedging

- fore-casting ?
- **BACK-CASTING !**
  ("Politicians' Lemma")

Calibration in Practice

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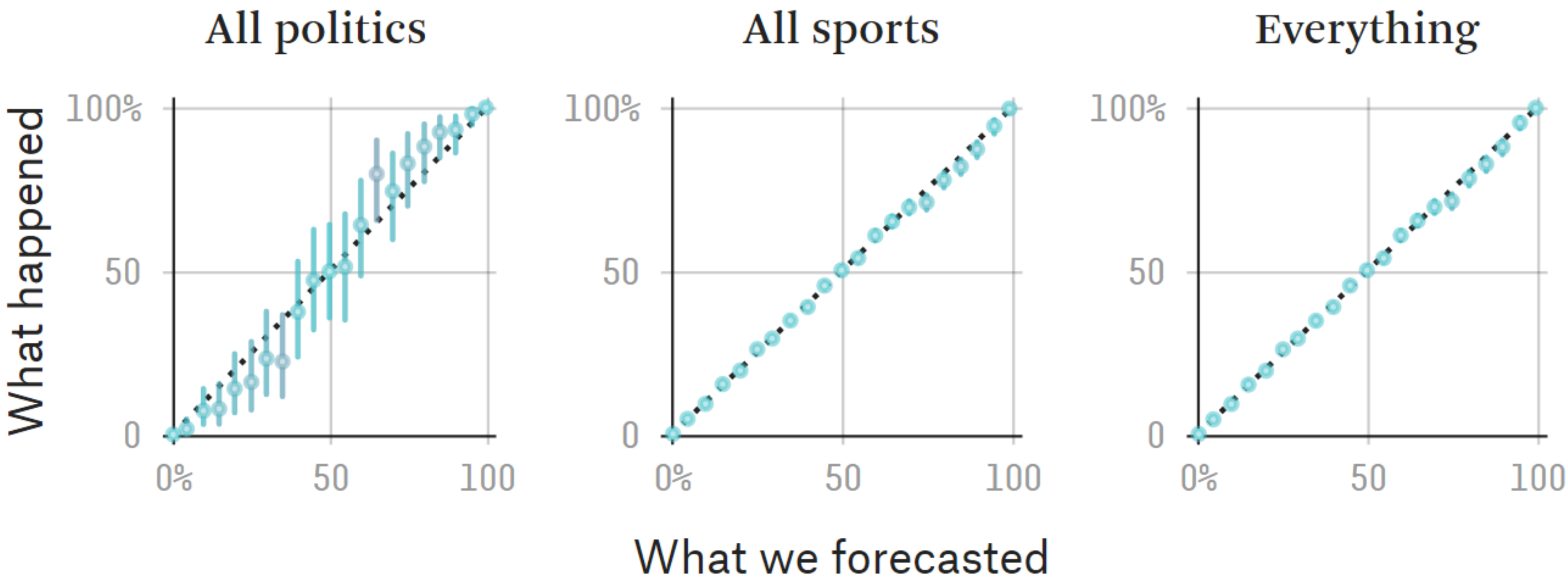
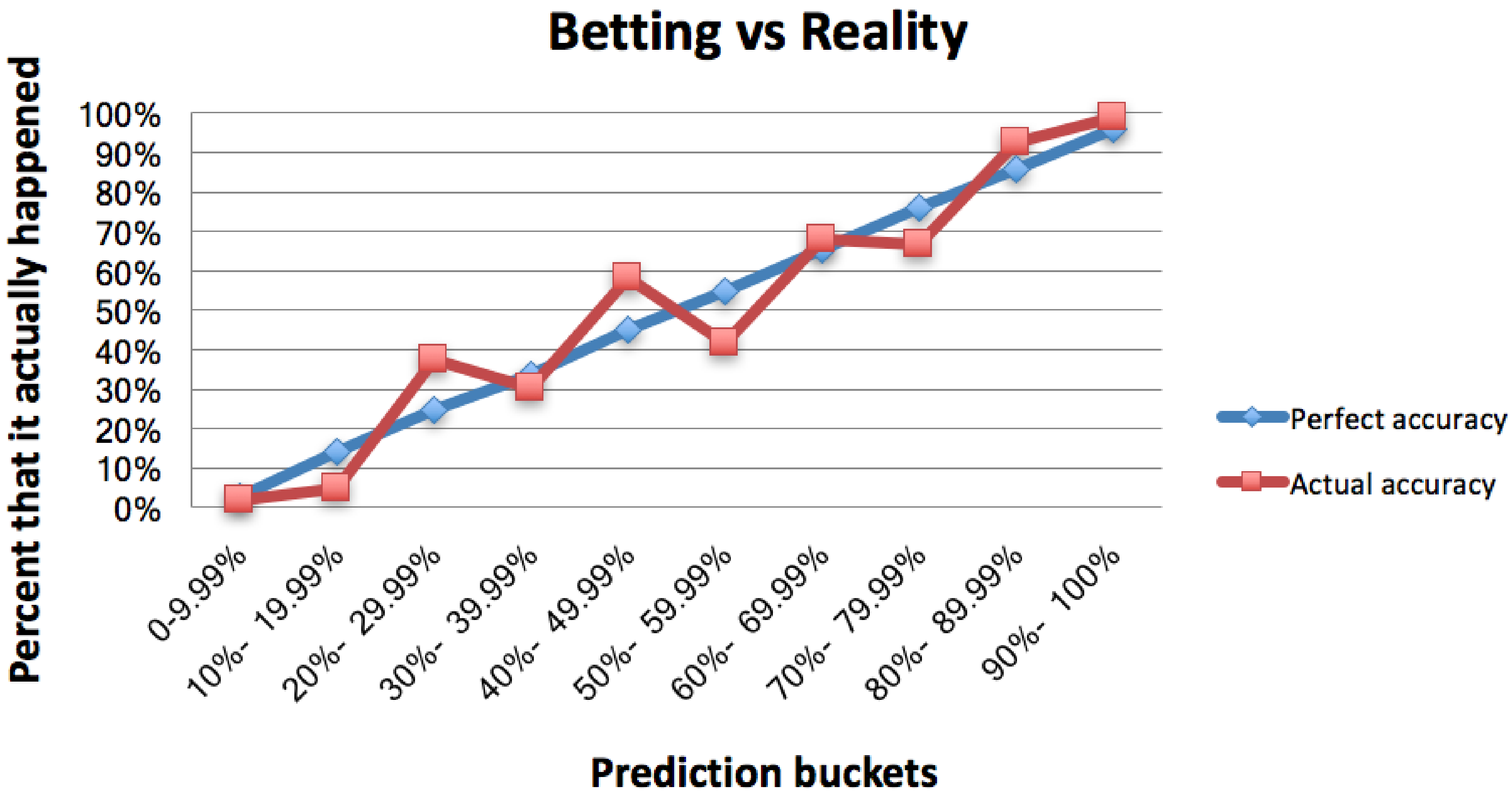


Calibration plots of FiveThirtyEight.com
(as of June 2019)

Calibration in Practice



Calibration plot of ElectionBettingOdds.com
(2016 – 2018)

Forecast-Hedging

Binning

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CONTINUOUS CALIBRATION:

- Each $w_i(c)$ is a continuous function of c ("continuous fractional binning")

Binning and Calibration

Binning and Calibration

- **GAP** of w_i at time t

$$G_t(w_i) := \sum_{s=1}^{t} w_i(c_s)(a_s - c_s)$$

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Deterministic Forecast-Hedging

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$\Rightarrow$ **Deterministic continuous** calibration

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for **every** $a_t \in A$

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Stochastic Forecast-Hedging

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- $\Rightarrow$ **Stochastic** classic calibration

Forecast-Hedging $\mapsto$ Calibration

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Fixed Point $\mapsto$ **deterministic** calibration

Forecast-Hedging $\mapsto$ Calibration

Fixed Point $\mapsto$ **deterministic** calibration

MiniMax $\mapsto$ **stochastic** calibration

Forecast-Hedging Tools

Outgoing Fixed Point Theorem

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- $\Leftrightarrow$ Brouwer's fixed-point theorem

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If f is bounded and $\varepsilon > 0$ then

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- support of η is at most $m + 2$ points
- if f is continuous it holds also for $\varepsilon = 0$

Outgoing Theorems

Outgoing FIXED POINT (FP):

$f : C \to \mathbb{R}^m$ continuous function

$\Rightarrow \quad \exists$ **POINT** $y \in C$

s.t. $f(y) \cdot (c - y) \leq 0$ for all $c \in C$

Outgoing MINIMAX (MM):

$f \to \mathbb{R}^m$ bounded function, $\varepsilon > 0$

$\Rightarrow \quad \exists$ **PROBABILITY DISTRIBUTION** $\eta \in \Delta(C)$

s.t. $\mathbb{E}_{y \sim \eta}[f(y) \cdot (c - y)] \leq \varepsilon$ for all $c \in C$

Calibration: FH Results

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Foster and Vohra 1998

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$\exists$ **DETERMINISTIC** procedure
that is **CONTINUOUSLY CALIBRATED**

(Foster and Kakade 2006, Foster and Hart 2018)

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(Foster and Kakade 2006, Foster and Hart 2018)

$\forall \varepsilon > 0, \rho > 0$: $\exists$ ρ-**LOCAL STOCHASTIC**
procedure that is ε-**CALIBRATED**

Foster 1999, Kakade and Foster 2004

Game Dynamics

Calibration and Game Dynamics

General n-person game

Calibration and Game Dynamics

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- Players ***forecast*** the play in the next period

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Foster and Vohra 1997

Continuously Calibrated Learning

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$\Rightarrow$ $1 - \varepsilon$ **OF THE TIME** the play is a **NASH ε-EQUILIBRIUM** in the long run (a.s.)

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In each period t, each player i:

1. runs the procedure **(F)** to get c_t
2. plays $g^i(c_t)$ given by **(P)**

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CONTINUOUSLY CALIBRATED LEARNING:

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CONTINUOUSLY CALIBRATED LEARNING:

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- ***Nash ε-equilibria are played at least $1-\varepsilon$ of the time*** in the long run (a.s.)

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Proof:

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$$\textbf{play}_t = g(c_t)$$

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  - $g(\textbf{play}_t) = g(g(c_t)) \approx g(c_t) = \textbf{play}_t$

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- **CONTINUOUS BEST REPLY**
  $\Rightarrow$ fixed point = **NASH EQUILIBRIUM**

Dynamics and Equilibrium

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Dynamics and Equilibrium

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Dynamics and Equilibrium

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Dynamics and Equilibrium

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(Hart and Mas-Colell 2003)

Summary

Calibration, Dynamics, Equilibria

MINIMAX universe

Calibration, Dynamics, Equilibria

MINIMAX universe

FIXEDPOINT universe

Calibration, Dynamics, Equilibria

MINIMAX universe | **FIXEDPOINT** universe

- ***stochastic***
  forecast-hedging

Calibration, Dynamics, Equilibria

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Calibration, Dynamics, Equilibria

MINIMAX universe	**FIXEDPOINT** universe
• ***stochastic*** forecast-hedging	• ***deterministic*** forecast-hedging
• *MM*-procedures	

Calibration, Dynamics, Equilibria

MINIMAX universe

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FIXEDPOINT universe

- ***deterministic*** forecast-hedging
- ***FP***-procedures

Calibration, Dynamics, Equilibria

MINIMAX universe

- ***stochastic*** forecast-hedging
- ***MM***-procedures
- ***classic*** calibration

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Calibration, Dynamics, Equilibria

MINIMAX universe	**FIXEDPOINT** universe
• ***stochastic*** forecast-hedging	• ***deterministic*** forecast-hedging
• ***MM***-procedures	• ***FP***-procedures
• ***classic*** calibration	• ***continuous*** calibration

Calibration, Dynamics, Equilibria

MINIMAX universe

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- ***MM***-procedures
- ***classic*** calibration
- ***correlated*** equilibria

FIXEDPOINT universe

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- ***FP***-procedures
- ***continuous*** calibration

Calibration, Dynamics, Equilibria

MINIMAX universe	**FIXEDPOINT** universe
stochastic forecast-hedging	*deterministic* forecast-hedging
MM-procedures	*FP*-procedures
classic calibration	*continuous* calibration
correlated equilibria	*Nash* equilibria

Calibration, Dynamics, Equilibria

MINIMAX universe	**FIXEDPOINT** universe
• ***stochastic*** forecast-hedging	• ***deterministic*** forecast-hedging
• ***MM***-procedures	• ***FP***-procedures
• ***classic*** calibration	• ***continuous*** calibration
• ***correlated*** equilibria	• ***Nash*** equilibria
• ***time-average***	

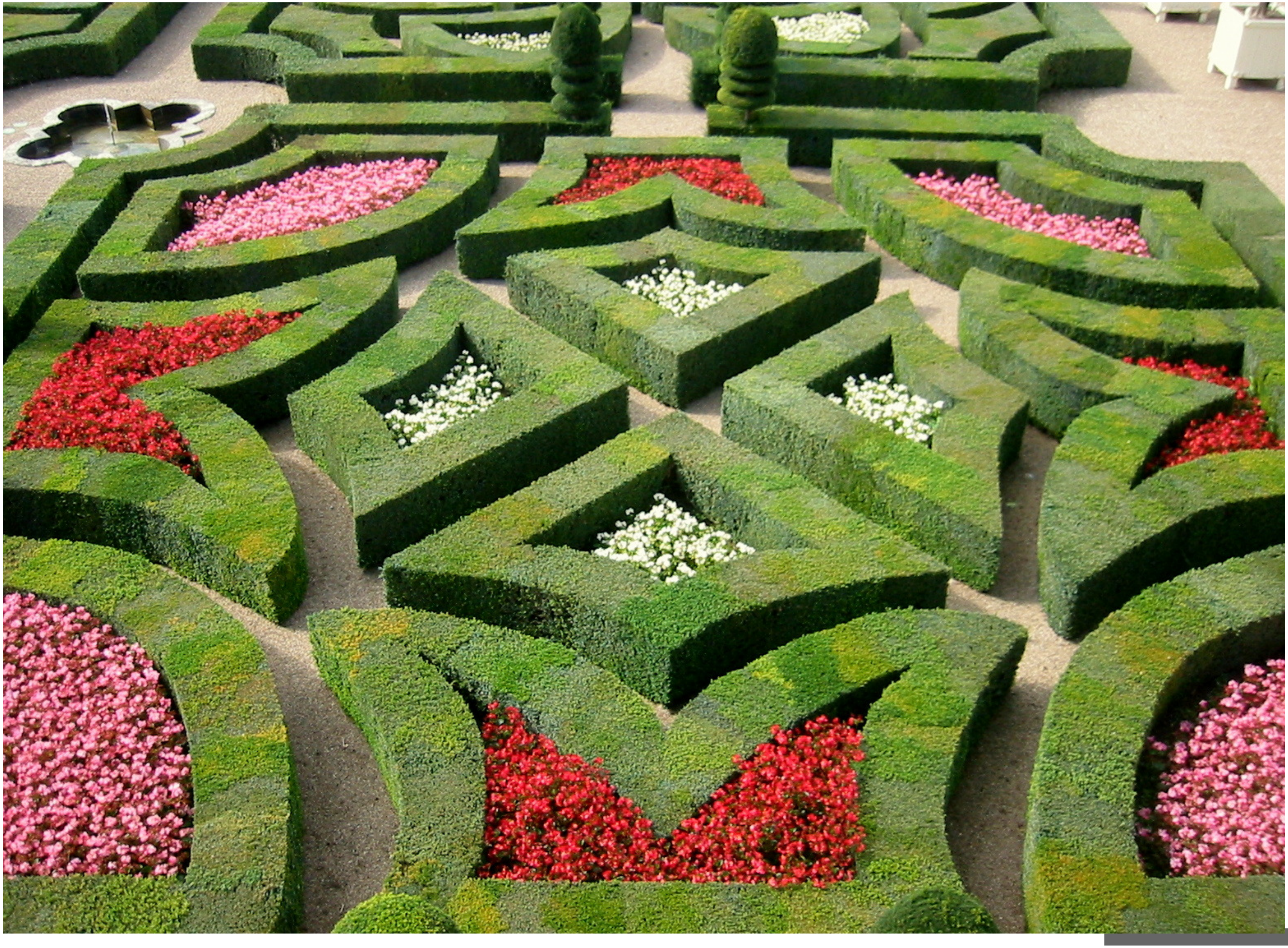
Calibration, Dynamics, Equilibria

MINIMAX universe	FIXEDPOINT universe
stochastic forecast-hedging	*deterministic* forecast-hedging
MM-procedures	*FP*-procedures
classic calibration	*continuous* calibration
correlated equilibria	*Nash* equilibria
time-average	*period-by-period*

Perfect Hedging

Perfect Hedging

Perfect Hedging



Perfect Hedging

Château de Villandry, 2005