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Discrete Colonel Blotto and General Lotto games

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Abstract A class of integer-valued allocation games—"General Lotto games"—is introduced and solved. The results are then applied to analyze the classical discrete "Colonel Blotto games"; in particular, optimal strategies are obtained for all symmetric Colonel Blotto games.

1 Introduction

There are two players, Player A and Player B. Player A is given A alabaster marbles to distribute any way he wants into K urns, and Player B is given B black marbles to (simultaneously) distribute into the same K urns. One urn is chosen at random (each urn is equally likely to be chosen); if it contains more alabaster marbles than black marbles, A wins; if it contains more black marbles than alabaster marbles, B wins; otherwise it is a draw. This two-person zero-sum game (a win is $+1$, a loss is -1, and a tie is 0), which we denote $\mathcal{B}(A, B; K)$, is known in the literature as a *Colonel Blotto* game: each urn represents a "battlefield," and the number of marbles in urn i corresponds to the number of "battalions" sent to battlefield i; see Borel (1921), Tukey (1949), Shubik (1982). This class of games is a prime example of "allocation"

Dedicated with great admiration to David Gale on his 85th birthday.

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games, where sides compete on different "fronts" and need to allocate their resources optimally among them. Some examples are lobbying and campaigning by political parties, research and development competitions among firms, and multi-unit and all-pay auctions.

How should the game be played? For example, when $A = 24$, $B = 18$, and $K = 8$? The problem turns out to be quite difficult, in part due to the integer restriction on the number of balls in each urn. Most of the literature has relaxed this condition; see Robertson (2006) for a complete solution of the "continuous" version and a survey of the literature.

Here we will proceed along a different route, one that respects the integer constraint. Specifically, we consider in Sect. 2 a variant of Colonel Blotto games—the "*General Lotto games*"—which we solve in Sect. 3: we find the value and optimal strategies. In Sect. 4 we show that certain optimal strategies of General Lotto games can be implemented in Colonel Blotto games. This yields, in particular, optimal strategies for all symmetric Colonel Blotto games (i.e., when $A = B$), as well as for other cases. We conclude with a discussion in Sect. 5.

2 Lotto Games

We consider two variants of the Colonel Blotto games, which we call "Colonel Lotto games" and "General Lotto games."

2.1 Colonel Lotto Games

Assume that the K urns are *indistinguishable*. This is easily seen to be equivalent to the following. Player A has K alabaster urns of his own, Player B has K black urns of his own, and, after each player distributes his marbles into his own urns, one alabaster urn and one black urn are chosen at random (all urns have the same probability of being chosen), and the contents of the two urns are compared to determine the winner.[1] This is a "symmetrized-across-urns" version of the Colonel Blotto game; we will refer to it as a *Colonel Lotto* game, and denote it $\mathcal{L}(A, B; K)$.

In both games a pure strategy of Player A is a *K-partition* $x = \langle x_1, x_2, \ldots, x_K \rangle$ *of* A, i.e., nonnegative integers $x_1, x_2, \ldots, x_K$ with $x_1 + x_2 + \cdots + x_K = A$, and a pure strategy of Player B is a K-partition $y = \langle y_1, y_2, \ldots, y_K \rangle$ of B. The payoff in the Colonel Blotto game is[2]

$$h_{\mathcal{B}}(x, y) = \frac{1}{K} \sum_{k=1}^{K} \operatorname{sign}(x_k - y_k),$$

1 Avrahami and Kareev (2005) have conducted a laboratory experiment on this specific version of the game.

2 $\operatorname{sign}(z) = 1$ when $z > 0$; $\operatorname{sign}(z) = -1$ when $z < 0$; and $\operatorname{sign}(0) = 0$.

whereas in the Colonel Lotto game it is

$$h_{\mathcal{L}}(x, y) = \frac{1}{K^2} \sum_{k=1}^{K} \sum_{\ell=1}^{K} \operatorname{sign}(x_k - y_\ell).$$

For a pure strategy x of Player A, let $\sigma(x)$ be the mixed strategy that gives probability $1/K!$ to each one of the $K!$ permutations of x; then[3] $h_{\mathcal{B}}(\sigma(x), y) = h_{\mathcal{L}}(x, y)$ for all pure strategies y of Player B. For a mixed strategy ξ of Player A, let $\sigma(\xi)$ be the mixed strategy obtained by replacing each pure x in the support of ξ by its corresponding[4] $\sigma(x)$; then $h_{\mathcal{B}}(\sigma(\xi), y) = h_{\mathcal{L}}(\xi, y)$ for all pure y, and so $h_{\mathcal{B}}(\sigma(\xi), \eta) = h_{\mathcal{L}}(\xi, \eta)$ for all mixed strategies η of Player B (we will refer to the strategies $\sigma(x)$ and $\sigma(\xi)$ as *symmetric across urns*). The same holds for Player B, and we thus have:

The Colonel Blotto game $\mathcal{B}(A, B; K)$ and the Colonel Lotto game $\mathcal{L}(A, B; K)$ have the same value. Moreover, the mapping σ maps the optimal strategies in the Colonel Lotto game onto the optimal strategies in the Colonel Blotto game that are symmetric across urns.[5]

2.2 General Lotto Games

Let $x = \langle x_1, x_2, \ldots, x_K \rangle$ be a pure strategy of Player A, i.e., a K-partition of A. We will view x as a random variable X whose values are $x_1, x_2, \ldots, x_K$ with probability $1/K$ each. For example, $x = \langle 0, 0, 0, 0, 5, 5, 5, 9 \rangle$ (for $A = 24$ and $K = 8$) yields $\mathbf{P}(X = 0) = 4/8 = 1/2$, $\mathbf{P}(X = 5) = 3/8$, and $\mathbf{P}(X = 9) = 1/8$; we will write this as[6] $X = (1/2)\mathbf{1}_0 + (3/8)\mathbf{1}_5 + (1/8)\mathbf{1}_9$. The expectation of X is $\mathbf{E}(X) = (1/K)\sum_{i=1}^{K} x_i = A/K$, which is the average number of marbles per urn. Similarly, let the random variable Y correspond to the strategy $y = \langle y_1, y_2, \ldots, y_K \rangle$ of Player B; then the (expected) payoff $h_{\mathcal{L}}(x, y)$ in the Colonel Lotto game equals

$$H(X, Y) := \mathbf{P}(X > Y) - \mathbf{P}(X < Y). \tag{1}$$

We now consider the generalization where X and Y are allowed to be *any* nonnegative integer-valued random variables with expectations $\mathbf{E}(X) = A/K = a$ and $\mathbf{E}(Y) = B/K = b$, respectively. That is, we remove the restriction that they can be derived from probability distributions on K-partitions.

For each $a, b > 0$ we thus define the game $\Gamma(a, b)$ where Player A chooses a (distribution of a) nonnegative integer-valued random variable X with expectation $\mathbf{E}(X) = a$, and Player B chooses a (distribution of a) nonnegative integer-valued

[3] We will write h also for the (bilinear) extension of the payoff function to pairs of mixed strategies.

[4] Equivalently, choose a random numbering of the K urns and use ξ.

[5] However, there may be optimal strategies in Colonel Blotto games that are not symmetric across urns.

[6] $\mathbf{1}_i$ denotes the Dirac measure which puts probability one on i. For simplicity, we will identify a random variable with its distribution.

random variable Y with $\mathbf{E}(Y) = b$, and the payoff is given by (1) with X and Y taken to be independent.[7] We will call $\Gamma(a, b)$ a *General Lotto* game.

In Sect. 3 we will prove that each General Lotto game has a minimax value; we do so by providing explicit optimal strategies for both players (see Fig. 1 for a summary).[8]

2.3 Continuous General Lotto Games

Before proceeding to the analysis of the General Lotto games, it is instructive to present a further generalization, which dispenses also with the integer-valued restriction. In a *Continuous General Lotto game*, which we denote $\Lambda(a, b)$, Player A chooses a (distribution of a) nonnegative random variable X with $\mathbf{E}(X) = a$, Player B chooses a (distribution of a) nonnegative random variable Y with $\mathbf{E}(Y) = b$, and the payoff is given by $H(X, Y)$ as defined in (1) with X and Y independent.

Theorem 1 *Let $a \geq b > 0$. The value of the Continuous General Lotto game $\Lambda(a, b)$ is*

$$\operatorname{val} \Lambda(a, b) = \frac{a-b}{a} = 1 - \frac{b}{a},$$

and the unique optimal strategies are $X^* = U(0, 2a)$ for Player A and $Y^* = (1 - b/a)\mathbf{1}_0 + (b/a)U(0, 2a)$ for Player B.[9]

When $a > b$ the strategies of Theorem 1 can be interpreted as follows. The stronger Player A plays a uniform distribution with expectation a, on the interval $(0, 2a)$. The weaker Player B "gives up" and plays 0 with probability $1 - b/a$, and with the remaining probability b/a he "matches" the stronger player (by playing the same uniform distribution on $(0, 2a)$); the probability b/a is chosen so that the expectation will indeed be b. The special case where the game is symmetric, i.e., $a = b$, has been solved by Bell and Cover (1980, Sect. 2); see also Myerson (1993) and Lizzeri (1999). The solution of the nonsymmetric case ($a > b$) is due to Sahuguet and Persico (2006). In the Appendix we will provide an elementary direct proof (the difficulty lies in showing the uniqueness of the optimal strategies; Sahuguet and Persico use a reduction to "all-pay auction" games and apply known results there, which makes the proof quite complex).

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[7] Equivalently, let the payoff be $\operatorname{sign}(X - Y)$ where X and Y are two independent draws from the two chosen distributions.

[8] Since the game $\Gamma(a, b)$ has infinitely many pure strategies, the classical Minimax Theorem for finite games does not apply to it. However, the existence of value can be shown also by finite-approximation arguments (as pointed out by Abraham Neyman). Finally, as the set of distributions of nonnegative integer-valued random variables X with a given expectation is already convex (and the payoff is linear in the distribution of X), no further mixtures are needed.

[9] Throughout this paper we identify a random variable with its distribution; thus $Y = \lambda\mathbf{1}_0 + (1 - \lambda)U$ means that $Y = 0$ with probability λ and $Y = U$ with probability $1 - \lambda$. The notation $U(c, d)$ stands for the uniform distribution on the interval (c, d); the cumulative distribution functions are thus $F_{X^*}(x) = x/(2a)$ for all $x \in [0, 2a]$, and $F_{Y^*}(y) = by/(2a^2)$ for all $y \in (0, 2a]$ and $F_{Y^*}(0) = 1 - b/a$.

3 Solution of the General Lotto Games

In this section we will solve the General Lotto games. We will assume throughout that *all random variables are nonnegative and integer-valued*. Every such random variable X is $\sum_{i=0}^{\infty} p_i \mathbf{1}_i$ where $p_i = \mathbf{P}(X = i)$ (so $p_i \geq 0$ for all i and $\sum_i p_i = 1$). Also, $\mathbf{E}(X) = \sum_{i=1}^{\infty} i\, \mathbf{P}(X = i) = \sum_{i=1}^{\infty} \mathbf{P}(X \geq i)$. For every Y we have

$$\begin{aligned} H(X, Y) &= \sum_{i=0}^{\infty} p_i \left[\mathbf{P}(i > Y) - \mathbf{P}(i < Y)\right] \\ &= \sum_{i=0}^{\infty} p_i \left[1 - \mathbf{P}(Y \geq i) - \mathbf{P}(Y \geq i+1)\right] \\ &= 1 - \sum_{i=0}^{\infty} p_i \left[\mathbf{P}(Y \geq i) + \mathbf{P}(Y \geq i+1)\right]. \end{aligned}$$

For every positive integer m we define three uniform distributions, each one with expectation m:

$$\begin{aligned} U^m &:= U(\{0, 1, \ldots, 2m\}) = \sum_{i=0}^{2m} \left(\frac{1}{2m+1}\right) \mathbf{1}_i; \\ U_{\mathrm{O}}^m &:= U(\{1, 3, \ldots, 2m-1\}) = \sum_{i=1}^{m} \left(\frac{1}{m}\right) \mathbf{1}_{2i-1}; \text{ and} \\ U_{\mathrm{E}}^m &:= U(\{0, 2, \ldots, 2m\}) = \sum_{i=0}^{m} \left(\frac{1}{m+1}\right) \mathbf{1}_{2i} \end{aligned}$$

(think of U_{O}^m and U_{E}^m as "uniform on ODD numbers" and "uniform on EVEN numbers," respectively; note that U^m is the average of U_{O}^m and U_{E}^m, with weights $m/(2m+1)$ and $(m+1)/(2m+1)$, respectively). For every Y we get

$$\begin{aligned} H(U_{\mathrm{O}}^m, Y) &= 1 - \frac{1}{m} \sum_{i=1}^{m} \left[\mathbf{P}(Y \geq 2i-1) + \mathbf{P}(Y \geq 2i)\right] \\ &= 1 - \frac{1}{m} \sum_{j=1}^{2m} \mathbf{P}(Y \geq j) \geq 1 - \frac{\mathbf{E}(Y)}{m}, \end{aligned} \tag{2}$$

with equality if and only if $\sum_{j=2m+1}^{\infty} \mathbf{P}(Y \geq j) = 0$, or $Y \leq 2m$; and

$$H(U_{\mathrm{E}}^m, Y) = 1 - \frac{1}{m+1} \sum_{i=0}^{m} \left[\mathbf{P}(Y \geq 2i) + \mathbf{P}(Y \geq 2i+1)\right]$$

$$
\begin{aligned}
&= 1 - \frac{1}{m+1}\left(\sum_{j=1}^{2m+1} \mathbf{P}(Y \geq j) + \mathbf{P}(Y \geq 0)\right) \\
&\geq 1 - \frac{\mathbf{E}(Y)+1}{m+1}, \qquad (3)
\end{aligned}
$$

with equality if and only if $\sum_{j=2m+2}^{\infty} \mathbf{P}(Y \geq j) = 0$, or $Y \leq 2m+1$.

Let $a \geq b > 0$; we will distinguish three cases in the analysis of $\Gamma(a, b)$:

- a is an integer (Theorem 2);
- a is not an integer and[10] $\lfloor a \rfloor < \lceil b \rceil$ (Theorem 3); and
- a is not an integer and $\lfloor a \rfloor \geq \lceil b \rceil$ (Theorem 4).

See the table of Fig. 1 at the end of this section for a summary of the results. As in the Continuous General Lotto games, the main difficulties lie in characterizing the optimal strategies.

Theorem 2 *Let $a \geq b > 0$ where a is an integer. Then the value of the General Lotto game $\Gamma(a, b)$ is*

$$\operatorname{val}\Gamma(a, b) = \frac{a-b}{a} = 1 - \frac{b}{a}.$$

The optimal strategies are as follows:

(i) *When $a = b$ the strategy X is optimal (for either player) if and only if*[11] $X \in \operatorname{conv}\{U_{\mathrm{O}}^a, U_{\mathrm{E}}^a\}$.
(ii) *When $a > b$ the strategy U_{O}^a is the unique optimal strategy of Player A.*
(iii) *When $a > b$ the strategies $(1 - b/a)\mathbf{1}_0 + (b/a)V$ with $V \in \operatorname{conv}\{U_{\mathrm{O}}^a, U_{\mathrm{E}}^a\}$ are optimal strategies of Player B.*
(iv) *Every optimal strategy Y of Player B satisfies $Y \leq 2a$ and*

$$1 - \frac{b}{a} \leq \mathbf{P}(Y = 0) \leq 1 - \frac{b}{a+1}.$$

Proof Using (2), (3), and $H(X, Y) = -H(Y, X)$ we get the following: in the game $\Gamma(a, a)$ (i.e., when $a = b$), for all X and Y with $\mathbf{E}(X) = \mathbf{E}(Y) = a$,

$$H(U_r^a, Y) \geq 0 \geq H(X, U_s^a) \qquad (4)$$

for $r, s \in \{\mathrm{O}, \mathrm{E}\}$. In the game $\Gamma(a, b)$ with $a > b$, for all X with $\mathbf{E}(X) = a$ and Y with $\mathbf{E}(Y) = b$ we get

$$H(U_{\mathrm{O}}^a, Y) \geq 1 - \frac{b}{a} \geq H\left(X, Y_r^{a,b}\right) \qquad (5)$$

[10] $\lfloor z \rfloor$ is the largest integer $\leq z$, and $\lceil z \rceil$ is the smallest integer $\geq z$.

[11] $\operatorname{conv}\{U_1, U_2\}$ denotes the convex hull of U_1 and U_2, i.e., the set of $\lambda U_1 + (1-\lambda)U_2$ for all $\lambda \in [0, 1]$.

for $r \in \{\text{O,E}\}$, where $Y_r^{a,b} = (1 - b/a)\mathbf{1}_0 + (b/a)U_r^a$, with the second inequality obtained as follows:

$$\begin{aligned} H\left(X, Y_r^{a,b}\right) &= \left(1 - \frac{b}{a}\right)\mathbf{P}(X > 0) + \left(\frac{b}{a}\right)H(X, U_r^a) \\ &= \left(1 - \frac{b}{a}\right)\mathbf{P}(X > 0) - \left(\frac{b}{a}\right)H(U_r^a, X) \\ &\leq \left(1 - \frac{b}{a}\right)1 - \left(\frac{b}{a}\right)0 = 1 - \frac{b}{a}. \end{aligned} \tag{6}$$

This proves that the value of $\Gamma(a, b)$ is $1 - b/a$, and that all the strategies above are optimal (and so are their convex combinations).

To prove (i), let X^0 be an optimal strategy[12] in $\Gamma(a, a)$, i.e., $H(X^0, Y) \geq 0$ for all Y with $\mathbf{E}(Y) = a$; therefore $H(X^0, U_r^a) = 0$ for $r \in \{\text{O,E}\}$ (since the U_r^a are optimal), which implies equality in (2), and so $X^0 \leq 2a$.

For every i, j with $0 \leq i \leq a \leq j \leq 2a$, let $T_{i,j} \equiv T_{i,j}^a$ be the distribution $\lambda\mathbf{1}_i + (1 - \lambda)\mathbf{1}_j$ with expectation a, i.e., $\lambda = (j - a)/(j - i)$ (when $i = a$ or $j = a$ this is just $\mathbf{1}_a$). The distribution U^a gives positive probability to both i and j, so we can express it as $U^a = \tau T_{i,j}^a + (1 - \tau)W$ for some $0 < \tau < 1$ and $W \geq 0$ with $\mathbf{E}(W) = a$ (indeed, take $\tau > 0$ so that both $\tau\lambda$ and $\tau(1 - \lambda)$ are $\leq 1/(2a + 1)$). Now $H(X^0, T_{i,j}^a) \geq 0$ and $H(X^0, W) \geq 0$ (since $T_{i,j}^a$ and W each have expectation a) and $H(X^0, U^a) = 0$ (since U^a is optimal), so we must have equality: $H(X^0, T_{i,j}^a) = 0$. Therefore $\lambda H(X^0, \mathbf{1}_i) + (1 - \lambda)H(X^0, \mathbf{1}_j) = 0$, or, denoting $w_i := H(X^0, \mathbf{1}_i)$

$$(j - a)w_i + (a - i)w_j = 0 \tag{7}$$

for every $0 \leq i \leq a \leq j \leq 2a$. Taking $i = a - 1$ gives $w_j = -(j - a)w_{a-1} = (a - j)w_{a-1}$ for all $j \geq a$, in particular $w_{a+1} = -w_{a-1}$; taking $j = a + 1$ gives $w_i = -(a - i)w_{a+1} = (a - i)w_{a-1}$ for every $i \leq a$, so $w_i = (a - i)w_{a-1}$ holds for all $0 \leq i \leq 2a$. Therefore

$$w_i - w_{i+1} = w_{a-1} \text{ for all } 0 \leq i \leq 2a - 1. \tag{8}$$

Let $q_i := \mathbf{P}(X^0 = i)$; then $w_i - w_{i+1} = \big(\mathbf{P}(X^0 > i) - \mathbf{P}(X^0 < i)\big) - \big(\mathbf{P}(X^0 > i + 1) - \mathbf{P}(X^0 < i + 1)\big) = q_i + q_{i+1}$, so (8) implies $q_i + q_{i+1} = q_{i+1} + q_{i+2}$, or $q_i = q_{i+2}$, for all $0 \leq i \leq 2a - 2$. Therefore $X^0 = \sum_{i=0}^{a} q_0\mathbf{1}_{2i} + \sum_{i=1}^{a} q_1\mathbf{1}_{2i-1} = ((a + 1)q_0)U_{\text{E}}^a + (aq_1)U_{\text{O}}^a$, or $X^0 \in \text{conv}\{U_{\text{O}}^a, U_{\text{E}}^a\}$, which completes the proof of (i).

To prove (ii), let X^0 be optimal for Player A in $\Gamma(a, b)$ when $a > b$. Equality in (6) for $X = X^0$ implies $\mathbf{P}(X > 0) = 1$ and $X^0 \leq 2a$ (recall (2)), or $1 \leq X^0 \leq 2a$. Therefore, for every Y with $\mathbf{E}(Y) = a$ we have $1 - b/a \leq H(X^0, (1 - b/a)\mathbf{1}_0 + (b/a)Y) = (1 - b/a) + (b/a)H(X^0, Y)$; hence $H(X^0, Y) \geq 0$. So X^0 is an optimal strategy in $\Gamma(a, a)$, and therefore $X^0 \in \text{conv}\{U_{\text{O}}^a, U_{\text{E}}^a\}$ by (i). But $\mathbf{P}(U_{\text{E}}^a = 0) > 0$ whereas $\mathbf{P}(X^0 = 0) = 0$, so $X^0 = U_{\text{O}}^a$, which completes the proof of (ii).

[12] For either player, since the game $\Gamma(a, a)$ is symmetric.

(iii) we have already seen in (6). To prove (iv), let Y^0 be an optimal strategy of Player B in $\Gamma(a, b)$. We must have equality in (2), so $Y^0 \le 2a$. Let $X = U(\{2, 4, \ldots, 2a-2\}) = \sum_{i=1}^{a-1}(1/(a-1))\mathbf{1}_{2i}$; then $\mathbf{E}(X) = a$ and

$$1 - \frac{b}{a} \ge H(X, Y^0) = 1 - \frac{1}{a-1}\sum_{i=1}^{a-1}\Big[\mathbf{P}(Y^0 \ge 2i) + \mathbf{P}(Y^0 \ge 2i+1)\Big]$$
$$= 1 - \frac{1}{a-1}\sum_{j=2}^{2a-1}\mathbf{P}(Y^0 \ge j) \ge 1 - \frac{1}{a-1}\Big(\mathbf{E}(Y^0) - \mathbf{P}(Y^0 \ge 1)\Big)$$
$$= 1 - \frac{1}{a-1}\Big(b - 1 + \mathbf{P}(Y^0 = 0)\Big),$$

from which it follows that $\mathbf{P}(Y^0 = 0) \ge 1 - b/a$.

Next, $H(T^a_{1,2a-1}, Y^0) = 1 - b/a$ (since, as in the proof of (i) above, the optimal strategy U^a_{O} of Player A can be expressed as $\tau T_{1,2a-1} + (1-\tau)W$ for some $0 < \tau < 1$ and W with $\mathbf{E}(W) = a$). Denoting $q_i := \mathbf{P}(Y^0 = i)$, we have

$$\frac{1}{2}(2q_0 + q_1 - 1) + \frac{1}{2}(1 - 2q_{2a} - q_{2a-1}) = 1 - \frac{b}{a}.$$

Also, $H(T^a_{1,2a}, Y^0) \le 1 - b/a$, i.e.,

$$\frac{a}{2a-1}(2q_0 + q_1 - 1) + \frac{a-1}{2a-1}(1 - q_{2a}) \le 1 - \frac{b}{a}.$$

Multiplying this inequality by $(2a-1)/(a-1)$ and subtracting the previous equation from it yields

$$\frac{a+1}{a-1}q_0 + \frac{a+1}{2(a-1)}q_1 + \frac{1}{2}q_{2a-1} - \frac{1}{a-1} \le \frac{a}{a-1}\left(1 - \frac{b}{a}\right);$$

hence $\mathbf{P}(Y^0 = 0) = q_0 \le 1 - b/(a+1)$ (we have used $q_1, q_{2a-1} \ge 0$), which completes the proof of (iv). □

When $a > b$ Player B may have additional optimal strategies beyond those in (iii); for example, when $a = 4$ and $b = 1$ (the value of $\Gamma(4, 1)$ is 3/4), the strategy $Y^0 = (25/32)\mathbf{1}_0 + (1/16)\mathbf{1}_2 + (1/32)\mathbf{1}_4 + (1/16)\mathbf{1}_5 + (1/16)\mathbf{1}_7$, which is not a convex combination of $(3/4)\mathbf{1}_0 + (1/4)U^4_{\mathrm{O}}$ and $(3/4)\mathbf{1}_0 + (1/4)U^4_{\mathrm{E}}$, is nevertheless optimal.[13]

We now come to the second case, where a is not an integer and $\lfloor a \rfloor < \lceil b \rceil$.

[13] Let $X = \sum_i p_i \mathbf{1}_i$ with $\mathbf{E}(X) = 4$ be a best reply to Y^*; then $X \le 8$, and a straightforward computation shows that $H(X, Y^*) - 3/4 = H(X, Y^*) - (1/2)\sum_i p_i - (1/16)\sum_i ip_i = -(23/32)p_0 - (1/32)p_4 \le 0$.

Theorem 3 *Let $a = m + \alpha$ and $b = m + \beta$ where $m \geq 0$ is an integer and $0 < \alpha, \beta < 1$. Then the value of the General Lotto game $\Gamma(a, b)$ is*

$$\operatorname{val}\Gamma(a,b) = \frac{a-b}{\lceil a \rceil} = \frac{\alpha - \beta}{m+1},$$

and the unique optimal strategies are $X^* = (1-\alpha)U_{\mathrm{E}}^m + \alpha U_{\mathrm{O}}^{m+1}$ for Player A and $Y^* = (1-\beta)U_{\mathrm{E}}^m + \beta U_{\mathrm{O}}^{m+1}$ for Player B.

Proof For every Y with $\mathbf{E}(Y) = b$ we have

$$\begin{aligned} H(X^*, Y) &= (1-\alpha)H(U_{\mathrm{E}}^m, Y) + \alpha H(U_{\mathrm{O}}^{m+1}, Y) \\ &\geq (1-\alpha)\left(1 - \frac{\mathbf{E}(Y)+1}{m+1}\right) + \alpha\left(1 - \frac{\mathbf{E}(Y)}{m+1}\right) = \frac{\alpha-\beta}{m+1} \end{aligned}$$

(by (3) and (2)). Similarly, $H(X, Y^*) = -H(Y^*, X) \leq -(\beta - \alpha)/(m+1)$ for every X with $\mathbf{E}(X) = a$ (interchange α and β in the previous inequality), showing that the value is indeed $(a - \beta)/(m+1)$.

Consider first the case where $a = b$, i.e., $0 < \alpha = \beta < 1$, and let X^0 be an optimal strategy in $\Gamma(a, a)$. We have just seen that $(1-\alpha)U_{\mathrm{E}}^m + \alpha U_{\mathrm{O}}^{m+1}$ is an optimal strategy; since it gives positive probability to all $0 \leq i \leq 2m+1$, it follows, as in the proof of Theorem 2 (i), that $H(X^0, T_{i,j}^a) = 0$ for all $0 \leq i \leq m$ and $m+1 \leq j \leq 2m+1$. Thus $(j-a)w_i + (a-i)w_j = 0$, from which we get (as in the proof there, taking $i = m$ and $j = m+1$; see (7) and (8)) that $w_i - w_{i+1} = w_m/\alpha$ for all $0 \leq i \leq 2m$. Therefore $\mathbf{P}(X^0 = i) = \mathbf{P}(X^0 = i+2)$ for all $0 \leq i \leq 2m-1$, and so $X^0 \in \operatorname{conv}\{U_{\mathrm{O}}^{m+1}, U_{\mathrm{E}}^m\}$. But $\mathbf{E}(U_{\mathrm{E}}^m) = m$, $\mathbf{E}(U_{\mathrm{O}}^{m+1}) = m+1$ and $\mathbf{E}(X^0) = a = m + \alpha$, so $X^0 = (1-\alpha)U_{\mathrm{E}}^m + \alpha U_{\mathrm{O}}^{m+1} = X^*$.

Consider next the case where $a > b$, i.e., $0 < \beta < \alpha < 1$, let X^0 be an optimal strategy of Player A. For every Z with $\mathbf{E}(Z) = a$, put $Y := (1-\beta/\alpha)U_{\mathrm{E}}^m + (\beta/\alpha)Z$; then $H(X^0, Y) \leq (1-\beta/\alpha)(\alpha/(m+1)) + (\beta/\alpha)H(X^0, Z) = (\alpha-\beta)/(m+1) + (\beta/\alpha)H(X^0, Z)$ (by (3)). But $\mathbf{E}(Y) = m + \beta = b$ and X^0 is optimal in $\Gamma(a, b)$, so $H(X^0, Y) \geq (\alpha - \beta)/(m+1)$; the two inequalities together imply, since $\beta > 0$, that $H(X^0, Z) \geq 0$ for all Z with $\mathbf{E}(Z) = a$. Therefore X^0 is an optimal strategy in $\Gamma(a, a)$, and thus $X^0 = X^*$.

Finally, let Y^0 be an optimal strategy of Player B in $\Gamma(a, b)$. Let $W^0 := \lambda Y^0 + (1-\lambda)U_{\mathrm{O}}^{m+1}$, where $\lambda = (1-\alpha)/(1-\beta) \in (0, 1)$; then $\mathbf{E}(W^0) = m + \alpha = a$, and, for every X with $\mathbf{E}(X) = a$ we have $H(X, Y^0) \leq (\alpha - \beta)/(m+1)$ (since Y^0 is optimal) and $H(X, U_{\mathrm{O}}^{m+1}) \leq -1 + a/(m+1) = -(1-\alpha)/(m+1)$ (by (2)), and so $H(X, W^0) = \lambda H(X, Y^0) + (1-\lambda)H(X, U_{\mathrm{O}}^{m+1}) \leq 0$. Therefore W^0 is optimal in $\Gamma(a, a)$, so $\lambda Y^0 + (1-\lambda)U_{\mathrm{O}}^{m+1} = W^0 = (1-\alpha)U_{\mathrm{E}}^m + \alpha U_{\mathrm{O}}^{m+1}$, from which it follows that $Y^0 = (1-\beta)U_{\mathrm{E}}^m + \beta U_{\mathrm{O}}^{m+1} = Y^*$. □

In the last case, a is not an integer and $\lfloor a \rfloor \geq \lceil b \rceil$.

Theorem 4 *Let $a = m + \alpha$ and $b \leq m$ where $m \geq 1$ is an integer and $0 < \alpha < 1$. Then the value of the General Lotto game $\Gamma(a, b)$ is*

$$\operatorname{val}\Gamma(a,b) = (1-\alpha)\frac{\lfloor a\rfloor - b}{\lfloor a\rfloor} + \alpha\frac{\lceil a\rceil - b}{\lceil a\rceil} = 1 - \frac{(1-\alpha)b}{m} - \frac{\alpha b}{m+1}.$$

The optimal strategies are as follows:

(i) *The strategy* $Y^* = (1-b/m)\mathbf{1}_0 + (b/m)U_{\mathrm{E}}^m$ *is the unique optimal strategy of Player B.*
(ii) *The strategy* $X^* = (1-\alpha)U_{\mathrm{O}}^m + \alpha U_{\mathrm{O}}^{m+1}$ *is an optimal strategy of Player A, and, when* $b = m$, *so are* $(1-\alpha)V + \alpha U_{\mathrm{O}}^{m+1}$ *for all* $V \in \operatorname{conv}\{U_{\mathrm{O}}^m, U_{\mathrm{E}}^m\}$.
(iii) *Every optimal strategy* X *of Player A satisfies* $X \le 2m+1$; *moreover, it also satisfies* $X \ge 1$ *when* $b < m$, *and*

$$\mathbf{P}(X=0) \le \frac{1-\alpha}{m+1}$$

when $b = m$.

Proof Let $v := 1-(1-\alpha)b/m - \alpha b/(m+1)$. For each Y with $\mathbf{E}(Y) = b$, (2) gives

$$\begin{aligned} H(X^*, Y) &= (1-\alpha)H(U_{\mathrm{O}}^m, Y) + \alpha H(U_{\mathrm{O}}^{m+1}, Y) \\ &\ge (1-\alpha)\left(1 - \frac{\mathbf{E}(Y)}{m}\right) + \alpha\left(1 - \frac{\mathbf{E}(Y)}{m+1}\right) = v. \end{aligned}$$

Next, for each X with $\mathbf{E}(X) = a$, (3) gives

$$\begin{aligned} H(X, Y^*) &= \left(1 - \frac{b}{m}\right)\mathbf{P}(X>0) + \left(\frac{b}{m}\right) H(X, U_{\mathrm{E}}^m) \\ &\le \left(1-\frac{b}{m}\right) - \left(\frac{b}{m}\right)\left(1 - \frac{\mathbf{E}(X)+1}{m+1}\right) = v. \end{aligned} \tag{9}$$

So the value is indeed v, and X^* and Y^* are optimal strategies.

To prove (i), let Y^0 be an optimal strategy of Player B. For every X with $\mathbf{E}(X) = m$, take $X' = (1-\alpha)X + \alpha U_{\mathrm{O}}^{m+1}$; then $\mathbf{E}(X') = a$ and so $H(X', Y^0) \le v = (1-\alpha)(1-b/m) + \alpha(1-b/(m+1))$ by the optimality of Y^0. Now $H(X', Y^0) \ge (1-\alpha)H(X, Y^0) + \alpha(1-b/(m+1))$ and $1-\alpha > 0$, so $H(X, Y^0) \le 1 - b/m$. Therefore Y^0 is optimal for Player B in $\Gamma(m, b)$. A similar argument (using $(1-\alpha)U_{\mathrm{O}}^m + \alpha X$ where $\mathbf{E}(X) = m+1$; recall that $\alpha > 0$) shows that Y^0 is also optimal in $\Gamma(m+1, b)$. Theorem 2 (iv) applied to both $\Gamma(m, b)$ and $\Gamma(m+1, b)$ implies that $Y^0 \le 2m$ and $\mathbf{P}(Y^0 = 0) = 1 - b/(m+1)$, so we can express Y^0 as $Y^0 = (1-b/(m+1))\mathbf{1}_0 + (b/(m+1))Z$ where $Z \ge 1$ and $\mathbf{E}(Z) = m+1$, so $Z = Z' + 1$ with $\mathbf{E}(Z') = m$.

For every X with $\mathbf{E}(X) = m$ we have $\mathbf{E}(X+1) = m+1$ and, therefore, since Y^0 is optimal in $\Gamma(m+1, b)$,

$$
\begin{aligned}
1-\frac{b}{m+1} &\geq H(X+1, Y^0) \\
&= \left(1-\frac{b}{m+1}\right)\mathbf{P}(X+1>0) + \left(\frac{b}{m+1}\right) H(X+1, Z'+1) \\
&= 1-\frac{b}{m+1} + \left(\frac{b}{m+1}\right) H(X, Z');
\end{aligned}
$$

hence[14] $H(X,Z') \leq 0$. Therefore Z' is optimal in $\Gamma(m,m)$, which implies by Theorem 2 (i) that $Z' \in \operatorname{conv}\{U_{\mathrm{O}}^m, U_{\mathrm{E}}^m\}$. We have seen above that $Y^0 \leq 2m$, so $Z \leq 2m$ and $Z' \leq 2m-1$, which implies that in fact $Z' = U_{\mathrm{O}}^m$ (since U_{E}^m, and thus all other elements of $\operatorname{conv}\{U_{\mathrm{O}}^m, U_{\mathrm{E}}^m\}$, give positive probability to $2m$). Hence $Y^0 = (1-b/(m+1))\mathbf{1}_0 + (b/(m+1))(U_{\mathrm{O}}^m + 1) = (1-b/m)\mathbf{1}_0 + (b/m)U_{\mathrm{E}}^m = Y^*$, which proves (i).

To prove (ii): We have already seen that X^* is an optimal strategy of Player A. When $b=m$, for every Y with $\mathbf{E}(Y)=m$ we have $H((1-\alpha)U_r^m + \alpha U_{\mathrm{O}}^{m+1}, Y) = (1-\alpha)\,0 + \alpha(1/(m+1)) = 1-(1-\alpha)(m/m) - \alpha(m/(m+1))$, so $(1-\alpha)U_r^m + \alpha U_{\mathrm{O}}^{m+1}$ is indeed optimal for Player A in $\Gamma(m+\alpha, m)$.

Finally, to show (iii), let X^0 be an optimal strategy of Player A. Equality in (9) implies that it must satisfy $X^0 \leq 2m+1$ (recall (3)) and, when $b<m$, also $\mathbf{P}(X^0 > 0) = 1$. When $b=m$, take $Y = (1/(m+1))\mathbf{1}_0 + (m/(m+1))U_{\mathrm{O}}^{m+1}$; then $\mathbf{E}(Y)=m$ and

$$
\begin{aligned}
\frac{\alpha}{m+1} &= \operatorname{val}\Gamma(m+\alpha, m) \leq H(X^0, Y) \\
&= \frac{1}{m+1}\mathbf{P}(X^0>0) + \frac{m}{m+1} H(X^0, U_{\mathrm{O}}^{m+1}) \\
&\leq \frac{1}{m+1}(1-\mathbf{P}(X^0=0)) - \frac{m}{m+1}\left(1-\frac{m+\alpha}{m+1}\right)
\end{aligned}
$$

(recall (2)), which yields $\mathbf{P}(X^0=0) \leq (1-\alpha)/(m+1)$. □

Again, there are additional optimal strategies for Player A; for example, $X = (1/2)\mathbf{1}_1 + (1/2)\mathbf{1}_2$ is optimal in $\Gamma(3/2, 1)$ and $X = (5/12)\mathbf{1}_1 + (1/4)\mathbf{1}_3 + (1/3)\mathbf{1}_4$ is optimal in $\Gamma(5/2, 1/2)$.

The table in Fig. 1 provides a summary of the results of this section: the first two rows correspond to Theorem 2, the third row to Theorem 3, and the last two rows to Theorem 4.

As we have seen, the main difference between the strong player and the weak player lies in the probability of "giving up," i.e., choosing 0 (see also Theorem 1 on the continuous version and the discussion following it). Our results yield precise bounds on these probabilities.[15]

[14] For the last equality we have used the fact that $H(X,Y)$ is a function of $X-Y$ only (see (1)).

[15] Interestingly, the experiments of Avrahami and Kareev (2005) have shown that the number of empty urns is a significant variable.

	Value	Optimal Strategies		
$a = m$, $b = m$	0	**A**	$*$	$U^m_{\mathrm{O/E}}$
		B	$*$	$U^m_{\mathrm{O/E}}$
$a = m$, $b < m$	$\frac{a-b}{a} = 1 - \frac{b}{m}$	**A**	$*$	U^m_{O}
		B		$\left(1 - \frac{b}{m}\right) \mathbf{1}_0 + \left(\frac{b}{m}\right) U^m_{\mathrm{O/E}}$
$a = m + \alpha$, $b = m + \beta$	$\frac{a-b}{\lceil a \rceil} = \frac{\alpha - \beta}{m+1}$	**A**	$*$	$(1-\alpha)\, U^m_{\mathrm{E}} + \alpha\, U^{m+1}_{\mathrm{O}}$
		B	$*$	$(1-\beta)\, U^m_{\mathrm{E}} + \beta\, U^{m+1}_{\mathrm{O}}$
$a = m + \alpha$, $b = m$	$\frac{a-b}{\lceil a \rceil} = \frac{\alpha}{m+1}$	**A**		$(1-\alpha)\, U^m_{\mathrm{O/E}} + \alpha\, U^{m+1}_{\mathrm{O}}$
		B	$*$	U^m_{E}
$a = m + \alpha$, $b < m$	$(1-\alpha)\frac{\lfloor a \rfloor - b}{\lfloor a \rfloor} + \alpha \frac{\lceil a \rceil - b}{\lceil a \rceil}$	**A**		$(1-\alpha)\, U^m_{\mathrm{O}} + \alpha\, U^{m+1}_{\mathrm{O}}$
		B	$*$	$\left(1 - \frac{b}{m}\right) \mathbf{1}_0 + \left(\frac{b}{m}\right) U^m_{\mathrm{E}}$

Fig. 1 Values and optimal strategies of the General Lotto games $\Gamma(a, b)$ (m denotes an integer; $0 < \alpha, \beta < 1$; a star (*) indicates that these are the *only* optimal strategies; and $U^m_{\mathrm{O/E}}$ stands for $\mathrm{conv}\{U^m_{\mathrm{O}}, U^m_{\mathrm{E}}\}$)

Corollary 5 *Let X^0 and Y^0 be optimal strategies of Player A and Player B, respectively, in the General Lotto game $\Gamma(a, b)$ with $a \geq b > 0$. Then $X^0, Y^0 \leq 2\lfloor a \rfloor + 1$ and*[16]

$$\left[1 - \frac{a}{\lceil b \rceil}\right]_+ \leq \mathbf{P}(X^0 = 0) \leq \left[1 - \frac{a}{\lfloor b \rfloor + 1}\right]_+ \quad \textit{and}$$
$$\left[1 - \frac{b}{\lceil a \rceil}\right]_+ \leq \mathbf{P}(Y^0 = 0) \leq \left[1 - \frac{b}{\lfloor a \rfloor + 1}\right]_+ .$$

Proof This is easily verified when the optimal strategies are fully characterized (* in Fig. 1). The remaining cases are covered in Theorems 2 (iv) and 4 (iii). [17] □

Thus, when b is not an integer we have $\lceil b \rceil = \lfloor b \rfloor + 1$ and so $\mathbf{P}(X^0 = 0) = [1 - a/\lceil b \rceil]_+$; similarly for a and Y^0. Also, $X^0 \geq 1$ when $a \geq \lfloor b \rfloor + 1$. Moreover, in all cases the bounds are attained (in particular, by the strategies in Fig. 1).

4 Colonel Blotto Games

Having solved the General Lotto games $\Gamma(a, b)$, we will now show how certain optimal strategies in these games can be implemented in the Colonel Blotto games $\mathcal{B}(A, B; K)$ (as well as in the Colonel Lotto games $\mathcal{L}(A, B; K)$).

[16] $[z]_+ = \max\{z, 0\}$ is the "positive part" of z.

[17] In fact, $\lfloor a \rfloor + \lceil a \rceil$ is a more precise bound on X^0 and Y^0.

Let $A \geq 1$ and $K \geq 2$ be integers. Recall (Sect. 2.2) that we identify a K-partition $x = \langle x_1, x_2, \ldots, x_K \rangle$ of A (i.e., $x_1+x_2+\cdots+x_K = A$, where the x_k are K nonnegative integers) with the *distribution* it generates, $X = \sum_{k=1}^{K}(1/K)\mathbf{1}_{x_k}$; note that $\mathbf{E}(X) = A/K$. A nonnegative integer-valued random variable Z will be called (A, K)-*feasible* if Z can be obtained from a probability distribution on K-partitions of A. That is, Z is a mixed strategy of Player A in the Colonel Blotto game $\mathcal{B}(A, B; K)$. Formally, it means that $Z = \sum_{i=1}^{n} \lambda_i X_i$, where each X_i is (the distribution of) a K-partition of A, each $\lambda_i > 0$, and $\sum_{i=1}^{n} \lambda_i = 1$. For example, let $Z = U_{\mathrm{E}}^3 = (1/4)\mathbf{1}_0 + (1/4)\mathbf{1}_2 + (1/4)\mathbf{1}_4 + (1/4)\mathbf{1}_6$; then Z is $(6, 2)$-feasible: put mass $1/2$ on the partition $\langle 0, 6 \rangle$ and $1/2$ on the partition $\langle 2, 4 \rangle$ (their distributions are, respectively, $(1/2)\mathbf{1}_0 + (1/2)\mathbf{1}_6$ and $(1/2)\mathbf{1}_2 + (1/2)\mathbf{1}_4$). However, Z is not $(9, 3)$-feasible, since the support of Z consists of even numbers only, whereas every 3-partition of 9 must contain an odd number.

We have

Proposition 6 *Let $A \geq 1$ and $K \geq 2$ be integers.*

(i) *If $A = mK$ where $m \geq 1$ is an integer, then U_{O}^m is (A, K)-feasible if and only if A and K have the same parity (i.e., both are even or both are odd).*
(ii) *If $A = mK$ where $m \geq 1$ is an integer, then U_{E}^m is (A, K)-feasible if and only if A is even.*
(iii) *If $A = mK + r$ where $m \geq 0$ and $1 \leq r \leq K - 1$ are integers, then*

$$\left(1 - \frac{r}{K}\right) U_{\mathrm{E}}^m + \left(\frac{r}{K}\right) U_{\mathrm{O}}^{m+1}$$

is (A, K)-feasible.

When $A/K = m$ is an integer, (i) and (ii) can be restated as follows. If K is even, both U_{O}^m and U_{E}^m are feasible; if K is odd, only one of them is feasible: U_{O}^m when A is odd and U_{E}^m when A is even. As we will see immediately below, it turns out that U_{O}^m and U_{E}^m are feasible except when this is ruled out by trivial parity considerations.

The proof of Proposition 6 provides explicit constructions of the appropriate distributions on partitions; a number of illustrative examples follow the Proof of Theorem 7.

Proof First, we note that the conditions of feasibility in (i) and (ii) are clearly necessary. Indeed, if A is the sum of K odd numbers then A has the same parity as K; hence U_{O}^m, whose support consists of odd numbers only, cannot be obtained from K-partitions of A when A and K have different parity. Similarly, if U_{E}^m, whose support consists of even numbers only, is (A, K)-feasible, then A is the sum of K even numbers, so it must be even.

We will now construct for each case an appropriate $\ell \times K$ matrix S, such that each row is a K-partition of A (i.e., all the row sums equal A), and the required distribution X is obtained by assigning equal probability of $1/\ell$ to each row. We will say that in this case S *implements* X (by K-partitions of A).

We first deal with (ii) and (iii); we distinguish two cases, according to the parity of K.

$$S^0 = \left[\begin{array}{ccc|ccc} 0 & \cdots & 0 & 2m & \cdots & 2m \\ 2 & \cdots & 2 & 2m-2 & \cdots & 2m-2 \\ \vdots & & \vdots & \vdots & & \vdots \\ 2m & \cdots & 2m & 0 & \cdots & 0 \end{array}\right]$$
$$\qquad\quad \underbrace{\qquad\qquad}_{k} \qquad \underbrace{\qquad\qquad\qquad}_{k}$$

Fig. 2 The matrix S^0 in Case 1

$$S^0 = \left[\begin{array}{ccc|ccc|c|c} 0 & \cdots & 0 & 4n & \cdots & 4n & 2n & 4n \\ 2 & \cdots & 2 & 4n-2 & \cdots & 4n-2 & 2n+2 & 4n-4 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 2n & \cdots & 2n & 2n & \cdots & 2n & 4n & 0 \\ 2n+2 & \cdots & 2n+2 & 2n-2 & \cdots & 2n-2 & 0 & 4n-2 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 4n & \cdots & 4n & 0 & \cdots & 0 & 2n-2 & 2 \end{array}\right]$$
$$\qquad \underbrace{\qquad\qquad\qquad}_{k} \qquad \underbrace{\qquad\qquad\qquad}_{k-1}$$

Fig. 3 The matrix S^0 in Case 2.1

Case 1 *K is even*, say $K = 2k$. For every $m \geq 0$, let $S^0 = (s^0_{ij})_{i=0,\ldots,m,\,j=1,\ldots,2k}$ be the $(m+1)\times K$ matrix of Fig. 2 (column 1 and column $k+1$ are repeated k times each): for each $i = 0, 1, \ldots, m$, put $s^0_{ij} = 2i$ for $j \leq k$ and $s^0_{ij} = 2m - 2i$ for $j \geq k+1$. The sum of each row of S^0 is $2mk = mK$, and each column is a permutation of $\{0, 2, \ldots, 2m\}$. Therefore each row is a K-partition of $A = mK$, and S^0 implements the distribution U^m_{E}: assigning probability $1/(m+1)$ to each row generates U^m_{E} in each column, hence U^m_{E} overall. This proves (ii) for K even and $A = mK$ even.

Next, for each $1 \leq r \leq K-1$, let S^r be the matrix obtained by adding 1 to all elements of S^0 in, say, the first r columns $1, 2, \ldots, r$. The sum of each row is now $Km + r$, so each row is a K-partition of $A = mK + r$. Assigning probability $1/(m+1)$ to each row generates the distribution $(r/K)U^{m+1}_{\mathrm{O}} + (1 - r/K)U^m_{\mathrm{E}}$ (since each one of the first r columns is a permutation of $\{1, 3, \ldots, 2m+1\}$ and thus yields U^{m+1}_{O}, and each one of the remaining $K - r$ columns yields U^m_{E}, as before). This proves (iii) for K even.

Case 2 *K is odd*, say $K = 2k+1$. We distinguish two subcases, according to the parity of m.

Case 2.1 *m is even*, say $m = 2n$, and $0 \leq r \leq K-1$. We start with the $(m+1) \times K$ matrix $S^0 = (s^0_{ij})_{i=0,\ldots,2n,\,j=1,\ldots,2k+1}$ of Fig. 3: for each $i = 0, 1, \ldots, 2n$, let $s^0_{ij} = 2i$ for $j \in \{1, \ldots, k\}$, $s^0_{ij} = 4n - 2i$ for $j \in \{k+1, \ldots, 2k-1\}$, $s^0_{i,2k} = 2n + 2i$ and $s^0_{i,2k+1} = 4n - 4i$, all modulo $4n + 2$. It can be verified that the sum in each row is $2n(2k+1) = mK$, and each column is a permutation of $\{0, 2, \ldots, 4n\}$. Therefore S^0 implements $U^{2n}_{\mathrm{E}} = U^m_{\mathrm{E}}$ by K-partitions of $A = mK$, which proves (ii) when K is odd and $A = mK$ is even. As in case 1, for each $1 \leq r \leq K-1$ we add 1 to all elements in the first r columns of S^0 to obtain the matrix S^r that implements $(r/K)U^{m+1}_{\mathrm{O}} + (1 - r/K)U^m_{\mathrm{E}}$, which proves (iii) here.

$$S^1 = \left[\begin{array}{ccc|ccc|c|c} 0 & \cdots & 0 & 4n-2 & \cdots & 4n-2 & 0 & 2n \\ 2 & \cdots & 2 & 4n-4 & \cdots & 4n-4 & 1 & 2n+1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 4n-2 & \cdots & 4n-2 & 0 & \cdots & 0 & 2n-1 & 4n-1 \\ 0 & \cdots & 0 & 4n-2 & \cdots & 4n-2 & 2n & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 4n-2 & \cdots & 4n-2 & 0 & \cdots & 0 & 4n-1 & 2n-1 \end{array}\right]$$
$$\qquad\qquad \underbrace{\qquad\qquad\qquad}_{k-1} \qquad \underbrace{\qquad\qquad\qquad}_{k}$$

Fig. 4 The matrix S^1 in Case 2.2

Case 2.2 m is odd, say $m = 2n - 1$, and $1 \leq r \leq K - 1$. Let $S^1 = (s^1_{ij})_{i=0,1,\ldots,4n-1,j=1,\ldots,2k+1}$ be the $(2m+2) \times K$ matrix of Fig. 4: for each $i = 0, 1, \ldots, 4n - 1$, put $s^1_{ij} = 2i$ for $j \in \{1, \ldots, k\}$, $s^1_{ij} = 4n - 2 - 2i$ for $j \in \{k + 1, \ldots, 2k - 1\}$, $s^1_{i,2k} = i$ and $s^1_{i,2k+1} = 2n + i$, all modulo $4n$. The sum in each row is $(2n - 1)(2k + 1) + 1 = mK + 1$; the distribution in each one of the first $2k - 1$ columns is $U_{\mathrm{E}}^{2n-1} = U_{\mathrm{E}}^m$, and in each one of the last two columns is $U^{4n-1} = (1/2)U_{\mathrm{E}}^{2n-1} + (1/2)U_{\mathrm{O}}^{2n} = (1/2)U_{\mathrm{E}}^m + (1/2)U_{\mathrm{O}}^{m+1}$. Therefore S^1 implements, by K-partitions of $A = mK + 1$, the distribution $((2k - 1)/K)U_{\mathrm{E}}^m + (2/K)((1/2)U_{\mathrm{E}}^m + (1/2)U_{\mathrm{O}}^{m+1}) = (1 - 1/K)U_{\mathrm{E}}^m + (1/K)U_{\mathrm{O}}^{m+1}$, which proves (iii) for $r = 1$. For each $2 \leq r \leq K - 1$, add 1 to all entries in the first $r - 1$ columns (so the last two columns are never changed), to obtain S^r that implements $(1 - r/K)U_{\mathrm{E}}^m + (r/K)U_{\mathrm{O}}^{m+1}$ by K-partitions of $A = mK + r$, completing the proof of (iii) in this case too.

We have completed the constructions for (ii) and (iii) in all cases. To prove (i), let K and $A = mK$ have the same parity. Then $A' = A - K = (m - 1)K$ is even, so U_{E}^{m-1} is (A', K)-feasible by (ii). Transform each K-partition of A' into a K-partition of $A = A' + K$ by adding 1 to every element of the partition; this transforms the distribution U_{E}^{m-1} into $U_{\mathrm{E}}^{m-1} + 1 = U_{\mathrm{O}}^m$ (in terms of the implementing matrices above, it corresponds to the matrix S^K obtained by adding 1 to all entries in all K columns of the matrix S^0 for K and $m - 1$). $\square$

Consider now a Colonel Blotto Game $\mathcal{B}(A, B; K)$ and the associated General Lotto game $\Gamma(a, b)$, where $a := A/K$ and $b := B/K$. An optimal strategy X of Player A in $\Gamma(a, b)$ guarantees a payoff of at least $\operatorname{val}\Gamma(a, b)$ against any strategy of Player B there, and so, *a fortiori*, against any strategy of Player B in $\mathcal{B}(A, B; K)$. Therefore, if such an X is feasible in $\mathcal{B}(A, B; K)$, i.e., if X is (A, K)-feasible, then $\operatorname{val}\mathcal{B}(A, B; K) \geq \operatorname{val}\Gamma(a, b)$. Similarly, for Player B, if an optimal strategy Y in $\Gamma(a, b)$ is (B, K)-feasible, then $\operatorname{val}\mathcal{B}(A, B; K) \leq \operatorname{val}\Gamma(a, b)$. If both X and Y are feasible, then $\operatorname{val}\mathcal{B}(A, B; K) = \operatorname{val}\Gamma(a, b)$, and X and Y (more precisely, their implementations by K-partitions) are optimal strategies in the Colonel Blotto game $\mathcal{B}(A, B; K)$.

We start with the symmetric case where $A = B$ (and the value is 0).

Theorem 7 *Proposition 6 provides optimal strategies for every symmetric Colonel Blotto game $\mathcal{B}(A, A; K)$.*

Proof If $A/K = m$ is an integer, at least one of U_{O}^{m} and U_{E}^{m} is (A, K)-feasible by Proposition 6 (i) and (ii), and we apply Theorem 2. Otherwise, $A/K = m + \alpha$ where $\alpha = r/K$ for some $1 \le r \le K - 1$, and then the strategy $(1 - \alpha)U_{\mathrm{E}}^{m} + \alpha U_{\mathrm{O}}^{m+1}$ is (A, K)-feasible by Proposition 6 (iii); apply Theorem 3. □

We illustrate this with some examples. First, let $A = 7$ and $K = 3$ (the case presented, without solution, in Borel 1921). Thus $m = 2$ and $r = 1$, and the Proof of Proposition 6, specifically Case 2.1 with $k = 1$ and $n = 1$, gives the matrices

$$S^0 = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 4 & 0 \\ 4 & 0 & 2 \end{bmatrix} \quad \text{and} \quad S^1 = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 4 & 0 \\ 5 & 0 & 2 \end{bmatrix},$$

so an optimal strategy in $\mathcal{B}(7, 7; 3)$ can be read from S^1 (since $r = 1$): it is

$$\frac{1}{3}\,\langle 1, 2, 4 \rangle + \frac{1}{3}\,\langle 0, 3, 4 \rangle + \frac{1}{3}\,\langle 0, 2, 5 \rangle$$

(it generates $(2/9)(\mathbf{1}_0 + \mathbf{1}_2 + \mathbf{1}_4) + (1/9)(\mathbf{1}_1 + \mathbf{1}_3 + \mathbf{1}_5) = (2/3)U_{\mathrm{E}}^{2} + (1/3)U_{\mathrm{O}}^{3}$).

Next, let $A = 7$ and $K = 4$ (so $m = 1$ and $r = 3$); Case 1 gives

$$S^0 = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 \end{bmatrix} \quad \text{and} \quad S^3 = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 3 & 3 & 1 & 0 \end{bmatrix},$$

therefore an optimal strategy in $\mathcal{B}(7, 7; 4)$ is

$$\frac{1}{2}\,\langle 1, 1, 2, 3 \rangle + \frac{1}{2}\,\langle 0, 1, 3, 3 \rangle$$

(it generates $(1/8)(\mathbf{1}_0 + \mathbf{1}_2) + (3/8)(\mathbf{1}_1 + \mathbf{1}_3) = (1/4)U_{\mathrm{E}}^{1} + (3/4)U_{\mathrm{O}}^{2}$).

Finally, when $A = 7$ and $K = 5$, Case 2.2 gives

$$S^1 = \begin{bmatrix} 0 & 2 & 2 & 0 & 2 \\ 2 & 0 & 0 & 1 & 3 \\ 0 & 2 & 2 & 2 & 0 \\ 2 & 0 & 0 & 3 & 1 \end{bmatrix} \quad \text{and} \quad S^2 = \begin{bmatrix} 1 & 2 & 2 & 0 & 2 \\ 3 & 0 & 0 & 1 & 3 \\ 1 & 2 & 2 & 2 & 0 \\ 3 & 0 & 0 & 3 & 1 \end{bmatrix},$$

and an optimal strategy in $\mathcal{B}(7, 7; 5)$ is thus[18]

$$\frac{1}{2}\,\langle 0, 1, 2, 2, 2 \rangle + \frac{1}{2}\,\langle 0, 0, 1, 3, 3 \rangle$$

(it generates $(3/10)(\mathbf{1}_0 + \mathbf{1}_2) + (2/10)(\mathbf{1}_1 + \mathbf{1}_3) = (3/5)U_{\mathrm{E}}^{1} + (2/5)U_{\mathrm{O}}^{2}$).

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[18] Another optimal strategy, $(1/2)\,\langle 0, 0, 2, 2, 3 \rangle + (1/2)\,\langle 0, 1, 1, 2, 3 \rangle$, is obtained by adding 1 to all entries in the *second* column of S^1.

The next three results deal with nonsymmetric Colonel Blotto games.

Proposition 8 *Let $m < A/K,\ B/K < m+1$ where m is an integer. The value of the Colonel Blotto game $\mathcal{B}(A, B; K)$ is*

$$\operatorname{val}\mathcal{B}(A, B; K) = \frac{A-B}{K(m+1)},$$

and optimal strategies for Player A and Player B are those of Proposition 6 (iii) that correspond to

$$\left(m+1-\frac{A}{K}\right)U_{\mathrm{E}}^{m} + \left(\frac{A}{K}-m\right)U_{\mathrm{O}}^{m+1},\ \textit{and}$$
$$\left(m+1-\frac{B}{K}\right)U_{\mathrm{E}}^{m} + \left(\frac{B}{K}-m\right)U_{\mathrm{O}}^{m+1},$$

respectively.

Proof Recall Theorem 3. □

In the following two cases we obtain bounds on the values of Colonel Blotto games.

Proposition 9 *Let $A > B$. If A/K is an integer and A and K have the same parity, then the value of the Colonel Blotto game $\mathcal{B}(A, B; K)$ satisfies*

$$\operatorname{val}\mathcal{B}(A, B; K) \geq \frac{A-B}{A}.$$

Proof Let $a := A/K$ and $b := B/K$. Proposition 6 (i) and Theorem 2 imply that U_{O}^{a}, the optimal strategy in $\Gamma(a, b)$, is feasible for Player A in $\mathcal{B}(A, B; K)$, so $\operatorname{val}\mathcal{B}(A, B; K) \geq \operatorname{val}\Gamma(a, b) = (a-b)/b = (A-B)/B$. □

Proposition 10 *Let $B/K \leq m \leq A/K < m+1$ where m is an integer. If B is even and B/m is an integer, then the value of the Colonel Blotto game $\mathcal{B}(A, B; K)$ satisfies*

$$\operatorname{val}\mathcal{B}(A, B; K) \leq 1-(1-\alpha)\frac{B}{mK} - \alpha\frac{B}{(m+1)K},$$

where $\alpha := A/K - m$.

Proof Put $a := A/K = m+\alpha$, $b := B/K$ and $K' := B/m$; by assumption K' is an integer and B is even. Therefore the strategy U_{E}^{m} is (B, K')-feasible by Proposition 6 (ii). From each K'-partition of B one obtains a K-partition of B by adding to it $K-K'$ zeroes;[19] thus the strategy $((K-K')/K)\mathbf{1}_0+(K'/K)U_{\mathrm{E}}^{m} = (1-b/m)\mathbf{1}_0+(b/m)U_{\mathrm{E}}^{m}$, which is optimal for Player B in $\Gamma(a, b)$ (by Theorem 2 when $\alpha = 0$ and by Theorem 4 when $\alpha > 0$), is (B, K)-feasible. Therefore $\operatorname{val}\mathcal{B}(A, B; K) \leq \operatorname{val}\Gamma(a, b)$. □

[19] $K' \leq K$ since $B/K \leq m$.

5 Discussion

We have introduced the General Lotto games as a generalization and technical tool for studying Colonel Blotto games. However, these games are clearly of interest in their own right, as natural models of optimal resource allocation in competitive environments.

For example, Continuous General Lotto games are used in various models of political competition (Myerson 1993; Lizzeri 1999; Sahuguet and Persico 2006; Dekel et al. 2004, and others). Requiring the allocations there to be integer-valued is only natural: it corresponds to having a minimum unit of exchange.

An interesting connection has been made to "all-pay auctions" (see, e.g., Appendix A in Sahuguet and Persico 2006). Auctions, particularly multi-object auctions, are natural instances of allocation games—so, again, the games studied here should be useful. Yet another connection is to tournaments (Groh et al. 2003).

We have not fully solved all Colonel Blotto games. However, it is clear that the methods we have used may be extended to cover various additional cases. First, one would need to extend the class of strategies that can be implemented by partitions, beyond those of Proposition 6. Second, it would be useful to find the additional optimal strategies of General Lotto games in those cases where we have not obtained complete characterizations (i.e., there is no * in the corresponding row of Fig. 1); this will provide additional candidates to be implemented by partitions in Colonel Blotto games.

A Appendix: Proof of Theorem 1

All random variables in this appendix will be assumed to be nonnegative. For every such Z,

$$\mathbf{E}(Z) = \int_0^\infty \mathbf{P}(Z \geq z)\,\mathrm{d}z$$

(see, e.g., Billingsley 1986, (21.9)). From (1) we get

$$1 - 2\mathbf{P}(Y \geq X) \leq H(X,Y) \leq 2\mathbf{P}(X \geq Y) - 1. \tag{10}$$

Proof of Theorem 1. For every Y with $\mathbf{E}(Y) = b$,

$$\begin{aligned}\mathbf{P}(Y \geq X^*) &= \frac{1}{2a}\int_0^{2a} \mathbf{P}(Y \geq x)\,\mathrm{d}x \leq \frac{1}{2a}\int_0^\infty \mathbf{P}(Y \geq x)\,\mathrm{d}x \\ &= \frac{1}{2a}\mathbf{E}(Y) = \frac{b}{2a},\end{aligned} \tag{11}$$

hence $H(X^*, Y) \geq 1-2\mathbf{P}(Y \geq X^*) \geq 1-b/a$. Similarly, for every X with $\mathbf{E}(X) = a$,

$$\begin{aligned} \mathbf{P}(X \geq Y^*) &= \left(1 - \frac{b}{a}\right)\mathbf{P}(X \geq 0) + \left(\frac{b}{a}\right)\frac{1}{2a}\int\limits_0^{2a} \mathbf{P}(X \geq y)\,\mathrm{d}y \\ &\leq 1 - \frac{b}{a} + \frac{b}{2a^2}\mathbf{E}(X) = 1 - \frac{b}{2a}, \end{aligned} \tag{12}$$

hence $H(X, Y^*) \leq 2\mathbf{P}(X \geq Y^*) - 1 \leq 1 - b/a$. The value of $\Lambda(a, b)$ is thus $1 - b/a$, and X^* and Y^* are optimal strategies.

Let X^0 be an optimal strategy of Player A, i.e., $H(X^0, Y) \geq 1 - b/a$, hence $\mathbf{P}(X^0 \geq Y) \geq 1 - b/(2a)$ (recall (10)), for all Y with $\mathbf{E}(Y) = b$. When $Y = Y^*$ we have equality, so (12) implies $\int_{2a}^{\infty} \mathbf{P}(X^0 \geq y)\,\mathrm{d}y = 0$, or

$$\mathbf{P}(X^0 > 2a) = 0. \tag{13}$$

We will now show that, for every $t \in [0, 2a]$,

$$\mathbf{P}(X^0 \geq t) \geq 1 - \frac{t}{2a}. \tag{14}$$

Indeed, when $t \in [b, 2a]$, take $Y = (1 - b/t)\mathbf{1}_0 + (b/t)\mathbf{1}_t$; then $\mathbf{P}(X^0 \geq Y) = (1 - b/t) + (b/t)\mathbf{P}(X^0 \geq t)$, and the inequality $\mathbf{P}(X^0 \geq Y) \leq 1 - b/(2a)$ yields (14). When $t \in [0, b)$, for any small $\varepsilon > 0$ take $Y = \lambda\mathbf{1}_t + (1 - \lambda)\mathbf{1}_{2a+\varepsilon}$ with $\lambda = (2a + \varepsilon - b)/(2a + \varepsilon - t)$; then $\mathbf{E}(Y) = b$, and $1 - b/(2a) \geq \mathbf{P}(X^0 \geq Y) = \lambda\mathbf{P}(X^0 \geq t)$ (recall (13)) yields (14). Now

$$a = \mathbf{E}(X^0) \geq \int\limits_0^{2a} \mathbf{P}(X^0 \geq t)\,\mathrm{d}t \geq \int\limits_0^{2a} \left(1 - \frac{t}{2a}\right)\,\mathrm{d}t = a,$$

so we must have equality in (14) for almost every $t \in [0, 2a]$, and thus for every $t \in [0, 2a]$ (take t' arbitrarily close to t), so $X^0 = X^*$.

Next, let Y^0 be an optimal strategy of Player B, i.e., $H(X, Y^0) \leq 1 - b/a$, hence $\mathbf{P}(Y^0 \geq X) \geq b/(2a)$ (recall (10)), for all X with $\mathbf{E}(X) = a$. When $X = X^*$ we have equality, so (11) implies

$$\mathbf{P}(Y^0 > 2a) = 0. \tag{15}$$

For every small $\varepsilon > 0$, let $X = U(\varepsilon, 2a - \varepsilon)$; then $\mathbf{E}(X) = a$ and

$$\begin{aligned} \frac{b}{2a} &\leq \mathbf{P}(Y^0 \geq X) = \frac{1}{2a - 2\varepsilon}\int\limits_{\varepsilon}^{2a-\varepsilon} \mathbf{P}(Y^0 \geq x)\,\mathrm{d}x \\ &\leq \frac{1}{2a - 2\varepsilon}\left(\mathbf{E}(Y^0) - \varepsilon\mathbf{P}(Y^0 \geq \varepsilon)\right), \end{aligned}$$

which implies that

$$\mathbf{P}(Y^0 \geq \varepsilon) \leq \frac{b}{a}. \tag{16}$$

We will now show that, for every $t \in (0, 2a]$,

$$\mathbf{P}(Y^0 \geq t) \geq \frac{b}{a}\left(1 - \frac{t}{2a}\right). \tag{17}$$

Indeed, when $t \in (a, 2a]$, take $X = \lambda \mathbf{1}_\varepsilon + (1-\lambda)\mathbf{1}_t$ with $\lambda = (t-a)/(t-\varepsilon)$; then $\mathbf{E}(X) = a$ and $b/(2a) \leq \mathbf{P}(Y^0 \geq X) = \lambda(b/a) + (1-\lambda)\mathbf{P}(Y^0 \geq t)$ (by (16)); as $\varepsilon \to 0$ we get (17). When $t \in (0, a)$, take $X = \lambda \mathbf{1}_t + (1-\lambda)\mathbf{1}_{2a+\varepsilon}$ with $\lambda = (a+\varepsilon)/(2a+\varepsilon-t)$; then $\mathbf{E}(X) = a$ and $b/(2a) \leq \mathbf{P}(Y^0 \geq X) = \lambda \mathbf{P}(Y^0 \geq t)$ (recall (15)), which yields (17) as $\varepsilon \to 0$. To complete the proof we proceed similarly to X^0 above: integrating (17) over t and using $\mathbf{E}(Y^0) = b$ implies equality in (17) for almost every t, thus for all t, so $Y^0 = Y^*$. □

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