The Dullness of Monotonic Mechanisms

Ran Ben Moshe Sergiu Hart Noam Nisan

The Hebrew University of Jerusalem

June 2022

Revenue-Maximizing Mechanisms for Selling Goods

- *Distant Past* (>40 years ago):
  - **One good:** *simple* (Myerson)
  - Buyer is willing to pay more ⇨ buyer pays more
- *Past* (>10 years ago):
  - **Two or more goods:** very *complex*

    "conceptual" complexity
  - Buyer is willing to pay more ⇨ buyer pays **less !**

    non-monotonicity (Hart-Reny)

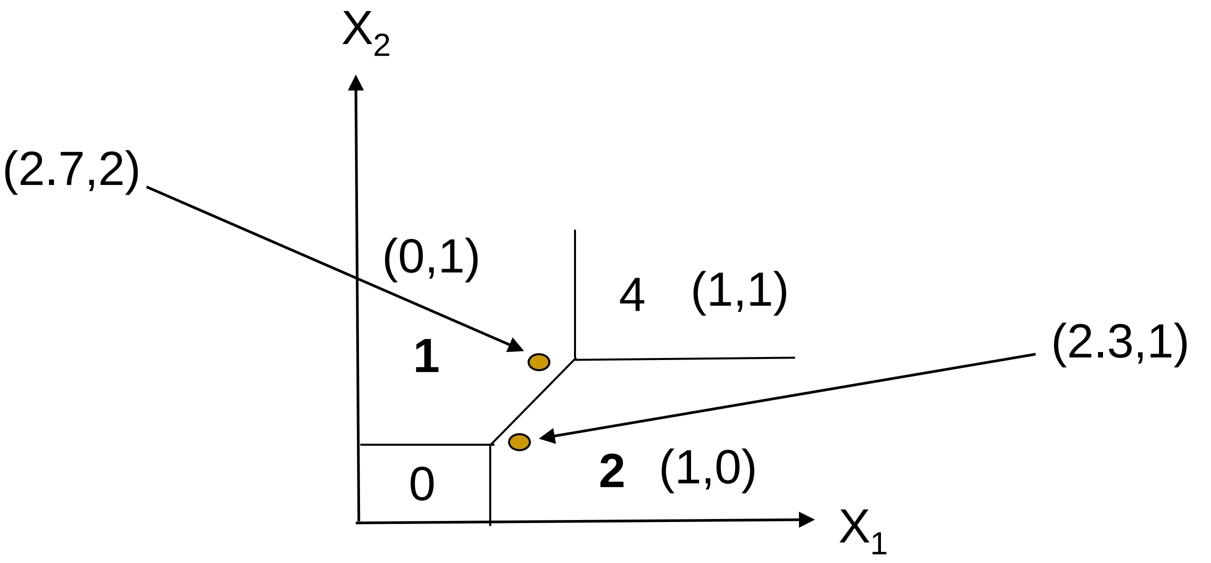
Present setting

- Single (Bayesian revenue maximizing) seller
- Single additive (risk-neutral) buyer
- *n* (indivisible) items / goods
- Buyer's valuation for items is drawn from a possibly correlated distribution $(X_1 \ldots X_n)$.
- All mechanisms are IC, IR

Monotonicity

Definition: A mechanism is **monotonic** if for every $x,y \in R_+^n$ with $x \leq y$ (component-wise) we have that $s(x) \leq s(y)$, where $s()$ is the payment to the seller.

The Deterministic 2-item case



Not Monotonic

Optimal for some distribution (Hart-Reny)

Monotonic vs Bundled revenues

Theorem: $MonRev(X_1 \ldots X_n) \leq n \cdot BRev(X_1 \ldots X_n)$

Proof:

$$MonRev(X_1 \ldots X_n) \leq$$
$$MonRev(X^{max} \ldots X^{max}) \leq (=)$$
$$Rev(X^{max} \ldots X^{max}) \leq (=)$$
$$n \cdot Rev(X^{max}) \leq$$
$$n \cdot Rev(X_1 + \ldots + X_n) =$$
$$n \cdot BRev(X_1 \ldots X_n)$$

Monotonic vs Bundled revenues

Theorem: $MonRev(X_1 \ldots X_n) \leq n \cdot BRev(X_1 \ldots X_n)$

Corollary: For some distribution there is an infinite gap between *MonRev* and *Rev* (for every $n \geq 2$)

Corollary: For some distribution there is a gap of $\Omega(2^n/n^2)$ between *MonRev* and *DRev* (for every $n \geq 2$)

(use Hart-Nisan 2013)

Monotonic vs Separate revenues

Theorem: $MonRev(X_1 \ldots X_n) \leq n \cdot SRev(X_1 \ldots X_n)$

Proof:

$$MonRev(X_1 \ldots X_n) \leq$$

$$MonRev(X^{max} \ldots X^{max}) \leq \ (=)$$

$$Rev(X^{max} \ldots X^{max}) \leq \ (=)$$

$$n \cdot Rev(X^{max}) \leq$$

$$n \cdot SRev(X_1 \ldots X_n)$$

Monotonic vs Simple revenues

$$MonRev(X_1 \ldots X_n) \leq n \cdot \min\{SRev(X_1 \ldots X_n),\ BRev(X_1 \ldots X_n)\}$$

Tightness

Theorem: $MonRev(X_1 \ldots X_n) \leq n \cdot BRev(X_1 \ldots X_n)$

Tight: $SRev(X_1 \ldots X_n) \geq n \cdot BRev(X_1 \ldots X_n)$
for some iid X_i (Hart-Nisan 2012)

Theorem: $MonRev(X_1 \ldots X_n) \leq n \cdot SRev(X_1 \ldots X_n)$

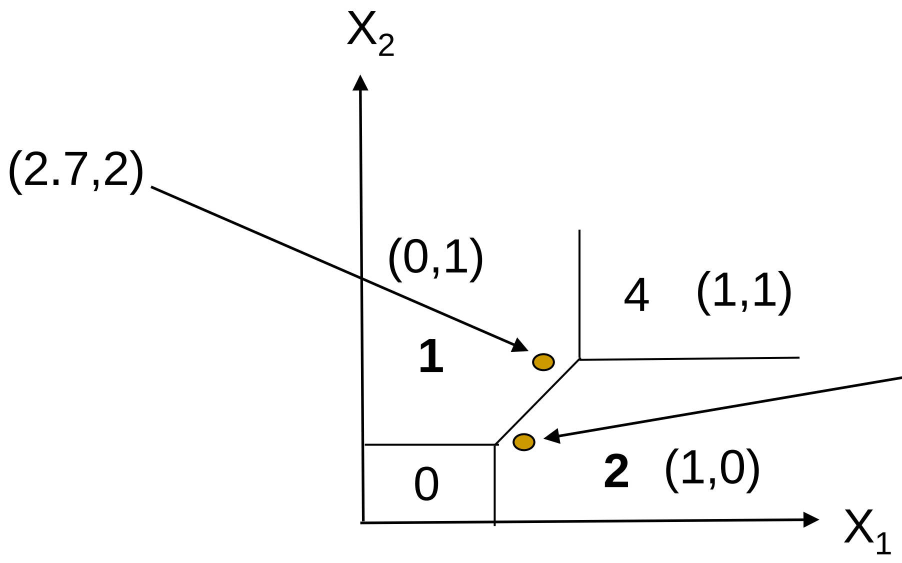
Best we know: $BRev(X_1 \ldots X_n) \geq \Omega(\log n) \cdot BRev(X_1 \ldots X_n)$
for some iid X_i (Hart-Nisan 2012)

Open problem

How large can the gap between *MonRev* and *SRev* be?

- at most n
- at least $O(\log n)$

The Deterministic 2-item case



Not Monotonic

Not "Allocation-Monotonic"

Monotonicity

Definition: A mechanism is **monotonic** if for every $x,y \in R_+^n$ with $x \leq y$ (component-wise) we have that $s(x) \leq s(y)$, where $s()$ is the payment to the seller.

Definition: A mechanism is **allocation-monotonic** if for every $x,y \in R_+^n$ with $x \leq y$ (component-wise) we have that $q(x) \leq q(y)$, where $q_i()$ is the allocation probability of the i'th good.

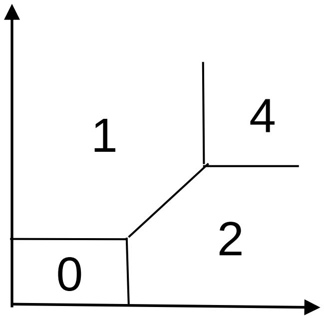
Monotonicity

Claim. Allocation-monotonicity ⇨ Monotonicity

Proof. If $q(y) \geq q(x)$ but $s(y) < s(x)$

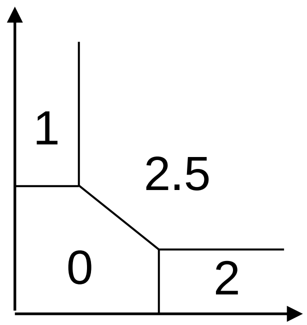
then $(q(y),s(y))$ is better than $(q(x),s(x))$

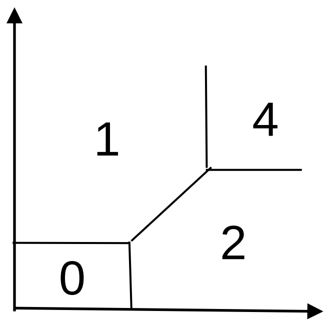
The Deterministic 2-item case



- Not Allocation-Monotonic
- Not Monotonic

The Deterministic 2-item case

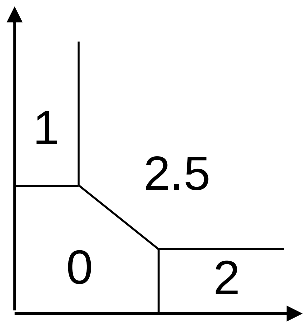


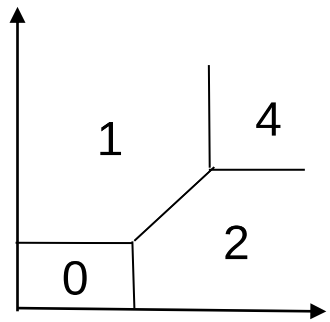


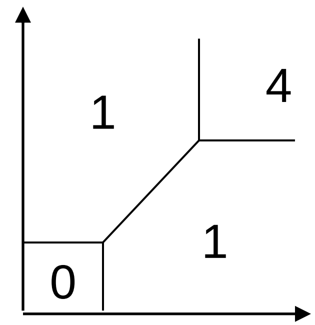
- Allocation-Monotonic
- Monotonic

- Not Allocation-Monotonic
- Not Monotonic

The Deterministic 2-item case





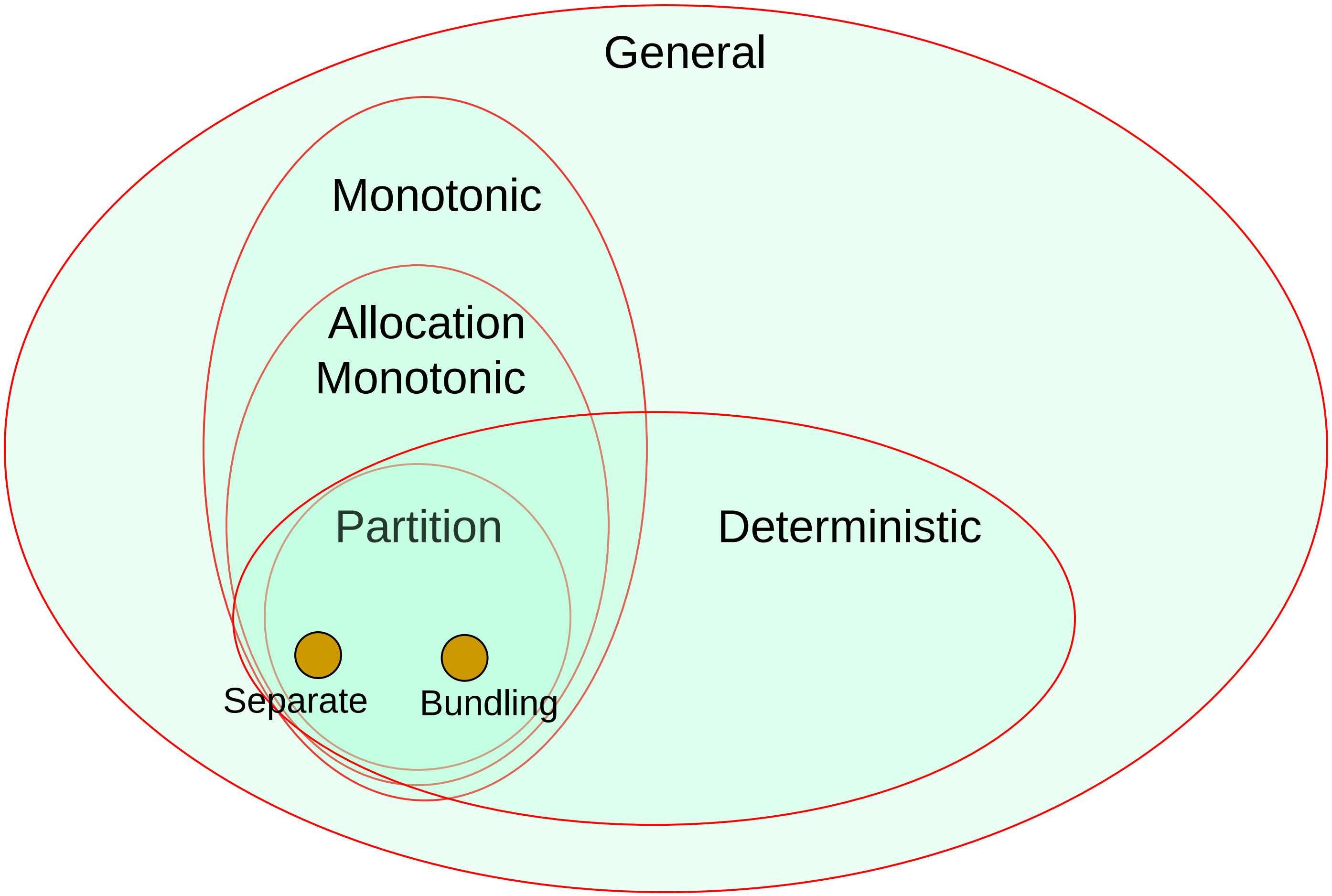


- Allocation-Monotonic
- Monotonic

- Not Allocation-Monotonic
- Not Monotonic

- Not Allocation-Monotonic
- Monotonic

Hierarchy of Mechanisms



Allocation-Monotonocity: Deterministic

Theorem: A deterministic mechanism (with the right tie-breaking) is allocation-monotonic if and only if its pricing function *p()* is submodular.

Pricing function: *p(S)* – the price you need to pay to get the subset *S* of items.

Corollary: $AMonDRev(X_1 \ldots X_n) \leq O(\log n) \cdot SRev(X_1 \ldots X_n)$

Proof: Chawla,Teng &Tzamos show this bound for "sybil-proof" mechanisms, a class that contains those with submodular pricing functions.

Alloc-Monotonicity $\not\Rightarrow$ submodularity

1-dimensional quadratic mechanism with parameter α>0:

$q(x)=\alpha\cdot x$ $\qquad s(x)=\alpha\cdot x^2/2$ $\qquad p(q)=\alpha^{-1}\cdot q^2/2$

General quadratic mechanism (with *A* positive definite matrix):

$q(x)=Ax$ $\qquad s(x) = x^t A x/2$ $\qquad p(q) = q^t A^{-1} q/2$

- *Allocation-monotonicity* ⇔ all off-diagonal entries of *A* are ≥ 0
- *Submodular pricing* ⇔ all off-diagonal entries of A^{-1} are ≤ 0

$$A = \begin{pmatrix} 6 & 3 & 1 \\ 3 & 6 & 3 \\ 1 & 3 & 6 \end{pmatrix} \qquad A^{-1} = \frac{1}{120}\begin{pmatrix} 27 & -15 & \mathbf{3} \\ -15 & 35 & -15 \\ \mathbf{3} & -15 & 27 \end{pmatrix}$$

Allocation-Monotonicity: General

Theorem:

$$AMonRev(X_1 \ldots X_n) \leq O(\log n) \cdot SRev(X_1 \ldots X_n)$$

Proof: Allocation-monotonicity

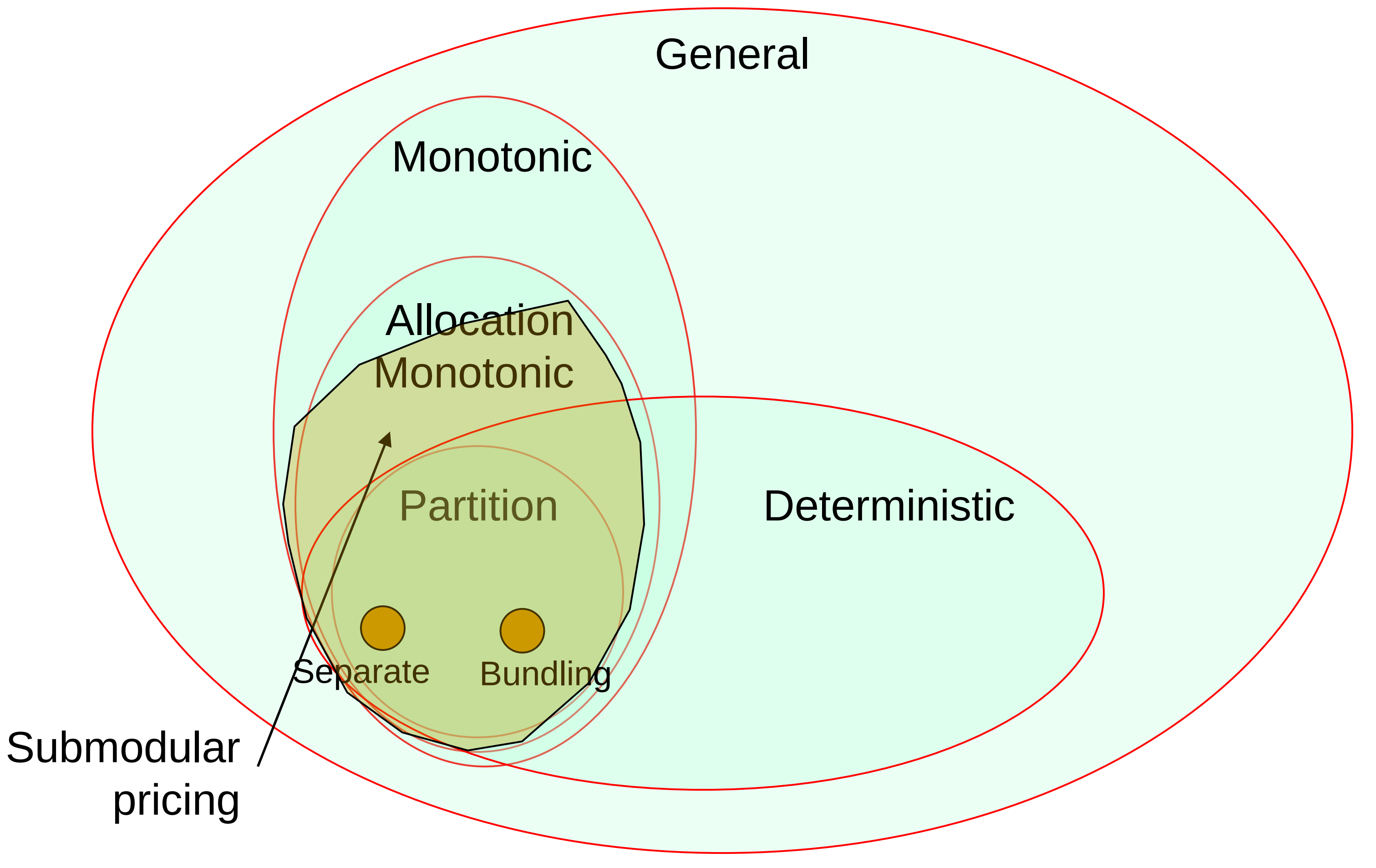
⇔ buyer payoff function *b()* is supermodular

⇨ pricing function is separately subadditive:

$$p(q) \leq \sum_i p(q_i)$$

which suffices for the Chawla, Teng & Tzamos approximation bound.

Hierarchy of Mechanisms



The symmetric deterministic case

Theorem:

$$SuperModSymDRev(X_1 \ldots X_n) \leq log(n) \cdot SRev(X_1 \ldots X_n)$$

$$SymDRev(X_1 \ldots X_n) \leq O(log^2 n) \cdot SRev(X_1 \ldots X_n)$$

(symmetric deterministic mechanisms are monotonic: Hart-Reny)

Future: Still-open problem

How large can the gap between *MonRev* and *SRev* be?

- at most *n*
- at least *O(log n)*

Thank You, Noam!