Game Theoretical Snapshots

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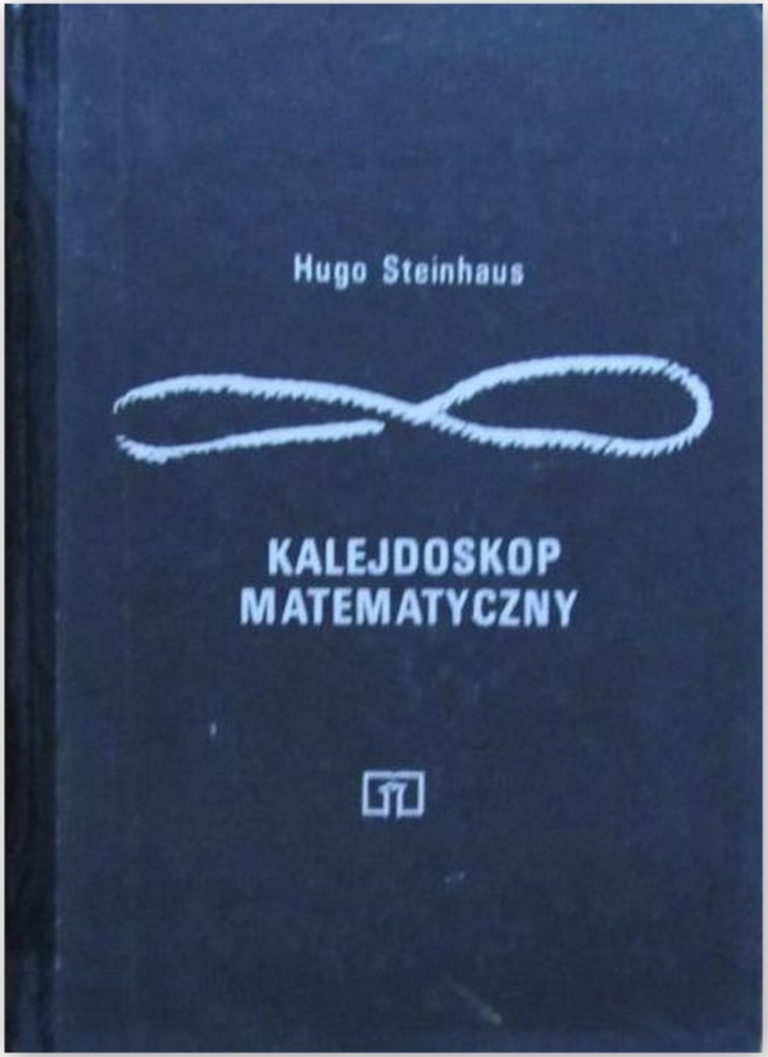
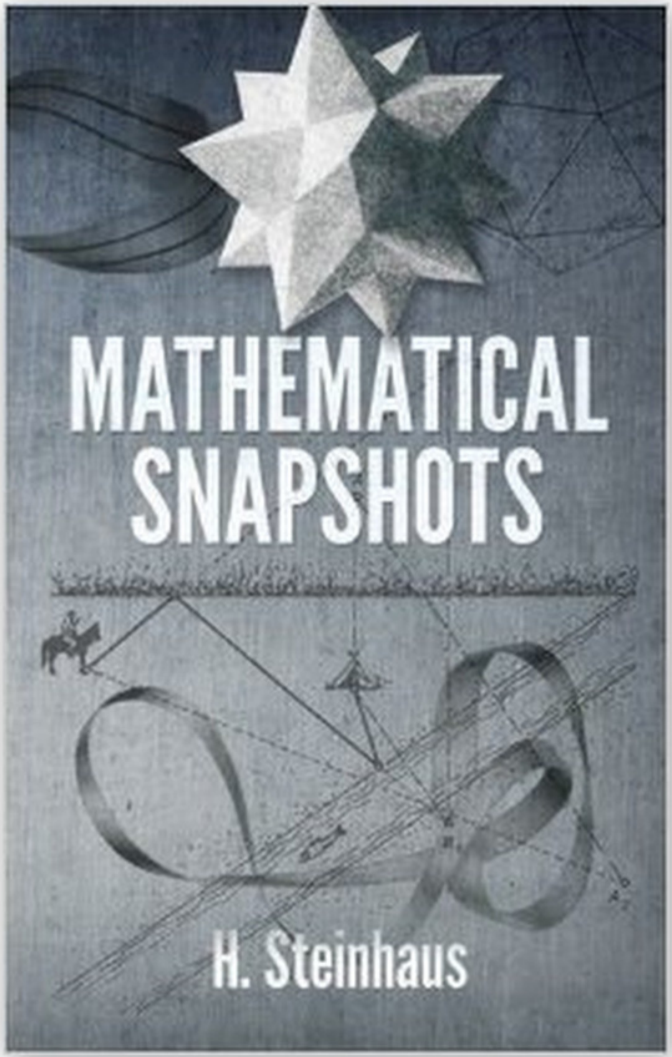
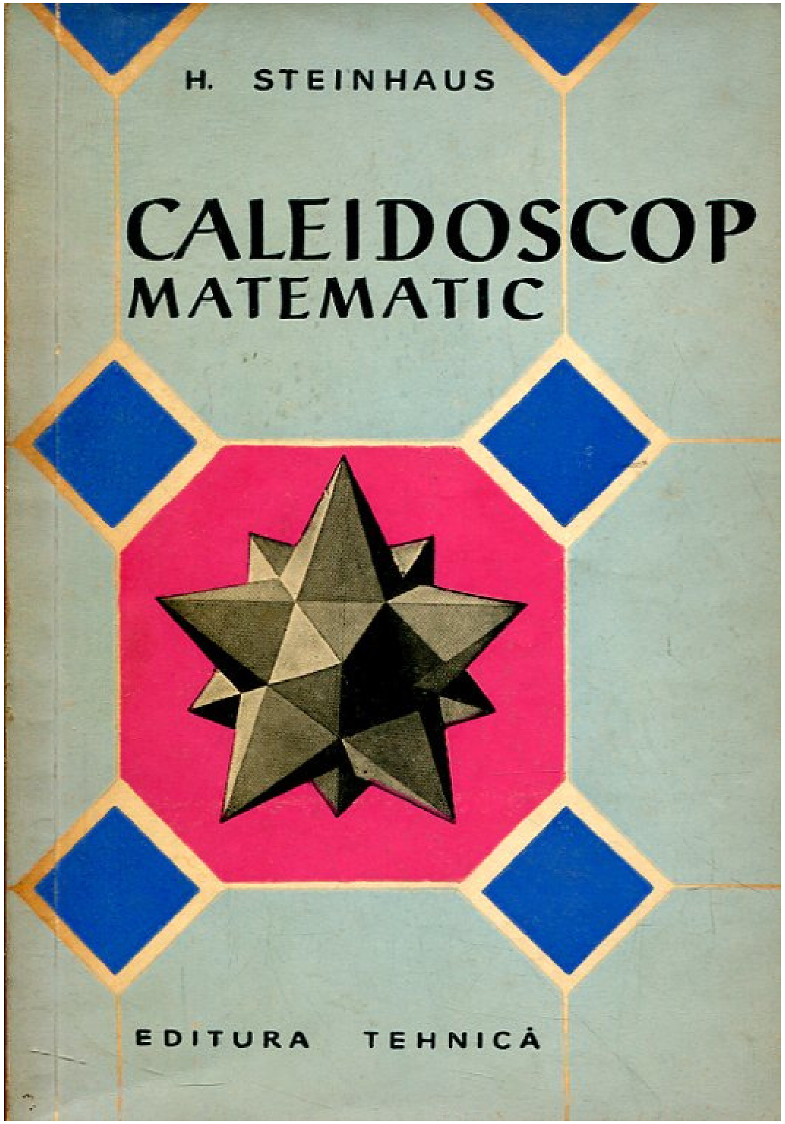
Mathematical Snapshots (1939)





Caleidoscop Matematic (1961)



Snapshot I

Snapshot I

Game Dynamics

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- Next week (in the Summer School)

Snapshot II

Snapshot II

Two(!) Good To Be True

Two(!) Good ...

Two(!) Good ...

- Haven't heard it yet ?

Two(!) Good ...

- Haven't heard it yet ?
- Two(!) bad ...

Snapshot III

Snapshot III

Blotto, Lotto ...

Snapshot III

Blotto, Lotto ...

... and All-Pay

Blotto, Lotto, and All-Pay

Blotto, Lotto, and All-Pay

- Sergiu Hart, *Discrete Colonel Blotto and General Lotto Games*, International Journal of Game Theory 2008
  `www.ma.huji.ac.il/hart/abs/blotto.html`
- Sergiu Hart, *Allocation Games with Caps: From Captain Lotto to All-Pay Auctions*, Center for Rationality 2014
  `www.ma.huji.ac.il/hart/abs/lotto.html`
- Nadav Amir, *Uniqueness of Optimal Strategies in Captain Lotto Games*, Center for Rationality 2015

Colonel Blotto Games

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- Player A has A aquamarine marbles

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  (two-person zero-sum game:
  win $= 1$, lose $= -1$, tie $= 0$)

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- The player who put more marbles in the selected urn **WINS** the game

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Colonel Lotto Games

- The number of aquamarine marbles in the selected urn is a random variable $X \geq 0$ with expectation $a = A/K$
- The number of blue marbles in the selected urn is a random variable $Y \geq 0$ with expectation $b = B/K$
- Payoff function:

$$H(X, Y) = \mathrm{P}[X > Y] - \mathrm{P}[X < Y]$$

Colonel Lotto Games

General Lotto Games

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- Player A chooses (the distribution of) a random variable $X \geq 0$ with expectation a
- Player B chooses (the distribution of) a random variable $Y \geq 0$ with expectation b

General Lotto Games

- Player A chooses (the distribution of) a random variable $\boldsymbol{X} \geq 0$ with expectation $\boldsymbol{a}$
- Player B chooses (the distribution of) a random variable $\boldsymbol{Y} \geq 0$ with expectation $\boldsymbol{b}$
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Theorem Let $a = b > 0$.

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Bell & Cover 1980, Myerson 1993, Lizzeri 1999

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Proof. Optimality of X^*:

$$\mathrm{P}[Y > X^*] = \int_0^{2a} \mathrm{P}[Y > x] \frac{1}{2a} \,\mathrm{d}x$$

$$\leq \frac{1}{2a} E[Y] = \frac{1}{2a} a = \frac{1}{2}$$

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Sahuguet & Persico 2006, Hart 2008

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Hart 2008, Dziubiński 2013

All-Pay Auction

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  worth v_B to Player B

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- **BOTH** players pay their bids

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Hart 2014

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Hart 2014, Amir 2015

Einav Hart, Avrahami, Kareev, and Todd 2015

Snapshot IV

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Complexity of Correlated Equilibria

Complexity of Correlated Equilibria

Complexity of Correlated Equilibria

- Sergiu Hart and Noam Nisan
  The Query Complexity of Correlated Equilibria
  Center for Rationality 2013
  www.ma.huji.ac.il/hart/abs/corr-com.html

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CORRELATED EQUILIBRIUM :

- $\mathbf{2}^{n}$ unknowns, $\geq \mathbf{0}$
- $\mathbf{2}n + \mathbf{1}$ linear inequalities

$\Rightarrow$ There is an algorithm for computing **CORRELATED EQUILIBRIA** with **COMPLEXITY** $=$ **POLY**$(\mathbf{2}^{n}) =$ **EXP**(n)

Correlated Equilibria

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BUT: **Regret-Matching** and other "no-regret" dynamics yield ϵ-**CORRELATED EQUILIBRIA** with high probability in $O(\log(n)/\epsilon^2)$ steps

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- **QUERY COMPLEXITY (QC)** $:=$ maximal number of payoff queries (out of $n \cdot 2^n$)

$\Rightarrow$ There are randomized algorithms for computing ϵ-**CORRELATED EQUILIBRIA** with **QC** $=$ **POLY**(n)

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  - no-regret dynamics

Correlated Equilibria (recall)

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- ***Exact*** CORRELATED EQUILIBRIA **?**

Correlated Equilibria

- There are randomized algorithms for computing ϵ-CORRELATED EQUILIBRIA with QC $=$ POLY(n) (Regret-Matching, ...)

- ***Exact*** **CORRELATED EQUILIBRIA ?**
- ***Deterministic*** algorithms **?**

Query Complexity of CE

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	Algorithm	
	Randomized	***Deterministic***
ε-CE		
exact CE		

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$[1] =$ Regret-Matching, No Regret

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[2] = Babichenko and Barman 2013

Query Complexity of CE

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	Randomized	***Deterministic***
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exact CE	**EXP**(n) [3]	**EXP**(n) [2]

$[1]$ = Regret-Matching, No Regret
$[2]$ = Babichenko and Barman 2013
$[3]$ = this paper

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  - dual decomposes into n problems

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- ***Question***: Why does this help ***only*** for approximate CE and randomized algorithms ?
- ***Question***: Complexity of ***Nash Equilibria*** ?

Snapshot V

Snapshot V

Smooth Calibration

Snapshot V

Smooth Calibration and Leaky Forecasts

Smooth Calibration and Leaky Forec

Smooth Calibration and Leaky Forec

- Dean Foster and Sergiu Hart
  Smooth Calibration, Leaky Forecasts, and Finite Recall
  2012 (in preparation)
  www.ma.huji.ac.il/hart/abs/calib-eq.html

Calibration

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Dawid 1982

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    with probability $1/2$ forecast $= 60\%$)

Foster and Vohra 1998

No Calibration

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Oakes 1985

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- **Main Result**:

> There exists a ***deterministic*** procedure that is **SMOOTHLY CALIBRATED**.

Smooth Calibration

- **SMOOTH CALIBRATION**: combine together the days when the forecast was ***close to*** p (*smooth* out the calibration score)
- **Main Result**:

There exists a ***deterministic*** procedure that is **SMOOTHLY CALIBRATED**.

Deterministic $\Longrightarrow$ result holds also when the forecasts are ***leaked***

Calibration

Calibration

- Set of **ACTIONS**: $A \subset \mathbb{R}^m$ (finite set)
- Set of **FORECASTS**: $C = \Delta(A)$
  - Example: $A = \{0, 1\}$, $C = [0, 1]$

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$$K_T = \frac{1}{T} \sum_{t=1}^{T} ||\bar{a}_t - c_t||$$

where

$$\bar{a}_t := \frac{\sum_{s=1}^{T} \mathbf{1}_{c_t = c_s} \, a_s}{\sum_{s=1}^{T} \mathbf{1}_{c_t = c_s}}$$

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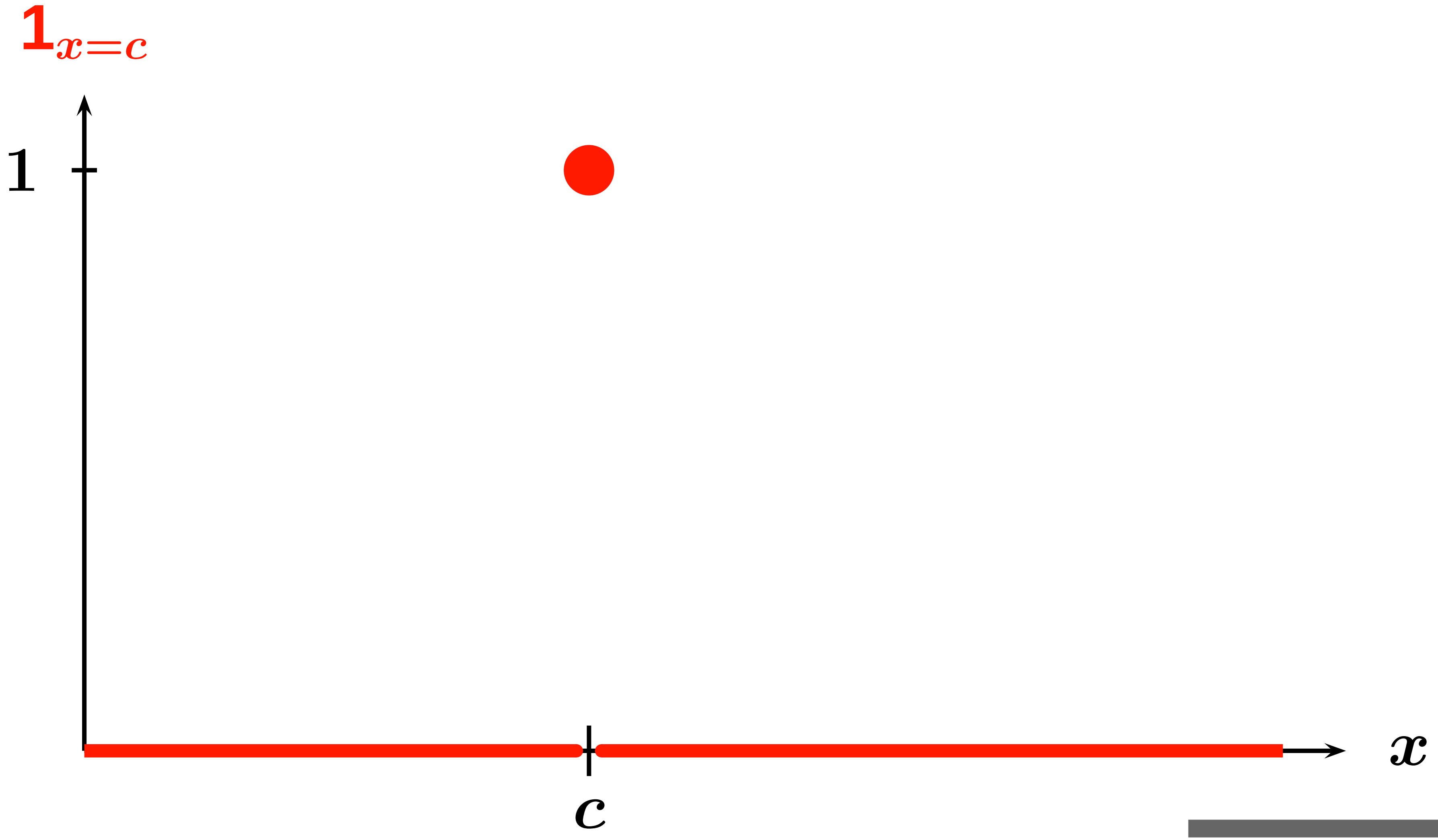
  - $\boldsymbol{\Lambda}(\boldsymbol{x}, \boldsymbol{c})$ = "weight" of $\boldsymbol{x}$ relative to $\boldsymbol{c}$
  - Example: $\boldsymbol{\Lambda}(\boldsymbol{x}, \boldsymbol{c}) = [\boldsymbol{\delta} - ||\boldsymbol{x} - \boldsymbol{c}||]_+ / \boldsymbol{\delta}$

Indicator Function

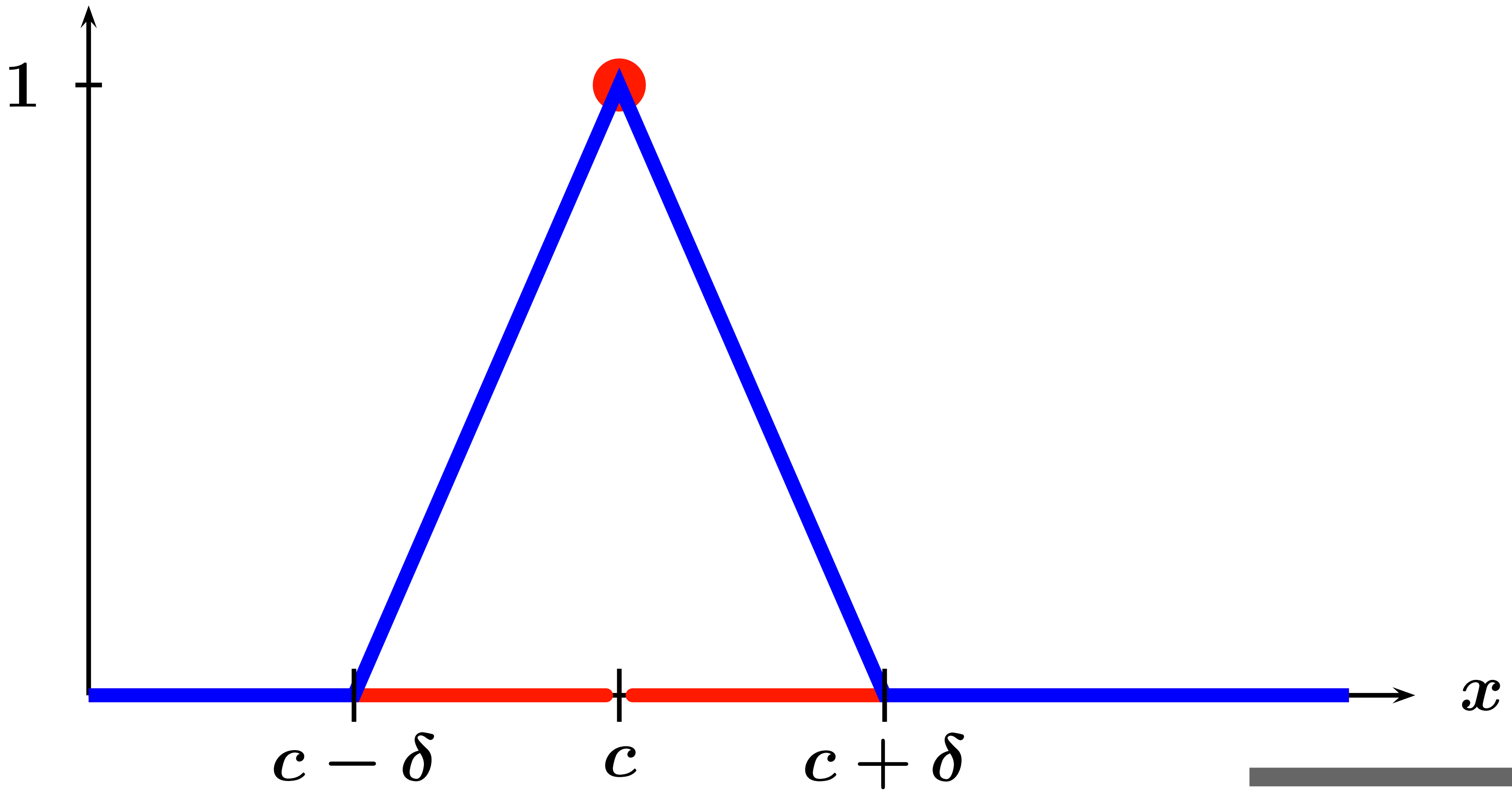
Indicator Function



Indicator and Λ Functions

$\Lambda(x,c)$

$\mathbf{1}_{x=c}$



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  - $\boldsymbol{\Lambda(x,c)}$ = "weight" of $\boldsymbol{x}$ relative to $\boldsymbol{c}$
- $\boldsymbol{\Lambda}$-CALIBRATION SCORE at time $\boldsymbol{T}$:

$$\boldsymbol{K_T^{\Lambda} = \frac{1}{T}\sum_{t=1}^{T} ||\bar{a}_t^{\Lambda} - c_t^{\Lambda}||}$$

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- A ***"smoothing function"*** is a Lipschitz function $\Lambda : C \times C \to [0, 1]$ with $\Lambda(c, c) = 1$ for every c.
  - $\Lambda(x, c)$ = "weight" of x relative to c
- Λ-CALIBRATION SCORE at time T:

$$K_T^{\Lambda} = \frac{1}{T} \sum_{t=1}^{T} ||\bar{a}_t^{\Lambda} - c_t^{\Lambda}||$$

$$\bar{a}_t^{\Lambda} = \frac{\sum_{s=1}^{T} \Lambda(c_s, c_t)\, a_s}{\sum_{s=1}^{T} \Lambda(c_s, c_t)}, \quad c_t^{\Lambda} = \frac{\sum_{s=1}^{T} \Lambda(c_s, c_t)\, c_s}{\sum_{s=1}^{T} \Lambda(c_s, c_t)}$$

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- Full monitoring, perfect recall

Smooth Calibration

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ε-SMOOTHLY CALIBRATED

Smooth Calibration

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if there is T_0 such that $K_T^\Lambda \le \varepsilon$ holds for:

- every $T \ge T_0$,
- every strategy of Player A, and
- every smoothing function Λ with Lipschitz constant $\le 1/\varepsilon$

Smooth Calibration: Result

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For every $\varepsilon > 0$

there exists a procedure

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- forecasts may be ***leaked***.

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- For ***game dynamics***:
  - Uncoupled, finite recall, dynamics that converge to the set of **CORRELATED EQUILIBRIA**
  - Uncoupled, finite recall, dynamics that are close most of the time to **NASH EQUILIBRIA**

Previous Work

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- Weak Calibration (deterministic):

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  - Kakade and Foster 2004 / 2008
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  - Foster and Kakade 2006
- Online Regression Problem:
  - Foster 1991, 1999
  - Vovk 2001
  - Azoury and Warmuth 2001
  - Cesa-Bianchi and Lugosi 2006

Snapshot VI

Snapshot VI

Evidence Games:

Snapshot VI

Evidence Games:
Truth and Commitment

Evidence Games

Evidence Games

- Sergiu Hart, Ilan Kremer, and Motty Perry
  Evidence Games: Truth and Commitment
  Center for Rationality 2015
  `www.ma.huji.ac.il/hart/abs/st-ne.html`

Q: "Do you deserve a pay raise?"

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- How can one determine the "right" reward, or punishment?
- How can one "separate" and avoid "unraveling" (Akerlof 70)?

Evidence Games: General Setup

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- **AGENT** who is informed

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- **PRINCIPAL** who takes decision but is uninformed

Evidence Games: General Setup

- **AGENT** who is informed
- **PRINCIPAL** who takes decision but is uninformed
- Agent **TRANSMITS** information to Principal (costlessly)

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Two Setups

- **GAME**: Principal decides ***after*** receiving Agent's message
- **MECHANISM**: Principal chooses a ***policy*** ***before*** Agent's message
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(obtained ***without commitment***)

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COINCIDE

Main Result: Equivalence

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$\uparrow$

Differs from signalling, screening, cheap-talk, ...

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$\Longrightarrow$ Agent can ***pretend*** to be a type that has ***less information (less evidence)***

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(T1) Revealing the whole truth is preferable when the reward is the same (lexicographic preference)

(T2) The whole truth is revealed with infinitesimal positive probability
(by mistake, or because the agent may be non-strategic, or ... [UK])

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- Persuasion games
  - **Glazer and Rubinstein 2004, 2006**

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- In EVIDENCE GAMES there is ***equivalence*** between EQUILIBRIUM (without commitment) and OPTIMAL MECHANISM (with commitment)

$\Rightarrow$ EQUILIBRIUM is ***constrained efficient*** (in the canonical case)

- The conditions of EVIDENCE GAMES are ***indispensable*** for this equivalence

"Do you swear to tell the truth, the whole truth, and nothing but the truth in the most entertaining way possible?"

Thanks !!

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