The hypoelliptic Laplacian

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In this series of three lectures, I will explain the theory of the hypoelliptic Laplacian. If X is a compact Riemannian manifold, and if $\mathcal{X}$ is the total space of its tangent bundle, there is a canonical interpolation between the classical Laplacian of X and the generator of the geodesic flow by a family of hypoelliptic operators $L_b^X|_{b>0}$ acting on $\mathcal{X}$. This interpolation extends to all the classical geometric Laplacians. There is a natural dynamical system counterpart, which interpolates between Brownian motion and the geodesic flow.

The hypoelliptic deformation preserves certain spectral invariants. like the Ray-Singer torsion, the holomorphic torsion and the eta invariants. In the case of locally symmetric spaces, the spectrum of the original Laplacian remains rigidly embedded in the spectrum of its deformation. This property has been used in the context of Selberg's trace formula. Another application of the hypoelliptic Laplacian is in complex Hermitian geometry, where the extra degrees of freedom provided by the hypoelliptic deformation can be used to solve a question which is unsolvable in the elliptic world.

In the first lecture, 'Hypoelliptic Laplacian, Brownian motion and the geodesic flow', I will give the structure of the hypoelliptic Laplacian. I will also describe a natural construction of the hypoelliptic Laplacian as a nonstandard Hodge Laplacian, and explain its connections with dynamical systems and an equation by Langevin.

In the second lecture 'Hypoelliptic Laplacian and the trace formula', I will concentrate on the case of symmetric spaces, and on applications to the evaluation of orbital integrals and to Selberg's trace formula. The Dirac operator of Kostant plays an important role in the constructions.

In the third lecture 'Hypoelliptic Laplacian, analytic torsion and RRG', I will explain applications to analytic torsion, and describe how the hypoelliptic Laplacian can be used in complex Hermitian geometry to establish a version of a theorem of Riemann-Roch-Grothendieck.