DRINFELD'S p-ADIC SYMMETRIC DOMAINS AND UNIFORMIZATION: *EXERCISES.*

1. A filtration on $\Omega^1 \otimes M_n$ by vector sub-bundles (after Schneider and Stuhler)

Let $M = M_n$ be the representation of $G = SL_2(K)$ on homogenous polynomials of degree n in the variables u and v given by

$$(\gamma P)(u,v) = P(au+bv, cu+dv)$$

if $\gamma^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $\mathcal{O}$ be the structure sheaf of the rigid analytic space $\mathfrak{X}$, and $\Omega = \Omega^1$ the sheaf of rigid 1-forms. Consider the following filtration on $\mathcal{O}(M) = \mathcal{O} \otimes_K M$:

$$\mathcal{O}(M) = F^0 \supset F^1 \supset \cdots \supset F^n \supset \{0\} = F^{n+1}$$

where

$$F^k = Span_{\mathcal{O}} \left\{ (u-zv)^k u^{n-k-l} v^l;\ 0 \le l \le n-k \right\}.$$

Show:

(1) The filtration is G-stable.

(2) $F^k = \mathcal{O} \cdot (u-zv)^k v^{n-k} \oplus F^{k+1}$ (as $\mathcal{O}$-modules, but not as G-modules).

(3) $\mathcal{O}(M)$ is free over $\mathcal{O}$ with basis $(u-zv)^k v^{n-k}$ $(0 \le k \le n)$.

(4) For any $m \in \mathbb{Z}$ let $\mathcal{O}(m)$ be the sheaf $\mathcal{O}$ with the *twisted* G-action $\gamma \cdot f = (cz+d)^m \left(f \circ \gamma^{-1}\right)$, where (c,d) is the bottom row of γ^{-1}. Then $f \mapsto f(z)(u-zv)^k v^{n-k}$ mod F^{k+1} is an isomorphism

$$\Theta_k : \mathcal{O}(n-2k) \simeq F^k / F^{k+1}.$$

(5) Similarly $F^{\cdot}\Omega = \Omega \otimes_{\mathcal{O}} F^{\cdot}$ is a G-stable filtration on $\Omega(M)$ and

$$f \mapsto f(z)(u-zv)^k v^{n-k} dz \ mod \Omega \otimes F^{k+1}$$

is an isomorphism

$$\Theta'_k : \mathcal{O}(n-2k-2) \simeq \Omega \otimes F^k / \Omega \otimes F^{k+1}.$$

(6) For $1 \le k \le n$, the G-homomorphism $d : \mathcal{O}(M) \to \Omega(M)$ maps F^k to $\Omega \otimes F^{k-1}$ (*Griffiths transversality*) and induces a commutative diagram

$$\begin{array}{ccc} \mathcal{O}(n-2k) & \stackrel{(-k)\cdot}{\to} & \mathcal{O}(n-2k) \\ \downarrow \Theta_k & & \downarrow \Theta'_{k-1} \\ F^k/F^{k+1} & \stackrel{d}{\simeq} & \Omega \otimes F^{k-1} / \Omega \otimes F^k \end{array}.$$

(7) There is a decomposition as a direct sum of abelian groups with G-action

$$\Omega(M) = d(F^1) \oplus (\Omega \otimes F^n).$$

(8) Let $n \geq 0$. There is a commutative diagram of additive (but not $\mathcal{O}$-linear) G-homomorphisms

$$\begin{array}{ccc} \mathcal{O}(n) & \overset{\frac{1}{n!}\left(\frac{d}{dz}\right)^{n+1}}{\longrightarrow} & \mathcal{O}(-n-2) \\ \downarrow \Theta_0 & & \downarrow \Theta'_n \\ \mathcal{O}(M)/F^1 & \overset{pr_2 \circ d}{\longrightarrow} & \Omega \otimes F^n \end{array},$$

where pr_2 is the projection on the second factor in the decomposition (7).

Note that the fact that the top arrow is a G-homomorphism is not easy to establish via a direct computation, since differentiation does not commute with the action of G. It is rather a consequence of the fact that the other three arrows commute with G. Deduce:

Corollary 1.1. *For $n \geq 0$, let $P(n)$ be the space of polynomials of degree at most n, with G-action induced from (4). Then there is an exact sequence of G-representations*

$$0 \to P(n) \to \mathcal{O}(\mathfrak{X})(n) \overset{\frac{1}{n!}\left(\frac{d}{dz}\right)^{n+1}}{\longrightarrow} \mathcal{O}(\mathfrak{X})(-n-2) \to H^1_{dR}(\mathfrak{X}; M_n) \to 0.$$

2. Automorphisms of the Drinfeld p-adic upper half plane

Let $\widehat{\mathfrak{X}}$ be the formal scheme underlying the Drinfeld upper half plane, and $\mathcal{K}$ the completion of the maximal unramified extension of $\mathbb{Q}_p$. Show that any automorphism of $\widehat{\mathfrak{X}}\widehat{\otimes}\mathcal{O}_{\mathcal{K}}$ over $Spf\mathcal{O}_{\mathcal{K}}$ which commutes with the action of $SL_2(\mathbb{Q}_p)$ is the identity.

Hints: (i) Such an automorphism φ must induce the identity on the Bruhat-Tits tree $\mathcal{T}$, hence preserves the irreducible components of the special fiber.

(ii) Show that φ is the identity on the special fiber.

(iii) Using induction on n, show that φ is the identity on $\widehat{\mathfrak{X}}\widehat{\otimes}\mathcal{O}_{\mathcal{K}}/p^n\mathcal{O}_{\mathcal{K}}$.

3. Semilinear Algebra (for the case of Drinfeld's upper half plane)

Notation: D is the unique quaternion algebra over $\mathbb{Q}_p$, $\tilde{K}$ the quadratic unramified extension of $\mathbb{Q}_p$ embedded in D, $\Pi \in D$, $\Pi^2 = p$, and $\mathcal{K}$ the completion of the maximal unramified extension fo $\mathbb{Q}_p$. Let σ be the arithmetic Frobenius automorphism of $\mathcal{K}$. $\mathcal{O}_D$ is the maximal order of D. Thus

$$\mathcal{O}_D = \mathcal{O}_{\tilde{K}} \oplus \mathcal{O}_{\tilde{K}}\Pi, \ \ \Pi a = \sigma(a)\Pi \quad (a \in \tilde{K}).$$

Let $N = D \otimes \mathcal{K} = N_0 \oplus N_1$, where $N_i = \left\{x \in D \otimes \mathcal{K} | \, (a \otimes 1)x = (1 \otimes \sigma^i(a))x\right\}$.

(i) $\mathcal{O}_N = \mathcal{O}_D \otimes \mathcal{O}_{\mathcal{K}}$ decomposes as $(\mathcal{O}_N \cap N_0) \oplus (\mathcal{O}_N \cap N_1)$. If $\mathcal{O}_{\tilde{K}} \otimes \mathcal{O}_{\mathcal{K}} = \mathcal{O}^{(0)}_{\mathcal{K}} \oplus \mathcal{O}^{(1)}_{\mathcal{K}}$ then

$$\mathcal{O}_N \cap N_i = \mathcal{O}^{(i)}_{\mathcal{K}} \oplus \Pi\mathcal{O}^{(i+1)}_{\mathcal{K}} \qquad (i mod 2).$$

(ii) Let the Verschiebung V be defined by

$$V(\delta \otimes \lambda) = \delta\Pi \otimes \sigma^{-1}(\lambda) \qquad (\delta \in D, \ \ \lambda \in \mathcal{K}).$$

Show that both (left) multiplication by Π and V map $\mathcal{O}_N \cap N_i$ to $\mathcal{O}_N \cap N_{i+1}$ and have the same image ($i = 0$ and 1 are both critical). Show that $\mathcal{O}_N \cap N_i / V(\mathcal{O}_N \cap N_{i-1})$ is one-dimensional over $\bar{\mathbb{F}}_p$.

(iii) If $N' \subset N$ is a sub-$\mathcal{K}$-vector space stable under $\tilde{K} \subset D$ then $N' = N'_0 \oplus N'_1$ where $N'_i = N' \cap N_i$. If furthermore N' is stable under D, $N'_1 = \Pi N'_0$.

(iv) Show that the $\mathcal{K}$-linear endomorphisms of N that commute with both D and V are naturally isomorphic to

$$\left(N^{\Pi V^{-1}}\right)^{opp} \cong N^{\Pi V^{-1}}.$$

(v) Show that $N^{\Pi V^{-1}} \cong M_2(\mathbb{Q}_p)$.
Remark: The pair of groups $(D^\times, GL_2(\mathbb{Q}_p))$ is a special case of the set-up considered in Chapter 1 [Rapoport-Zink].

4. Supersingular elliptic curves and special formal $\mathcal{O}_D$-modules

Let H be the quaternion algebra over $\mathbb{Q}$ ramified at ∞ and p. Let $\mathcal{O}_H$ be some maximal order in H. Fix (E, ι) where E is a supersingular elliptic curve over $\mathbb{F}_{p^2}$ and $\iota : \mathcal{O}_H \simeq End(E)$. By Deuring, (E, ι) exists. Let E^σ be the Frobenius twist of E, $\iota^\sigma : \mathcal{O}_H \simeq End(E^\sigma)$ and $\Pi : E \to E^\sigma$ the Frobenius morphism. Write Π also for the Frobenius morphism of E^σ to $E = E^{(\sigma^2)}$. Consider $A = E \times E^\sigma$. Find $End(A)$. Embed $\mathcal{O}_{\tilde{K}}$ in $\mathcal{O}_H \otimes \mathbb{Z}_p$ and let $\mathcal{O}_D = \mathcal{O}_{\tilde{K}}[\Pi]$. Show that the formal group $\widehat{A}$ is a model for the special formal height 4 $\mathcal{O}_D$-module which was denoted in class $\mathbb{X}$.

5. Isomorphism types of special formal height 4 $\mathcal{O}_D$-modules

(i) Using Drinfeld's modular interpretation of $\widehat{\mathfrak{X}}\widehat{\otimes}\mathcal{O}_{\mathcal{K}}$ show that the isomorphism types of special formal height 4 $\mathcal{O}_D$-modules over $\bar{\mathbb{F}}_p$ are in bijection with two copies of $GL_2(\mathbb{F}_p)\backslash\mathbb{P}^1(\bar{\mathbb{F}}_p)$ joined at the unique $\mathbb{F}_p$-rational point.
(ii) Show that the unique $\mathbb{F}_p$-rational point corresponds to $\mathbb{X}$.
(iii) Characterize the two "components" in terms of critical indices for the Cartier module.
Note: This is only a characterization over $\bar{\mathbb{F}}_p$. The problem of classifying isomorphism types over an arbitrary base is not representable.

6. Drinfeld's p-adic symmetric domain in dimension 2

This is a vaguely phrased exercise but a very good one. Try to understand the Bruhat-Tits building as well as you can for GL_3 $(d = 2)$. Mimic the construction of the formal scheme $\widehat{\mathfrak{X}}$ (warning: the irreducible components of the special fibers will not be $\mathbb{P}^{2\prime}s$ this time, but rather their blow-ups at the $q^2 + q + 1$ rational points). Describe the pre-images under reduction $r^{-1}(|\sigma|)$ for open simplices σ of the building.

While the case $d = 1$ is too special, if you understand well the case $d = 2$, you are likely to understand the general case.