DRINFELED’S $p$-ADIC SYMMETRIC DOMAINS AND UNIFORMIZATION: EXERCISES.

1. A filtration on $\Omega^1 \otimes M_n$ by vector sub-bundles (after Schneider and Stuhler)

Let $M = M_n$ be the representation of $G = SL_2(K)$ on homogenous polynomials of degree $n$ in the variables $u$ and $v$ given by

$$ (\gamma P)(u,v) = P(au + bv, cu + dv) $$

if $\gamma^{-1} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$. Let $\mathcal{O}$ be the structure sheaf of the rigid analytic space $X$, and $\Omega = \Omega^1$ the sheaf of rigid 1-forms. Consider the following filtration on $\mathcal{O}(M) = \mathcal{O} \otimes_K M$:

$$ \mathcal{O}(M) = F^0 \supset F^1 \supset \cdots \supset F^n \supset \{0\} = F^{n+1} $$

where

$$ F^k = \text{Span}_\mathcal{O} \{(u-zv)^k u^{n-k-l} v^l; 0 \leq l \leq n-k\}. $$

Show:

1. The filtration is $G$-stable.
2. $F^k = \mathcal{O}(u-zv)^k v^{n-k} \oplus F^{k+1}$ (as $\mathcal{O}$-modules, but not as $G$-modules).
3. $\mathcal{O}(M)$ is free over $\mathcal{O}$ with basis $(u-zv)^k v^{n-k}$ (0 $\leq k \leq n$).
4. For any $m \in \mathbb{Z}$ let $\mathcal{O}(m)$ be the sheaf $\mathcal{O}$ with the *twisted* $G$-action $\gamma \cdot f = (cz+d)^m (f \circ \gamma^{-1})$, where $(c, d)$ is the bottom row of $\gamma^{-1}$. Then $f \mapsto f(z)(u-zv)^k v^{n-k}$ mod $F^{k+1}$ is an isomorphism

$$ \Theta_k : \mathcal{O}(n-2k) \simeq F^k / F^{k+1}. $$

5. Similarly $F^k \Omega = \Omega \otimes_F F^k$ is a $G$-stable filtration on $\Omega(M)$ and

$$ f \mapsto f(z)(u-zv)^k v^{n-k} dz \mod \Omega \otimes F^{k+1} $$

is an isomorphism

$$ \Theta'_k : \mathcal{O}(n-2k-2) \simeq \Omega \otimes F^k / \Omega \otimes F^{k+1}. $$

6. For $1 \leq k \leq n$, the $G$-homomorphism $d : \mathcal{O}(M) \to \Omega(M)$ maps $F^k$ to $\Omega \otimes F^{k-1}$ (*Griffiths transversality*) and induces a commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}(n-2k) & \xrightarrow{(-k)} & \mathcal{O}(n-2k) \\
\downarrow \Theta_k & & \downarrow \Theta'_{k-1} \\
F^k / F^{k+1} & \xrightarrow{d} & \Omega \otimes F^{k-1} / \Omega \otimes F^k
\end{array}
$$

7. There is a decomposition as a direct sum of abelian groups with $G$-action

$$ \Omega(M) = d(F^1) \oplus (\Omega \otimes F^n). $$
(8) Let \( n \geq 0 \). There is a commutative diagram of additive (but not \( \mathcal{O} \)-linear) \( G \)-homomorphisms

\[
\begin{array}{ccl}
\mathcal{O}(n) & \overset{\hat{\psi}((\psi_1)^{n+1})}{\rightarrow} & \mathcal{O}(-n - 2) \\
\downarrow \Theta_0 & & \downarrow \Theta'_0 \\
\mathcal{O}(M)/F^1 & \overset{pr_2^\text{ord}}{\rightarrow} & \Omega \otimes F^n
\end{array}
\]

where \( pr_2 \) is the projection on the second factor in the decomposition (7).

Note that the fact that the top arrow is a \( G \)-homomorphism is not easy to establish via a direct computation, since differentiation does not commute with the action of \( G \). It is rather a consequence of the fact that the other three arrows commute with \( G \). Deduce:

**Corollary 1.1.** For \( n \geq 0 \), let \( P(n) \) be the space of polynomials of degree at most \( n \), with \( G \)-action induced from (4). Then there is an exact sequence of \( G \)-representations

\[
0 \rightarrow P(n) \rightarrow \mathcal{O}(\mathfrak{X})(n) \overset{\hat{\psi}((\psi_1)^{n+1})}{\rightarrow} \mathcal{O}(\mathfrak{X})(-n - 2) \rightarrow H^1_{dR}(\mathfrak{X}; M_n) \rightarrow 0.
\]

2. **Automorphisms of the Drinfeld \( p \)-adic upper half plane**

Let \( \hat{\mathfrak{X}} \) be the formal scheme underlying the Drinfeld upper half plane, and \( \mathcal{K} \) the completion of the maximal unramified extension of \( \mathbb{Q}_p \). Show that any automorphism of \( \hat{\mathfrak{X}} \otimes \mathcal{O}_\mathcal{K} \) over \( Spf \mathcal{O}_\mathcal{K} \) which commutes with the action of \( SL_2(\mathbb{Q}_p) \) is the identity.

**Hints:** (i) Such an automorphism \( \varphi \) must induce the identity on the Bruhat-Tits tree \( T \), hence preserves the irreducible components of the special fiber.

(ii) Show that \( \varphi \) is the identity on the special fiber.

(iii) Using induction on \( n \), show that \( \varphi \) is the identity on \( \hat{\mathfrak{X}} \otimes \mathcal{O}_\mathcal{K}/p^n\mathcal{O}_\mathcal{K} \).

3. **Semilinear Algebra (for the case of Drinfeld’s upper half plane)**

**Notation:** \( D \) is the unique quaternion algebra over \( \mathbb{Q}_p \), \( \hat{K} \) the quadratic unramified extension of \( \mathbb{Q}_p \) embedded in \( D \), \( \Pi \in D \), \( \Pi^2 = p \), and \( \mathcal{K} \) the completion of the maximal unramified extension to \( \mathbb{Q}_p \). Let \( \sigma \) be the arithmetic Frobenius automorphism of \( \mathcal{K} \). \( \mathcal{O}_D \) is the maximal order of \( D \). Thus

\[
\mathcal{O}_D = \mathcal{O}_\hat{K} \otimes \mathcal{O}_\mathcal{K} \Pi, \quad \Pi a = \sigma(a) \Pi \quad (a \in \hat{K}).
\]

Let \( N = D \otimes \mathcal{K} = N_0 \oplus N_1 \), where \( N_i = \{ x \in D \otimes \mathcal{K} | (a \otimes 1)x = (1 \otimes \sigma^i(a))x \} \).

(i) \( \mathcal{O}_N = \mathcal{O}_D \otimes \mathcal{O}_\mathcal{K} \) decomposes as \( (\mathcal{O}_N \cap N_0) \oplus (\mathcal{O}_N \cap N_1) \). If \( \mathcal{O}_\hat{K} \otimes \mathcal{O}_\mathcal{K} = \mathcal{O}_K^{(0)} \oplus \mathcal{O}_K^{(1)} \) then

\[
\mathcal{O}_N \cap N_i = \mathcal{O}_K^{(i)} \oplus \Pi \mathcal{O}_K^{(i+1)} \quad (\text{mod}2).
\]

(ii) Let the Verschiebung \( V \) be defined by

\[
V(\delta \otimes \lambda) = \delta \Pi \otimes \sigma^{-1}(\lambda) \quad (\delta \in D, \ \lambda \in \mathcal{K}).
\]

Show that both (left) multiplication by \( \Pi \) and \( V \) map \( \mathcal{O}_N \cap N_i \) to \( \mathcal{O}_N \cap N_{i+1} \) and have the same image \((i = 0 \text{ and } 1 \text{ are both critical})

(iii) If \( N' \subset N \) is a sub-\( \mathcal{K} \)-vector space stable under \( \hat{K} \subset D \) then \( N' = N_0' \oplus N_1' \) where \( N_i' = N' \cap N_i \). If furthermore \( N' \) is stable under \( D, \ N_1' = \Pi N_0' \).
(iv) Show that the $K$-linear endomorphisms of $N$ that commute with both $D$ and $V$ are naturally isomorphic to
\[ (N^\dagger)^{-1} \cong N^\dagger. \]

(v) Show that $N^\dagger \cong M_2(Q_p)$.

Remark: The pair of groups $(D^\times, GL_2(Q_p))$ is a special case of the set-up considered in Chapter 1 [Rapoport-Zink].

4. Supersingular elliptic curves and special formal $\mathcal{O}_D$-modules

Let $H$ be the quaternion algebra over $\mathbb{Q}$ ramified at $\infty$ and $p$. Let $\mathcal{O}_H$ be some maximal order in $H$. Fix $(E, \iota)$ where $E$ is a supersingular elliptic curve over $\mathbb{F}_{p^2}$ and $\iota : \mathcal{O}_H \cong \text{End}(E)$. By Deuring, $(E, \iota)$ exists. Let $E^\sigma$ be the Frobenius twist of $E$, $\iota^\sigma : \mathcal{O}_H \cong \text{End}(E^\sigma)$ and $\Pi : E \rightarrow E^\sigma$ the Frobenius morphism. Write $\Pi$ also for the Frobenius morphism of $E^\sigma$ to $E = E^{(\sigma^2)}$. Consider $A = E \times E^\sigma$. Find $\text{End}(A)$. Embed $\mathcal{O}_K$ in $\mathcal{O}_H \otimes \mathbb{Z}_p$ and let $\mathcal{O}_D = \mathcal{O}_K[\Pi]$. Show that the formal group $\hat{A}$ is a model for the special formal height 4 $\mathcal{O}_D$-module which was denoted in class $X$.

5. Isomorphism types of special formal height 4 $\mathcal{O}_D$-modules

(i) Using Drinfeld’s modular interpretation of $\hat{X} \otimes \mathcal{O}_K$ show that the isomorphism types of special formal height 4 $\mathcal{O}_D$-modules over $\bar{\mathbb{F}}_p$ are in bijection with two copies of $GL_2(F_p) \backslash \mathbb{P}^1(\bar{\mathbb{F}}_p)$ joined at the unique $F_p$-rational point.

(ii) Show that the unique $F_p$-rational point corresponds to $X$.

(iii) Characterize the two “components” in terms of critical indices for the Cartier module.

Note: This is only a characterization over $\bar{\mathbb{F}}_p$. The problem of classifying isomorphism types over an arbitrary base is not representable.

6. Drinfeld’s $p$-adic symmetric domain in dimension 2

This is a vaguely phrased exercise but a very good one. Try to understand the Bruhat-Tits building as well as you can for $GL_3 (d = 2)$. Mimic the construction of the formal scheme $\hat{X}$ (warning: the irreducible components of the special fibers will not be $\mathbb{P}^2$’s this time, but rather their blow-ups at the $q^2 + q + 1$ rational points). Describe the pre-images under reduction $r^{-1}([\sigma])$ for open simplices $\sigma$ of the building.

While the case $d = 1$ is too special, if you understand well the case $d = 2$, you are likely to understand the general case.