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The analogue of the Dedekind eta function for CY manifolds I

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Abstract. This is the first of series of articles in which we are going to study the regularized determinants of the Laplacians of Calabi-Yau metrics acting on $(0, q)$ forms on the moduli space of CY manifolds with a fixed polarization.

It is well known that in the case of elliptic curves, the Kronecker limit formula gives an explicit formula for the regularized determinant of the flat metric with fixed volume on the elliptic curves. The following formula holds in this case

$$\exp\left(\frac{d}{ds}\zeta(s) \mid s = 0\right) = \det \Delta_{(0,1)}(\tau) = (\operatorname{Im} \tau)^2 |\eta(\tau)|^4, \tag{1}$$

where $\eta(\tau)$ is the Dedekind eta function. It is well known fact that $\eta(\tau)^{24}$ is a cusp automorphic form of weight 12 related to the discriminant of the elliptic curve.

Formula (1) implies that there exists a holomorphic section of some power of the line bundle of the classes of cohomologies of $(1, 0)$ forms of the elliptic curves over the moduli space with an L^2 norm equal to $\det \Delta_{(0,1)}(\tau)$. This section is $\eta(\tau)^{24}$. The purpose of this project is to find the relations between the regularized determinants of CY metric on $(0, 1)$ forms and algebraic discriminants of CY manifolds.

In this paper we will establish the local analogue of the formula (1) for CY manifolds.

Contents

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1. Introduction

1.1. General comments. The regularized determinants play an important role in different branches of Physics and Mathematics. The explicit formulas for the regularized determinants have many important applications both in Mathematics and Physics. They are well known. This is the first of a series of papers in which we will study the regularized determinant of the Laplacians of CY metrics on $(0,q)$ forms. We will first review the computation and the main results about the regularized determinant of the flat metric on the elliptic. We will describe its relations to the discriminant locus and then we will describe our program how to generalize the results in the case of elliptic curves to CY manifolds.

An easy computation shows that the zeta function of the Laplace operator of the flat metric on an elliptic curve with periods $(1,\tau)$ is given by the expression

$$E(s)=\left(\frac{1}{2\pi}\right)^{2s}\sum_{\substack{n,m\in\mathbb{Z}\\(n,m)\neq(0,0)}}\frac{1}{|n+m\tau|^{2s}}$$

where $\tau\in\mathbb{C}$, $\operatorname{Im}\tau>0$ and ' means that the sum is over all pair of integers $(m,n)\neq(0,0)$. The computation of the regularized determinant in the case of the flat metric on an elliptic curve is based on the Kronecker limit formula. See [28]. It states that $E(s)$ has a meromorphic continuation in $\mathbb{C}$ with only one pole at $s=1$ and

$$\exp\left(-\frac{d}{ds}E(s)|_{s=0}\right)=\frac{1}{(\operatorname{Im}\tau)^2|\eta|^4} \tag{2}$$

where η is the Dedekind eta function. Thus formula (2) implies that

$$\det \Delta_{\tau,1} = \exp\left(\frac{d}{ds}E(s)|_{s=0}\right) = (\operatorname{Im}\tau)^2|\eta|^4. \tag{3}$$

Let us denote by $G_k(\tau)$ the Eisenstein series of the lattice spanned by 1 and τ. Then the elliptic curve $E_\tau = \mathbb{C}/(n + m\tau)$ is given by

$$y^2 = 4x^3 - g_2x - g_3,$$

where $g_2 = 60G_2$ and $g_3 = 140G_3$. The formula for the discriminant $\Delta(\tau)$ of the elliptic curve is given by

$$\Delta(\tau) = \frac{g_2^3(\tau) - 27g_3^2(\tau)}{1728g_2^3(\tau)}. \tag{4}$$

It is a well known formula that $\Delta(\tau) = \eta(\tau)^{24}$. Thus the formula for the regularized determinant of the Laplacian is related to the discriminant of the elliptic curve E_τ.

In this paper we will use the following method of deriving (3). See [15] and [16]. Compute the Hessian of $\log\det\Delta_{\tau,1}$ and show that

$$dd^c \log\det\Delta_{\tau,1} = -dd^c \log \operatorname{Im}\tau = dd^c \log\langle\omega_\tau, \omega_\tau\rangle. \tag{5}$$

Thus (5) implies that $\log\det\Delta_{\tau,1}$ is a potential of the Poincaré metric on the upper half plane $\mathfrak{h}$, which is the Teichmüller space of marked elliptic curves. Notice that

$$\operatorname{Im}\tau = -\frac{\sqrt{-1}}{2}\int_E \omega_\tau \wedge \overline{\omega_\tau} = \langle\omega_\tau, \omega_\tau\rangle,$$

where ω_τ are holomorphic one forms on the elliptic curves normalized as follows:

$$\int_{\gamma_0} \omega_\tau = 1, \tag{6}$$

where γ_0 is one of the generators of $H_1(E, \mathbb{Z})$. This means that: From the relation (5) we derive that we have

$$\det\Delta_{\tau,1} = \langle\omega_\tau, \omega_\tau\rangle|\eta(\tau)|^4 = (\operatorname{Im}\tau)^2|\eta|^4, \tag{7}$$

where $\eta(\tau)$ is a holomorphic function.

We will outline how (7) implies that $\eta(\tau)$ is the Dedekind eta function. This can be done in several steps. The first step is to prove that $\det\Delta_{\tau,1}$ is a bounded function on the moduli space of the elliptic curves. The next step is to prove that $\langle\omega_\tau, \omega_\tau\rangle = \operatorname{Im}\tau$ has a logarithmic growth near the infinity of $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{h}$. This will imply that $|\eta(\tau)|$ must vanish at infinity. It is not difficult to see that

$$\mathrm{SL}_2(\mathbb{Z})/[\mathrm{SL}_2(\mathbb{Z}), \mathrm{SL}_2(\mathbb{Z})] \approxeq \mathbb{Z}/12\mathbb{Z}. \tag{8}$$

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The normalization (6) implies that $\langle\omega_\tau, \omega_\tau\rangle = \operatorname{Im}\tau$ is an automorphic form of weight -2. Thus (7) and (8) imply that $\eta(\tau)^{24}$ will be a cusp form of weight 12 up to a constant. This will prove that $\eta(\tau)$ is the Dedekind eta function. This fact intuitively represents the following observation, when the metric degenerates then the discrete spectrum of the Laplacian in the limit turns into a continuous spectrum which contains the zero. Thus the regularized determinant must vanish at the points of the compactified moduli space that corresponds to singular manifolds. $\det\Delta_{\tau,1}$ is a bounded function on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{h}$.

One of the observation in this paper is that the normalization of the period of the holomorphic form ω_τ (6) is related to the choice of a maximal unipotent element of the mapping class group as follows. It was proved in [22] that maximal unipotent elements in the mapping class group correspond to a monodromy operator T acting on the middle cohomology group of a generic fibre of a family of projective algebraic varieties over the unit disk with only one singular fibre over the origin of the disk. Grothendieck proved that we always have:

$$(T^N - \mathrm{id})^{n+1} = 0$$

where N is some positive integer and n is the complex dimension of the fibre of the family. Maximal unipotent elements correspond to Jordan blocks of dimension one. In case of CY manifolds it was proved in [22] that if T has maximal index of unipotency then T has a unique Jordan block of dimension $n+1$. Thus once we choose the unipotent element of the mapping class group we can associate to it a unique up to a sign primitive class of homology in the middle homology group which corresponds to the invariant vanishing cycle. The cycle γ_0 that appears in (6) is the vanishing invariant cycle associated with the maximal unipotent element.

It is a classical fact that the Dedekind eta function is related to the algebraic discriminant of the equation that defines the elliptic curve. Thus the arguments that we provided to prove that the absolute value of the holomorphic function that appeared in (7) is in fact the analytic analogue of the discriminant. This follows from the fact that $\det\Delta_{\tau,1}$ can be interpreted as the L^2 norm of a holomorphic section of some power of the line bundle of the holomorphic one forms over the moduli space of elliptic curves. In our next publications we will generalized these arguments for CY manifolds.

This paper is the first one in the realization of the above described program. In it we will compute $dd^c \log\det\Delta_{\tau,1}$. We will show that locally in the Teichmüller space of the polarized CY manifolds we have

$$dd^c \log\det\Delta_{\tau,1} = dd^c \log\langle\omega_\tau, \omega_\tau\rangle, \tag{9}$$

where ω_τ is a family of holomorphic n forms. Thus (9) implies

$$\det\Delta_{\tau,1} = \langle\omega_\tau, \omega_\tau\rangle|\eta(\tau)|^2. \tag{10}$$

In fact by proving (10) we generalized (7) for higher dimensional CY manifolds.

In the case of elliptic curves formula (5) implies that $\log\det\Delta_{\tau,1}$ is the potential of the Poincaré metric on the upper half plane which is the Teichmüller space of elliptic curves. In

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[32] we proved that $dd^c \log\langle\omega_\tau, \omega_\tau\rangle$ gives the imaginary part of the Weil-Petersson metric on the Teichmüller space of polarized CY manifolds. Thus by establishing formula (9), we proved that the logarithm of the regularized determinant $\det \Delta_{\tau,1}$ forms is the potential of the Weil-Petersson metric.

1.2. Outline of the main ideas. The ideas that we used in this paper are similar to the ideas used in [6] to established variational formulas for regularized determinants on vector bundles on Riemann surfaces. We need first to fix the coordinates of the local deformation space of the fix CY manifold M_0 with a fix polarization class. In fact we use the coordinates $(\tau^1, \dots, \tau^N)$ introduced in [32]. These coordinates were later used in string theory. See [3]. The coordinates $(\tau^1, \dots, \tau^N)$ depend on the choice of a basis of harmonic forms $\phi_1, \dots, \phi_N$ of $H^1(M_0, T^{1,0})$. In these coordinates we have the following local expression for the family $\overline{\partial_\tau}$ on M_0:

$$\overline{\partial_{i,\tau}} = \frac{\bar{\partial}}{\overline{\partial z^i}} - \sum_{k=1}^{N} \tau^k \phi^j_{k,\bar{i}} \frac{\partial}{\partial z^j} + O(\tau^2),$$

where $(z^1, \dots, z^n)$ are local coordinates on M_0. Thus we get that $\overline{\partial_\tau}$ depends holomorphically on τ and

$$\frac{\partial}{\partial \tau^i} \overline{\partial_\tau} = -\phi_i \circ \partial_0 + O(\tau). \tag{11}$$

From the uniqueness of the solution of Calabi conjecture by Yau we know that once we fix the polarization class then we obtain a family of Calabi-Yau metrics g_τ such that $\operatorname{Im} g_\tau = L$. One of our results is that $\operatorname{Im} g_\tau$ is a constant symplectic form. From here and the expression for $\overline{\partial_\tau}^*$ we obtain that $\overline{\partial_\tau}^*$ depends anti holomorphically on τ.

After these preliminary results we are ready to compute the Hessian of $\log \det \Delta_{\tau,q}$. If we know the expressions for $\dfrac{\partial^2}{\partial \tau^i \overline{\partial \tau^j}} \log \Delta''_{\tau,q}$, where $\Delta''_{\tau,q} = \Delta_{\tau,q}|_{\operatorname{Im} \bar{\partial}^*}$ then we know how to compute $\dfrac{\partial^2}{\partial \tau^i \overline{\partial \tau^j}} \log \Delta_{\tau,q}$. So we will describe the ideas of the computations of $\dfrac{\partial^2}{\partial \tau^i \overline{\partial \tau^j}} \log \Delta''_{\tau,q}$. Since by definition of the regularized determinants of the Laplacians are given by the formula $\Delta''_{\tau,q} = \exp\left(-\dfrac{d}{ds} \zeta_{\Delta''_{\tau,q}}(s)\right)\Big|_{s=0}$ all our computations are based on the expression

$$\zeta_{\Delta''_{\tau,q}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr}\left(\frac{1}{4\pi t^n} \exp(-t\Delta''_{\tau,q})\right) t^{s-1}\, dt. \tag{12}$$

From (11) and (12) we obtain

$$\begin{aligned} &\frac{\bar{\partial}}{\overline{\partial \tau^i}} \left(\zeta''_{q,\tau}(s)\right)|_{\tau=0} \\ &\quad = \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr}\left(\exp(-t\Delta''_{0,q}) \circ \Delta''_{0,q} \circ \partial_0^{-1} \circ \mathscr{F}'(q+1, \overline{\phi_i}) \circ \overline{\partial_0}\right) t^s\, dt, \end{aligned} \tag{13}$$

where $\phi_i \in \mathrm{H}^1(M, \Theta)$ are the Kodaira-Spencer classes viewed as bundle maps:

$$\phi_i : C^\infty(M, \Omega_M^{1,0}) \to C^\infty(M, \Omega_M^{0,1})$$

and $\mathscr{F}(q+1, \phi_i)$ are the map induced by

$$\phi_i \wedge \mathrm{id}_{q-1} : C^\infty(M, \Omega_M^{0,q}) \to C^\infty(M, \Omega_M^{1,0} \otimes \Omega_M^{0,q-1})$$

on $C^\infty(M, \Omega_M^{0,q+1})$ and restricted in $\mathrm{Im}\,\bar{\partial}$. From (13) we obtain

$$(14)\qquad \frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\left(\zeta''_{\tau,q}(s)\right)$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\left(\left(\frac{\partial}{\partial\tau^j}\exp(-t\Delta''_{\tau,q})\right) \circ (\Delta''_{\tau,q}) \circ \partial_\tau^{-1} \circ F'(q+1, \overline{\phi_i}) \circ \overline{\partial_0}\right) t^s\, dt$$
$$+ \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\left(\exp(-t\Delta''_{\tau,q}) \circ \frac{\partial}{\partial\tau^j}(\Delta''_{\tau,q}) \circ \partial_\tau^{-1} \circ F'(q+1, \overline{\phi_i}) \circ \overline{\partial_0}\right) t^s\, dt.$$

Following the ideas in [6] to integrate by parts (14) we derive that:

$$(15)\qquad \frac{\partial^2}{\partial\tau^i\overline{\partial\tau^j}}\zeta''_{\tau,q}(s) = \frac{s(1-s)}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\big(\exp(-t\Delta'_{\tau,q+1})\mathscr{F}(q+1, \phi_i \circ \overline{\phi_j})\big) t^{s-1}\, dt.$$

By using the short term asymptotic expansion of $\exp(-t\Delta'_{\tau,q+1})$ we derive that

$$(16)\qquad \mathrm{Tr}\big(\exp(-t\Delta'_{\tau,q+1})\mathscr{F}(q+1, \phi_i \circ \overline{\phi_j})\big) = \sum_{k=-n}^{1} \frac{\alpha_{q,k}}{t^k} + \alpha_{q,0} + \psi_q(t).$$

Combining (15) and (16) we get that

$$\frac{d}{ds}\left(\frac{\partial^2}{\partial\tau^i\overline{\partial\tau^j}}\zeta''_{\tau,q}(s)\right)\Bigg|_{s=0} = \frac{\partial^2}{\partial\tau^i\overline{\partial\tau^j}}\log\det\Delta''_{\tau,q} = \alpha_{0,q}.$$

In this paper we gave an explicit formula for $\alpha_{0,q}$. By using the expressions for $\alpha_{0,1}$ and $\alpha_{0,n-1}$ we show that $d \circ d^c(\log\det\Delta_{\tau,1}) = -\mathrm{Im}\,\mathrm{WP}$, where $\mathrm{Im}\,\mathrm{WP}$ is the imaginary part of the Weil-Petersson metric. This can be viewed as the generalization of the results in [15] and [18].

1.3. Organization of the paper. This article is organized as follows.

In Section 2 we introduce some basic notions about zeta functions of Laplacians on Riemannian manifolds. We review the results from [32].

In Section 3 we review the basic properties of the Weil-Petersson metric on the moduli of CY manifolds. See also [25].

In Section 4 we review the theory of moduli of CY manifolds following [23] and also metrics with logarithmic singularities on vector bundle following Mumford's article [?]. We

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prove that the L^2 metric on the dualizing line bundle over the moduli space of CY manifolds is a good metric in the sense of Mumford. This implies that the volumes of the moduli spaces of CY manifolds are rational numbers.

In Section 5 we review some facts about the Hilbert spaces of the $(0,q)$ forms and their isospectral identifications which we used in the paper. We also study traces the operators acting on the L^2 sections of some vector bundle induced by some global C^∞ section its endormorphisms composed with the heat kernel. We study the short term expansions of these operators and especially the constant term of the expansion.

In Section 6 we establish the variational formulas for the zeta functions of the Laplacians and its regularized determinants. We also showed that

$$dd^c \log\det \Delta_{\tau,1} = -\operatorname{Im} \mathrm{WP}.$$

In Section 7 we prove that for Kähler-Ricci flat manifold N with $H^0(N,\Omega^n_M) = 0$ we have

$$dd^c \log\det \Delta_{\tau,n} = -\operatorname{Im} \mathrm{WP}.$$

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2. Preliminary material

2.1. Basic notions. Let M be an n-dimensional Kähler manifold with a zero canonical class. Suppose that $H^k(M,\mathcal{O}_M) = 0$ for $1 \leqq k < n$. *Such manifolds are called Calabi-Yau manifolds.* A pair (M,L) will be called a polarized CY manifold if M is a CY manifold and $L \in H^2(M,\mathbb{Z})$[2)] is a fixed class such that it represents the imaginary part of a Kähler metric on M.

Yau's celebrated theorem asserts the existence of a unique Ricci flat Kähler metric g on M such that the cohomology class $[\operatorname{Im}(g)] = L$. (See [34].) From now on we will consider polarized CY manifolds of odd dimension. The polarization class L determines the CY metric g uniquely. We will denote by

$$\Delta_q = \bar{\partial}^* \circ \bar{\partial} + \bar{\partial} \circ \bar{\partial}^*$$

the associated Laplacians that act on smooth $(0,q)$ forms on M for $0 \leqq q \leqq n$. $\bar{\partial}^*$ is the adjoint operator of $\bar{\partial}$ with respect to the CY metric g.

The regularized determinants are defined as follows: Let (M,g) be an n-dimensional Riemannian manifold. Let

[2)] Notice that $H^{1,1}(M,\mathbb{R}) = H^2(M,\mathbb{R})$ since $H^2(M,\mathcal{O}_M) = 0$ for CY manifolds.

$$\Delta_q = dd^* + d^*d$$

be the Laplacian acting on the space of q forms on M. We recall that the spectrum of the Laplacian Δ_q is positive and discrete. Thus the non zero eigen values of Δ_q are

$$0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n \leqq \cdots.$$

We define the zeta function of Δ_q as follows:

$$\zeta_q(s) = \sum_{i=1}^{\infty} \lambda_i^{-s}.$$

It is known that $\zeta_q(s)$ is a well defined analytic function for $\mathrm{Re}(s) \gg C$, it has a meromorphic continuation in the complex plane and 0 is not a pole of $\zeta_q(s)$. Define

$$\det(\Delta_q) = \exp\left(-\frac{d}{ds}(\zeta_q(s))|_{s=0}\right).$$

The determinant of these operators Δ_q, defined through zeta function regularization, will be denoted by $\det(\Delta_q)$.

The Hodge decomposition theorem asserts that

$$\Gamma(M, \Omega_M^{0,q}) = \mathrm{Im}(\bar{\partial}) \oplus \mathrm{Im}(\bar{\partial}^*)$$

for $1 \leqq q \leqq \dim_{\mathbb{C}} M - 1$. The restriction of Δ_q on $\mathrm{Im}(\bar{\partial})$ will be denoted by Δ_q' and $\Delta_q' = \bar{\partial} \circ \bar{\partial}^*$ and the restriction of Δ_q on $\mathrm{Im}(\bar{\partial}^*)$ will be denoted by Δ_q'' and $= \bar{\partial}^* \circ \bar{\partial}$. Hence we have

$$\mathrm{Tr}\big(\exp(-t\Delta_q)\big) = \mathrm{Tr}\big(\exp(-t\Delta_q')\big) + \mathrm{Tr}\big(\exp(-t\Delta_q'')\big).$$

This implies that

$$\zeta_q(s) = \sum_{k=1}^{\infty} \lambda_k^{-s} = \zeta_q'(s) + \zeta_q''(s),$$

where $\lambda_k > 0$ are the positive eigen values of Δ_q, and $\zeta_q'(s)$ and $\zeta_q''(s)$ are the zeta functions of Δ_q' and Δ_q''. From here and the definition of the regularized determinant we obtain that

$$\log\det(\Delta_q) = \log\det(\Delta_q') + \log\det(\Delta_q'').$$

It is a well known fact that the action of Δ_q'' on $\mathrm{Im}\,\bar{\partial}^*$ is isospectral to the action of Δ_{q+1}' on $\mathrm{Im}\,\bar{\partial}$, which means that the spectrum of Δ_q'' is equal to the spectrum of Δ_{q+1}'. So we have the equality

$$\det(\Delta_q'') = \det(\Delta_{q+1}').$$

Notation 2. Let f be a map from a set A to a set B and let g be a map from the set B to the set C, then the compositions of those two maps we will denote by $f \circ g$.

2.2. Basic notions about complex structures. Let M be an even dimensional C^∞ manifold. We will say that M has an almost complex structure if there exists a section $I \in C^\infty\big(M, \mathrm{Hom}(T^*, T^*)\big)$ such that $I^2 = -\mathrm{id}$. T is the tangent bundle and T^* is the cotangent bundle on M. This definition is equivalent to the following one: Let M be an even dimensional C^∞ manifold. Suppose that there exists a global splitting of the complexified cotangent bundle

$$T_M^* \otimes \mathbb{C} = \Omega_M^{1,0} \oplus \Omega_M^{0,1},$$

where $\Omega_M^{0,1} = \overline{\Omega_M^{1,0}}$. Then we will say that M has an almost complex structure. We will say that an almost complex structure is an integrable one, if for each point $x \in M$ there exists an open set $U \subset M$ such that we can find local coordinates $z^1, \dots, z^n$, such that $dz^1, \dots, dz^n$ are linearly independent in each point $m \in U$ and they generate $\Omega_M^{1,0}|_U$.

Definition 3. Let M be a complex manifold. Let $\phi \in \Gamma\big(M, \mathrm{Hom}(\Omega_M^{1,0}, \Omega_M^{0,1})\big)$, then we will call ϕ a Beltrami differential.

Since

$$\Gamma\big(M, \mathrm{Hom}(\Omega_M^{1,0}, \Omega_M^{0,1})\big) \backsimeq \Gamma(M, \Omega_M^{0,1} \otimes T_M^{1,0}),$$

we deduce that locally ϕ can be written as follows:

$$\phi|_U = \sum \phi_{\bar{\alpha}}^{\beta} \overline{dz}^{\alpha} \otimes \frac{\partial}{\partial z^{\beta}}.$$

From now on we will denote by A_ϕ the following linear operator:

$$A_\phi = \begin{pmatrix} \mathrm{id} & \phi(\tau) \\ \overline{\phi(\tau)} & \mathrm{id} \end{pmatrix}.$$

We will consider only those Beltrami differentials ϕ such that $\det(A_\phi) \neq 0$. The Beltrami differential ϕ defines an integrable complex structure on M if and only if the following equation holds:

$$\bar{\partial}\phi = \frac{1}{2}[\phi, \phi], \tag{17}$$

where

$$[\phi, \phi]|_U := \sum_{\nu=1}^{n} \sum_{1 \leqq \alpha < \beta \leqq n} \left(\sum_{\mu=1}^{n} \big(\phi_{\bar{\alpha}}^{\mu} (\partial_\mu \phi_{\bar{\beta}}^{\nu}) - \phi_{\bar{\beta}}^{\mu} (\partial_\nu \phi_{\bar{\alpha}}^{\nu}) \big) \right) \overline{dz}^{\alpha} \wedge \overline{dz}^{\beta} \otimes \frac{\partial}{dz^{\nu}} \tag{18}$$

(See [21].)

2.3. Kuranishi space and flat local coordinates. Kuranishi proved the following theorem:

Theorem 4. *Let* $\{\phi_i\}$ *be a basis of harmonic* $(0,1)$ *forms of* $\mathrm{H}^1(M, T^{1,0})$ *on a Hermitian manifold* M. *Let* G *be the Green operator and let* $\phi(\tau^1, \dots, \tau^N)$ *be defined as follows*:

$$\phi(\tau^1, \dots, \tau^N) = \sum_{i=1}^{N} \phi_i \tau^i + \frac{1}{2} \bar{\partial}^* G[\phi(\tau^1, \dots, \tau^N), \phi(\tau^1, \dots, \tau^N)]. \tag{19}$$

There exists $\varepsilon > 0$ *such that if* $\tau = (\tau^1, \dots, \tau^N)$ *satisfies* $|\tau_i| < \varepsilon$ *then* $\phi(\tau^1, \dots, \tau^N)$ *is a global* C^∞ *section of the bundle* $\Omega_M^{(0,1)} \otimes T^{1,0}$. (*See* [21].)

Based on Theorem 4, we proved in [32] the following theorem:

Theorem 5. *Let* M *be a* CY *manifold and let* $\{\phi_i\}$ *be a basis of harmonic* $(0,1)$ *forms with coefficients in* $T^{1,0}$, *i.e.*

$$\{\phi_i\} \in \mathrm{H}^1(M, T^{1,0}),$$

then the equation (17) *has a solution in the form*:

$$\begin{aligned} \phi(\tau^1, \dots, \tau^N) &= \sum_{i=1}^{N} \phi_i \tau^i + \sum_{|I_N| \geqq 2} \phi_{I_N} \tau^{I_N} \\ &= \sum_{i=1}^{N} \phi_i \tau^i + \frac{1}{2} \bar{\partial}^* G[\phi(\tau^1, \dots, \tau^N), \phi(\tau^1, \dots, \tau^N)] \end{aligned}$$

and $\bar{\partial}^* \phi(\tau^1, \dots, \tau^N) = 0$, $\phi_{I_N} \lrcorner \omega_M = \partial \psi_{I_N}$ *where* $I_N = (i_1, \dots, i_N)$ *is a multi-index*,

$$\phi_{I_N} \in C^\infty(M, \Omega_M^{0,1} \otimes T^{1,0}), \quad \tau^{I_N} = (\tau^i)^{i_1} \dots (\tau^N)^{i_N}$$

and if for some $\varepsilon > 0$ $|\tau^i| < \varepsilon$ *then* $\phi(\tau) \in C^\infty(M, \Omega_M^{0,1} \otimes T^{1,0})$ *where* $i = 1, \dots, N$. (*See* [31] *and* [32].)

It is a standard fact from Kodaira-Spencer-Kuranishi deformation theory that for each

$$\tau = (\tau^1, \dots, \tau^N) \in \mathscr{K}$$

as in Theorem 5 the Beltrami differential $\phi(\tau^1, \dots, \tau^N)$ defines a new integrable complex structure on M. This means that the points of $\mathscr{K}$, where

$$\mathscr{K} : \{\tau = (\tau^1, \dots, \tau^N) \,|\, |\tau^i| < \varepsilon\}$$

defines a family of operators $\bar{\partial}_\tau$ on the C^∞ family

$$\mathscr{K} \times M \to M$$

and $\bar{\partial}_\tau$ are integrable in the sense of Newlander-Nirenberg. Moreover it was proved by Kodaira, Spencer and Kuranishi that we get a complex analytic family of CY manifolds $\pi : \mathscr{X} \to \mathscr{K}$, where as C^∞ manifold $\mathscr{X} \backsimeq K \times M$. The family

$$\pi : \mathscr{X} \to \mathscr{K} \tag{20}$$

is called the Kuranishi family. The operators $\bar{\partial}_\tau$ are defined as follows:

Definition 6. Let $\{\mathscr{U}_i\}$ be an open covering of M, with local coordinate system in $\mathscr{U}_i$ given by $\{z_i^k\}$ with $k = 1, \ldots, n = \dim_{\mathbb{C}} M$. Assume that $\phi(\tau^1, \ldots, \tau^N)|_{\mathscr{U}_i}$ is given by:

$$\phi(\tau^1, \ldots, \tau^N) = \sum_{j,k=1}^{n} \big(\phi(\tau^1, \ldots, \tau^N)\big)_{\bar{j}}^{k} d\bar{z}^j \otimes \frac{\partial}{\partial z^k}.$$

Then we define

$$(\bar{\partial})_{\tau,\bar{j}} = \frac{\bar{\partial}}{\partial \overline{z^j}} - \sum_{k=1}^{n} \big(\phi(\tau^1, \ldots, \tau^N)\big)_{\bar{j}}^{k} \frac{\partial}{\partial z^k}. \tag{21}$$

Definition 7. The coordinates $\tau = (\tau^1, \ldots, \tau^N)$ defined in Theorem 5, will be fixed from now on and will be called the flat coordinate system in $\mathscr{K}$.

2.4. Family of holomorphic forms. In [32] the following theorem is proved:

Theorem 8. *There exists a family of holomorphic forms ω_τ of the Kuranishi family* (20) *such that*

$$\begin{aligned} (22)\quad & \langle [\omega_\tau], [\omega_\tau] \rangle \\ &= 1 - \sum_{i,j} \langle \omega_0 \lrcorner \phi_i, \omega_0 \lrcorner \phi_j \rangle \tau^i \overline{\tau^j} + \sum_{i,j} \langle \omega_0 \lrcorner (\phi_i \wedge \phi_k), \omega_0 \lrcorner (\phi_j \wedge \phi_l) \rangle \tau^i \overline{\tau^j} \tau^k \overline{\tau^l} + O(\tau^5) \\ &= 1 - \sum_{i,j} \tau^i \overline{\tau^j} + \sum_{i,j} \langle \omega_0 \lrcorner (\phi_i \wedge \phi_k), \omega_0 \lrcorner (\phi_j \wedge \phi_l) \rangle \tau^i \overline{\tau^j} \tau^k \overline{\tau^l} + O(\tau^5) \end{aligned}$$

and

$$\langle [\omega_\tau], [\omega_\tau] \rangle \leqq \langle [\omega_0], [\omega_0] \rangle.$$

3. Weil-Petersson metric

3.1. Basic properties. It is a well known fact from Kodaira-Spencer-Kuranishi theory that the tangent space $T_{\tau,\mathscr{K}}$ at a point $\tau \in \mathscr{K}$ can be identified with the space of harmonic $(0,1)$ forms with values in the holomorphic vector fields $\mathrm{H}^1(M_\tau, T)$. We will view each element $\phi \in \mathrm{H}^1(M_\tau, T)$ as a point wise linear map from $\Omega_{M_\tau}^{(1,0)}$ to $\Omega_{M_\tau}^{(0,1)}$. Given ϕ_1 and $\phi_2 \in \mathrm{H}^1(M_\tau, T)$, the trace of the map

$$\phi_1 \circ \overline{\phi_2} : \Omega_{M_\tau}^{(0,1)} \to \Omega_{M_\tau}^{(0,1)}$$

at the point $m \in M_\tau$ with respect to the metric g is simply:

$$\mathrm{Tr}(\phi_1 \circ \overline{\phi_2}) = \sum_{k,l,m=1}^{n} (\phi_1)_{\bar{l}}^{k} \overline{(\phi_2)_{\bar{k}}^{m}} g^{\bar{l},k} g_{k,\bar{m}}. \tag{23}$$

Definition 9. We will define the Weil-Petersson metric on $\mathscr{K}$ via the scalar product:

$$\langle \phi_1, \phi_2 \rangle = \int_M \mathrm{Tr}(\phi_1 \circ \overline{\phi_2}) \,\mathrm{vol}(g). \tag{24}$$

We proved in [32] that the coordinates

$$\tau = (\tau^1, \dots, \tau^N)$$

as defined in Definition 7 are flat in the sense that the Weil-Petersson metric is Kähler and in these coordinates we have that the components $g_{i,\bar{j}}$ of the Weil Petersson metric are given by the following formulas:

$$g_{i,\bar{j}} = \delta_{i,\bar{j}} + R_{i,\bar{j},l,\bar{k}}\tau^l\overline{\tau^k} + O(\tau^3).$$

Very detailed treatment of the Weil-Petersson geometry of the moduli space of polarized CY manifolds can be found in [24] and [25]. In those two papers important results are obtained.

3.2. Infinitesimal deformation of the imaginary part of the WP metric.

Theorem 10. *Near each point τ_0 of the Kuranishi space $\mathscr{K}$, the imaginary part* $\operatorname{Im}(g)$ *of the* CY *metric g has the following expansion in the coordinates $\tau := (\tau^1, \dots, \tau^N)$:*

$$\operatorname{Im}(g)(\tau, \bar{\tau}) = \operatorname{Im}(g)(\tau_0) + O\big((\tau - \tau_0)^2\big).$$

Proof. Without loss of generality we can assume that $\tau_0 = 0$. In [32] we proved that the forms

$$\theta^k_\tau = dz^k + \sum_{l=1} \big(\phi(\tau^1, \dots, \tau^N)^k_{\bar{l}}\big)\overline{dz^l} \tag{25}$$

for $k = 1, \dots, n$ form a basis of $(1,0)$ forms relative to the complex structure defined by $\tau \in \mathscr{K}$ in $\mathscr{U} \subset M$. Let

$$\operatorname{Im}(g_\tau) = \sqrt{-1}\left(\sum_{1 \leqq k \leqq l \leqq n} g_{k,\bar{l}}(\tau, \bar{\tau})\theta^k_\tau \wedge \overline{\theta^l_\tau}\right) \tag{26}$$

and

$$g_{k,\bar{l}}(\tau, \bar{\tau}) = g_{k,\bar{l}}(0) + \sum_{i=1}^N \Big(\big(g_{k,\bar{l}}(1)\big)_i\tau^i + \big(g'_{k,\bar{l}}(1)\big)_{\bar{i}}\overline{\tau^i}\Big) + O(2). \tag{27}$$

We get the following expression for $\operatorname{Im}(g_\tau)$ in terms of dz^i and $\overline{dz^j}$, by substituting the expressions for θ^k_τ from (25) and the expressions for $g_{k,\bar{l}}(\tau, \bar{\tau})$ from formula (27) in the formula (26):

$$\begin{aligned}
\operatorname{Im}(g_\tau) &= \sqrt{-1}\left(\sum_{1 \leqq k \leqq l \leqq n} g_{k,\bar{l}}(\tau, \bar{\tau})\theta^k_\tau \wedge \overline{\theta^l_\tau}\right) = \sqrt{-1}\left(\sum_{1 \leqq k \leqq l \leqq n} g_{k,\bar{l}}(0) dz^k \wedge \overline{dz^l}\right) \\
&+ \sqrt{-1}\left(\sum_{i=1}^N \tau^i\left(\sum_{1 \leqq k \leqq l \leqq n}\left(\big(g_{k,\bar{l}}(1)\big)_i dz^k \wedge \overline{dz^l} + \sum_{m=1}^n (g_{k,\bar{m}}\overline{\phi^m_{i,\bar{l}}} - g_{l,\bar{m}}\overline{\phi^m_{i,\bar{k}}}) dz^k \wedge dz^l\right)\right)\right) \\
&+ \frac{1}{\sqrt{-1}}\sum_{i=1}^N \overline{\tau^i}\overline{\left(\sum_{1 \leqq k \leqq l \leqq n}\left(\big(g_{k,\bar{l}}(1)\big)_i dz^k \wedge \overline{dz^l} + \sum_{m=1}^n (g_{k,\bar{m}}\overline{\phi^m_{i,\bar{l}}} - g_{l,\bar{m}}\overline{\phi^m_{i,\bar{k}}}) dz^k \wedge dz^l\right)\right)}.
\end{aligned}$$

On page 332 of [32] the following result is proved:

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Lemma 11. *Let $\phi \in \mathrm{H}^1(M,T)$ be a harmonic form with respect to the* CY *metric g. Let*

$$\phi|_U = \sum_{k,l=1}^{n} \phi_{\bar{k}}^{l} \overline{dz}^k \otimes \frac{\partial}{\partial z^l},$$

then

$$\phi_{\bar{k},\bar{l}} = \sum_{j=1}^{n} g_{j,\bar{k}} \phi_{\bar{l}}^{j} = \sum_{j=1}^{n} g_{j,\bar{l}} \phi_{\bar{k}}^{j} = \phi_{\bar{l},\bar{k}}.$$

From Lemma 11 we conclude that

$$\sum_{m=1}^{n} (g_{k,\bar{m}} \overline{\phi_{i,\bar{l}}^{m}} - g_{l,\bar{m}} \overline{\phi_{i,\bar{k}}^{m}}) = 0. \tag{28}$$

From (28) we get the following expression for $\mathrm{Im}(g_\tau)$:

$$\begin{aligned} \mathrm{Im}(g_\tau) = {} & \sqrt{-1}\Bigg(\sum_{1 \leqq k \leqq l \leqq n} g_{k,\bar{l}}(0) dz^k \wedge \overline{dz^l} \Bigg) \\ & + \sqrt{-1}\Bigg(\sum_{i=1}^{N} \tau^i \Bigg(\sum_{1 \leqq k \leqq l \leqq n} \big(g_{k,\bar{l}}(1)\big)_i dz^k \wedge \overline{dz^l} \Bigg)\Bigg) \\ & + \sqrt{-1}\Bigg(\sum_{i=1}^{N} \overline{\tau^i \sum_{1 \leqq k \leqq l \leqq n} \big(g_{k,\bar{l}}(1)\big)_i dz^k \wedge \overline{dz^l}} \Bigg) + O(2). \end{aligned} \tag{29}$$

Let us define the $(1,1)$ forms ψ_i as follows:

$$\psi_i = \sqrt{-1}\Bigg(\sum_{1 \leqq k \leqq l \leqq n} \big(g_{k,\bar{l}}(1)\big)_i dz^k \wedge \overline{dz^l} \Bigg). \tag{30}$$

We derive the following formula, by substituting in the expression (29) the expression given by (30):

$$\mathrm{Im}(g_\tau) = \mathrm{Im}(g_0) + \sum_{i=1}^{N} \tau^i \psi_i + \sum_{i=1}^{N} \overline{\tau^i \psi_i} + O(\tau^2). \tag{31}$$

From the fact that the class of the cohomology of the imaginary part of the CY metric is fixed, i.e. $[\mathrm{Im}(g_\tau)] = [\mathrm{Im}(g_0)] = L$, and (31) we deduce that each ψ_i is an exact form, i.e.

$$\psi_i = \sqrt{-1}\partial\bar{\partial} f_i, \tag{32}$$

where f_i are globally defined functions on M. Our theorem will follow if we prove that $\psi_i = 0$.

Lemma 12. $\psi_i = 0$.

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Proof. In [32] we proved that

$$\det(g_\tau) = \bigwedge^n \operatorname{Im}(g_\tau) = \det(g_0) + O(2) \tag{33}$$

in the flat coordinates $(\tau^1, \dots, \tau^N)$. We deduce from the expressions (31) and (32), by direct computations that:

$$\det(g_\tau) = \det(g_0) + \sqrt{-1}\sum_{i=1}^{N} \tau^i \left(\sum_{k,l} g^{\bar{l},k} \partial_k \overline{\partial}_l (f_i) \right) + \frac{1}{\sqrt{-1}} \sum_{i=1}^{N} \overline{\tau^i} \overline{\left(\sum_{k,l} g^{\bar{l},k} \partial_k \overline{\partial}_l (f_i) \right)} + O(2). \tag{34}$$

Combining (33) and (34) we obtain that for each i we have:

$$\sum_{k,l} g^{\bar{l},k} \partial_k \overline{\partial}_l (f_i) = \Delta(f_i) = 0,$$

where Δ is the Laplacian of the metric g. From the maximum principle, we deduce that all f_i are constants. Formula (32) implies that $\psi_i = 0$. Lemma 12 is proved. $\square$

Theorem 10 follows directly from Lemma 12. Theorem 10 is proved. $\square$

Corollary 13. *The imaginary part* $\operatorname{Im} g_\tau$ *of the* CY *metric is a constant symplectic form on the moduli space* $\mathfrak{M}_L(M)$.

Corollary 14. *The following formulas are true*:

$$\frac{\partial}{\partial \tau_i} (\overline{\partial_\tau})^* = 0 \quad \text{and} \quad \frac{\bar{\partial}}{\overline{\partial \tau_i}} (\partial_\tau)^* = 0. \tag{35}$$

Proof. We know from Kähler geometry that $(\overline{\partial_\tau})^* = [\Lambda_\tau, \partial_\tau]$, where Λ_τ is the contraction with $(1,1)$ vector field:

$$\Lambda_\tau = \frac{\sqrt{-1}}{2} \sum_{k,l=1}^{n} g_\tau^{\bar{k},l} (\theta_\tau^l)^* \wedge (\overline{\theta_\tau^k})^* \tag{36}$$

on M_τ and $(\theta_\tau^l)^*$ is $(1,0)$ vector field on M_τ dual to the $(1,0)$ form

$$\theta_\tau^i = dz^i + \sum_{j=1}^{N} \tau^j \left(\sum_{k=1}^{n} (\phi_j)_{\bar{k}}^{i} \overline{dz}^k \right).$$

Corollary 13 implies that $\dfrac{\partial}{\partial \tau_i}(\Lambda_\tau) = 0$. On the other hand, ∂_τ depends antiholomorphically on τ, i.e. it depends on $\bar{\tau} = (\overline{\tau_1}, \dots, \overline{\tau_N})$. So we deduce that

$$\frac{\partial}{\partial \tau_i}((\overline{\partial_\tau})^*) = \left(\left[\frac{\partial}{\partial \tau_i}(\Lambda_\tau), \partial_\tau \right] + \left[\Lambda_\tau, \frac{\partial}{\partial \tau_i}(\partial_\tau) \right] \right) = 0.$$

Exactly in the same way we prove that $\dfrac{\bar{\partial}}{\overline{\partial \tau_i}}(\partial_\tau)^* = 0$. Corollary 14 is proved. $\square$

4. Moduli of CY manifolds

4.1. Basic construction.

Definition 15. We will define the Teichmüller space $\mathscr{T}(M)$ of a CY manifold M as follows: $\mathscr{T}(M) := \mathscr{I}(M)/\mathrm{Diff}_0(M)$, where

$$\mathscr{I}(M) := \{\text{all integrable complex structures on } M\}$$

and $\mathrm{Diff}_0(M)$ is the group of diffeomorphisms isotopic to identity. The action of the group $\mathrm{Diff}(M_0)$ is defined as follows: Let $\phi \in \mathrm{Diff}_0(M)$ then ϕ acts on integrable complex structures on M by pull back, i.e. if

$$I \in C^\infty\big(M, \mathrm{Hom}\big(T(M), T(M)\big)\big),$$

then we define $\phi(I_\tau) = \phi^*(I_\tau)$.

Definition 16. We will call a pair $(M; \gamma_1, \ldots, \gamma_{b_n})$ a marked CY manifold where M is a CY manifold and $\{\gamma_1, \ldots, \gamma_{b_n}\}$ is a basis of $H_n(M, \mathbb{Z})/\mathrm{Tor}$.

Remark 17. Let $\mathscr{K}$ be the Kuranishi space. It is easy to see that if we choose a basis of $H_n(M, \mathbb{Z})/\mathrm{Tor}$ in one of the fibres of the Kuranishi family $\pi : \mathscr{M} \to \mathscr{K}$ then all the fibres will be marked, since as a C^∞ manifold $\mathscr{X}_{\mathscr{K}} \approxeq M \times \mathscr{K}$.

In [23] the following theorem was proved:

Theorem 18. *There exists a family of marked polarized* CY *manifolds*

$$\mathscr{Z}_L \to \tilde{T}(M), \tag{37}$$

which possesses the following properties:

(a) *It is effectively parametrized.*

(b) *The base has dimension $h^{n-1,1}$.*

(c) *For any marked* CY *manifold M of fixed topological type for which the polarization class L defines an imbedding into a projective space $\mathbb{CP}^N$, there exists an isomorphism of it* (*as a marked* CY *manifold*) *with a fibre M_s of the family $\mathscr{Z}_L$.*

Corollary 19. *Let $\mathscr{Y} \to X$ be any family of marked polarized* CY *manifolds, then there exists a unique holomorphic map $\phi : X \to \tilde{T}(M)$ up to a biholomorphic map ψ of M which induces the identity map on $H_n(M, \mathbb{Z})$.*

From now on we will denote by $\mathscr{T}(M)$ the irreducible component of the Teichmüller space that contains our fixed CY manifold M.

Definition 20. We will define the mapping class group $\Gamma_1(M)$ of any compact C^∞ manifold M as follows: $\Gamma_1(M) = \mathrm{Diff}_+(M)/\mathrm{Diff}_0(M)$, where $\mathrm{Diff}_+(M)$ is the group of

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diffeomorphisms of M preserving the orientation of M and $\mathrm{Diff}_0(M)$ is the group of diffeomorphisms isotopic to identity.

Definition 21. Let $L \in H^2(M, \mathbb{Z})$ be the imaginary part of a Kähler metric. Let $\Gamma_2 := \{\phi \in \Gamma_1(M) \mid \phi(L) = L\}$.

It is a well know fact that the moduli space of polarized algebraic manifolds $\mathscr{M}_L(M) = \mathscr{T}(M)/\Gamma_2$. In [23] the following fact was established:

Theorem 22. *There exists a subgroup of finite index Γ_L of Γ_2 such that Γ_L acts freely on $\mathscr{T}(M)$ and $\Gamma \backslash \mathscr{T}(M) = \mathfrak{M}_L(M)$ is a non-singular quasi-projective variety. Over $\mathfrak{M}_L(M)$ there exists a family of polarized* CY *manifolds $\mathscr{M} \to \mathfrak{M}_L(M)$.*

Remark 23. Theorem 22 implies that we constructed a family of non-singular CY manifolds $\pi : \mathscr{X} \to \mathfrak{M}_L(M)$ over a quasi-projective non-singular variety $\mathfrak{M}_L(M)$. Moreover it is easy to see that $\mathscr{X} \subset \mathbb{CP}^N \times \mathfrak{M}_L(M)$. So $\mathscr{X}$ is also quasi-projective. From now on we will work only with this family.

5. Hilbert spaces and trace class operators

5.1. Preliminary material.

Definition 24. We will denote by $L^2_{0,q}\big(\mathrm{Im}(\bar{\partial}^*)\big)$ the Hilbert subspace in $L^2(M, \Omega_M^{(0,q)})$ which is the L^2 completion of $\bar{\partial}^*$ exact forms in $C^\infty(M, \Omega_M^{(0,q)})$ for $q \geqq 0$. In the same manner we will denote by $L^2_{1,q-1}\big(\mathrm{Im}(\partial)\big)$ the Hilbert subspace in $L^2(M, \Omega_M^{(1,q-1)})$ which is the L^2 competition of the ∂ exact $(1, q-1)$ forms in $C^\infty(M, \Omega_M^{(1,q-1)})$ for $q > 0$. All the completions are with respect to the scalar product on the bundles $\Omega_M^{p,q}$ defined by the CY metric g.

Let $\phi(\tau^1, \ldots, \tau^N)$ be a solution of the equation (17):

$$\bar{\partial}\phi(\tau^1, \ldots, \tau^N) = \frac{1}{2}[\phi(\tau^1, \ldots, \tau^N), \phi(\tau^1, \ldots, \tau^N)]$$

which is guaranteed by Theorem 5. From Definition 3 of the Beltrami differential, we know that the Beltrami differential $\phi(\tau^1, \ldots, \tau^N)$ defines a linear fibrewise map

$$\phi(\tau^1, \ldots, \tau^N) : \Omega_M^{(1,0)} \to \Omega_M^{(0,1)}.$$

So

$$\phi(\tau^1, \ldots, \tau^N) \in C^\infty\big(M, \mathrm{Hom}(\Omega_M^{(1,0)}, \Omega_M^{(0,1)})\big). \tag{38}$$

Definition 25. We define the following maps between vector bundles

$$\phi \wedge \mathrm{id} : \Omega_M^{(1,q-1)} \to \Omega_M^{(0,q)}$$

as

$$\phi(dz^i \wedge \alpha) = \phi(dz^i) \wedge \alpha$$

for each $1 \leqq q \leqq n$. Clearly each fibre wise linear map $\phi \wedge \mathrm{id}_{q-1}$ defines a natural linear operator

$$F(q,\phi) : L^2(M, \Omega_M^{(1,q-1)}) \to L^2(M, \Omega_M^{(0,q)})$$

between the Hilbert spaces. The restriction of the linear operator $F(q,\phi)$ on the subspace $\mathrm{Im}(\partial) \subset L^2(M, \Omega_M^{(1,q-1)})$ to $\mathrm{Im}(\bar{\partial}) \subset L^2(M, \Omega_M^{(0,q)})$ will be denoted by $F'(q,\phi)$. The restriction of the linear operator $F(q,\phi)$ on the subspace $\mathrm{Im}(\partial^*) \subset L^2(M, \Omega_M^{(1,q-1)})$ to $\mathrm{Im}(\bar{\partial}^*) \subset L^2(M, \Omega_M^{(0,q)})$ will be denoted by $F''(q,\phi)$. Let ϕ and ψ be two Kodaira Spencer classes and let

$$\phi \circ \bar{\psi} : L^2(M, \Omega_M^{(0,1)}) \to L^2(M, \Omega_M^{(0,1)})$$

be a fibrewise linear map given by

$$\phi \circ \bar{\psi}|_U := \sum_{\alpha,\beta=1}^{n} (\phi \circ \bar{\psi})_{\bar{\beta}}^{\bar{\alpha}} \overline{dz^\beta} \otimes \frac{\partial}{\overline{\partial z^\alpha}}. \tag{39}$$

We define the fibrewise bundle maps

$$(\phi \circ \bar{\psi}) \wedge \mathrm{id}_{q-1} : \Omega_M^{0,q} \to \Omega_M^{0,q}, \tag{40}$$

as follows:

$$\big((\phi \circ \bar{\psi}) \wedge \mathrm{id}_{q-1}\big)(\omega) := \left(\sum_{\alpha,\beta=1}^{n} (\phi \circ \bar{\psi})_{\bar{\beta}}^{\bar{\alpha}} \overline{dz^\beta} \otimes \frac{\bar{\partial}}{\overline{\partial z^\alpha}}\right) \lrcorner \omega, \tag{41}$$

where $\lrcorner$ means contraction of tensors and ω is some global form of type $(0,q)$ on M. We will define for the linear operators

$$\mathscr{F}'(q, \phi \circ \bar{\psi}) : L^2_{0,q}\big(\mathrm{Im}(\bar{\partial})\big) \to L^2_{0,q}\big(\mathrm{Im}(\bar{\partial})\big) \tag{42}$$

and

$$\mathscr{F}''(q, \phi \circ \bar{\psi}) : L^2_{0,q}\big(\mathrm{Im}(\bar{\partial})^*\big) \to L^2_{0,q}\big(\mathrm{Im}(\bar{\partial})^*\big) \tag{43}$$

as the restriction of the operators $\big((\phi \circ \bar{\psi}) \wedge \mathrm{id}_{q-1}\big)$ on $L^2_{0,q}\big(\mathrm{Im}(\bar{\partial})\big)$ and $L^2_{0,q}\big(\mathrm{Im}(\bar{\partial})^*\big)$ respectively.

Remark 20. It is a standard fact that we can choose globally $\bar{\partial}$ closed forms $\omega_1, \dots, \omega_N$ of type $(0,q)$ such that at each point $z \in M$ they span the fibre $\Omega_{M,z}^{0,q}$. We can deduce directly from the definitions of the operators $F'(q,\phi)$, $\mathscr{F}'(q, \phi \circ \bar{\psi})$ and $F'(q, \bar{\psi} \circ \phi)$ and the existence of the forms $\omega_1, \dots, \omega_N$ that the operators $F'(q,\phi)$, $\mathscr{F}'(q, \phi \circ \bar{\psi})$ and $F'(q, \bar{\psi} \circ \phi)$ pointwise will be represented by matrices of dimensions $\binom{n}{q}$, $\binom{n}{q}$ and $n \times \binom{n}{q-1}$.

5.2. Trace class operators (see [2]). Let H be a Hilbert space with a orthonormal basis e_i. An operator A is a *Hilbert-Schmidt* operator if

$$\|A\|_{\mathrm{HS}}^2 = \sum_i \|Ae_i\|^2 = \sum_{ij} |\langle Ae_i, e_j\rangle|^2 < \infty$$

is finite. The number $\|A\|_{\mathrm{HS}}^2$ is called the Hilbert-Schmidt norm of A. If A is Hilbert-Schmid so is its adjoint A^* and $\|A\|_{\mathrm{HS}}^2 = \|A^*\|_{\mathrm{HS}}^2$. If U is a bounded operator on H and A is an Hilbert-Schmidt, then $U \circ A$ and $A \circ U$ are Hilbert-Schmidt operators and $\|U \circ A\|_{\mathrm{HS}} \leqq \|A \circ U\|_{\mathrm{HS}}$.

In this paper we will consider the Hilbert spaces of the square integrable sections of the bundles $\Omega_M^{0,q} \otimes |\Lambda_M|^{1/2}$ on M, where $|\Lambda_M|$ is the trivial density bundles generated by the volume form of the CY metric.

An operator K with square-integrable kernel

$$k(w,z) \in \Gamma_{L^2}\big(M \times M, (\Omega_M^{0,q} \otimes |\Lambda_M|^{1/2} \boxtimes \Omega_M^{0,q} \otimes |\Lambda_M|^{1/2})\big)$$

is Hilbert-Schmidt, and

$$\|K\|_{\mathrm{HS}}^2 = \int_{(w,z) \in M \times M} \mathrm{Tr}\big(k(w,z)^* k(w,z)\big). \tag{44}$$

Formula (44) follows from the definition of the Hilbert-Schmidt norm

$$\|K\|_{\mathrm{HS}}^2 = \sum_{i,j} |\langle Ke_i, e_j\rangle|^2.$$

An operator K is said to be *trace class* if it has the form $A \circ B$, where A and B are Hilbert-Schmidt. For such operators, the sum

$$\mathrm{Tr}\, K = \sum_i \langle Ke_i, e_i\rangle$$

is absolutely summable and $\mathrm{Tr}\, K$ is independent of the choice of the orthonormal basis in H and is called the trace of K.

5.3. Adiabatic limits, heat kernels and traces. In this subsection we study the traces of operators which are compositions of the heat kernel with operators induced by endomorphisms of some vector bundle. We will use some of the results from [6] and will adopt them to our situation.

Let h be a metric on a vector bundle E over M. Let Δ_h be the Laplacian on E. It is a well known fact that the operator $\exp(-t\Delta_h)$ can be represented by an integral kernel:

$$k_t(w,z,\tau) = \sum_j \exp(-t\lambda_j)\phi_j(w) \otimes \phi_j(z),$$

where λ_j and ϕ_j are the eigen values and the eigen sections of the Laplace operator Δ_h on some vector bundle E on M. $k_t(w,z,\tau)$ is an operator of trace class. We know that the following formula holds for the short term asymptotic expansion of $\mathrm{Tr}\big(k_t(w,z,\tau)\big)$

$$\operatorname{Tr}\big(k_t(w,z,\tau)\big) = \frac{\alpha_{-n}}{t^n} + \cdots + \frac{\alpha_{-1}}{t} + \alpha_0 + O(t).$$

Let E be a holomorphic vector bundle over M, let $\phi \in C^\infty\big(M, \operatorname{Hom}(E,E)\big)$. It is easy to see that the operator $\exp(-\Delta_h) \circ \phi$ is of trace class and its trace has an asymptotic expansion

$$\operatorname{Tr}\big(k_t(w,z,\tau) \circ \phi\big) = \frac{\beta_{-n}(\phi)}{t^n} + \cdots + \frac{\beta_{-1}(\phi)}{t} + \beta_0(\phi) + O(t) \tag{45}$$

according to [2]. We will study the following problem in this section:

Problem 27. Find an explicit expression for $\beta_0(\phi)$.

Definition 28. We define the function $k_\tau^d(w,z,t)$ in a neighborhood of the diagonal Δ in $M \times M$ as follows: Let ρ_τ be the injectivity radius on M_τ. Let $d_\tau(w,z)$ be the distance between the points w and z on M_τ with respect to CY metric g_τ. We suppose that $|\tau| < \varepsilon$. Let δ be such that $\delta > \rho_\tau$. Then we define the function $k_\tau^d(w,z,t)$ as a C^∞ function on $M \times M$ using partition of unity by using the functions

$$k_t^d(w,z,\tau) = \begin{cases} \dfrac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\dfrac{d_\tau^2(w,z)}{4t}\right) & \text{if } d_\tau(w,z) < \rho_\tau, \\ 0 & \text{if } d_\tau(w,z) > \delta. \end{cases} \tag{46}$$

defined on the opened balls around countable points (w_k, z_k) on $M \times M$ with injectivity radius ρ_τ.

Let E be a holomorphic vector bundle on M with a Hermitian metric h on it and let $\mathscr{P}_\tau(w,z)$ be the parallel transport of the bundle E along the minimal geodesic joining the point w and z with respect to natural connection on E induced by the metric h on E. It was proved in [2] on page 87 that we can represent the operator $\exp(-t\Delta_h)$ by an integral kernel $k_t(w,z,\tau)$, where

$$k_t(w,z,\tau) = k_t^d(w,z,\tau)\big(\mathscr{P}_\tau(w,z) + O(t)\big) \tag{47}$$

and $\Delta_h := \bar{\partial}_h^* \circ \bar{\partial}$.

Definition 29. We will define the kernel $k_t^\#(w,z,\tau)$ as the matrix operator defined by

$$k_t^\#(w,z,\tau) = k_t^d(w,z,\tau)\mathscr{P}_\tau(w,z), \tag{48}$$

where

$$k_t(w,z,\tau) = k_t^\#(w,z,\tau) + \varepsilon_t(w,z,\tau). \tag{49}$$

Let us define

$$\Upsilon_t(\phi,\tau,z) := \int_M \operatorname{Tr}\Big(\big(k_{t/2}^d(w,z,\tau)\big)^* \circ \big(k_{t/2}^\#(w,z,\tau) \circ \phi\big)\Big) \operatorname{vol}(g)_w. \tag{50}$$

Proposition 30. *We have*

$$\lim_{t\to 0} \int_M \operatorname{Tr}\big(\varepsilon_t(w,z,0)\circ\phi\big)\operatorname{vol}(g)_w = 0. \tag{51}$$

Proof. The Definition 29 of $k_\tau^{\#}(w,z,t)$ and the arguments from [6], page 260 imply that $\varepsilon_t(w,z,0)$ is bounded and tends to zero away from the diagonal, as t tends to zero. From here we deduce that

$$\lim_{t\to 0} \int_M \operatorname{Tr}\big(\varepsilon_t(w,z,0)\circ\phi\big)\operatorname{vol}(g)_w = 0$$

uniformly in z. Proposition 30 is proved. □

Lemma 31. *Let E be a holomorphic vector bundle over M, let $\phi\in C^\infty\big(M,\operatorname{Hom}(E,E)\big)$, then $\lim_{t\to 0}\Upsilon_t(\phi,\tau,z)$ exists and*

$$\lim_{t\to 0}\Upsilon_t(\phi,\tau,z) = \operatorname{Tr}(\phi|_{E_z}). \tag{52}$$

Proof. We have:

$$\begin{aligned}\lim_{t\to 0}\Upsilon_t(\phi,0,z) &= \lim_{t\to 0}\int_M \operatorname{Tr}\Big(\big(k_{t/2}^d(w,z,\tau)\big)^*\circ\big(k_{t/2}^{\#}(w,z,\tau)\circ\phi\big)\Big)\operatorname{vol}(g)_w \\ &= \lim_{t\to 0}\int_M \operatorname{Tr}\left(\frac{1}{(4\pi t)^{\frac{n}{2}}}\exp\left(-\frac{d_0^2(w,z)}{4t}\right)\circ\mathscr{P}_0(w,z)\circ\phi\right)\operatorname{vol}(g)_w.\end{aligned} \tag{53}$$

Using the fact that

$$\lim_{t\to 0}\left(\frac{1}{(4\pi t)^{\frac{n}{2}}}\exp\left(-\frac{d_\tau^2(w,z)}{4t}\right)\right) = \delta(z-w), \tag{54}$$

$$\lim_{w\to z}\mathscr{P}_0(w,z) = \mathrm{id} \tag{55}$$

and the explicit formula (53) for $\Upsilon_t(\phi,\tau,z)$ we obtain that

$$\lim_{t\to 0}\Upsilon_t(\phi,\tau,z) = \int_M \operatorname{Tr}\big(\delta(z-w)\circ\phi\big)\operatorname{vol}(g)_\omega = \operatorname{Tr}\big(\phi(z)|_{E_z}\big). \tag{56}$$

Lemma 31 is proved. □

Theorem 32. *Let $\phi\in C^\infty\big(M,(\Omega_M^{0,q})^*\otimes\Omega_M^{0,q}\big)$ then the operator $\exp(-t\Delta_h)\circ\phi$ for $t>0$ is of trace class and its trace is given by the formula*

$$\operatorname{Tr}\big(\exp(-t\Delta_h)\circ\phi\big) = \int_M \Upsilon_t(\phi,\tau,z)\operatorname{vol}(g)_z + \Phi(t), \tag{57}$$

where the short term asymptotic of $\Phi(t)$ is given by

$$\Phi(t) = \sum_{k=1}^{N_0>0}\frac{a_{-k}}{t^k} + O(t). \tag{58}$$

Proof. The proof of Theorem 32 is based on the facts that

$$\exp(-t\Delta_h) \circ \phi = \exp\left(-\frac{t}{2}\Delta_h\right) \circ \exp\left(-\frac{t}{2}\Delta_h \circ \phi\right), \tag{59}$$

and the operators $\exp\left(-\frac{t}{2}\Delta_h\right)$ and $\exp\left(-\frac{t}{2}\Delta_h \circ \phi\right)$ can be represented by C^∞ kernels $k_1(z,w,t)$ and $k_\phi(z,w,t)$.

As we pointed out the operators defined by the kernels $k_1(z,w,t)$ and $k_\phi(z,w,t)$ are Hilbert-Schmidt operators. Thus since the operator $\exp(-t\Delta_h) \circ \phi$ is a product of two Hilbert-Schmidt operators it is of trace class. On the other hand the definition of the trace of the operator $\exp(-t\Delta_h) \circ \phi$ implies that

$$\begin{aligned}\operatorname{Tr}\big(\exp(-t\Delta_h) \circ \phi\big) &= \left\langle \exp\left(-\frac{t}{2}\Delta_h\right)^*, \exp\left(-\frac{t}{2}\Delta_h\right) \circ \phi \right\rangle \\ &= \int\limits_{(z,w)\in M\times M} \operatorname{Tr}\big((k_1(z,w,t))^* \circ k_\phi(z,w,t)\big)\operatorname{vol}(g_{z,w}).\end{aligned} \tag{60}$$

From the definitions of the function $\Upsilon_t(\phi,\tau,z)$ and the operator $\varepsilon_t(w,z,\tau)$ we deduce that

$$\begin{aligned}&\operatorname{Tr}\big(\exp(-t\Delta_h) \circ \phi\big) \\ &= \big\langle \big(k_{t/2}(w,z,\tau)\big), k_{t/2}(w,z,\tau) \circ \phi \big\rangle \\ &= \int\limits_M \left(\int\limits_M \big(k_{t/2}(w,z,\tau)\big)^* \circ \big(k_{t/2}(w,z,\tau) \circ \phi\big)\operatorname{vol}(g)_w \right)\operatorname{vol}(g)_z \\ &= \int\limits_M \Upsilon_t(\phi,\tau,z)\operatorname{vol}(g)_z \\ &\quad + \int\limits_M \left(\int\limits_M \operatorname{Tr}\big(\big(\varepsilon_{t/2}(w,z,\tau)\big)^* \circ k^{\#}_{t/2}(w,z,\tau) \circ \phi\big)\operatorname{vol}(g)_\omega \right)\operatorname{vol}(g)_z \\ &\quad + \int\limits_M \left(\int\limits_M \operatorname{Tr}\big(k^{\#}_{t/2}(w,z,\tau) \circ \varepsilon_{t/2}(w,z,\tau) \circ \phi\big)\operatorname{vol}(g)_\omega \right)\operatorname{vol}(g)_z.\end{aligned} \tag{61}$$

Lemma 33. *Let*

$$\Phi_1(t) := \int\limits_M \left(\int\limits_M \operatorname{Tr}\big(\big(\varepsilon_{t/2}(w,z,\tau)\big)^* \circ k^{\#}_{t/2}(w,z,\tau) \circ \phi\big)\operatorname{vol}(g)_\omega \right)\operatorname{vol}(g)_z$$

and

$$\Phi_2(t) := \int\limits_M \left(\int\limits_M \operatorname{Tr}\big(k^{\#}_{t/2}(w,z,\tau) \circ \varepsilon_{t/2}(w,z,\tau) \circ \phi\big)\operatorname{vol}(g)_\omega \right)\operatorname{vol}(g)_z, \tag{62}$$

then we have

$$\Phi_1(t) = \sum_{k=1}^{N_0>0} \frac{b_{-k}}{t^k} + O(t) \quad \text{and} \quad \Phi_2(t) = \sum_{k=1}^{N_0>0} \frac{c_{-k}}{t^k} + O(t). \tag{63}$$

Proof. Let

$$k^{\#}_{t/2}(w,z,\tau)=\sum_{k=1}^{N_0>0}\frac{\mathscr{B}_{-k}(w,z)}{t^k}+\mathscr{B}_0(w,z)+\sum_{k=1}\mathscr{B}_k(w,z)t^k \tag{64}$$

be the short term asymptotic expansion of the operator $k^{\#}_{t/2}(w,z,\tau)$. We know that

$$\lim_{t\to 0}\varepsilon_t(w,z,\tau)=0 \tag{65}$$

away from the diagonal $\Delta\subset M\times M$. Combining (64) and (65) with the definitions of operators $k^{\#}_{t/2}(w,z,\tau)\circ\varepsilon_{t/2}(w,z,\tau)\circ\phi$ and $\left(\varepsilon_{t/2}(w,z,\tau)\right)^*\circ k^{\#}_{t/2}(w,z,\tau)\circ\phi$ we obtain that

$$\begin{aligned}(66)\qquad & k^{\#}_{t/2}(w,z,\tau)\circ\varepsilon_{t/2}(w,z,\tau)\circ\phi\\ &=\sum_{k=1}^{N_0>0}\frac{\mathscr{B}_{-k}(w,z)\circ\varepsilon_{t/2}(w,z,\tau)\circ\phi}{t^k}+\mathscr{B}_0(w,z)\circ\varepsilon_{t/2}(w,z,\tau)\circ\phi+O(t)\end{aligned}$$

and

$$\begin{aligned}(67)\qquad & \left(\varepsilon_{t/2}(w,z,\tau)\right)^*\circ k^{\#}_{t/2}(w,z,\tau)\circ\phi\\ &=\sum_{k=1}^{N_0>0}\frac{\left(\varepsilon_{t/2}(w,z,\tau)\right)^*\circ\mathscr{B}_{-k}(w,z)}{t^k}+\left(\varepsilon_{t/2}(w,z,\tau)\right)^*\circ\mathscr{B}_0(w,z)+O(t).\end{aligned}$$

Combining (66), (67) with (65) we get that

$$\lim_{t\to 0}\mathscr{B}_0(w,z)\circ\varepsilon_{t/2}(w,z,\tau)\circ\phi=\lim_{t\to 0}\left(\varepsilon_{t/2}(w,z,\tau)\right)^*\circ\mathscr{B}_0(w,z)=0$$

away from the diagonal. From here we obtain that

$$\lim_{t\to 0}\int_M \mathrm{Tr}\left(\mathscr{B}_0(w,z)\circ\varepsilon_{t/2}(w,z,\tau)\circ\phi\right)\mathrm{vol}(g)=0$$

and

$$\lim_{t\to 0}\int_M \mathrm{Tr}\left(\left(\varepsilon_{t/2}(w,z,\tau)\right)^*\circ\mathscr{B}_0(w,z)\right)\mathrm{vol}(g)=0.$$

Lemma 33 is proved. □

Theorem 32 follows directly from Lemma 33 and (61). □

Theorem 34. *We have the following expression for* $\beta_0(\phi)$ *from* (45):

$$\beta_0(\phi)=\lim_{t\to 0}\int_M \Upsilon_t(\phi,\tau,z)\,\mathrm{vol}(g)_z=\int_M \mathrm{Tr}(\phi)\,\mathrm{vol}(g). \tag{68}$$

Proof. Theorem 34 follows directly from Theorem 32, Lemma 31 and the definition of $\Upsilon_t(\phi,\tau,z)$. □

5.4. Explicit formulas.

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Theorem 35. *Let $\mathscr{F}'(q, \phi \circ \overline{\psi})$ be given by the formula* (42). *Then for $t > 0$ and $q \geqq 1$ the following equality of the traces of the respective operators holds*

$$\begin{aligned} \text{(69)} \qquad & \operatorname{Tr}\bigl(\exp\bigl((-t\Delta''_{q-1}) \circ \bar{\partial}^{-1} \circ \mathscr{F}'(q, \phi \circ \overline{\psi}) \circ \bar{\partial}\bigr)\bigr) \\ & = \operatorname{Tr}\bigl(\exp(-t\Delta'_q) \circ \bigl(\mathscr{F}'(q, \phi \circ \overline{\psi})\bigr)\bigr). \end{aligned}$$

Proof. From in [2], Proposition 2.45, page 96 it follows directly that the operators

$$\exp(-t\Delta''_{q-1}) \circ \bar{\partial}^{-1} \circ \mathscr{F}'(q, \phi \circ \overline{\psi}) \circ \bar{\partial}, \quad \exp(-t\Delta'_q) \circ \mathscr{F}'(q, \phi \circ \overline{\psi})$$

are of trace class since the operators $\exp(-t\Delta''_{q-1})$ have smooth kernels for $q \geqq 1$. We know from [2], Proposition 2.45 that we have the following formula:

$$\text{(70)} \qquad \operatorname{Tr}(DK) = \operatorname{Tr}(DA)$$

where D is a differential operator and A is an operator with a smooth kernel. By using (70) and the fact that the operators Δ_q and $\bar{\partial}$ commute we derive Theorem 35. Theorem 35 is proved. □

Remark 36. From Definition 25 of the operator $\mathscr{F}'(q, \phi_i \circ \overline{\phi_j})$ and Remark 26 we know that it can be represented pointwise by a matrix which we will denote by $\mathscr{F}'\bigl(q, (\phi_i \circ \overline{\phi_j})\bigr)$. Since $k_t^{\#}(z, w, 0)$ is also a matrix of the same dimension as the operator $\mathscr{F}'(q, \phi_i \circ \overline{\phi_j})$ we get that the operator $k_t^{\#}(z, w, 0) \circ \mathscr{F}'(q, \phi_i \circ \overline{\phi_j})$ will be represented pointwise by the product of finite dimensional matrices. So the integral

$$\int_M \operatorname{Tr}\bigl(\bigl(k^{\#}_{t/2}(z, w, 0)\bigr)^* \circ k^{\#}_{t/2}(z, w, 0) \circ \mathscr{F}'(q, \phi_i \circ \overline{\phi_j})\bigr) \operatorname{vol}(g)$$

makes sense for $t > 0$.

Theorem 37. *Let*

$$\begin{aligned} \text{(71)} \qquad & \operatorname{Tr}\bigl(k_t(w, z, \tau) \circ \mathscr{F}'\bigl(q, (\phi_i \circ \overline{\phi_j})\bigr)\bigr) \\ & = \frac{\beta_{-n}(\phi_i \circ \overline{\phi_j})}{t^n} + \cdots + \frac{\beta_{-1}(\phi_i \circ \overline{\phi_j})}{t} + \beta_0(\phi_i \circ \overline{\phi_j}) + O(t) \end{aligned}$$

be the short term asymptotic. Then the following limit

$$\lim_{t \to 0} \int_M \left(\int_M \operatorname{Tr}\bigl(\bigl(k^{\#}_{t/2}(z, w, 0)\bigr)^* \circ k^{\#}_{t/2}(z, w, 0) \circ \mathscr{F}'(q, \phi_i \circ \overline{\phi_j})\bigr) \operatorname{vol}(g)_w \right) \operatorname{vol}(g)$$

exists and

$$\begin{aligned} \text{(72)} \qquad & \beta_0(\phi_i \circ \overline{\phi_j}) \\ & = \lim_{t \to 0} \int_M \left(\int_M \operatorname{Tr}\bigl(\bigl(k^{\#}_{t/2}(z, w, 0)\bigr)^* \circ k^{\#}_{t/2}(z, w, 0) \circ \mathscr{F}'(q, \phi_i \circ \overline{\phi_j})\bigr) \operatorname{vol}(g)_w \right) \operatorname{vol}(g) \\ & = \int_M \operatorname{Tr}\bigl(\mathscr{F}'(q, \phi_i \circ \overline{\phi_j})\bigr) \operatorname{vol}(g) < \infty. \end{aligned}$$

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Proof. Formulas (57) and (58) in Theorem 32 imply that

$$\beta_0(\phi_i \circ \overline{\phi_j}) = \lim_{t\to 0} \int_M \Upsilon_t(\mathscr{F}'(q, \phi_i \circ \overline{\phi_j}), \tau, z) \operatorname{vol}(g)_z.$$

Theorem 34 implies that

$$\beta_0(\phi_i \circ \overline{\phi_j}) = \lim_{t\to 0} \int_M \Upsilon_t(\mathscr{F}'(q, \phi_i \circ \overline{\phi_j}), \tau, z) \operatorname{vol}(g)_z = \operatorname{Tr}(\mathscr{F}'(q, \phi_i \circ \overline{\phi_j}))|_z. \tag{73}$$

Formula (73) implies formula (72). Theorem 37 is proved. □

6. The variational formulas

6.1. Preliminary formulas.

Lemma 6.1. *The following formulas are true for* $1 \leqq q \leqq n$:

$$\frac{\partial}{\partial \tau^i}(\overline{\partial_\tau})|_{\tau=0} = -F(q, \phi_i) \circ \partial \tag{74}$$

and

$$\frac{\bar{\partial}}{\overline{\partial \tau^i}}(\partial_\tau)|_{\tau=0} = -F(q, \overline{\phi_i}) \circ \bar{\partial}. \tag{75}$$

Proof. From the expression of $\bar{\partial}_\tau$ given in Definition 6:

$$\bar{\partial}_\tau = \frac{\bar{\partial}}{\overline{\partial z^j}} - \sum_{m=1}^{N} \left(\sum_{k=1}^{n} (\phi_m)_{\bar{j}}^k \frac{\partial}{\partial z^k} \right) \tau^m + O(\tau^2),$$

we conclude that

$$\frac{\partial}{\partial \tau^i}(\overline{\partial_\tau})|_{\tau=0} = -\sum_{k=1}^{N} (\phi_i)_{\bar{j}}^k \frac{\partial}{\partial z^k}. \tag{76}$$

Formula (75) is proved in the same way as formula (76). Lemma 38 follows directly from Definition 25 of the linear operators $F'(q, \phi)$ and $F''(q, \phi)$. □

Corollary 39. *The following formulas are true for* $1 \leqq q \leqq n$:

$$\left(\frac{\partial}{\partial \tau^i}(\Delta''_{\tau,q})|_{\operatorname{Im} \bar{\partial}_\tau^*} \right)\Bigg|_{\tau=0} = -\Delta''_{0,q} \circ \overline{\partial_0}^{-1} \circ F(q, \phi_i) \circ \partial_0,$$

and

$$\left(\frac{\bar{\partial}}{\overline{\partial \tau^j}}(\Delta''_{\tau,q})|_{\operatorname{Im} \bar{\partial}_\tau^*} \right)\Bigg|_{\tau=0} = -\Delta''_{0,q} \circ \partial_0^{-1} \circ F'(q+1, \overline{\phi_i}) \circ \overline{\partial_0}. \tag{77}$$

Proof. From the standard facts of Kähler geometry we obtain that on $\operatorname{Im}\bar{\partial}^*$ in $\Omega_M^{0,q}$ we have

$$\Delta''_{\tau,q}|_{\operatorname{Im}\bar{\partial}^*} = \Lambda_\tau \circ \partial_\tau \circ \overline{\partial_\tau} = \Lambda_\tau \circ \overline{\partial_\tau} \circ \partial_\tau. \tag{78}$$

We know from (74) and (75) that

$$\frac{\partial}{\partial \tau^j}(\overline{\partial_\tau})|_{\tau=0} = -F'(q+1,\phi_j) \circ \partial_0, \quad \frac{\partial}{\partial \tau^j}(\Lambda_\tau) = \frac{\partial}{\partial \tau^j}(\partial_\tau) = 0. \tag{79}$$

Combining (79), (78) we obtain:

$$\begin{aligned} \frac{\partial}{\partial \tau^j}(\Delta''_{\tau,q})|_{\tau=0} &= \frac{\partial}{\partial \tau^j}(\Lambda_\tau \circ \overline{\partial_\tau} \circ \overline{\partial_\tau})|_{\tau=0} \\ &= \left(\overline{\partial_\tau}^* \circ \frac{\partial}{\partial \tau^j}(\overline{\partial_\tau})\right)\bigg|_{\tau=0} = -\overline{\partial_0}^* \circ F'(q+1,\phi_j) \circ \partial_0. \end{aligned} \tag{80}$$

Thus on $\operatorname{Im}\bar{\partial}^*$ we have

$$\partial_\tau^* = \Delta''_{\tau,q} \circ \overline{\partial_\tau}^{-1}. \tag{81}$$

Substituting (81) in (80) we obtain the first formula in (77). In the same manner we obtain the second formula in (77). Corollary 39 is proved. □

6.2. The computation of the antiholomorphic derivative of $\zeta''_{\tau,q-1}(s)$. First we will compute the antiholomorphic derivative of $\zeta''_{\tau,q}(s)$.

Theorem 40. *The following formula is true for $t > 0$:*

$$\begin{aligned} &\frac{\bar{\partial}}{\partial \overline{\tau^i}}(\zeta''_{q,\tau}(s))|_{\tau=0} \\ &\quad = \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr}\bigl(\exp\bigl(-t(\Delta''_{0,q})\bigr) \circ \Delta''_{0,q} \circ \partial_0^{-1} \circ \mathscr{F}'(q+1,\overline{\phi_i}) \circ \overline{\partial_0}\bigr) t^s \, dt. \end{aligned}$$

Proof. For the proof of Theorem 40 we will need the following lemma:

Lemma 41. *The following formula is true for $t > 0$ and $0 < q < n$:*

$$\begin{aligned} &\frac{\bar{\partial}}{\partial \overline{\tau^i}}\bigl(\operatorname{Tr}\bigl(\exp(-t\Delta''_{\tau,q})\bigr)\bigr)|_{\tau=0} \\ &\quad = t \operatorname{Tr}\bigl(\exp(-t\Delta''_{0,q}) \circ \Delta''_{\tau,q} \circ \partial^{-1} \circ \mathscr{F}'(q+1,\overline{\phi_i}) \circ \bar{\partial}\bigr)|_{\tau=0}. \end{aligned} \tag{82}$$

Proof. Direct computations based on [6], Proposition 9.38, page 304 show that:

$$\frac{\bar{\partial}}{\partial \overline{\tau^i}}\bigl(\operatorname{Tr}\bigl(\exp(-t\Delta''_{\tau,q})\bigr)\bigr)|_{\tau=0} = -t\left(\exp(-t\Delta''_{\tau,q}) \circ \frac{\bar{\partial}}{\partial \overline{\tau^i}}(\Delta''_{\tau,q})\right)\bigg|_{\tau=0}. \tag{83}$$

See also [3], Theorem 2.48, page 98. Formulas (74) and (75) in Lemma 38 imply that

$$\frac{\bar\partial}{\overline{\partial\tau^i}}(\partial_\tau) = -F'\big(q+1,\overline{\phi_i(\tau)}\big)\circ\bar\partial \tag{84}$$

and on $\operatorname{Im}\bar\partial^*$ we have

$$\frac{\bar\partial}{\overline{\partial\tau^i}}(\partial_\tau^*) = \frac{\bar\partial}{\overline{\partial\tau^i}}(\Lambda\circ\overline{\partial_\tau}) = \left(\frac{\bar\partial}{\overline{\partial\tau^i}}\Lambda\right)\circ\overline{\partial_\tau} + \Lambda\circ\frac{\bar\partial}{\overline{\partial\tau^i}}(\overline{\partial_\tau}) = 0. \tag{85}$$

The last equality follows from Corollaries 38 and 14. On Kähler manifolds we know that $\partial^*\circ\partial + \partial\circ\partial^* = \bar\partial^*\circ\bar\partial + \bar\partial\circ\bar\partial^*$. So we deduce that

$$\Delta''_{\tau,q} = (\partial_\tau^*\circ\partial_\tau + \partial_\tau\circ\partial_\tau^*)|_{\operatorname{Im}\overline{\partial_\tau}^*} = \partial_\tau^*\circ\partial_\tau|_{\operatorname{Im}\overline{\partial_\tau}^*}.$$

Thus from formulas (84) and (85) it follows:

$$\frac{\bar\partial}{\overline{\partial\tau^i}}(\Delta''_{\tau,q}) = \left(\partial_\tau^*\circ\frac{\bar\partial}{\overline{\partial\tau^i}}(\partial_\tau)\right) = -\partial_\tau^*\circ F'\big(q+1,\overline{\phi_i(\tau)}\big)\circ\bar\partial. \tag{86}$$

By substituting in (83) the expression from (86) we obtain:

$$\frac{\bar\partial}{\overline{\partial\tau^i}}\big(\operatorname{Tr}\big(\exp(-t\Delta''_{\tau,q})\big)\big)|_{\tau=0} = t\operatorname{Tr}\big(\exp(-t\Delta''_q)\circ\partial_0^*\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\big). \tag{87}$$

The operator ∂_τ^* is well defined on the space of $C^\infty(0,q)$ forms on M_τ. So the following formula is true on $\operatorname{Im}\bar\partial_\tau^*$:

$$\partial_\tau^* = (\Delta''_{\tau,q})\circ(\partial_\tau)^{-1}. \tag{88}$$

Substituting the expression for ∂_τ^* in formula (86) in (87), we deduce formula (82). Lemma 41 is proved. $\square$

The end of the proof of Theorem 40. The definition of the zeta function implies that

$$\begin{aligned}\frac{\bar\partial}{\overline{\partial\tau^i}}\big(\zeta_{\Delta''_{\tau,q}}(s)\big)|_{\tau=0} &= \frac{\bar\partial}{\overline{\partial\tau^i}}\left(\frac{1}{\Gamma(s)}\int_0^\infty \operatorname{Tr}\big(\exp\big(-t(\Delta''_{\tau,q})\big)\big)t^{s-1}\,dt\right)\Bigg|_{\tau=0} \\ &= \frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{\bar\partial}{\overline{\partial\tau^i}}\big(\operatorname{Tr}\exp\big(-t(\Delta''_{\tau,q})\big)\big)\right)\Bigg|_{\tau=0} t^{s-1}\,dt.\end{aligned} \tag{89}$$

Substituting in (89) the expression for $\frac{\bar\partial}{\overline{\partial\tau^i}}\big(\operatorname{Tr}\big(\exp\big(-t(\Delta''_{\tau,q})\big)\big)\big)$ given by (82) we obtain:

$$\begin{aligned}&\frac{\bar\partial}{\overline{\partial\tau^i}}\big(\zeta_{\Delta''_{\tau,q}}(s)\big)|_{\tau=0} \\ &\quad= \frac{1}{\Gamma(s)}\int_0^\infty \operatorname{Tr}\big(\big(\exp\big(-t(\Delta''_{0,q})\big)\big)\circ(\Delta''_{0,q})\circ\partial^{-1}\circ F'(q+1,\overline{\phi_i})\circ\bar\partial\big)t^s\,dt.\end{aligned} \tag{90}$$

Theorem 40 is proved. $\square$

6.3. The computation of the Hessian of $\zeta''_{\tau,q}(s)$.

Theorem 42. *The following formula holds*:

$$(91)\qquad \frac{\partial^2}{\partial\tau^j\overline{\partial\tau^i}}\left(\zeta''_{\tau,q}(s)\right)|_{\tau=0} = \frac{(1-s)s}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\left(\left(\exp\left(-t(\Delta'_{0,q+1})\right)\right)\circ\mathscr{F}'(q+1,\phi_j\circ\overline{\phi_i})\right)t^{s-1}\,dt.$$

Proof. The facts that the operators

$$\frac{\partial}{\partial\tau^i}(\overline{\partial_\tau}) = \left(-\phi_i + O(\tau)\right)\circ\partial_0$$

depend holomorphically on τ and the operator ∂_τ^{-1} depends antiholomorphically imply that the operators $\partial_\tau^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}$ depend antiholomorphically on the coordinates $\tau=(\tau^1,\dots,\tau^N)$. By using the explicit formula (90) for the antiholomorphic derivative of $\zeta''_{\tau,q}(s)$ and that

$$\frac{\partial}{\partial\tau^j}\left(\partial_\tau^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)=0$$

we derive

$$(92)\qquad \frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\left(\zeta''_{\tau,q}(s)\right) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\left(\left(\frac{\partial}{\partial\tau^j}\exp\left(-t(\Delta''_{\tau,q})\right)\right)\circ(\Delta''_{\tau,q})\circ\partial_\tau^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s\,dt + \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\left(\left(\exp\left(-t(\Delta''_{\tau,q})\right)\right)\circ\frac{\partial}{\partial\tau^j}(\Delta''_{\tau,q})\circ\partial_0^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s\,dt.$$

Lemma 43. *We have the following expression*:

$$(93)\qquad \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\left(\left(\frac{\partial}{\partial\tau^j}\left(\exp\left(-t(\Delta''_{\tau,q})\right)\right)\right)\circ(\Delta''_{\tau,q})\circ\partial_\tau^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s\,dt = \frac{-s}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\left(\left(\exp(-t\Delta''_{0,q})\right)\circ\frac{\partial}{\partial\tau^j}(\Delta''_{\tau,q})\circ\partial_\tau^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s\,dt.$$

Proof. Direct computations show that

$$(94)\qquad \frac{1}{\Gamma(s)}\left(\int_0^\infty \mathrm{Tr}\left(\left(\frac{\partial}{\partial\tau^j}\left(\exp\left(-t(\Delta''_{\tau,q})\right)\right)\right)\circ(\Delta''_{\tau,q})\circ\partial_\tau^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s\,dt\right)\Bigg|_{\tau=0} = \frac{-1}{\Gamma(s)}\int_0^\infty\left(\frac{d}{dt}\mathrm{Tr}\left(\left(\frac{\partial}{\partial\tau^j}\left(\exp\left(-t(\Delta''_{\tau,q})\right)\right)\right)\Bigg|_{\tau=0}\circ\partial_0^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)\right)t^s\,dt.$$

By integrating by parts the right-hand side of formula (94) we deduce that:

$$(95)\qquad \frac{-1}{\Gamma(s)}\int_0^\infty\left(\frac{d}{dt}\operatorname{Tr}\left(\left(\frac{\partial}{\partial\tau^j}\left(\exp\left(-t(\Delta''_{\tau,q})\right)\right)\right)\Big|_{\tau=0}\circ\partial_0^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)\right)t^s\,dt$$
$$=\frac{s}{\Gamma(s)}\int_0^\infty\operatorname{Tr}\left(\left(\frac{\partial}{\partial\tau^j}\left(\exp\left(-t(\Delta''_{\tau,q})\right)\right)\right)\Big|_{\tau=0}\circ\partial_0^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^{s-1}\,dt.$$

Direct computations of the right-hand side of (95) show that:

$$\frac{s}{\Gamma(s)}\int_0^\infty\operatorname{Tr}\left(\left(\frac{\partial}{\partial\tau^j}\left(\exp(-t\Delta''_{\tau,q})\right)\right)\Big|_{\tau=0}\circ\partial_0^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^{s-1}\,dt$$
$$=\frac{-s}{\Gamma(s)}\int_0^\infty\operatorname{Tr}\left(\left(\exp(-t\Delta''_{\tau,q})\right)\circ\frac{\partial}{\partial\tau^j}(\Delta''_{\tau,q})|_{\tau=0}\circ\partial_0^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s\,dt.$$

Formula (93) is proved. □

Substituting in (92) the expression from (93) for

$$\frac{1}{\Gamma(s)}\int_0^\infty\operatorname{Tr}\left(\left(\frac{\partial}{\partial\tau^j}\left(\exp\left(-t(\Delta''_{\tau,q})\right)\right)\right)\circ(\Delta''_{\tau,q})\circ\partial_\tau^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s\,dt$$

we get:

$$(96)\qquad \frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\left(\zeta''_{q-1,\tau}(s)\right)|_{\tau=0}$$
$$=\frac{1-s}{\Gamma(s)}\int_0^\infty\operatorname{Tr}\left(\left(\exp\left(-t(\Delta''_{\tau,q})\right)\right)\circ\frac{\partial}{\partial\tau^j}(\Delta''_{\tau,q})|_{\tau=0}\circ\partial_0^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s\,dt.$$

Lemma 44.

$$(97)\qquad \int_0^\infty\operatorname{Tr}\left(\left(\exp\left(-t(\Delta''_{\tau,q})\right)\right)\circ\frac{\partial}{\partial\tau^j}(\Delta''_{\tau,q})|_{\tau=0}\circ\partial_0^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s\,dt$$
$$=s\int_0^\infty\operatorname{Tr}\left(\left(\exp\left(-t(\Delta''_{\tau,q})\right)\right)|_{\tau=0}\circ(\overline{\partial_0})^{-1}\circ\mathscr{F}'(q+1,\phi_j\circ\overline{\phi_i})\circ\overline{\partial_0}\right)t^{s-1}\,dt.$$

Proof. Substituting the expression of (77) for

$$\frac{\partial}{\partial\tau^j}(\Delta''_{\tau,q})|_{\tau=0}=-\Delta''_{0,q}\circ(\overline{\partial_0})^{-1}\circ F'(q+1,\phi_j)\circ\overline{\partial_0}$$

in the expression of

$$\int_0^\infty\operatorname{Tr}\left(\left(\exp\left(-t(\Delta''_{\tau,q})\right)\right)\circ\frac{\partial}{\partial\tau^j}(\Delta''_{\tau,q})|_{\tau=0}\circ\partial_0^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s\,dt$$

we obtain:

$$(98)\qquad \int_0^\infty \mathrm{Tr}\left(\left(\exp\left(-t(\Delta''_{\tau,q})\right)\right)\circ\frac{\partial}{\partial\tau^j}(\Delta''_{\tau,q})|_{\tau=0}\circ\partial_0^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s\,dt$$

$$= -\int_0^\infty \mathrm{Tr}\left(\left(\exp\left(-t(\Delta''_{0,q})\right)\right)\circ\Delta''_{0,q}\circ\partial_0^{-1}\circ F'(q+1,\overline{\phi_i})\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s\,dt.$$

Simple observations show that

$$(99)\qquad \int_0^\infty \mathrm{Tr}\left(\left(\exp\left(-t(\Delta''_{0,q})\right)\right)\circ\Delta''_{0,q}\circ\partial_0^{-1}\circ F'(q+1,\overline{\phi_i})\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s\,dt$$

$$= -\int_0^\infty \mathrm{Tr}\left(\frac{d}{dt}\exp\left(-t(\Delta''_{0,q})\right)\partial_0^{-1}\circ F'(q+1,\overline{\phi_i})\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s\,dt.$$

By integrating by parts the right-hand side of (99) we obtain:

$$-\int_0^\infty \mathrm{Tr}\left(\frac{d}{dt}\exp\left(-t(\Delta''_{0,q})\right)\partial_0^{-1}\circ F'(q+1,\overline{\phi_i})\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s\,dt$$

$$= s\int_0^\infty \mathrm{Tr}\left(\exp\left(-t(\Delta''_{0,q})\right)\partial_0^{-1}\circ F'(q+1,\overline{\phi_i})\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^{s-1}\,dt.$$

Thus we derive formula (97), i.e.

$$\int_0^\infty \mathrm{Tr}\left(\left(\exp\left(-t(\Delta''_{\tau,q})\right)\right)\circ\frac{\partial}{\partial\tau^j}(\Delta''_{\tau,q})|_{\tau=0}\circ\partial_0^{-1}\circ F'(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s\,dt$$

$$= s\int_0^\infty \mathrm{Tr}\left(\left(\exp\left(-t(\Delta''_{\tau,q})\right)\right)|_{\tau=0}\circ(\overline{\partial_0})^{-1}\circ\mathscr{F}'(q+1,\phi_j\circ\overline{\phi_i})\circ\overline{\partial_0}\right)t^{s-1}\,dt.$$

Lemma 44 is proved. □

Substituting the expression of (97) in the expression (96) we get the following equality:

$$(100)\qquad \frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\left(\zeta''_{q,\tau}(s)\right)|_{\tau=0}$$

$$= \frac{(1-s)s}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\left(\left(\exp\left(-t(\Delta''_{0,q})\right)\right)\circ\overline{\partial_0}^{-1}\circ\mathscr{F}'(q+1,\phi_j\circ\overline{\phi_i})\circ\overline{\partial_0}\right)t^{s-1}\,dt.$$

Applying Theorem 35 we deduce Theorem 42. □

6.4. The computations of the Hessian of $\log\det\Delta_{\tau,q}$.

Theorem 45. *The following formula is true*:

$$\frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}(\log\det\Delta''_{\tau,q})|_{\tau=0} \tag{101}$$

$$= -\lim_{t\to 0}\int_M\left(\int_M \operatorname{Tr}\left((4\pi t)^{-\frac{n}{2}}\exp\left(-\frac{d_0^2(w,z)}{4t}\right)\circ\mathscr{P}\right.\right.$$

$$\left.\left.\circ\mathscr{F}'(q+1,\phi_i\circ\overline{\phi_j})\right)\operatorname{vol}(g)_w\right)\operatorname{vol}(g)_z$$

$$= -\int_M \operatorname{Tr}\big(\mathscr{F}'\big(q+1,(\phi_i\circ\overline{\phi_j})\big)\big)\operatorname{vol}(g).$$

Proof. $\zeta''_{\tau,q}(s)$ is obtained from the meromorphic continuation of

$$\frac{1}{\Gamma(s)}\int_0^\infty \operatorname{Tr}\big(\exp(-t\Delta''_{\tau,q})\big)t^{s-1}\,dt.$$

Thus it is a meromorphic function on $\mathbb{C}$ well defined at 0. So we get that $\zeta''_{\tau,q}(s) = \mu_0(\tau) + \mu_1(\tau)s + O(s^2)$. From here we deduce

$$\frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\big(\zeta''_{\tau,q}(s)\big)|_{\tau=0} = \frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\mu_0(\tau)|_{\tau=0} + \left.\left(\frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\mu_1(\tau)\right)\right|_{\tau=0} s + O(s^2) \tag{102}$$

$$= \alpha_0 + \alpha_1 s + O(s^2).$$

Thus from the definition of the regularized determinant

$$\log\det(\Delta''_{\tau,q}) = \left.\left(\frac{d}{ds}\big(-\zeta''_{\tau,q}(s)\big)\right)\right|_{s=0}$$

we see that

$$\frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}(\log\det\Delta''_{\tau,q})|_{\tau=0} = \frac{d}{ds}\left.\left(\frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\big(-\zeta''_{\tau,q}(s)\big)|_{\tau=0}\right)\right|_{s=0} = -\alpha_1. \tag{103}$$

Combining formula (91)

$$\frac{\partial^2}{\partial\tau^j\overline{\partial\tau^i}}\big(\zeta''_{\tau,q}(s)\big)|_{\tau=0} = \frac{s}{\Gamma(s)}\int_0^\infty \operatorname{Tr}\big(\big(\exp\big(-t(\Delta'_{0,q+1})\big)\big)\circ\mathscr{F}'(q+1,\phi_j\circ\overline{\phi_i})\big)t^{s-1}\,dt$$

$$-\frac{s^2}{\Gamma(s)}\int_0^\infty \operatorname{Tr}\big(\big(\exp\big(-t(\Delta'_{0,q+1})\big)\big)\circ\mathscr{F}'(q+1,\phi_j\circ\overline{\phi_i})\big)t^{s-1}\,dt$$

with the short term expansion:

$$\operatorname{Tr}\big(\big(\exp\big(-t(\Delta'_{0,q+1})\big)\big)\circ\mathscr{F}'(q+1,\phi_j\circ\overline{\phi_i})\big) = \sum_{k=-n}^{1}\frac{v_k}{t^k} + v_0 + \psi(t) \tag{104}$$

where

$$\psi(t) = \mathrm{Tr}\big(\big(\exp\big(-t(\Delta'_{0,q+1})\big)\big) \circ \mathscr{F}'(q+1, \phi_j \circ \overline{\phi_i})\big) - \sum_{k=-n}^{1} \frac{v_k}{t^k} + v_0$$

we obtain:

$$(105) \qquad \frac{\partial^2}{\partial \tau^j \partial \overline{\tau^i}} \big(\zeta''_{\tau,q}(s)\big)|_{\tau=0}$$

$$= \frac{s}{\Gamma(s)} \left(\int_0^1 \left(\sum_{k=-n}^{1} \frac{v_k}{t^k} \right) t^{s-1}\, dt + v_0 \int_0^1 t^{s-1}\, dt + \int_0^1 \psi(t) t^{s-1}\, dt \right)$$

$$+ \frac{s}{\Gamma(s)} \left(\int_1^\infty \mathrm{Tr}\big(\big(\exp\big(-t(\Delta'_{0,q+1})\big)\big) \circ \mathscr{F}'(q+1, \phi_j \circ \overline{\phi_i})\big) t^{s-1}\, dt \right)$$

$$- \frac{s^2}{\Gamma(s)} \left(\int_0^1 \left(\sum_{k=-n}^{1} \frac{v_k}{t^k} \right) t^{s-1}\, dt + v_0 \int_0^1 t^{s-1}\, dt + \int_0^1 \psi(t) t^{s-1}\, dt \right)$$

$$- \frac{s^2}{\Gamma(s)} \left(\int_1^\infty \mathrm{Tr}\big(\big(\exp\big(-t(\Delta'_{0,q+1})\big)\big) \circ \mathscr{F}'(q+1, \phi_j \circ \overline{\phi_i})\big) t^{s-1}\, dt \right).$$

By using formula (105) we will prove the following lemma:

Lemma 46. *We have the following formula*:

$$\frac{\partial^2}{\overline{\partial \tau^j} \partial \tau^i} (\log \det \Delta''_{\tau,q})|_{\tau=0} = -\int_M \mathrm{Tr}\, \mathscr{F}'(q+1, \phi_i \circ \overline{\phi_j})\, \mathrm{vol}(g) = -\alpha_1.$$

Proof. [2], Lemma 9.34, page 300, or direct computations show that for $|s| < \varepsilon$ we have the following identity:

$$(106) \qquad \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\big(\big(\exp\big(-t(\Delta'_{0,q})\big)\big) \circ \mathscr{F}'(q+1, \phi_j \circ \overline{\phi_i})\big) t^{s-1}\, dt$$

$$= \frac{1}{\Gamma(s)} \left(\int_0^1 \left(\sum_{k=-n}^{1} \frac{v_k}{t^k} \right) t^{s-1}\, dt + v_0 \int_0^1 t^{s-1}\, dt + \int_0^1 \psi(t) t^{s-1}\, dt \right)$$

$$+ \frac{1}{\Gamma(s)} \int_1^\infty \mathrm{Tr}\big(\big(\exp\big(-t(\Delta'_{0,q})\big)\big) \circ \mathscr{F}'(q+1, \phi_j \circ \overline{\phi_i})\big) t^{s-1}\, dt = \frac{v_0}{s} + \kappa + O(s).$$

Combining the expression in (106) with the standard fact $\frac{s}{\Gamma(s)} = s^2 + O(s^3)$ we obtain from formulas (105) and (106) that for $|s| < \varepsilon$

$$(107) \qquad \frac{\partial^2}{\partial \tau^j \overline{\partial \tau^i}} \big(\zeta''_{\tau,q}(s)\big)|_{\tau=0} = v_0 s + O(s^2).$$

Thus according to (103) and (107)

$$\frac{d}{ds}\left(\frac{\partial^2}{\partial\tau^j\overline{\partial\tau^i}}\left(\zeta''_{\tau,q}(s)\right)|_{\tau=0}\right)\Bigg|_{s=0} = v_0 = \alpha_1. \tag{108}$$

Applying Theorem 34 to formula (108) we deduce that

$$\alpha_1 = v_0 = \int_M \operatorname{Tr}\big(\mathscr{F}'(q+1,\phi_i\circ\overline{\phi_j})\big)\operatorname{vol}(g).$$

Lemma 46 is proved. □

Lemma 46 implies directly Theorem 45. □

6.5. Some applications of the variational formulas.

Theorem 47. *The following identity holds*:

$$dd^c(\log\det\Delta_{\tau,1}) = dd^c(\log\det\Delta'_{\tau,1}\det\Delta''_{\tau,1}) = -\operatorname{Im}\mathrm{W.P.}$$

Proof. The proof of Theorem 47 is based on the following formulas which hold for Kähler manifolds:

$$\Delta_\partial = \Delta_{\bar\partial},\quad \partial^* = -*\bar\partial*\quad\text{and}\quad \bar\partial^* = -*\partial*, \tag{109}$$

where $*$ is the Hodge star operator. See [21], page 95. On CY manifolds we have the duality $*:\Omega_M^{0,q}\approxeq\Omega_M^{0,n-q}$ induced by the Hodge star operator $*$ of a CY metric and the holomorphic n form. Using this duality direct check shows that on CY manifolds we have

$$*(\operatorname{Im}\overline{\partial_q}) = \operatorname{Im}(\overline{\partial_{n-q}}^*)\quad\text{and}\quad *(\operatorname{Im}\overline{\partial_q}^*) = \operatorname{Im}(\overline{\partial_{n-q}}). \tag{110}$$

Formulas (109) and (110) imply that we have $\det\Delta'_{\bar\partial,q} = \det\Delta'_{\partial,q} = \det\Delta'''_{\partial,n-q} = \det\Delta'''_{\bar\partial,n-q}$.

Lemma 48. *We have the following relations between the operators on a* CY *manifold*:

$$\operatorname{Tr}\big(\exp(-t\Delta'_{\tau,1})\circ\mathscr{F}'(1,\phi_i\circ\overline{\phi_j})\big) = \operatorname{Tr}\big(\exp(-t\Delta''_{\tau,1})\circ\mathscr{F}''(n-1,\phi_i\circ\overline{\phi_j})\big)$$

and

$$\operatorname{Tr}\big(\exp(-t\Delta''_{\tau,1})\circ\mathscr{F}''(1,\phi_i\circ\overline{\phi_j})\big) = \operatorname{Tr}\big(\exp(-t\Delta'_{\tau,1})\circ\mathscr{F}'(n-1,\phi_i\circ\overline{\phi_j})\big) \tag{111}$$

by identifying the Hilbert spaces $\operatorname{Im}\bar\partial\subset L^2(M,\Omega_M^{0,1})$ *and* $\operatorname{Im}\bar\partial^*\subset L^2(M,\Omega_M^{0,1})$ *with* $\operatorname{Im}\partial^*\subset L^2(M,\Omega_M^{0,n-1})$ *and* $\operatorname{Im}\partial\subset L^2(M,\Omega_M^{0,n-1})$ *respectively by using* (110).

Proof. We will need the following propositions to prove Lemma 48:

Proposition 49. *Let* $\{\omega_i\}$ *be an orthonormal basis at* $\Omega_x^{1,0}$. *Let* $\bar\omega_n = \overline{\omega_1}\wedge\cdots\wedge\overline{\omega_n}$ *be the antiholomorphic volume form. Then for* $\alpha \neq \beta$ *we have*

$$(\phi_i\circ\overline{\phi_j}\wedge\mathrm{id}_{n-2})\left(\bar\omega_n\lrcorner\frac{\bar\partial}{\overline{\partial z^\alpha}}\right) = \sum_\beta(\phi_i\circ\overline{\phi_j})_{\bar\beta}^{\bar\alpha}\left(\bar\omega_n\lrcorner\frac{\bar\partial}{\overline{\partial z^\alpha}}\wedge\frac{\bar\partial}{\overline{\partial z^\beta}}\right)\wedge\overline{dz^\alpha}. \tag{112}$$

Proof. Formula 112 follows directly from the definition of the linear operator $(\phi_i \circ \overline{\phi_j} \wedge \mathrm{id}_{n-2})$. $\square$

Proposition 50. *Let us define* $(\phi_i \circ \overline{\phi_j} \wedge \mathrm{id}_{n-2})^* : \Omega_M^{0,n-1} \to \Omega_M^{0,n-1}$ *as follows*:

$$(\phi_i \circ \overline{\phi_j} \wedge \mathrm{id}_{n-2})^*(*\overline{\omega_k}) = *\big((\phi_i \circ \overline{\phi_j})(\overline{\omega_k})\big) \tag{113}$$

Let us denote by $M(\overline{\phi_j} \circ \phi_i)$ *and* $M\big((\phi_i \circ \overline{\phi_j} \wedge \mathrm{id}_{n-2})^*\big)$ *the matrices of the operators* $\overline{\phi_j} \circ \phi_i$ *and* $(\phi_i \circ \overline{\phi_j} \wedge \mathrm{id}_{n-2})^*$ *in the orthonormal bases with respect to the* CY *metric. Then fibrewise we have the equality*

$$M(\overline{\phi_j} \circ \phi_i) = M\big((\phi_i \circ \overline{\phi_j} \wedge \mathrm{id}_{n-2})^*\big). \tag{114}$$

Proof. We need to compute the matrix of the operator $(\phi_i \circ \overline{\phi_j} \wedge \mathrm{id}_{n-2})^*$ in the orthonormal basis $\overline{\omega_{i_1}} \wedge \cdots \wedge \overline{\omega_{i_{n-k}}}$ and compare it with the matrix of the operator $\overline{\phi_j} \circ \phi_i$ of the bundle $\Omega_M^{1,0}$ written in the orthonormal basis $\{\omega_i\}$.

Let $\{\omega_i\}$ be an orthonormal basis at $\Omega_x^{1,0}$. According to Lemma 11 the operators $\phi_i : \Omega_x^{1,0} \to \Omega_x^{0,1}$ in the orthonormal basis $\{\omega_i\}$ and $\{\overline{\omega_i}\}$ are given by symmetric matrices. From here (114) follows directly. Indeed from the relations of the elements of the matrices of the operators ϕ_i in an orthonormal basis $\phi^{\alpha}_{i,\bar\beta} = \phi^{\beta}_{i,\bar\alpha}$ we obtain

$$(\phi_i \circ \overline{\phi_j})^{\bar\beta}_{\bar\alpha} = \sum_{\mu=1}^{n} \phi^{\mu}_{i,\bar\alpha}\overline{\phi^{\beta}_{j,\bar\mu}} = \sum_{\mu=1}^{n} \phi^{\alpha}_{i,\bar\mu}\overline{\phi^{\mu}_{j,\bar\beta}} = \sum_{\mu=1}^{n} \overline{\phi^{\mu}_{j,\bar\beta}}\phi^{\alpha}_{i,\bar\mu} = (\overline{\phi_j} \circ \phi_i)^{\alpha}_{\beta}. \tag{115}$$

From the definition of the operator $(\phi_i \circ \overline{\phi_j} \wedge \mathrm{id}_{n-2})^*$ given by (113) we get:

$$\begin{aligned}(\phi_i \circ \overline{\phi_j} \wedge \mathrm{id}_{n-2})^*(*\overline{\omega_k}) &= *\big((\phi_i \circ \overline{\phi_j})(\overline{\omega_k})\big) = *\left(\sum_{k,l=1}^{n} (\phi_i \circ \overline{\phi_j})^l_k(\overline{\omega_l})\right) \\ &= \sum_{k,l=1}^{n} (\phi_i \circ \overline{\phi_j})^l_k\big(*(\overline{\omega_l})\big) = \sum_{k,l=1}^{n} (\phi_i \circ \overline{\phi_j})^l_k(\overline{\omega_1} \wedge \cdots \wedge \overline{\omega_{l-1}} \wedge \overline{\omega_{l+1}} \wedge \cdots \wedge \overline{\omega_n}).\end{aligned} \tag{116}$$

Combining (115) and (116) we get

$$\begin{aligned}(\phi_i \circ \overline{\phi_j} \wedge \mathrm{id}_{n-2})^*(*\overline{\omega_k}) &= \sum_{l=1}^{n}(\overline{\phi_j} \circ \phi_i)^k_l(\overline{\omega_1} \wedge \cdots \wedge \overline{\omega_{l-1}} \wedge \overline{\omega_{l+1}} \wedge \cdots \wedge \overline{\omega_n}) \\ &= \sum_{l=1}^{n}(\phi_i \circ \overline{\phi_j})^l_k(\overline{\omega_1} \wedge \cdots \wedge \overline{\omega_{k-1}} \wedge \overline{\omega_{k+1}} \wedge \cdots \wedge \overline{\omega_n}).\end{aligned}$$

By using the Calabi-Yau metric and the holomorphic volume form we can identify the dual of $\Omega_M^{0,n-1}$ with $\Omega_M^{1,0}$. This identification is given by

$$\overline{\omega_1} \wedge \cdots \wedge \overline{\omega_{l-1}} \wedge \overline{\omega_{l+1}} \wedge \cdots \wedge \overline{\omega_n} \to \omega_l.$$

Thus from (112) we get

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$$(117)\qquad *\big((\phi_i\circ\overline{\phi_j})(\overline{\omega_k})\big) = (\phi_i\circ\overline{\phi_j}\wedge \mathrm{id}_{n-2})^*(*\overline{\omega_k}) = \sum_{l=1}^{n}(\overline{\phi_j}\circ\phi_i)_l^k(*\overline{\omega_k}) = \sum_{l=1}^{n}(\overline{\phi_j}\circ\phi_i)_l^k\omega_k.$$

From (117) and (115) we conclude Proposition 50. □

Corollary 51. *Formula* (117) *implies that the composition of the complex conjugation with the Hodge star operator $*$ identifies the restriction of the image $(\overline{\phi_j}\circ\phi_i)(\operatorname{Im}\partial)$ on $\operatorname{Im}\partial$ in $C^\infty(M,\Omega_M^{1,0})$ with the restriction of the image $(\phi_i\circ\overline{\phi_j}\wedge\mathrm{id}_{n-2})(\operatorname{Im}\bar\partial^*)$ on $\operatorname{Im}\bar\partial^*$ in $C^\infty(M,\Omega_M^{0,n-1})$ and the restriction of $(\overline{\phi_j}\circ\phi_i)(\operatorname{Im}\partial^*)$ on $\operatorname{Im}\partial^*$ in $C^\infty(M,\Omega_M^{1,0})$ with the restriction of $(\phi_i\circ\overline{\phi_j}\wedge\mathrm{id}_{n-2})(\operatorname{Im}\bar\partial)$ on $\operatorname{Im}\bar\partial$ in $C^\infty(M,\Omega_M^{0,n-1})$.*

From (117), (110), and the identification $\Omega_M^{1,0}$ with $\Omega_M^{0,n-1}$ we deduce that we can identify $(\overline{\phi_j}\circ\phi_i)\operatorname{Im}\partial^*$ in $C^\infty(M,\Omega_M^{1,0})$ with $(\phi_i\circ\overline{\phi_j}\wedge\mathrm{id}_{n-2})\operatorname{Im}\bar\partial$ in $C^\infty(M,\Omega_M^{0,n-1})$ and $(\overline{\phi_j}\circ\phi_i)\operatorname{Im}\partial$ in $C^\infty(M,\Omega_M^{1,0})$ with $(\phi_i\circ\overline{\phi_j}\wedge\mathrm{id}_{n-2})\operatorname{Im}\bar\partial^*$ in $C^\infty(M,\Omega_M^{0,n-1})$. Since on a Kähler manifold we have that $\Delta_\partial=\Delta_{\bar\partial}$, (111) is established. Lemma 48 is proved. □

From (111) we deduce that

$$(118)\qquad \operatorname{Tr}\big(\exp(-t\Delta'_{\tau,1})\mathscr{F}'(1,\phi_i\circ\overline{\phi_j})+\exp(-t\Delta''_{\tau,1})\mathscr{F}''(1,\phi_i\circ\overline{\phi_j})\big) = \int_M \operatorname{Tr}\big(\exp(-t\Delta_{\tau,1})(\phi_i\circ\overline{\phi_j})\big)\,\mathrm{vol}.$$

We know that

$$(119)\qquad \operatorname{Tr}\big(\mathscr{F}'(1,\phi_i\circ\overline{\phi_j})+\mathscr{F}''(1,\phi_i\circ\overline{\phi_j})\big)=\int_M \operatorname{Tr}(\phi_i\circ\overline{\phi_j})\,\mathrm{vol}=\mathrm{W.P.}$$

Combining (118), (119) with Theorem 45 we deduce that

$$(120)\qquad dd^c(\log\det\Delta_{\tau,1}) = dd^c\log(\Delta'_{\tau,1}\times\Delta''_{\tau,1}) = -\operatorname{Im}\mathrm{W.P.}$$

Theorem 47 is proved. □

Theorem 52. *The relative dualizing sheaf $\pi_*(\omega_{\mathscr{X}/\mathfrak{M}_L(M)}):=\mathscr{L}$ is a trivial C^∞ line bundle.*

Proof. It is a well known fact that a complex line bundle on a complex manifold $\mathfrak{M}(M)$ is topologically trivial if and only if its first Chern class is zero. According to [32] the first Chern class of the relative dualizing sheaf $\pi_*(\omega_{\mathscr{X}/\mathfrak{M}_L(M)}):=\mathscr{L}$ is the minus imaginary part of the Weil-Petersson metric on $\overline{\mathfrak{M}}(M)$. We observe that the regularized determinant of the Laplacian of the CY metric is a well defined function on $\mathfrak{M}(M)$. Theorem 47 implies that $dd^c(\log\det\Delta_{\tau,1})$ is minus the imaginary part of the Weil-Petersson metric on $\overline{\mathfrak{M}}(M)$. Thus the first Chern class of the relative dualizing sheaf $\pi_*(\omega_{\mathscr{X}/\mathfrak{M}_L(M)}):=\mathscr{L}$ is represented by the zero class of cohomology. This proves that the relative dualizing sheaf $\pi_*(\omega_{\mathscr{X}/\mathfrak{M}_L(M)}):=\mathscr{L}$ is topologically trivial on $\mathfrak{M}(M)$. Theorem 52 is proved. □

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7. The computation on Kähler-Ricci flat manifold with a non trivial canonical bundle

Theorem 53. *Suppose that M is an n complex dimensional Kähler-Ricci flat manifold such that $(K_M)^{\otimes n} \approxeq \mathcal{O}_M$. Then*

$$dd^c(\log \det \Delta_{\tau,n}) = -\operatorname{Im} \mathrm{W.P.} \tag{121}$$

Proof. The proof of Theorem 53 is based on Theorem 45 which shows that we need to compute

$$\int_M \operatorname{Tr}\big(\mathscr{F}'\big(n, (\phi_i \circ \overline{\phi_j})\big)\big) \operatorname{vol}(g).$$

Here we are using the fact that $\det \Delta_{\tau,n} = \det \Delta''_{\tau,n-1}$. Since we assumed that $H^n(M, \Omega^n_M) = 0$ we get that

$$\int_M \operatorname{Tr}\big(\mathscr{F}'\big(n, (\phi_i \circ \overline{\phi_j})\big)\big) \operatorname{vol}(g) = \int_M \operatorname{Tr}\big(\big((\phi_i \circ \overline{\phi_j}) \wedge \mathrm{id}_{n-1}\big)\big) \operatorname{vol}(g).$$

Direct computation using elementary linear algebra shows that if $A : V \to V$ is a linear operator acting on an n dimensional vector space then the trace of the operator $A \wedge \mathrm{id}_{n-1}$ acting on $\bigwedge^n V$ is given by the formula:

$$\operatorname{Tr}(A \wedge \mathrm{id}_{n-1}) = \operatorname{Tr}(A). \tag{122}$$

From formula (122) and the definition of the Weil-Petersson metric given by (24) we derive (121). Theorem 53 is proved. □

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