Geometry of Riemannian manifolds with the biggest possible eigenfunctions
Joint work in part with C. Sogge

Zabrodsky Lecture 2, March 25, 2014

Sizes of eigenfunctions

My first Zabrodsky lecture was about shapes of eigenfunctions, visualized by their nodal sets. This lecture is about sizes of eigenfunctions. It is of interest for Schrödinger operators but we only consider L^2 normalized eigenfunctions

$$\Delta \varphi_\lambda = -\lambda^2 \varphi_\lambda, \quad \int_M |\varphi_\lambda|^2 dV = 1$$

of compact Riemannian manifolds (M, g).

How should we measure the size? The simplest way is by L^p norms

$$||\varphi_\lambda||_{L^p}^p = \int_M |\varphi_\lambda|^p dV.$$

Measuring eigenfunction growth

The eigenvalues $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \uparrow \infty$ form a discrete increasing sequence, and we denote an orthonormal basis of eigenfunctions with increasing eigenvalue by $\{\varphi_{\lambda_j}(x)\}$, or by φ_j for short.

The eigenvalues may have multiplicity > 1 and L^p norms depend very much on which basis we pick. So we consider the maximal value the L^p norm in the eigenspace (minimal is also of much interest).

Denote the eigenspaces by

$$V_\lambda = \{\varphi : \Delta\varphi = -\lambda^2\varphi\}.$$

We measure the growth rate of L^p norms by

$$L^p(\lambda, g) = \sup_{\varphi \in V_\lambda : ||\varphi||_{L^2} = 1} ||\varphi||_{L^p}. \tag{1}$$

Universal upper bounds on L^∞ norms of eigenfunctions

One of the simplest L^p norms to consider is the L^∞ norm,

$$||\varphi_\lambda||_{L^\infty} = \sup_M |\varphi_\lambda(x)|.$$

There exist a bound which holds for all (M, g): A classical result of Avakumovic, Levitan, H"ormander states:

$$||\varphi_\lambda||_{L^\infty} \leq C_g \lambda^{\frac{m-1}{2}}, \ \ (m = \dim M).$$

It is sharp in the sense that there exist (M, g) and sequences of eigenfunctions saturating (achieving) the bound. But such (M, g) are very rare.

Sogge universal upper bounds on L^p norms of eigenfunctions

THEOREM

(Sogge, 1985)

$$\sup_{\varphi \in V_\lambda} \frac{\|\varphi\|_p}{\|\varphi\|_2} = O(\lambda^{\delta(p)}), \quad 2 \leq p \leq \infty \tag{2}$$

where

$$\delta(p) = \begin{cases} n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}, & \frac{2(n+1)}{n-1} \leq p \leq \infty \\ \frac{n-1}{2}(\frac{1}{2} - \frac{1}{p}), & 2 \leq p \leq \frac{2(n+1)}{n-1}. \end{cases} \tag{3}$$

Note that there is a 'phase transition' at $p = \frac{2(n+2)}{n-1}$. Again, there exist (M, g) and sequences of eigenfunctions for which they are sharp, but again such (M, g) are very rare.

Extremals

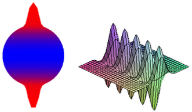
The upper bounds are sharp in the class of all (M, g) and are saturated on the round sphere:

- For $p > \frac{2(n+1)}{n-1}$, zonal (rotationally invariant) spherical harmonics saturate the L^p bounds. Such eigenfunctions also occur on surfaces of revolution.

- For L^p for $2 \leq p \leq \frac{2(n+1)}{n-1}$ the bounds are saturated by highest weight spherical harmonics, i.e. Gaussian beam along a stable elliptic geodesic. Such eigenfunctions also occur on surfaces of revolution.

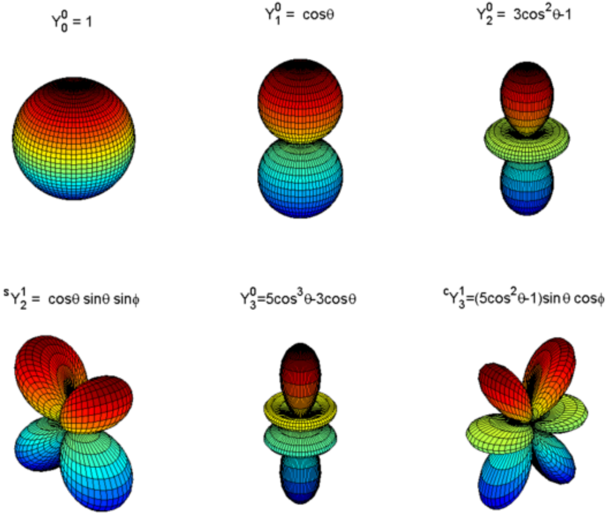
(i)Zonal eigenfunction (ii) Gaussian beam

The left image is a zonal spherical harmonic of degree N on S^2: it has high peaksof height $\sqrt{N}$ at the north and south poles. The right image is a Gaussian beam: its height along the equator is $N^{1/4}$ and then it has Gaussian decay transverse to the equator.



The zonal has high L^p norm due to its high peaks on balls of radius $\frac{1}{N}$. The balls are so small that they do not have high L^p norms for small p. The Gaussian beams are not as high but they are relatively high over an entire geodesic.

Graphics of spherical haromonics



(M, g) with maximal eigenfunction growth

Definition: Say that (M, g) has maximal L^p eigenfunction growth if it possesses a sequence of eigenfunctions $\varphi_{\lambda_{j_k}}$ which saturates the L^p bounds. When $p = \infty$ we say that it has maximal sup norm growth.

Problems

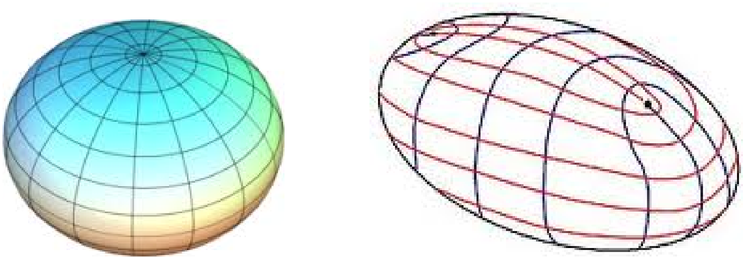
- Characterize (M, g) with maximal L^∞ eigenfunction growth. The same sequence of eigenfunctions should saturate all L^p norms with $p \geq \frac{2(n+1)}{n-1}$.
- Characterize (M, g) with maximal L^p eigenfunction growth for $2 \leq p \leq \frac{2(n+1)}{n-1}$.
- In recent work, Sogge-Z related growth of nodal volumes to behavior of L^1 norms of eigenfunctions. Characterize (M, g) for which $||\varphi_\lambda||_{L^1} \geq C > 0$.

Poles and self-focal points

- The zonal spherical harmonics on S^2 or a surface of revolution peak at the "poles". We call a point $p \in (M, g)$ a pole if every geodesics leaving p is a closed geodesic: the geodesic $\gamma_{p,\xi}$ at p in direction ξ returns to p at a fixed time T and $\gamma'_{p,\xi}(0) = \gamma'_{p,\xi}(T)$. The fixed points of a surface of revolution are poles; all points of S^2 are poles.

- We call a point p a *self-focal point* or *blow-down point* if all geodesics leaving p loop back to p at a common time T. They do not have to be closed geodesics. Examples: umbilic points of ellipsoids; foci of ellipses;

Pole of a surface of revolution and umbilic points on an ellipsoid



Known examples of (M, g) with maximal eigenfunction growth

- The only known examples of (M, g) with maximal eigenfunction growth for high L^p norms have poles. In fact, the only known examples are zonal eigenfunctions on surfaces of revolution (and their higher dimensional analogues).
- The only known examples of eigenfunctions saturating low L^p bounds are Gaussian beams. It is a very interesting problem to prove that any (M, g) with maximal L^p growth of eigenfunctions for low p must have a stable elliptic geodesic. We will not discuss them further in this talk.

Conjectures

- If (M,g) has maximal L^∞ eigenfunction growth, i.e. a sequence $\{\varphi_{j_k}\}$ such that $||\varphi_{j_k}||_\infty \geq C_g \lambda_{j_k}^{\frac{n-1}{2}}$, then (M,g) posses a pole, i.e. a point such that that all geodesics leaving the point return to the point as smoothly closed geodesics.

- f (M,g) has maximal L^p eigenfunction growth for $p < p_n = \frac{2(n+1)}{n-1}$, i.e. a sequence $\{\varphi_{j_k}\}$ such that $||\varphi_{j_k}||_p \geq C_g \lambda_{j_k}^{\delta(p)}$, then (M,g) has an elliptic closed geodesic.

First characterization of (M, g) of maximal eigenfunction growth

THEOREM

(Sogge-Z, 2002) Suppose (M, g) is a C^∞ Riemannian manifold with maximal eigenfunction growth, i.e. having a sequence $\{\varphi_{\lambda_{j_k}}\}$ of eigenfunctions which achieves (saturates) the bound $||\varphi_\lambda||_{L^\infty} \leq \lambda^{(n-1)/2}$.
Then there must exist a point $x \in M$ for which the set

$$\mathcal{L}_x = \{\xi \in S^*_x M : \exists T : \exp_x T\xi = x\} \tag{4}$$

of directions of geodesic loops at x has positive measure in $S^*_x M$. Here, exp *is the exponential map, and the measure $|\Omega|$ of a set Ω is the one induced by the metric g_x on $T^*_x M$. For instance, the poles x_N, x_S of a surface of revolution (S^2, g) satisfy $|\mathcal{L}_x| = 2\pi$.*

Real analytic (M, g) of maximal eigenfunction growth

THEOREM

(Sogge-Z, 2002) Suppose (M, g) is a C^ω Riemannian manifold with maximal eigenfunction growth, i.e. having a sequence $\{\varphi_{\lambda_{j_k}}\}$ of eigenfunctions which achieves (saturates) the bound $||\varphi_\lambda||_{L^\infty} \leq \lambda^{(n-1)/2}$.
Then there must exist a self-focal point, i.e. a point $x \in M$ for which the set

$$\mathcal{L}_x = \{\xi \in S_x^* M : \exists T : \exp_x T\xi = x\} = S_x^* M. \tag{5}$$

In dimension 2, M must be a topological sphere.

Are these results sharp?

No!

Counter-example: a tri-axial ellipsoid (three distinct axes, so not a surface of revolution). The umbilic points are self-focal points. However, the eigenfunctions which maximize the sup-norm only have L^∞ norms of order $\lambda^{\frac{n-1}{2}} / \log \lambda$.

They are only a logarithm away from maximizing the sup norm estimate, but the logarithm is quite important– even on a compact hyperbolic surface one has no better sup-norm bound than for these eigenfunctions.

Additional necessary conditions

For simplicity we assume henceforth that (M, g) is real analytic. Then $\mathcal{L}_x = S^*_x M$ when $|\mathcal{L}_x| > 0$.

At a self-focal point, with least common return time T_x we define the first return map

$$\Phi_x := G^{T_x} : S^*_x M \to S^*_x M.$$

Definition: If x is self-focal and $\Phi_x = Id$, then x is a pole (as above). If $\Phi_x \neq Id$ then we say that Φ_x is twisted.

Conjecture : If (M, g) has maximal eigenfunction growth, it must have a pole.

Ellipsoid

In the case of round S^n, the first return map $\Phi_x = Id$ for all x. Thus every direction is recurrent.

In the case of the triaxial ellipsoid satisfy, $|\mathcal{R}_x| = 0$ for each $x \in M$ including the umbilic points wher $|\mathcal{L}_x| = 1$. The reason is that Φ_x at the umbilic points is a circle map with two fixed points corresponding to the two closed geodesics through x. One is stable, one is unstable. Under iterations, $\Phi_x^n(\xi)$ tends to the stable fixed point for any $\xi \in S_x^* M$ except the unstable fixed point. Hence the only recurrent point is the stable fixed point.

New result (2013-2014)

THEOREM

(Z, Sogge 2013) If (M, g) is real analytic and has maximal eigenfunction growth, then it possesses a self-focal point whose first return map Φ_x has an invariant function $f \in L^2(S^*_x M, \mu_x)$, where μ_x is the Euclidean surface measure. Equivalently, it has an invariant measure $|f|^2 d\mu_x$ which is absolutely continuous with respect to $d\mu_x$

Invariant means that $f \circ \Phi_x = f$.

Dynamical problems

As mentioned earlier, we wonder if maximal eigenfunction growth implies existence of a self-focal points for which $\Phi_x = Id$.

Does this follow in the real analytic case from existence of an invariant L^2 function? Not in general (e.g. irrational rotations of S^1), but Φ_x has additional properties, e.g. periodic points, possibly fixed points.

No (M, g) with $\dim M \geq 3$ with self-focal points are known except when $\Phi_x = Id$. I.e. there are no known generalizations of umbilic points of ellipsoids in dimension two. Do any exist?

If $\Phi_x : S^*_x M \to S^*_x M$ has an invariant function, does it have a real analytic (or at least smooth) invariant function?

Recurrent loop directions

The result above implies an earlier improvement on the result on existence of self-focal points. It involves the notions of conservative and dissipative part of the first return map

$$\Phi_x = G_x^{T_x} : S_x^* M \to S_x^* M.$$

We define the set of *recurrent loop directions* to be the subset

$$\mathcal{R}_x = \{\xi \in S_x^* M : \xi \in \omega(\xi)\},$$

where $\omega(\xi)$ denotes the ω-limit set, i.e. the limit points of the orbit $\{(G_x^{T_x})^n \xi : n \in \mathbb{Z}_+\}$. Equivalently, $\xi \in \mathcal{L}_x^\infty$ belongs to $\mathcal{R}_x$ if infinitely many iterates, $(G_x^{T_x})^n \xi$, $n \in \mathbb{Z}$, belong to Γ, whenever Γ is a neighborhood of ξ in $S_x^* M$. Finally, we say that Φ_x has *recurrence* if $|\mathcal{R}_x| > 0$.

Max efn growth can only occur if the recurrent directions have positive measure

Theorem

(Sogge-Toth-Z (2011)
Suppose (M, g) is a C^∞ Riemannian manifold with maximal eigenfunction growth, i.e. having a sequence $\{\varphi_{\lambda_{j_k}}\}$ of eigenfunctions which achieves (saturates) the bound $||\varphi_\lambda||_{L^\infty} \leq \lambda^{(n-1)/2}$.
Then there must exist a point $x \in M$ for which the set $\mathcal{L}_x$ has positive measure and also $\mathcal{R}_x$ has positive measure. I.e. there must exist a point for which the first return map Φ_x has recurrence.

This already rules out ellipsoids, for which Φ_x is transient (dissipative in Hopf's classification).

Sketch of proof

The proof is based on a study of the kernel of the solution operator $\cos t\sqrt{\Delta}$ of the wave equation on (M, g), which is 'dual' to the spectral projections kernal $\Pi_{[0,\lambda]}$.

Squares of eigenfunctions are JUMPS in $\Pi_{[0,\lambda]}$ as λ increases and crosses an eigenvalue. Thus maximal growth of eigenfunctions implies that the jumps have to be of maximal size at some points x_λ which may vary with λ.

Because the x_λ might move around, we need more than estimates on the remainder at one point. We need uniform "little oh" bounds which are valid as x varies. But the structure of loops at x varies erratically with x, and 'almost-loops' cause as many problems as loops.

L^∞ estimates of eigenfunctions

Define:

$$N(\lambda, x) := \sum_{\lambda_\nu \leq \lambda} |\varphi_\nu(x)|^2.$$

The pointwise Weyl law gives,

$$N(\lambda, x) = (2\pi)^{-n} \int_{p(x,\xi)\leq\lambda} d\xi + R(\lambda, x)$$

with uniform remainder bounds

$$|R(\lambda, x)| \leq C\lambda^{n-1}, \quad x \in M.$$

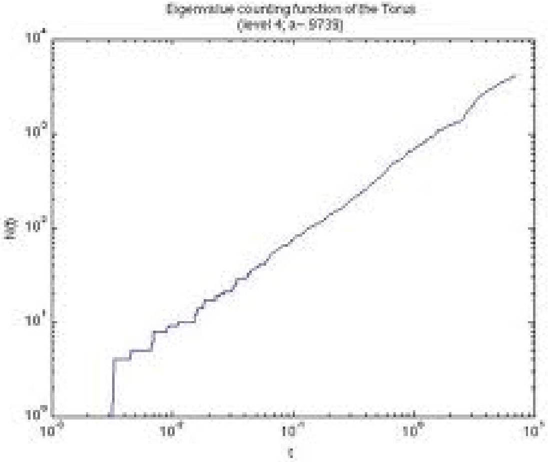
Since the integral in the local Weyl law is a continuous function of λ and since the spectrum of the Laplacian is discrete, this immediately gives

$$\sum_{\lambda_\nu = \lambda} |\varphi_\nu(x)|^2 \leq 2C\lambda^{n-1}$$

which in turn yields (for any (M, g))

$$L^\infty(\lambda, g) = 0(\lambda^{\frac{n-1}{2}}). \tag{6}$$

Weyl sum



Almost no loops at any point implies a small remainder at all points

THEOREM

Let $R(\lambda, x)$ denote the remainder for the local Weyl law at x. Then

$$R(\lambda, x) = o(\lambda^{n-1}) \text{ if } |\mathcal{L}_x| = 0. \tag{7}$$

Additionally, if $|\mathcal{L}_x| = 0$ then, given $\varepsilon > 0$, there is a neighborhood $\mathcal{N}$ of x and a $\Lambda =< \infty$, both depending on ε so that

$$|R(\lambda, y)| \leq \varepsilon \lambda^{n-1}, \; y \in \mathcal{N}, \; \lambda \geq \Lambda. \tag{8}$$

The first statement was proved earlier by Safarov et al and Ivrii. Our proof was modeled on Ivrii's little $o(\lambda^{n-1})$ bound for the remainder in the Weyl law when the set of geodesics has measure zero.

Structure of first return map on loops

Conclusion: maximal eigenfunction growth forces existence of points with positive measures of loops. In the real analytic case, there must exist points from which all geodesics loop back.

But we know that even with such self-focal points, the eigenfunctions need not have maximal size. We want to understand the structure of the first return maps, i.e. whether the geodesics are closed or not.

Complication: there might exist many – infinitely many? – such self-focal points.

Ergodic theory of first return map

We cannot prove that maximal eigenfunction growth forces existence of a pole. But we can prove that the first return map preserves an L^1 invariant measure– which might imply the point is a pole (?). Outline of proof:

- $|\varphi_j(x)|^2$ can only achieve the maximum possible size at or very near a self-focal point.
- If the first return map has no invariant L^1 measures at the self-focal point, then $|\varphi_j(x)|^2$ cannot achieve the maximum possible size right at the self-focal point.
- Using the smoothness of the remainder in Weyl's law, we also prove that if it does not have the maximal order of magnitude at the self-focal point, then it cannot be maximally large very near it either.

More detailed summary

- There only exist a finite number M_T of self-focal points p with $\Phi_p \neq Id$ and first return time $\leq T$ on $S_p^* M$ in the real analytic case.
- Ergodic theory shows that if Φ_p has no L^2-invariant function, then $R(\lambda, p) = o(\lambda^{m-1})$. Thus, no invariant L^1 measures implies small remainders at self-focal points.
- Oscillatory integral analysis shows that $R(\lambda, x) = o(\lambda^{m-1})$ uniformly if $r(x, p) \geq \lambda^{-\frac{1}{2}} \log \lambda$ for all self-focal points p.
- Remaining step: $R(\lambda, x) = o(\lambda^{m-1})$ uniformly if $r(x, p) \leq \lambda^{-\frac{1}{2}} \log \lambda$ assuming that $R(\lambda, p) = o(\lambda^{m-1})$.

More details for the experts

Let $\hat{\rho} \in C_0^\infty$ be an even function with $\hat{\rho}(0) = 1$, $\rho(\lambda) > 0$ for all $\lambda \in \mathbb{R}$, and $\hat{\rho}_T(t) = \hat{\rho}(\frac{t}{T})$.

The classical cosine Tauberian method to determine Weyl asymptotics + remainder is to study

$$\rho_T * dN(\lambda, x) = a_0(x)\lambda^{n-1} + \lambda^{n-1} R(\lambda, x, T), \tag{9}$$

where $a_0(x)$ is a smooth density ($=$ a constant C_m in our case). We use large $T \iff$ long time behavior of the geodesic flow.

Oscillatory integral formula

By the usual parametrix construction: there exist phases $\tilde{t}_j$ and amplitudes such that

$$\rho_T * dN(\lambda, x) = \int_{\mathbb{R}} \hat{\rho}(\frac{t}{T}) e^{i\lambda t} U(t, x, x) dt$$

$$\simeq \lambda^{n-1} \sum_j \int_{S_x^* M} e^{i\lambda \tilde{t}_j(x,\xi)} \left(\hat{\rho}(\frac{t}{T}) a_{j0} \right) |d\xi| + O(\lambda^{n-2}),$$

where the remainder is uniform in x.

Uses long time parametrix for $U(t) = e^{it\sqrt{\Delta}}$, polar coordinates in $T^* M$, and stationary phase in $dtdr$. $\tilde{t}_j$ is the value of the phase $\varphi_j(t, x, x, \xi)$ at the critical point.

Perron-Frobenius operator

Following Safarov, at a self-focal point x define

$$U_x : L^2(S_x^* M, d\mu_x) \to L^2(S_x^* M, d\mu_x), \quad U_x f(\xi) := \ f(\Phi_x(\xi))\sqrt{J_x(\xi)}. \tag{10}$$

Here, J_x is the Jacobian of the map Φ_x, i.e. $\Phi_x^*|d\xi| = J_x(\xi)|d\xi|$.
Also define

$$U_x^{\pm}(\lambda) = e^{i\lambda T_x^{\pm}} U_x^{\pm}. \tag{11}$$

Pointwise Weyl asymptotics: Safarov pre-trace formula

If $\hat{\rho} = 0$ in a neighborhood of $t = 0$ then

$$\rho' * N(\lambda, x) = \lambda^{n-1} \sum_{k \in \mathbb{Z} \setminus 0} \int_{\mathcal{L}_x} \hat{\rho}(kT(\xi)) \overline{U_x(\lambda)^k \cdot 1} d\xi + o_x(\lambda^{n-1}).$$

Here, $\mathcal{L}_x = \{\xi \in S^*_x M : \exp_x \xi = x\}$ is the set of loops at x. Recall that

$$U_x f(\xi) = f(\Phi_x(\xi)) \sqrt{J_\Phi},$$

and that $U_x(\lambda) f = e^{i\lambda \tilde{t}(x,\xi)} U_x f$.
In the real analytic case, $\mathcal{L}_x = S^*_x M$ or has measure zero. In the latter case the integral is zero. The remainder is NON-UNIFORM in x, so we cannot use this directly and must study its proof.

Ergodic theory at the self-focal points

Here is the main result showing that $R(\lambda, x)$ is small at the self-focal points if there do not exist invariant L^2 functions.

PROPOSITION

Assume that U_x has no invariant L^2 function for any x. Then, for all $\eta > 0$, there exists T independent of x so that for every self-focal point x,

$$\frac{1}{T}\left|\int_{\mathcal{L}_x}\sum_{k=1}^{\infty}\hat{\rho}\left(\frac{T_x^{(k)}(\xi)}{T}\right)U_x^k\cdot 1|d\xi|\right| \leq \eta. \qquad (12)$$

Real analyticity implies the number of self-focal points with twisted Φ_x is finite

Two key points:

- As mentioned above, in the real analytic case, there can only exist a finite number $M(T)$ of self-focal points with return time $\leq T$ if there are no points with $\Phi_x = Id$. Thus, the ergodic theory is uniform! But it only applies at self-focal points. This is the only place that we really use real analyticity.

- Far from focal points, the measure of "almost critical points" (i.e. loops) is small and we can neglect these.

Uniform $o(\lambda^{n-1})$ outside of small balls around self-focal points

. We surround each by a ball $B_\delta(x_j)$ with δ to be chosen later. In $M\backslash\bigcup_{j=1}^{M} B_\delta(x_j)$, there do not exist almost any critical directions nor almost critical directions. For sufficiently large T (the first return time) and for sufficiently small δ, the remainders $R(\lambda, x)$ are uniformly small in this set. This just follows from the fact that there are almost no critical points.

Summing up

- No eigenfunction $\varphi_j(x)$ can be maximally large at a point x which is $\geq \lambda_j^{-\frac{1}{2}} \log \lambda_j$ away from the self-focal points.
- When there are no invariant measures, it also cannot be large at a self-focal point.
- Using the jump of the remainder, we also see that if it is not large at a self-focal point, it cannot be maximally large very near it either.

Final remarks

- We think we have used as much of the spectral theory as possible. To improve the result, we need better dynamical results– does having L^2 eigenfunctions of the first return map Φ_x imply that the geodesics from x are closed?
- There might not exist real analytic (M, g) of dim > 3 with self-focal points and for which Φ_x has L^2 eigenfunctions and which are twisted, i.e. $\Phi_x \neq Id$.
- How to approach maximal growth of L^p norms for small p? We cannot just study remainder terms in the Weyl law.