p-adic Banach space representations of p-adic groups

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Part I
Background from p-adic functional analysis

Throughout this course K is a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, $|\ |$ denotes the absolute value on K, $o \subseteq K$ the ring of integers, $\pi \in o$ a prime element, and $k := o/\pi o$ the residue class field.

1 Locally convex K-vector spaces

Let V be a K-vector space throughout this section. The vector space V in particular is an o-module so that we can speak about o-submodules of V.

Definition 1.1. *A lattice L in V is an o-submodule which satisfies the condition that for any vector $v \in V$ there is a nonzero scalar $a \in K^\times$ such that $av \in L$.*

Let $(L_j)_{j \in J}$ be a nonempty family of lattices in the K-vector space V such that we have:

(lc1) For any $j \in J$ and any $a \in K^\times$ there exists a $k \in J$ such that $L_k \subseteq aL_j$;

(lc2) for any two $i, j \in J$ there exists a $k \in J$ such that $L_k \subseteq L_i \cap L_j$.

The second condition implies that the intersection of two subsets $v + L_i$ and $v' + L_j$ either is empty or contains a subset of the form $w + L_k$. This means that the subsets $v + L_j$ for $v \in V$ and $j \in J$ form the basis of a topology on V which will be called the *locally convex topology on V defined by the family (L_j)*. For any vector $v \in V$ the subsets $v + L_j$, for $j \in J$, form a fundamental system of open and closed neighborhoods of v in this topology.

Definition 1.2. *A locally convex K-vector space is a K-vector space equipped with a locally convex topology.*

Since on a nonzero K-vector space the scalar multiplication cannot be continuous for the discrete topology we see that the discrete topology is not locally convex.

Definition 1.3. *A subset $B \subseteq V$ in a locally convex K-vector space V is called bounded if for any open lattice $L \subseteq V$ there is an $a \in K$ such that $B \subseteq aL$.*

It is almost immediate that any finite set is bounded, and that any finite union of bounded subsets is bounded.

Lemma 1.4. *Let $B \subseteq V$ be a bounded subset in a locally convex K-vector space V; then the closure of the o-submodule of V generated by B is bounded.*

Proof. Let $L \subseteq V$ be an open lattice and $a \in K$ such that $B \subseteq aL$. Since aL is a closed o-submodule it necessarily contains the closed o-submodule generated by B. $\square$

2 Banach spaces

A (nonarchimedean) norm $\| \ \|$ on a K-vector space V is a function $\| \ \| : V \longrightarrow \mathbb{R}$ such that

i. $\|av\| = |a| \cdot \|v\|$ for any $a \in K$ and $v \in V$,

ii. $\|v + w\| \leq \max(\|v\|, \|w\|)$ for any $v, w \in V$, and

iii. $\|v\| = 0$ implies that $v = 0$.

On the one hand, a normed K-vector space $(V, \| \ \|)$ will always be considered as a metric space with respect to the metric $d(v, w) := \|v - w\|$. It therefore is a Hausdorff topological K-vector space. On the other hand, this metric topology also is the locally convex topology given by the family of lattices $(\pi^n L)_{n \geq 0}$ where $L := \{v \in V : \|v\| \leq 1\}$ denotes the unit ball in $(V, \| \ \|)$.

Definition 2.1. *A locally convex K-vector space is called a K-Banach space if its topology can be defined by a norm and if the corresponding metric space is complete.*

We mention the following two facts.

Proposition 2.2. *Assume V to be a Hausdorff locally convex K-vector space; then the topology of V can be defined by a norm if and only if there is a bounded open lattice in V.*

Proposition 2.3. *The only locally convex and Hausdorff topology on a finite dimensional vector space K^n is the one defined by the norm $\|(a_1, \ldots, a_n)\| := \max_{1 \leq i \leq n} |a_i|$.*

Examples:

1. Let X be any set; then

$$\ell^\infty(X) := \text{all bounded functions } \phi : X \to K$$

with pointwise addition and scalar multiplication and the norm

$$\|\phi\|_\infty := \sup_{x \in X} |\phi(x)|$$

is a K-Banach space. The following vector subspaces are closed and therefore Banach spaces in their own right:

- $c_0(X) :=$ subspace of all $\phi \in \ell^\infty(X)$ such that for any $\epsilon > 0$ there are at most finitely many $x \in X$ with $|\phi(x)| \geq \epsilon$;
  e.g., $c_0(\mathbb{N})$ is the space of all zero sequences in K.
- $C(X) := \{$all continuous functions $\phi : X \to K\}$ provided X is a compact topological space.

2. Let $L \subseteq K$ be a complete subfield, $r \in |L^\times|$, and $a \in L^n$ a fixed point. Let $B := B_r(a) := \{x \in L^n : \|x - a\| \leq r\}$ denote the closed polydisk of radius r around a. By the first example the K-vector space $\mathcal{A}_K(B)$ of all power series

$$f(x) = f(x_1, \ldots, x_n) = \sum_{\underline{i}} c_{\underline{i}} (x - a)^{\underline{i}} \qquad \text{with } c_{\underline{i}} \in K \text{ and } \lim_{|\underline{i}| \to \infty} |c_{\underline{i}}| r^{|\underline{i}|} = 0$$

is a Banach space with respect to the norm $\|f\| = \max_{\underline{i}} |c_{\underline{i}}| r^{|\underline{i}|}$. Here $\underline{i} = (i_1, \ldots, i_n)$ is a multi-index, $|\underline{i}| := i_1 + \ldots + i_n$, and $(x-a)^{\underline{i}} := (x_1 - a_1)^{i_1} \cdot \ldots \cdot (x_n - a_n)^{i_n}$. In fact, $\mathcal{A}_K(B)$ is the algebra of all K-valued *rigid-analytic* functions on B, i.e., all power series which converge for any point of B with coordinates in an algebraic closure of L.

A very basic fact about Banach spaces is the following so called *open mapping theorem.*

Proposition 2.4. *Every surjective continuous linear map $f : V \longrightarrow W$ between two Banach spaces is open.*

Corollary 2.5. *i. Let V be a Banach space, and let $U \subseteq V$ be a closed vector subspace; then V/U with the quotient topology is a Banach space as well.*

ii. Any continuous linear bijection between two K-Banach spaces is a topological isomorphism.

3 Vector spaces of linear maps

In this section V and W will denote two locally convex K-vector spaces. It is straightforward to see that

$$\mathcal{L}(V, W) := \{f : V \longrightarrow W \text{ continuous and linear}\}$$

again is a K-vector space. We describe a general technique to construct locally convex topologies on $\mathcal{L}(V, W)$. For this we choose a nonempty family $\mathcal{B}$ of bounded subsets of V which is closed under finite unions. For any $B \in \mathcal{B}$ and any open lattice $M \subseteq W$ the subset

$$\mathcal{L}(B, M) := \{f \in \mathcal{L}(V, W) : f(B) \subseteq M\}$$

is a lattice in $\mathcal{L}(V, W)$: It is clear that $\mathcal{L}(B, M)$ is an o-submodule. If $f \in \mathcal{L}(V, W)$ is any continuous linear map then, by the boundedness of B, there has to be an $a \in K^\times$ such that $B \subseteq af^{-1}(M)$. This means that $f(B) \subseteq aM$ or equivalently that $a^{-1}f \in \mathcal{L}(B, M)$.

The family of all these lattices $\mathcal{L}(B, M)$ is nonempty and satisfies (lc1) and (lc2). The corresponding locally convex topology is called the $\mathcal{B}$-topology. We write

$$\mathcal{L}_{\mathcal{B}}(V, W) := \mathcal{L}(V, W) \qquad \text{equipped with the } \mathcal{B}\text{-topology.}$$

Examples:

1. Let $\mathcal{B}$ be the family of all finite subsets of V. The corresponding $\mathcal{B}$-topology is called the *weak topology* or the topology of pointwise convergence. We write $\mathcal{L}_s(V, W) := \mathcal{L}_{\mathcal{B}}(V, W)$.

2. Let $\mathcal{B}$ be the family of all bounded subsets in V. The corresponding $\mathcal{B}$-topology is called the *strong topology* or the topology of bounded convergence. We write $\mathcal{L}_b(V, W) := \mathcal{L}_{\mathcal{B}}(V, W)$.

If W is Hausdorff both locally convex vector spaces $\mathcal{L}_s(V, W)$ and $\mathcal{L}_b(V, W)$ are Hausdorff.

It is technically important to know what the bounded subsets in $\mathcal{L}_{\mathcal{B}}(V, W)$ are. There always are some obvious ones.

Definition 3.1. *A subset $H \subseteq \mathcal{L}(V, W)$ is called equicontinuous if for any open lattice $M \subseteq W$ there is an open lattice $L \subseteq V$ such that $f(L) \subseteq M$ for every $f \in H$.*

Lemma 3.2. *Every equicontinuous subset $H \subseteq \mathcal{L}(V, W)$ is bounded in every $\mathcal{B}$-topology.*

Proof. Let $\mathcal{L} \subseteq \mathcal{L}_{\mathcal{B}}(V, W)$ be any open lattice. There is an open lattice $M \subseteq W$ and a $B \in \mathcal{B}$ such that $\mathcal{L} \supseteq \mathcal{L}(B, M)$. Since H is equicontinuous we find an open lattice $L \subseteq V$ such that $f(L) \subseteq M$ for any $f \in H$. Furthermore, there is an $a \in K^{\times}$ such that $B \subseteq aL$. Hence $f(B) \subseteq aM$ for any $f \in H$ which shows that $H \subseteq a\mathcal{L}(B, M) \subseteq a\mathcal{L}$. $\square$

Proposition 3.3. *(Banach-Steinhaus) Suppose that V is a Banach space; then in $\mathcal{L}_{\mathcal{B}}(V, W)$, for any $\mathcal{B}$-topology which is finer than the weak topology, the bounded subsets coincide with the equicontinuous subsets.*

4 Dual spaces

We now specialize the content of the previous section to the case $W = K$. The vector space $V' := \mathcal{L}(V, K)$ is called the *dual space* of V; more precisely, we call the locally convex vector spaces $V'_s := \mathcal{L}_s(V, K)$ and $V'_b := \mathcal{L}_b(V, K)$ the *weak* and *strong dual*, respectively.

Proposition 4.1. *(Hahn-Banach) Let $(U, \| \ \|)$ be a normed K-vector space and $U_0 \subseteq U$ be a vector subspace; for any linear form $\ell_0 : U_0 \longrightarrow K$ such that $|\ell_0(v)| \leq \|v\|$ for any $v \in U_0$ there is a linear form $\ell : U \longrightarrow K$ such that $\ell | U_0 = \ell_0$ and $|\ell(v)| \leq \|v\|$ for any $v \in U$.*

Proof. This is proved exactly as in functional analysis over $\mathbb{C}$. For this proof one needs that the intersection of any decreasing sequence of closed balls in the field K is nonempty. In our case this is immediate from the fact that the topology of K is locally compact. (**But this no longer holds true for general nonarchimedean fields.**) $\square$

One of the important applications is the following result.

Proposition 4.2. *For any Banach space V the linear map*

$$\begin{aligned} \delta : V &\longrightarrow (V'_s)'_s \\ v &\longmapsto \delta_v(\ell) := \ell(v) \end{aligned}$$

is a continuous bijection.

The map δ in the above proposition is called a *duality map* for V. It rarely is a topological isomorphism. In contrast to this the analogous duality map $\delta : V \longrightarrow (V'_b)'_b$ is an isometry of Banach spaces but which rarely is surjective. The way out of this dilemma proceeds as follows.

Lemma 4.3. *For any Banach space $(V, \| \ \|)$ the o-submodule*

$$V^d := \{\ell \in V'_b : |\ell(v)| \leq \|v\| \text{ for any } v \in V\}$$

is compact in V'_s.

Proof. One checks that

$$\begin{aligned} V^d &\hookrightarrow \prod_{\|v\|\leq 1} o \\ \ell &\longmapsto (\ell(v))_v \end{aligned}$$

is a topological and closed embedding. As a direct product of compact spaces the right hand side is compact. □

Obviously V^d is a linear-topological and torsionfree o-module. On the other hand, if we start with a linear-topological compact and torsionfree o-module M then

$$M^d := \text{ all continuous } o\text{-linear maps } f : M \longrightarrow K$$

equipped with the norm

$$\|f\| := \max_{m\in M} |f(m)|$$

is a K-Banach space.

Proposition 4.4. *(Schikhof) The functors $V \longmapsto V^d$ and $M \longmapsto M^d$ are quasi-inverse anti-equivalences between the category of K-Banach spaces $(V, \|\ \|)$ with norm decreasing linear maps and the category of linear-topological compact and torsionfree o-modules.*

Proof. One easily checks that the o-linear map

$$\begin{aligned} M &\longrightarrow (M^d)'_s \\ m &\longmapsto \ell_m(f) := f(m) \end{aligned}$$

is a closed embedding. Using the Hahn-Banach Prop. 4.1 one then shows that this map induces a topological isomorphism $M \xrightarrow{\cong} M^{dd}$. It follows that the functor $M \longmapsto M^d$ is fully faithful. Using the structure theory of Banach spaces over K (cf. [NFA] §10) one finally establishes that this functor is essentially surjective. □

5 Distributions

For any topological space M and any locally convex K-vector space W we let $C(M, W)$ denote the space of W-valued continuous functions, with the topology of uniform convergence on compact sets. If $(V, \|\ \|_V)$ is a K-Banach space and if M is compact then $C(M, V)$, with the norm given by

$$\|f\| = \max_{x\in M} \|f(x)\|_V ,$$

is a Banach space as well.

Let $D^c(M, K)$ denote the vector space of continuous distributions on M which, by definition, is the dual to $C(M, K)$. The Dirac distributions δ_x, for $x \in M$, defined by $\delta_x(f) := f(x)$ are elements of $D^c(M, K)$. The following statement can be viewed as "integration".

Proposition 5.1. *Let W be a quasi-complete (meaning that any bounded closed subset of W is complete) and Hausdorff locally convex K-vector space, and assume M to be compact; then evaluation on Dirac distributions gives a K-linear isomorphism*

$$\mathcal{L}(D^c(M, K), W) \xrightarrow{\cong} C(M, W) .$$

If M is compact, then $D^c(M,K)_b$, equipped with the strong topology, is a Banach space for the usual dual norm

$$\|\ell\| = \sup_{0 \neq f \in C(M,K)} \frac{|\ell(f)|}{\|f\|} \ .$$

However, in the light of Prop. 4.4 another topology is more important. We know from Lemma 4.3 that the unit ball $D^c(M,K)_0$ in $D^c(M,K)_b$ is compact for the weak topology.

Definition 5.2. *Suppose that M is compact. We let $D^c(M,K)$ be the continuous dual to $C(M,K)$ equipped with the "bounded-weak" topology. This is, by definition, the finest locally convex topology such that the inclusion of the unit ball $D^c(M,K)_0$, with its weak topology, is continuous.*

We now suppose that $M = G$ is a profinite group. Then $D^c(G,K)$ becomes a topological algebra. This is most apparent through an alternative construction of $D^c(G,K)$:

(i) We have the completed group ring

$$o[[G]] = \varprojlim o[G/H]$$

where the limit is over open normal subgroups H of G. It carries the projective limit topology for which it is a torsionfree compact o-module and a topological ring.

(ii) Choose a cofinal sequence H_n of open normal subgroups of G. Any element μ of $o[[G]]$ is a projective limit of $\mu_n = \sum_{G/H_n} a_g g \in o[G/H_n]$, and any continuous function f on G may be uniformly approximated by a sequence f_n of locally constant functions that are right H_n-invariant. There is a well-defined integration pairing

$$\int_G f d\mu = \lim_{n\to\infty} \sum_{g \in G/H_n} a_g f_n(g).$$

This pairing gives a map $o[[G]] \to D^c(G,K)$, and its image can be identified with the unit ball in $D^c(G,K)$. In fact $o[[G]]$ is naturally the unit ball in $D^c(G,K)$ equipped with its weak topology.

(iii) $D^c(G,K) = K \otimes_o o[[G]]$. As stated in Def. 5.2, we give $D^c(G,K)$ the finest locally convex topology such that the inclusion of $o[[G]]$ is continuous.

Proposition 5.3. *The ring $D^c(G,K)$ is a K-algebra with a separately continuous multiplication. (Separately continuous means that the map $y \to x \cdot y$ is continuous for fixed x, and $x \to x \cdot y$ is continuous for fixed y).*

6 An example

Let $G = \mathbb{Z}_p$. The binomials

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-(n-1))}{n!} \ .$$

make sense for any p-adic integer x.

Theorem 6.1. *(Mahler) Any $f \in C(\mathbb{Z}_p, K)$ has a unique representation*

$$f = \sum_{n=0}^{\infty} T_n(f) \binom{x}{n}$$

where the coefficients $T_n(f) \in K$ go to zero as $n \to \infty$. Conversely any such series converges uniformly to a continuous function. We have $\|f\| = \max_n |T_n(f)|$.

Thm. 6.1 gives an explicit isomorphism between $C(\mathbb{Z}_p, K)$ and $c_0(\mathbb{N}_0)$. Consequently the dual $D^c(\mathbb{Z}_p, K)$, as a vector space, is the space $\ell^\infty(\mathbb{N}_0)$ of bounded sequences, and the elements of the unit ball $o[[\mathbb{Z}_p]]$ may be represented as sums

$$\mu = \sum b_n T_n \qquad \text{with } |b_n| \leq 1.$$

The linear maps T_n give the coefficients of f in the expansion of Thm. 6.1.

To compute the ring structure on $D^c(\mathbb{Z}_p, K)$ we use Fourier theory. Any distribution is determined by its values on the dense subspace of locally constant functions in $C(\mathbb{Z}_p, K)$, and the characters of finite order

$$\chi_\zeta(x) = \zeta^x \qquad \text{for } \zeta \text{ a } p\text{-power root of } 1$$

span the locally constant functions. We compute that

$$T_n(\zeta^x) = T_n(((\zeta - 1) + 1)^x) = T_n(\sum \binom{x}{i} (\zeta - 1)^i) = (\zeta - 1)^n$$

and

$$T_n * T_m(\zeta^x) = T_n^{(x)} T_m^{(y)}(\zeta^{x+y}) = T_n(\zeta^x) T_m(\zeta^y) = (\zeta - 1)^{m+n} = T_{n+m}(\zeta^x) \ .$$

It follows inductively that $T_n = T_1^n$ and so, writing $T_1 = T$, we obtain that

$$o[[\mathbb{Z}_p]] = o[[T]]$$

is the formal power series ring in one variable over o.

Part II

Banach space representations

Let V be a K-Banach space, and G a locally compact and totally disconnected group. Examples of such groups are $\mathrm{GL}_n(K)$, $\mathrm{SL}_n(K)$, and $\mathrm{PGL}_n(K)$. More generally, the group of K-valued points of any algebraic group over K is of this kind. Compact subgroups like $\mathrm{GL}_n(o)$ are of particular importance.

Definition 6.2. *A K-Banach space representation of G (on V) is a G-action by continuous linear automorphisms such that the map $G \times V \to V$ giving the action is continuous.*

Example: Assume G to be compact, and let V be $C(G, K)$. Then the (left) translation action ${}^g f(h) = f(g^{-1}h)$ is such a representation.

7 Modules over the distribution algebra

The main tool for studying Banach space representations is to make the problem algebraic by viewing the representations as modules over the distribution algebra. Suppose that G is a profinite group.

Proposition 7.1. *The G-action on any Banach space representation V extends uniquely to a separately continuous $D^c(G,K)$-module structure on V. Moreover, G-equivariant continuous linear maps extend to module homomorphisms.*

This is an application of "integration". Using the Banach-Steinhaus Prop. 3.3 one shows that $\mathcal{L}_s(V,V)$ is quasi-complete. Hence Prop. 5.1 applies and gives the "integration map"

$$C^{an}(G,\mathcal{L}_s(V,V)) \to \mathcal{L}(D^c(G,K),\mathcal{L}_s(V,V)) \ .$$

In fact it is more useful to consider, not the spaces V, but their dual spaces V'. Via the transpose action these, too, are modules for the distribution algebras.

8 The duality functor

We let $\mathrm{Ban}(K)$ denote the category of K-Banach space with continuous linear maps and $\mathrm{Ban}(K)^{\leq 1}$ the category of normed K-Banach spaces with norm decreasing linear maps. We remark that $\mathrm{Ban}(K)$ can be reconstructed from $\mathrm{Ban}(K)^{\leq 1}$ by "localization in $\mathbb{Q}$"; this simply means that, for any two Banach spaces V and W, one has

$$\mathrm{Hom}_{\mathrm{Ban}(K)}(V,W) = \mathrm{Hom}_{\mathrm{Ban}(K)^{\leq 1}}(V,W) \otimes \mathbb{Q} \ .$$

By Schikhof's Prop. 4.4 the functor

$$\begin{aligned} \mathrm{Ban}(K)^{\leq 1} &\xrightarrow{\sim} \mathcal{M}(o) \\ V &\longmapsto V^d = \{\ell \in V' : \|\ell\| \leq 1\} \end{aligned}$$

is an anti-equivalence of categories where $\mathcal{M}(o)$ denotes the category of linear-topological compact and torsionfree o-modules.

Let now G be a profinite group. In section 5 we have explained the identification

$$D^c(G,K) = K \otimes_o o[[G]]$$

It is an important point that the completed group ring $o[[G]]$ as an o-module is linear-topological, compact, and torsionfree, i.e., lies in the category $\mathcal{M}(o)$.

Another important technical point (as already indicated in Def. 5.2) is to consider on $\mathcal{L}(V,W)$, for two Banach spaces V and W, not the natural Banach space topology but the bounded weak topology which is the finest locally convex topology which restricts to the weak topology on some open lattice in $\mathcal{L}(V,W)$. We write $\mathcal{L}_{bs}(V,W)$ in this case. Observe that by the above equivalence of categories we have a natural linear isomorphism

$$\mathcal{L}(V,W) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{M}(o)}(W^d,V^d) \otimes \mathbb{Q} \ .$$

On the other hand, for two modules M and N in $\mathcal{M}(o)$, the natural topology to consider on $\mathrm{Hom}_{\mathcal{M}(o)}(M,N)$ is the topology of compact convergence.

Proposition 8.1. *The bounded weak topology on $\mathcal{L}(V,W)$ induces the topology of compact convergence on* $\mathrm{Hom}_{\mathcal{M}(o)}(W^d, V^d)$.

Let now $\mathrm{Ban}_G(K)$ denote the category of K-Banach space representations of G with continuous linear and G-equivariant maps. By Prop. 7.1 (which was based on Prop. 5.1), to give a continuous representation of G on the Banach space V is the same as to give a continuous algebra homomorphism

$$K \otimes_o o[[G]] \longrightarrow \mathcal{L}_s(V,V) \ .$$

Because $o[[G]]$ is compact this furthermore is the same as to give a continuous algebra homomorphism

$$K \otimes_o o[[G]] \longrightarrow \mathcal{L}_{bs}(V,V) \ .$$

Remark 8.2. *On any Banach space representation V of G there is a G-invariant defining norm.*

Proof. Let $L \subseteq V$ be any bounded open lattice. By the continuity of the G-action there is an open normal subgroup $H \subseteq G$ and an open lattice $L_0 \subseteq L$ such that $H \cdot L_0 \subseteq L$. The G-invariant intersection

$$L_1 := \bigcap_{g \in G} gL$$

contains the finite intersection $\bigcap_{g \in G/H} gL_0$ and therefore is an open lattice in V. The corresponding "gauge"

$$p_{L_1}(v) := \inf_{v \in aL_1} |a|$$

is a G-invariant norm on V which defines the topology. □

Going back to the above discussion we therefore see, by using Prop. 8.1, that to give a continuous representation of G on the Banach space V is the same as to give a continuous module structure

$$o[[G]] \times V^d \longrightarrow V^d \ .$$

If we let $\mathcal{M}(o[[G]])$ denote the category of all continuous (left) $o[[G]]$-modules such that the underlying o-module lies in $\mathcal{M}(o)$ then we have established the following result.

Theorem 8.3. *The functor*

$$\begin{aligned} \mathrm{Ban}_G(K) &\xrightarrow{\sim} \mathcal{M}(o[[G]]) \otimes \mathbb{Q} \\ V &\longmapsto V' = V^d \otimes \mathbb{Q} \end{aligned}$$

is an anti-equivalence of categories.

9 Admissibility

In order to motivate what is to come we want to mention the following two pathologies of Banach space representations.

– In general there exist non-isomorphic topologically irreducible Banach space representations V and W of G for which nevertheless there is a nonzero G-equivariant continuous linear map $V \longrightarrow W$.

– Even such a simple commutative group like $G = \mathbb{Z}_p$ has infinite dimensional topologically irreducible Banach space representations.

It is clear that in order to avoid such pathologies we have to impose an additional finiteness condition on our Banach space representations. For this purpose we now assume that G is a compact p-adic Lie group (cf. [pALG]). We have the following fundamental result by Lazard in [Laz].

Theorem 9.1. *The rings $o[[G]]$ and $K \otimes_o o[[G]]$ are noetherian.*

The most natural finiteness condition one can impose on modules over a ring is to be finitely generated. If the ring is noetherian then the finitely generated modules over it form a nice abelian category. In view of Thm. 8.3 we therefore propose the following definition.

Definition 9.2. *A K-Banach space representation V of G is called admissible if the dual V' as a $K \otimes_o o[[G]]$-module is finitely generated.*

We let $\mathrm{Ban}_G^a(K)$ denote the full subcategory in $\mathrm{Ban}_G(K)$ of all admissible Banach space representations. We also let $\mathcal{M}_{fg}(o[[G]])$, resp. $\mathcal{M}_{fg}(K \otimes_o o[[G]])$, be the category of finitely generated and o-torsionfree $o[[G]]$-modules, resp. finitely generated $K \otimes_o o[[G]]$-modules. We remark that because of the anti-involution $g \longmapsto g^{-1}$ there is no need to distinguish between left and right modules.

Since the ring $o[[G]]$ is compact and noetherian the following facts are more or less exercises.

Lemma 9.3. *i. A finitely generated $o[[G]]$-module M carries a unique Hausdorff topology - its canonical topology - such that the action $o[[G]] \times M \longrightarrow M$ is continuous.*

ii. Any submodule of a finitely generated $o[[G]]$-module is closed in the canonical topology

iii. Any $o[[G]]$-linear map between two finitely generated $o[[G]]$-modules is continuous for the canonical topologies.

It follows that equipping a module in $\mathcal{M}_{fg}(o[[G]])$ with its canonical topology induces a fully faithful embedding

$$\mathcal{M}_{fg}(o[[G]]) \longrightarrow \mathcal{M}(o[[G]]) .$$

This then in turn induces a fully faithful embedding

$$\mathcal{M}_{fg}(K \otimes_o o[[G]]) \longrightarrow \mathcal{M}(o[[G]]) \otimes \mathbb{Q} .$$

Together with Thm. 8.3 we obtain the following.

Theorem 9.4. *The functor*

$$\begin{aligned} \mathrm{Ban}_G^a(K) &\xrightarrow{\sim} \mathcal{M}_{fg}(K \otimes_o o[[G]]) \\ V &\longmapsto V' \end{aligned}$$

is an anti-equivalence of categories.

In particular the category of admissible K-Banach space representations of G is abelian and is completely algebraic in nature. One also deduces easily the following consequences.

Corollary 9.5. *i. The functor $V \longmapsto V'$ induces a bijection between the set of isomorphism classes of topologically irreducible admissible K-Banach space representations of G and the set of isomorphism classes of simple $K \otimes_o o[[G]]$-modules.*

ii. Any nonzero G-equivariant continuous linear map between two topologically irreducible admissible K-Banach space representations of G is an isomorphism.

Our category $\mathrm{Ban}_G^a(K)$ in particular avoids the pathology 1. But also the pathology 2 disappears. We claim that for the group $G = \mathbb{Z}_p$ any topologically irreducible admissible Banach space representation is finite dimensional over K. By the above corollary this reduces to the assertion that any simple $K \otimes_o o[[\mathbb{Z}_p]]$-module is finite dimensional which is equivalent to any maximal ideal in $K \otimes_o o[[\mathbb{Z}_p]]$ being of finite codimension. But we know from section 6 that $K \otimes_o o[[\mathbb{Z}_p]] = K \otimes_o o[[T]]$ is a power series ring in one variable. By Weierstrass preparation any ideal in this ring is generated by a polynomial.

Another indication that admissibility is the correct concept is the fact that it can be characterized in an intrinsic way.

Proposition 9.6. *A K-Banach space representation V of G is admissible if and only if there is a G-invariant bounded open lattice $L \subseteq V$ such that, for any open normal subgroup $H \subseteq G$, the k-vector space $(L/\pi L)^H$ of H-invariant elements in $L/\pi L$ is finite dimensional.*

Proof. Let us first assume that V' is finitely generated over $K \otimes_o o[[G]]$. There is then a finitely generated $o[[G]]$-submodule $M \subseteq V'$ such that $V' = K \otimes_o M$. After equipping M with its canonical topology we have $V = M^d$. Moreover $L := \mathrm{Hom}_o^{cont}(M, o)$ is a G-invariant bounded open lattice in V. One checks that $L/\pi L = \mathrm{Hom}_o^{cont}(M, k)$ and hence that

$$(1) \qquad (L/\pi L)^H = \mathrm{Hom}_o^{cont}(M, k)^H = \mathrm{Hom}_o^{cont}(M/I_H M, k)$$

where I_H denotes the kernel of the projection map $o[[G]] \longrightarrow k[G/H]$. Hence $(L/\pi L)^H$ is finite dimensional.

On the other hand fix now an open normal subgroup $H \subseteq G$ which is pro-p (there is a fundamental system of such) and let $L \subseteq V$ be a G-invariant bounded open o-submodule such that $(L/\pi L)^H$ is finite dimensional. The technical point is to check that the G-invariant o-submodule $M := \{\ell \in V' : |\ell(v)| \leq 1 \text{ for any } v \in L\}$ in V'_s is compact. Since L is bounded we have $V' = K \otimes_o M$. So the identities (1) apply correspondingly and we obtain that $\mathrm{Hom}_o^{cont}(M/I_H M, k)$ is finite dimensional. But since I_H is finitely generated as a right ideal the submodule $I_H M$ is the image of finitely many copies $M \times \ldots \times M$ under a continuous map and hence is closed in M. By Pontrjagin duality applied to the compact o-module $M/I_H M$ the latter is finite dimensional over k. The Nakayama lemma over $o[[G]]$ (observe that I_H is contained in the Jacobson radical of $o[[G]]$) finally says that M is finitely generated over $o[[G]]$ and hence that V' is finitely generated over $K \otimes_o o[[G]]$. □

The above proof shows that the condition of this proposition in fact only needs to be checked for a single open normal subgroup $H \subseteq G$ which is pro-p. We also point out that the condition in this proposition is exactly analogous to Harish Chandra's admissibility condition for smooth representations. In fact, it says that $L/\pi L$ is an admissible smooth representation of G over the residue class field k.

We now drop the condition that the group G is compact. A Banach space representation of an arbitrary finite dimensional p-adic Lie group G is called admissible if it is admissible as a representation of every compact open subgroup $H \subseteq G$. Obviously this again gives rise to an abelian category.

Part III

The p-adic Satake isomorphism

All what follows can be done for a general split reductive group over some finite extension of $\mathbb{Q}_p$. But, for simplicity, we restrict to the case of the group $G := \mathrm{GL}_{d+1}(\mathbb{Q}_p)$ for some integer $d \geq 1$. It contains $G_0 := \mathrm{GL}_{d+1}(\mathbb{Z}_p)$ as an open and maximal compact subgroup.

Let $\mathrm{val} : K^\times \longrightarrow \mathbb{R}$ be the unique additive valuation such that $\mathrm{val}(\mathbb{Q}_p^\times) = \mathbb{Z}$, and put $|a|_p := p^{-\mathrm{val}(a)}$.

10 The smooth spherical principal series

The traditional local Langlands theory considers *smooth* G-representations. This is a linear G-action on a K-vector space V such that the stabilizer in G of each vector in V is an open subgroup. In this section we recall the classification of those irreducible smooth G-representations which have a nonzero G_0-fixed vector.

As a kind of universal object we have the smooth representation

$$\mathrm{ind}_{G_0}^G(1) := \text{ all finitely supported functions } f : G/G_0 \longrightarrow K$$

on which G acts by left translations. The characteristic function ϵ_0 of G_0 is a G_0-fixed vector which generates $\mathrm{ind}_{G_0}^G(1)$ as a G-representation. Hence any irreducible quotient V of $\mathrm{ind}_{G_0}^G(1)$ must have a nonzero G_0-fixed vector. On the other hand, the Frobenius reciprocity

$$\mathrm{Hom}_G(\mathrm{ind}_{G_0}^G(1), V) = \mathrm{Hom}_{G_0}(1, V) = V^{G_0} := \text{ } G_0\text{-fixed vectors in } V$$

tells us that any irreducible smooth G-representation with a nonzero G_0-fixed vector must be a quotient of $\mathrm{ind}_{G_0}^G(1)$.

The determination of the irreducible quotients of $\mathrm{ind}_{G_0}^G(1)$ is based on the study of its endomorphism ring.

Definition 10.1. *The Satake-Hecke algebra is the endomorphism ring*

$$\mathcal{H}(G, G_0) := \mathrm{End}_G(\mathrm{ind}_{G_0}^G(1)) \ .$$

Any character $\chi : \mathcal{H}(G, G_0) \longrightarrow K$ leads to the quotient G-representation

$$V_\chi := K_\chi \otimes_{\mathcal{H}(G,G_0)} \mathrm{ind}_{G_0}^G(1)$$

which we might hope to be irreducible.

Lemma 10.2. *The map*

$$\begin{aligned} \mathcal{H}(G, G_0) &\xrightarrow{\cong} \text{all finitely supported functions } \psi : G_0\backslash G/G_0 \longrightarrow K \\ A &\longmapsto \psi_A(g) := A(\epsilon_0)(g^{-1}) \end{aligned}$$

is bijective.

Proof. With ϵ_0 also $A(\epsilon_0)$ must be G_0-invariant and hence a function on double cosets. Since ϵ_0 generates $\mathrm{ind}_{G_0}^G(1)$ the operator A is determined by its value $A(\epsilon_0)$. The surjectivity is left as an exercise. $\square$

To explain Satake's actual computation of the algebra $\mathcal{H}(G, G_0)$ we need further notation. Let $T \subseteq G$ be the torus of diagonal matrices and $N \subseteq P \subseteq G$ be the subgroup of unipotent lower triangular and all lower triangular matrices, respectively. Hence $P = TN$. The Weyl group W of G is the subgroup of permutation matrices. It normalizes T by permuting the diagonal entries. We put $T_0 := G_0 \cap T$ and $N_0 := G_0 \cap N$. The quotient $\Lambda := T/T_0$ is a free abelian group of rank $d+1$ and can naturally be identified with the cocharacter group $X_*(T)$ of T. Let $\lambda : T \longrightarrow \Lambda$ denote the projection map. The conjugation action of W on T induces a W-action Λ; they are denoted by $t \longmapsto {}^w t$ and $\lambda \longmapsto {}^w\lambda$, respectively. We also need the torus T' dual to T. Its K-valued points are given by

$$T'(K) := \mathrm{Hom}(\Lambda, K^\times) \ .$$

The group ring $K[\Lambda]$ of Λ over K naturally identifies with the ring of algebraic functions on the torus T'. It carries an obvious induced W-action.

Theorem 10.3. *(Satake) The map*

$$\begin{aligned} S_1 : \mathcal{H}(G, G_0) &\xrightarrow{\cong} K[\Lambda]^{W,\gamma_1} \\ \psi &\longmapsto \sum_{t \in T/T_0} \Big(\sum_{n \in N/N_0} \psi(tn) \Big) \lambda(t) \end{aligned}$$

is an isomorphism of K-algebras; here the W-invariants on the group ring $K[\Lambda]$ are formed with respect to a certain twisted W-action, see below.

It follows that the algebra $\mathcal{H}(G, G_0)$ is commutative and that its K-valued characters are parametrized by the γ_1-twisted W-orbits in the dual torus $T'(K) = \mathrm{Hom}(\Lambda, K^\times)$ (at least in the limit over all K). To make good for the twist we let $\lambda_i \in \Lambda$ denote the coset of the diagonal matrix having p at the place i and 1 elsewhere on the diagonal and we use the embbeding

$$\begin{aligned} \mathrm{Hom}(\Lambda, K^\times) &\longrightarrow \mathrm{GL}_{d+1}(K) \\ \zeta &\longmapsto \begin{pmatrix} p^0\zeta(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & p^d\zeta(\lambda_{d+1}) \end{pmatrix} \end{aligned}$$

of the dual torus $T'(K)$ into the dual group $G'(K) := \mathrm{GL}_{d+1}(K)$. The twisted W-action on the left side corresponds to the usual conjugation by W on the right side.

Corollary 10.4. *In the limit over K the characters $\chi = \chi_{\mathcal{O}}$ of $\mathcal{H}(G, G_0)$ correspond bijectively to the semisimple conjugacy classes $\mathcal{O}$ in the dual group G'.*

Proposition 10.5. *Each G-representation $V_{\chi_{\mathcal{O}}}$ is of finite length and has a unique irreducible quotient $V_{\mathcal{O}}$. These $V_{\mathcal{O}}$ are pairwise nonisomorphic and constitute, up to isomorphism, all irreducible smooth G-representations which have a nonzero G_0-fixed vector.*

Why are we interested in this part of the smooth theory? Quite obviously the vector space $\mathrm{ind}_{G_0}^G(1)$ carries the G-invariant sup-norm on functions. The corresponding completion then is a K-Banach space representation $B_{G_0}^G(1)$ of G. Similarly the Satake-Hecke algebra $\mathcal{H}(G, G_0)$ carries, via Lemma 10.2, a submultiplicative sup-norm as well and therefore completes to a K-Banach algebra $\mathcal{B}(G, G_0)$. One checks that in fact

$$\mathrm{End}_G^{cont}(B_{G_0}^G(1)) = \mathcal{B}(G, G_0) \ .$$

So using a continuous character $\chi : \mathcal{B}(G, G_0) \longrightarrow K$ we may introduce the quotient Banach space representation

$$B_{1,\chi} := K_\chi \widehat{\otimes}_{\mathcal{B}(G,G_0)} B^G_{G_0}(1)$$

of G. It has the additional feature that it is "unitary", i. e., it carries a G-invariant norm. But it must be noted immediately that the obligation to use the completed tensor product in the definition of $B_{1,\chi}$ causes a **fundamental problem**: There is no a priori reason why $B_{1,\chi}$ should be nonzero! Hence the only thing we can do in general is to compute the Banach algebra $\mathcal{B}(G, G_0)$. But before we do this we want to enrich the situation.

11 Rational representations of GL_{d+1}

Let E be the (finite dimensional) $\mathbb{Q}_p$-vector space which underlies a rational representation of the $\mathbb{Q}_p$-algebraic group GL_{d+1}. The word "rational" means that the action is given by polynomial functions in the matrix entries. The general theory says that for the torus T any such E decomposes into eigenspaces where the corresponding eigencharacters are algebraic. An algebraic character of T is a homomorphism of the form

$$\begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_{d+1} \end{pmatrix} \longmapsto \prod_{i=1}^{d+1} t_i^{a_i}$$

with $(a_1, \ldots, a_{d+1}) \in \mathbb{Z}^{d+1}$. Suppose that E is irreducible. Then among all the eigencharacters of T occurring in E there is a unique dominant one ξ, i. e. one for which the corresponding integers satisfy $a_1 \leq \ldots \leq a_{d+1}$. We write $E = E_\xi$ which is justified by the following main result of the theory (cf. [Hum]).

Theorem 11.1. *The map $\xi \longmapsto E_\xi$ is a bijection between the set of all dominant algebraic characters of T and the set of isomorphism classes of irreducible rational representations of* GL_{d+1}.

12 The p-adic Satake isomorphism

We fix a dominant algebraic character ξ of T and let $E = E_\xi$ be the corresponding rational GL_{d+1}-representation. The tensor product

$$\mathrm{ind}^G_{G_0}(1) \otimes_{\mathbb{Q}_p} E = \mathrm{ind}^G_{G_0}(E|G_0)$$

is, for the diagonal action, a G-representation over K but which no longer is smooth. On the other hand, E contains a rational $\mathrm{GL}_{d+1}/\mathbb{Z}_p$-lattice. The corresponding norm $\| \ \|$ on E is G_0-invariant and will be fixed in the following. This allows us, by viewing the elements in the above tensor product as certain compactly supported functions $f : G \longrightarrow E$ to introduce a G-invariant sup-norm $\| \ \|$ on $\mathrm{ind}^G_{G_0}(E|G_0)$. The corresponding completion $B^G_{G_0}(E)$ is a K-Banach space representation of G.

As far as the commuting algebra is concerned the situation has not changed (before completion) by the following result. It is a formal consequence of the fact that E is even absolutely irreducible for the derived action of the Lie algebra of G.

Lemma 12.1. *Let $v \in E$ be any nonzero vector and consider the map $\rho_v : G \longrightarrow E$ which sends g to $g^{-1}v$; the map*

$$\begin{aligned} \mathrm{End}_G(\mathrm{ind}_{G_0}^G(E|G_0)) &\xrightarrow{\cong} \mathcal{H}(G, G_0) \\ A &\longmapsto \frac{A(\epsilon_0 \rho_v)}{\rho_v} \end{aligned}$$

is a well defined isomorphism of K-algebras which does not depend on the choice of v.

But the operator norm $\|\psi\|_\xi$ of an element $\psi \in \mathcal{H}(G, G_0)$ viewed as an endomorphism of the normed vector space $\mathrm{ind}_{G_0}^G(E|G_0)$ is different from the sup-norm considered in the last section. Instead we have

$$\|\psi\|_\xi = \sup_{g \in G}\{|\psi(g)| \cdot \text{operator norm of } g \text{ on } E\} \ .$$

The $\| \ \|_\xi$-completion $\mathcal{B}_\xi(G, G_0)$ of $\mathcal{H}(G, G_0)$ is a K-Banach algebra which satisfies

$$\mathrm{End}_G^{cont}(B_{G_0}^G(E)) = \mathcal{B}_\xi(G, G_0) \ .$$

As before, for a continuous character $\chi : \mathcal{B}_\xi(G, G_0) \longrightarrow K$ we may introduce the unitary quotient Banach space representations

$$B_{\xi,\chi} := K_\chi \widehat{\otimes}_{\mathcal{B}_\xi(G,G_0)} B_{G_0}^G(E)$$

of G.

To actually compute $\| \ \|_\xi$ we introduce the "antidominant" submonoid

$$T^{--} := \{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_{d+1} \end{pmatrix} : |t_1| \le \ldots \le |t_{d+1}|\}$$

of T. The Cartan decomposition says that T^{--}/T_0 is a system of representatives for the double cosets of G_0 in G.

Lemma 12.2. *For any $\psi \in \mathcal{H}(G, G_0)$ we have*

$$\|\psi\|_\xi := \sup_{t \in T^{--}} |\psi(t)\xi(t)| \ .$$

This is proved by computing the operator norm of any $t \in T^{--}$ on E in terms of the eigenspace decomposition of E with respect to T.

Next we introduce the adapted Satake map

$$\begin{aligned} S_\xi : \mathcal{H}(G, G_0) &\longrightarrow K[\Lambda] \\ \psi &\longmapsto \sum_{t \in T/T_0} p^{\mathrm{val}(\xi(t))}\big(\sum_{n \in N/N_0} \psi(tn)\big)\lambda(t) \ . \end{aligned}$$

Since $p^{\mathrm{val} \circ \xi}$ defines a character of Λ it is clear that with S_1 also S_ξ is a homomorphism of algebras. To describe its image we also have to adapt the twist in the W-action. The map

$$\begin{aligned} \gamma_\xi : \quad W \times \Lambda &\longrightarrow \mathbb{Q}_p^\times \\ (w, \lambda(t)) &\longmapsto \delta_P^{1/2}(\frac{{}^w t}{t}) p^{\mathrm{val}(\xi(\frac{{}^w t}{t}))} \ , \end{aligned}$$

where δ_P is the modulus character of P, is a cocycle on W with values in $T'(\mathbb{Q}_p)$. It gives rise to the γ_ξ-twisted W-action defined by

$$\begin{aligned} W \times K[\Lambda] &\longrightarrow K[\Lambda] \\ (w, \sum_\lambda c_\lambda \lambda) &\longmapsto \sum_\lambda \gamma(w,\lambda) c_\lambda {}^w\lambda \ . \end{aligned}$$

The following is now a straightforward consequence of Satake's Thm. 10.3.

Corollary 12.3. *The map $S_\xi : \mathcal{H}(G, G_0) \xrightarrow{\cong} K[\Lambda]^{W,\gamma_\xi}$ is an isomorphism of algebras.*

We also introduce the norm

$$\|\sum_\lambda c_\lambda \lambda\|_{\gamma_\xi} := \sup_{\lambda \in \Lambda} |\gamma_\xi(w,\lambda) c_\lambda|$$

on $K[\Lambda)$ where for each λ the $w \in W$ has to be chosen in such a way that ${}^w\lambda \in \lambda(T^{--})$. Using various additional properties of the cocycle γ_ξ one checks:

- The norm $\| \ \|_{\gamma_\xi}$ is submultiplicative.
- The γ_ξ-twisted W-action is isometric for $\| \ \|_{\gamma_\xi}$.

It follows that the completion $K\langle\Lambda;\xi\rangle$ of $K[\Lambda]$ is a K-Banach algebra, that the γ_ξ-twisted W-action extends to $K\langle\Lambda;\xi\rangle$, and that the corresponding invariants form a Banach subalgebra $K\langle\Lambda;\xi\rangle^{W,\gamma_\xi}$.

By carefully redoing Satake's argument while keeping track of the norms one now establishes the following main result.

Theorem 12.4. *The map S_ξ extends to an isometric isomorphism of K-Banach algebras*

$$\mathcal{B}_\xi(G, G_0) \xrightarrow{\cong} K\langle\Lambda;\xi\rangle^{W,\gamma_\xi} \ .$$

To compute the algebra $K\langle\Lambda;\xi\rangle$ we introduce the subset

$$T'_\xi(K) := \{\zeta \in T'(K) : |\zeta(\lambda)| \le |\gamma_\xi(w,\lambda)| \text{ for any } \lambda \in \Lambda \text{ and } w \in W \text{ s. t. } {}^w\lambda \in \lambda(T^{--})\}$$

of the dual torus T'.

Proposition 12.5. *i. $T'_\xi(K)$ is the set of K-valued points of an open $\mathbb{Q}_p$-affinoid subdomain T'_ξ in the torus T'.*

ii. The Banach algebra $K\langle\Lambda;\xi\rangle$ is naturally isomorphic to the ring of K-valued analytic functions on the affinoid domain T'_ξ;

In fact, we can be completely explicit. Let $(a_1, \ldots, a_{d+1})$ be the increasing sequence of integers which defines the dominant character ξ. We now use the embedding

$$\begin{aligned} \iota_\xi : T'(K) = \mathrm{Hom}(\Lambda, K^\times) &\longrightarrow G'(K) \\ \zeta &\longmapsto \begin{pmatrix} \zeta_1 & & 0 \\ & \ddots & \\ 0 & & \zeta_{d+1} \end{pmatrix} \end{aligned}$$

with $\zeta_i := p^{i-1+a_i}\zeta(\lambda_i)$. In these coordinates T'_ξ is the rational subdomain of all $(\zeta_1, \ldots, \zeta_{d+1})$ such that

$$\prod_{i\in I} |\zeta_i| \leq |p|^{|I|(|I|-1)/2+\sum_{i=1}^{|I|} a_i}$$

for any proper nonempty subset $I \subseteq \{1, \ldots, d+1\}$ and

$$\prod_{i=1}^{d+1} |\zeta_i| = |p|^{d(d+1)/2+\sum_{i=1}^{d+1} a_i} \ .$$

As before the point of these coordinates is that the γ_ξ-twisted W-action on ζ becomes the usual permutation action on the ζ_i.

Geometrically the above equations can best be visualized by using the map

$$\begin{aligned} val : T'(K) = \mathrm{Hom}(\Lambda, K^\times) &\longrightarrow \mathbb{R}^{d+1} \\ \zeta &\longmapsto \big(\mathrm{val}(\zeta(\lambda_1)), \ldots, \mathrm{val}(\zeta(\lambda_{d+1}))\big) \ . \end{aligned}$$

We let W act on $\mathbb{R}^{d+1}$ by permutation of the standard coordinates, and we put

$$z_\xi := (a_1, \ldots, a_{d+1}) + (0, 1, \ldots, d) \in \mathbb{R}^{d+1} \ .$$

Lemma 12.6. *$T'_\xi(K)$ is the preimage under the map "val" of the convex hull of the points ${}^w z_\xi - z_\xi$, for $w \in W$, in $\mathbb{R}^{d+1}$.*

For the purposes of the next section we need another reformulation. For any point $z = (z_1, \ldots, z_{d+1}) \in \mathbb{R}^{d+1}$ we let z^{dom} denote the point with the same coordinates z_i but rearranged in increasing order. Moreover, if $z_1 \leq \ldots \leq z_{d+1}$ then we let $\mathcal{P}(z)$ denote the convex polygon in the plane through the points

$$(0,0), (1, z_1), (2, z_1 + z_2), \ldots, (d+1, z_1 + \ldots + z_{d+1}) \ .$$

Lemma 12.7. *T'_ξ is the subdomain of all $\zeta \in T'$ such that $\mathcal{P}((val(\zeta) + z_\xi)^{dom})$ lies above $\mathcal{P}(z_\xi)$ and both polygons have the same endpoint.*

13 Crystalline Galois representations

A filtered K-isocrystal is a triple $\underline{D} = (D, \varphi, Fil^{\cdot} D)$ consisting of a finite dimensional K-vector space D, a K-linear automorphism φ of D – the "Frobenius" – , and an exhaustive and separated decreasing filtration $Fil^{\cdot} D$ on D by K-subspaces. In the following we fix the dimension of D to be equal to $d+1$ and, in fact, the vector space D to be the $d+1$-dimensional standard vector space $D = K^{d+1}$. We then may think of φ as being an element in the dual group $G'(K) = GL_{d+1}(K)$. The Frobenius type $\mathcal{O}(\underline{D})$ of $\underline{D}$ by definition is the conjugacy class of the semisimple part of φ in $G'(K)$. On the other hand, the (filtration) type $type(\underline{D}) \in \mathbb{Z}^{d+1}$ is the sequence $(b_1, \ldots, b_{d+1})$, written in increasing order, of the break points b of the filtration $Fil^{\cdot} D$ each repeated $dim_K\, gr^b D$ many times. We put

$$t_H(\underline{D}) := \sum_{b\in\mathbb{Z}} b \cdot dim_K\, gr^b\, D \ .$$

Then $(d+1, t_H(\underline{D}))$ is the endpoint of the polygon $\mathcal{P}(type(\underline{D}))$. We put

$$t_N(\underline{D}) := \mathrm{val}(\det(\varphi)) \ .$$

Definition 13.1. *The filtered K-isocrystal $\underline{D}$ is called weakly admissible if $t_H(\underline{D}) = t_N(\underline{D})$ and $t_H(\underline{D}') \leq t_N(\underline{D}')$ for any filtered K-isocrystal $\underline{D}'$ corresponding to a φ-invariant K-subspace $D' \subseteq D$ with the induced filtration.*

By a theorem of Colmez and Fontaine ([CF]) there is a fully faithful functor $\underline{D} \longmapsto V_{cris}(\underline{D})$ from the category of weakly admissible filtered K-isocrystals into the category of continuous representations of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ of the field $\mathbb{Q}_p$ in finite dimensional K-vector spaces. The Galois representations in the image of this functor are called *crystalline*.

Proposition 13.2. *Let $\zeta \in T'(K)$ and let ξ be a dominant character of T; then $\zeta \in T'_\xi(K)$ if and only if there is a weakly admissible filtered K-isocrystal $\underline{D}$ such that $type(\underline{D}) = z_\xi$ and $\iota_\xi(\zeta) \in \mathcal{O}(\underline{D})$.*

Proof. Let us first suppose that there exists a filtered K-isocrystal $\underline{D}$ with the asserted properties. Then $\mathcal{P}(type(\underline{D})) = \mathcal{P}(z_\xi)$ is the Hodge polygon of $\underline{D}$ and $\mathcal{P}((val(\zeta) + z_\xi)^{dom})$ is its Newton polygon. The weak admissibility of $\underline{D}$ implies that its Newton polygon lies above its Hodge polygon with both having the same endpoint. Lemma 12.7 therefore says that $\zeta \in T'_\xi(K)$.

We now assume vice versa that $\zeta \in T'_\xi(K)$. We let φ_{ss} be the semisimple automorphism of the standard vector space D given by the diagonal matrix with diagonal entries $(\zeta_1, \ldots, \zeta_{d+1})$. Let $D = D_1 + \ldots + D_m$ be the decomposition of D into the eigenspaces of φ_{ss}. We now choose the Frobenius φ on D in such a way that φ_{ss} is the semisimple part of φ and that any D_j is φ-indecomposable. In this situation D has only finitely many φ-invariant subspaces D' and each of them is of the form $D' = D'_1 + \ldots + D'_m$ with D'_j one of the finitely many φ-invariant subspaces of D_j. By construction the Newton polygon of (D, φ) is equal to $\mathcal{P}((val(\zeta)+z_\xi)^{dom})$. To begin with consider any filtration $Fil^{\cdot}D$ of type z_ξ on D and put $\underline{D} := (D, \varphi, Fil^{\cdot}D)$. The corresponding Hodge polygon then is $\mathcal{P}(z_\xi)$. By Lemma 12.7 the first polygon lies above the second and both have the same endpoint. The latter already says that

$$t_H(\underline{D}) = t_N(\underline{D}) \ .$$

It remains to be seen that we can choose the filtration $Fil^{\cdot}D$ in such a way that $t_H(\underline{D}') \leq t_N(\underline{D}')$ holds true for any of the above $\underline{D}'$. Since these only are finitely many conditions any generic filtration will do. $\square$

Altogether the picture is as follows. Let $\chi : \mathcal{B}_\xi(G, G_0) \longrightarrow K$ be a continuous character. On the one hand we have the unitary Banach space representation $B_{\xi,\chi}$ of G which hopefully is nonzero. On the other hand χ corresponds to the W-orbit of a $\zeta \in T'_\xi(K)$ (at least after enlarging K) and therefore, by Prop. 13.2 and the Colmez-Fontaine theorem, to a whole family of crystalline Galois representations over K. The main reason for obtaining a whole family is that ζ only prescribes the jumps in the filtration of the isocrystal not the actual filtration. The obvious speculation would be that $\mathcal{B}_\xi(G, G_0)$ still is a rather big Banach space representation whose topologically irreducible constituents should be in some kind of correspondence to this family of Galois representations.

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14 Exercises

Exercise 14.1. *1. For a lattice $L \subseteq V$ the natural map*

$$\begin{aligned} K \otimes_o L &\xrightarrow{\cong} V \\ a \otimes v &\longmapsto av \end{aligned}$$

is a bijection.

2. *The preimage of a lattice under a K-linear map again is a lattice.*

3. *The intersection $L \cap L'$ of two lattices $L, L' \subseteq V$ again is a lattice.*

Exercise 14.2. *1. If V is locally convex then addition $V \times V \xrightarrow{+} V$ and scalar multiplication $K \times V \xrightarrow{\cdot} V$ are continuous maps.*

2. *V is Hausdorff if and only if for any nonzero vector $v \in V$ there is a $j \in J$ such that $v \notin L_j$.*

Exercise 14.3. *If V and W are normed vector spaces then the topology on $\mathcal{L}_b(V, W)$ is defined by the operator norm*

$$\|f\| := \sup\{\frac{\|f(v)\|}{\|v\|} : v \in V \setminus \{0\}\}.$$

With W also $\mathcal{L}_b(V, W)$ is a Banach space.

Exercise 14.4. *Show by explicit computation that $c_0(X)^{dd} = c_0(X)$.*

Exercise 14.5. *Let M be a compact and totally disconnected topological space; then the subspace $C^\infty(M, K)$ of K-valued locally constant functions is dense in the Banach space $C(M, K)$.*

Exercise 14.6. *Let V be an admissible Banach space representation of the compact p-adic Lie group G; then, for any G-invariant open lattice $L \subseteq V$, the smooth G-representation $L/\pi L$ over k is admissible.*

Exercise 14.7. *Admissibility can be tested on a single compact open subgroup $H \subseteq G$.*

Exercise 14.8. *Complete the proof of Lemma 10.2, and work out the formula for the multiplication on the right hand side.*

Exercise 14.9. *Call a point $z \in \mathbb{R}^{d+1}$ dominant if $z = z^{dom}$ holds true. Furthermore define a partial order on $\mathbb{R}^{d+1}$ by*

$$(z_1, \ldots, z_{d+1}) \leq (z'_1, \ldots, z'_{d+1})$$

if and only if

$$z_{d+1} \leq z'_{d+1}\ ,\ \ z_d + z_{d+1} \leq z'_d + z'_{d+1}\ ,\ \ \ldots,\ \ z_2 + \ldots + z_{d+1} \leq z'_2 + \ldots + z'_{d+1}$$

and

$$z_1 + \ldots + z_{d+1} = z'_1 + \ldots + z'_{d+1}\ .$$

Show the following equivalences:

i. For any two dominant points z and z' we have $z \leq z'$ if and only if the polygon $\mathcal{P}(z)$ lies above the polygon $\mathcal{P}(z')$ and both have the same endpoints.

ii. For any dominant point z the set $\{x : x^{dom} \leq z\}$ coincides with the convex hull of the points ${}^{w}z$ for $w \in W$.

Exercise 14.10. *Let $d = 1$ and $\xi = (a_1, a_2)$ be a dominant character. Let φ be a linear automorphism on $D = K^2$ which has two different eigenvalues $\zeta_1, \zeta_2 \in K$. Consider filtrations on D with break points a_1 and $a_2 + 1$. Work out the conditions on ζ_1, ζ_2 such that there exists a weakly admissible such filtration. In this case determine, up to isomorphism, all weakly admissible such filtrations.*