Period Domains and their cohomology

1) Show that Fil_k^K is not abelian.

2) Let $V, W \in \mathrm{Fil}_k^K$. Prove that

(i) $\mu(V \otimes W) = \mu(V) + \mu(W)$

(ii) $\mu(V^*) = -\mu(V)$.

3) Let $f : V \longrightarrow W$ be a morphism in Fil_k^K such that the underlying map of vector spaces is a bijection. Prove that $\deg(V) \leq \deg(W)$ with equality iff f is an isomorphism.

4) Let $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$ be a short exact sequence in Fil_k^K. Then the sequence $\mu(V'), \mu(V), \mu(V'')$ is either strictly increasing or strictly decreasing or stationary. In particular we have

$$\min\{\mu(V'), \mu(V'')\} \leq \mu(V) \leq \max\{\mu(V'), \mu(V'')\}.$$

5) Let $K = k$. Determine the semi-stable objects in Fil_k^K.

6) Prove that

(i) V is semi-stable iff $\mu(V) \leq \mu(W)$ for all quotients W of V.

(ii) V is semi-stable $\Leftrightarrow V^*$ is semi-stable.

6) Let $(V, \mathcal{F}) \in \mathrm{Fil}_k^K$. Let $c \in \mathbb{Z}$. Consider the filtration $\mathcal{F}(c)$ on V_K given by

$$\mathcal{F}(c)^x = \mathcal{F}^{x-c}.$$

The object $V(c) := (V, \mathcal{F}(c))$ is called the *twist of* V *by* c. Then $\mu(V(c)) = \mu(V) + c$ and V is semi-stable if and only if $V(c)$ is semi-stable.

7) Let $V, W \in \mathrm{Fil}_k^K$ both be semi-stable. If $\mu(V) > \mu(W)$, then $\mathrm{Hom}(V, W) = 0$. If $\mu(V) = \mu(W) = \mu$, then for every $f \in \mathrm{Hom}(V, W)$ the objects $\mathrm{Ker}\, f$, $\mathrm{Coker}\, f$, and $\mathrm{Im} f$ are semi-stable with the same slope μ. In particular, the full subcategory of Fil_k^K consisting of semi-stable objects of slope μ is abelian, and stable under extensions.

8) Let $V, W \in \mathrm{Fil}_k^K$ such that $V \otimes W$ is semi-stable. Prove that V and W are semi-stable.

9) Let $\nu = (n-1, -1, \ldots, -1) \in (\mathbb{Z}^n)_+$. Show that $\boldsymbol{\mathcal{F}}(\nu)^{ss} = \Omega_k^{n-1}$.

10) Let $\nu = (\nu_1, \nu_2, \nu_3) \in (\mathbb{Z}^3)_+$. Determine $\boldsymbol{\mathcal{F}}(\nu)^{ss}$.

11) Let $N = (E_{\frac{1}{2}})^2$ and $\nu = (1, 1, 0, 0)$. Determine $\boldsymbol{\mathcal{F}}(N, \nu)^{\mathrm{wa}}$.

12) Let $N \in \mathrm{Isoc}(L)$ and $\nu \in (\mathbb{Z}^n)_+$. Decide when $\boldsymbol{\mathcal{F}}(N,\nu)^{ss} \neq \emptyset$.

13) Let $\nu = (\nu_1, \nu_2, \nu_3) \in (\mathbb{Z}^3)_+$ with $\nu_1 > \nu_2 > \nu_3$, and let $N = E_{\frac{1}{2}} \oplus E_0$. Determine the non-empty HN-strata.

14) Let $\nu = (\nu_1, \nu_2, \nu_3) \in (\mathbb{Z}^3)_+$. Determine $H_c^*(\boldsymbol{\mathcal{F}}(\nu)^{ss})$.