Canonical Subgroups over Hilbert Modular Varieties

Joint work with Payman Kassaei (King’s College, London)

Eyal Goren

McGill University

Minerva school on $p$-adic methods in arithmetic algebraic geometry, Jerusalem, Spring 2009.
1 Introduction

2 Stratifications of the special fibers

3 Valuations on $X_{\text{rig}}, Y_{\text{rig}}$

4 Construction of the canonical subgroup
The Problem. Associate in a natural way to a $g$-dimensional abelian variety $A/R$, $[R : \mathbb{Z}_p] < \infty$, with real multiplication an invariant isotropic subgroup $H$ such that $pH = \{0\}$, and $\# H = p^g$.

Motivation. Properties of the $U$-operator on overconvergent modular forms; used to prove classicality results for modular forms, study of $p$-adic families of modular forms, special values of $L$-functions, and modularity of Galois representations. Cf. Besser’s talk.

If $A$ has ordinary reduction: classical $\check{\square}$
(There is a unique way to lift the kernel of Frobenius $\text{Fr}: \overline{A} \to \overline{A}^{(p)}$, where $\overline{A} = A \pmod{pR}$.)

Therefore, the problem is:

- Extend this to non-ordinary abelian varieties;
- Do it in families.
Reformulation

For appropriate moduli schemes $X, Y$ over $\mathbb{Z}_p$, $X_{\text{rig}}, Y_{\text{rig}}$ the associated (Raynaud) generic fibers, we have a diagram,

\[(A, H) \sim \sim \sim \sim \sim Y_{\text{rig}} \sim \sim \sim \sim \sim X_{\text{rig}},\]

and the section $s$ exists over the ordinary locus.

The Problem. Extend $s$ “as much as possible”.
The main theorem

Theorem (G. - Kassaei)

Let \( \{ \tilde{h}_\beta \}_{\beta \in \mathbb{B}} \) be (Zariski local) lifts of the partial Hasse invariants. Let \( \mathcal{U} \subset \mathcal{X}_{\text{rig}} \) be

\[
\mathcal{U} = \{ P : \nu(\tilde{h}_\beta(P)) + p\nu(\tilde{h}_{\sigma^{-1}\circ \beta}(P)) < p, \ \forall \beta \in \mathbb{B} \}.
\]

There exists a section \( s^\dagger : \mathcal{U} \to \mathcal{Y}_{\text{rig}}, \) extending the section \( s \) on the ordinary locus.
What comes into the proof?

1. Stratifications of $\overline{X}$, $\overline{Y}$ (the special fibers).
2. Study of $\pi: \overline{Y} \to \overline{X}$ on completed local rings.
3. “Dissection” of $Y_{\text{rig}}$, the generic fiber of $Y$, using $g$ different valuations.
   \((g = [L : \mathbb{Q}], \text{where } L \text{ is the totally real field acting.})\)

Remark. *The structure suggests strategy should be applicable to many Shimura varieties of PEL type.*
1 Introduction

2 Stratifications of the special fibers

- Stratification of $\overline{X}$
- Stratification of $\overline{Y}$
- Functoriality of local models

3 Valuations on $X_{\text{rig}}, Y_{\text{rig}}$

4 Construction of the canonical subgroup
Notation

- $L$ - totally real field, $[L : \mathbb{Q}] = g$.
- $p$ unramified in $L$.

$$\mathcal{B} = \text{Hom}(L, \mathbb{Q}_p^{ur}) = \bigsqcup_{p | p} \mathcal{B}_p \circlearrowleft \sigma.$$  
($\sigma = \text{Frobenius, lift of } x \mapsto x^p.$)

- For $S \subseteq \mathcal{B}$, let $S^c = \mathcal{B} \setminus S$ and
  $$\ell(S) = \{\sigma^{-1} \circ \beta : \beta \in S\}, \quad r(S) = \{\sigma \circ \beta : \beta \in S\}.$$  

- $\kappa = \text{minimal field } \supseteq \mathcal{O}_L/p, \forall p | p$.  

- $\mathcal{O}_L \otimes_{\mathbb{Z}} W(\kappa) \cong \bigoplus_{\beta \in \mathcal{B}} W(\kappa)_{\beta}$

induces a decomposition of any $\mathcal{O}_L \otimes W(\kappa)$-module.
**Moduli Spaces**

\( X/W(\kappa) \) parameterizes \( A = (A, \iota, \alpha, \lambda_A)/S \), where \( S \) is a \( W(\kappa) \)-scheme and:

\[
A \to S \text{ abelian scheme, of rel. dim' n } g, \quad \iota : \mathcal{O}_L \hookrightarrow \text{End}_S(A),
\]

\( \alpha = \text{rigid } \Gamma_{00}(N) \)-level structure.

\( \lambda_A : (a, a^+) \cong (\mathcal{P}_A, \mathcal{P}_A^+) \) a polarization: \( A \otimes a \cong A^t \),

\( \mathcal{P}_A = \text{Hom}_{\mathcal{O}_L}(A, A^t)^{\text{sym}} \) with the positive cone of polarizations.

\( Y/W(\kappa) \) parameterizes \( (A, H) \) such that:

\( H \) is killed by \( p \), degree \( p^g \), \( \mathcal{O}_L \)-invariant, isotropic

Equivalently,

\[
(f : A \to B),
\]

such that \( \text{deg}(f) = p^g, \text{Ker}(f) \subseteq A[p], f^*\mathcal{P}_B = p\mathcal{P}_A. \)

Atkin-Lehner: \( w(f : A \to B) = (f^t : B \to A), \quad f \circ f^t = p \)
Invariants for $X$

For $A/k$, $k \supseteq \kappa$ perfect. Let

$$\alpha_A = \ker(Fr_A) \cap \ker(Ver_A).$$

Decomposition of the Dieudonné modules:

$$\mathbb{D}(A[p]) = \bigoplus_{\beta \in \mathbb{B}} k^2 \supseteq \mathbb{D}(\alpha_A) = \bigoplus_{\beta \in \mathbb{B}} \mathbb{D}(\alpha_A)_\beta$$

0 or 1 dim’l

The type of $A$ is

$$\tau(A) = \{ \beta \in \mathbb{B} : \mathbb{D}(\alpha_A)_\beta \neq \{0\} \}.$$ 

Define strata of $X$,

$$W_\tau \leftrightarrow \{ A : \tau(A) = \tau \} \quad \text{(locally closed)},$$

$$Z_\tau \leftrightarrow \{ A : \tau(A) \supseteq \tau \} \quad \text{(closed)}.$$
Theorems (G. - Oort)

1. \( \overline{W_\tau} = Z_\tau = \bigsqcup_{\tau' \supseteq \tau} W_{\tau'} \). So \( \{ W_\tau : \tau \subseteq \mathbb{B} \} \) is a stratification of the moduli space \( \overline{X} \) by \( 2^g \) strata.

2. \( W_\tau \) is non-singular, quasi-affine of dimension \( g - \# \tau \).

3. \( \exists h_\beta \), a Hilbert modular form of weight \( p \cdot \sigma^{-1} \circ \beta - \beta \), such that \( (h_\beta) = Z_\beta \).
   (In classical terms: weight \((0, \ldots, 0, p, -1, 0 \ldots, 0)\).)

4. \( \hat{\mathcal{O}}_{\overline{X}, P} \cong k[[t_\beta : \beta \in \mathbb{B}]] \) and if \( h_\beta(P) = 0 \) then we may identify \( h_\beta \) with \( t_\beta \).

5. The kernel of the \( q \)-expansion map on the graded ring of Hilbert modular forms modulo \( p \) is the ideal \( \langle h_\beta - 1 : \beta \in \mathbb{B} \rangle \).
Invariants for $\overline{Y}$

Given $A \xleftarrow{f} \xrightarrow{f^t} B$, we have

$$\bigoplus \beta \text{Lie}(A)_\beta \xrightarrow{\bigoplus \beta \text{Lie}(f)_\beta} \bigoplus \beta \text{Lie}(B)_\beta.$$ 

Define

1. $\varphi = \varphi(f) = \{ \beta \in B : \text{Lie}(f)_{\sigma^{-1} \circ \beta} = 0 \}$,
2. $\eta = \eta(f) = \{ \beta \in B : \text{Lie}(f^t)_\beta = 0 \}$,
3. $I = \ell(\varphi) \cap \eta$ (the “critical indices”).

Properties:

1. $(\varphi \triangle \eta)^c \supseteq \tau(A) \supseteq \varphi \cap \eta$.
2. $\eta \supseteq \ell(\varphi)^c$. 
A pair \((\varphi, \eta)\) (for \(\varphi, \eta \subseteq B\)) is admissible if

\[ \eta \supseteq \ell(\varphi)^c. \]

Exist 3\(g\) such pairs.

Define strata in \(\overline{Y}\):

\[
W_{\varphi, \eta} \leftrightarrow \{(f: A \to B) : \varphi(f) = \varphi, \eta(f) = \eta\} \quad \text{(loc. closed)},
\]

\[
Z_{\varphi, \eta} \leftrightarrow \{(f: A \to B) : \varphi(f) \supseteq \varphi, \eta(f) \supseteq \eta\} \quad \text{(closed)}. 
\]
Theorems

1. \( \bar{W}_{\varphi, \eta} = Z_{\varphi, \eta} = \bigsqcup_{(\varphi', \eta') \geq (\varphi, \eta)} W_{\varphi', \eta'} \) and so \( \{ W_{\varphi, \eta} \} \) is a stratification of \( \bar{Y} \) with \( 3^g \) strata.

2. \( W_{\varphi, \eta} \) and \( Z_{\varphi, \eta} \) are non-singular, equi-dimensional of dimension \( 2g - (\# \varphi + \# \eta) \).

3. There are \( 2^g \) maximal strata, given by \( Z_{\varphi, \ell(\varphi)c}, \varphi \subseteq \mathbb{B} \).

4. There are \( 2^r \) horizontal components, where \( r = \# \{ p \mid p \} \).

Two of which are
\[ \bar{Y}_F = Z_{\mathbb{B}, \emptyset} \quad \text{and} \quad (A, \text{Ker}(\text{Fr}_A)) \]
\[ \bar{Y}_V = Z_{\emptyset, \mathbb{B}} \quad \text{and} \quad (A, \text{Ker}(\text{Ver}_A)) \]

5. \( w(Z_{\varphi, \eta}) = Z_{r(\eta), \ell(\varphi)} \).

6. \( \pi(Z_{\varphi, \eta}) = Z_{\varphi \cap \eta} \).

If \( C \subseteq Z_{\varphi, \eta} \) is an irreducible component then
\[ C \cap \bar{Y}_F \cap \bar{Y}_V \neq \emptyset. \]
Let $\overline{Q} \in \overline{Y}$ be a closed $k$-point, then

$$\hat{\mathcal{O}}_{\overline{Y}, \overline{Q}} \cong \frac{W(k)[[\{x_\beta : \beta \in I\}, \{y_\beta : \beta \in I\}, \{z_\beta : \beta \in I^c\}]]}{\langle\{x_\beta y_\beta - p : \beta \in I\}\rangle}.$$ 

Moreover, the variables can be chosen so that: If

$$\varphi \supseteq \varphi' \supseteq \varphi - r(I), \quad \eta \supseteq \eta' \supseteq \eta - I,$$

and $(\varphi', \eta')$ is admissible, write

$$\varphi' = \varphi - J, \quad \eta' = \eta - K,$$

then $Z_{\varphi', \eta'}$ is described in $\hat{\mathcal{O}}_{\overline{Y}, \overline{Q}}$ by the ideal

$$\langle\{x_\beta : \beta \in I - K\}, \{y_\beta : \beta \in I - \ell(J)\}\rangle.$$ 

Moreover, if $\overline{Q} \in Z_{\varphi', \eta'}$ then $(\varphi', \eta')$ are as above.
Key Lemma

Let $\beta \in \varphi \cap \eta = \tau$, $\pi(Q) = \overline{P}$,

$$\pi^* : \hat{O}_{X, P} \longrightarrow \hat{O}_{Y, Q}.$$

Then:

$$\pi^*(t_\beta) = \begin{cases} 
ux_\beta + vy_{\sigma^{-1} \circ \beta} & \sigma \circ \beta \in \varphi, \quad \sigma^{-1} \circ \beta \in \eta, \\
ux_\beta & \sigma \circ \beta \in \varphi, \quad \sigma^{-1} \circ \beta \notin \eta, \\
v y_{\sigma^{-1} \circ \beta} & \sigma \circ \beta \notin \varphi, \quad \sigma^{-1} \circ \beta \in \eta, \\
0 & \sigma \circ \beta \notin \varphi, \quad \sigma^{-1} \circ \beta \notin \eta,
\end{cases}$$

where $u, v$ are units.
Ideas coming into the proof

- Study the situation on components of $\text{Spf}(\mathring{\mathcal{O}_Y}, \mathring{Q})$; they correspond to strata $Z_{\varphi', \eta'}$ passing through $\mathring{Q}$.
- Gain data on $\pi^*(t_{\beta})$; roughly, $\pi^*(t_{\beta}) = ux_{\beta}^M + vy_{\sigma}^N$.  
- Globalize so as to be able to study these expressions on components of $Z_{\varphi', \eta'}$ but at other points then $\mathring{Q}$.
- Reduce to computation at a “special superspecial point” (using that any component $C$ of any strata $Z_{\varphi, \eta}$ intersects $\overline{Y_F} \cap \overline{Y_V}$).
1 Introduction

2 Stratifications of the special fibers

3 Valuations on $X_{\text{rig}}, Y_{\text{rig}}$

4 Construction of the canonical subgroup
\( \mathcal{X}_{\text{rig}}, \mathcal{Y}_{\text{rig}} \) are the rigid spaces associated to \( X^\wedge \bar{X}, Y^\wedge \bar{Y} \). Given \( P \in \mathcal{X}_{\text{rig}} \), get \( \bar{P} = \text{sp}(P) \in \bar{X} \). The variables \( t_\beta \in \hat{O}_{X, \bar{P}} \) are functions on \( \text{sp}^{-1}(\bar{P}) = \text{residue "disc" about } P \). Let

\[
\nu(P) = (\nu_\beta(P)), \quad \nu_\beta(P) = \begin{cases} 
\nu(t_\beta(P)) & \beta \in \tau(\bar{P}), \\
0 & \text{else}.
\end{cases}
\]

\( (\nu(x) = \min(\text{val}(x), 1). ) \) For \( Q \in \mathcal{Y}_{\text{rig}}, \bar{Q} = \text{sp}(Q) \in \bar{Y} \), let

\[
\nu(Q) = (\nu_\beta(Q)), \quad \nu_\beta(Q) = \begin{cases} 
1 & \beta \in \eta(\bar{Q}) - I(\bar{Q}), \\
\nu(x_\beta(P)) & \beta \in I(\bar{Q}), \\
0 & \beta \notin I(\bar{Q}).
\end{cases}
\]

\( \nu(P), \nu(Q) \) belong to the valuation cube \( \Theta = [0, 1]^B \).
\( \nu(Q) + \nu(wQ) = 1 \) (easy!).
The Cube Theorem

Parameterize the “open faces” of $\Theta$ by $a = (a_\beta), a_\beta \in \{0, 1, *\}$. For such $a$ define

$$\varphi(a) = \{\beta \in \mathbb{B} : a_\beta \neq 0\}, \quad \eta(a) = \{\beta \in \mathbb{B} : a_{\sigma^{-1}\beta} \neq 1\}.$$  

There is a 1 : 1 order-reversing correspondence

$$\{\text{open faces of } \Theta\} \leftrightarrow \{\text{strata } W_{\varphi, \eta}\}.$$

$$F_a \leftrightarrow W_{\varphi(a), \eta(a)}$$

- $\nu(Q) \in F_a \iff \overline{Q} \in W_{\varphi(a), \eta(a)}$.
- $\nu(Q) \in \text{Star}(F_a) \iff \overline{Q} \in Z_{\varphi(a), \eta(a)}$.

The open faces of $\Theta$ produce a “dissection” of $Y_{\text{rig}}$:

$$F_a \leftrightarrow \{Q : \nu(Q) \in F_a\}.$$
1 Introduction

2 Stratifications of the special fibers

3 Valuations on $X_{\text{rig}}, Y_{\text{rig}}$

4 Construction of the canonical subgroup
The canonical subgroup theorem

\[ \mathcal{U} := \{ P \in X_{\text{rig}} : \nu_\beta(P) + p\nu_{\sigma^{-1}\circ\beta}(P) < p, \forall \beta \in \mathbb{B} \} \]

\[ \mathcal{V} := \{ Q \in Y_{\text{rig}} : \nu_\beta(Q) + p\nu_{\sigma^{-1}\circ\beta}(Q) \leq p, \forall \beta \in \mathbb{B} \} \]

Theorem

\[ \pi(\mathcal{V}) = \mathcal{U} \text{ and there is a section} \]

\[ s^\dagger : \mathcal{U} \rightarrow \mathcal{V}, \]

extending the canonical section on the ordinary locus.
Ideas in the proof

- Define for \( p | p \),
  \[
  \mathcal{V}_p := \{ Q : \lambda_\beta(Q) < p, \forall \beta \in \mathbb{B}_p \}
  \]
  \[
  \mathcal{W}_p := \{ Q : \lambda_\beta(Q) > p, \forall \beta \in \mathbb{B}_p \}
  \]

We first show that:
- \( \mathcal{U} \) is admissible.
- \( \pi^{-1}(\mathcal{U}) = \mathcal{V} \bigsqcup \mathcal{W} \), admissible disjoint union, where
  \[
  \mathcal{W} = \bigcup_{\emptyset \neq S \subseteq \{ p | p \}} \left[ \bigcap_{p \in S} \mathcal{W}_p \cap \bigcap_{p \notin S} \mathcal{V}_p \right].
  \]
  This uses the notion of tubular neighborhoods (cf. Grosse-Kloenne’s talk) and our strata on \( \overline{Y} \).
- \( \pi|_{\pi^{-1}(\mathcal{U})} \) is finite-flat.
- The connected components of \( \mathcal{V} \) are in bijection with those of \( \mathcal{U} \).
- We calculate that \( \pi|_{\mathcal{V}} \) has degree 1 by restricting to
  \( \text{sp}^{-1}(\mathcal{W}_B, \emptyset) \subseteq \text{sp}^{-1}(\overline{Y}_F) \).
Further properties

- We can determine what happens under

  \[ P \mapsto s^\dagger(P) \mapsto w \circ s^\dagger(P) \mapsto \pi \circ w \circ s^\dagger(P), \]

  and so can iterate the construction to construct higher level canonical subgroups \((\subseteq A[p^n])\).

- Can prove functorial behavior relative to morphisms between Hilbert modular varieties. In particular,
  (i) decent to \(\mathbb{Q}_p\) of the canonical subgroup, and
  (ii) prove \(\mathcal{U}\) is maximal (in a suitable sense) for the construction of the canonical subgroup.

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