Minerva School 2009, Jerusalem
Rigid Cohomology (EgK)

Let R be a complete discrete valuation ring, $\pi \in R$ a uniformizer, $R/(\pi) = k = \mathbb{F}_q$ the field with q elements, $K = \mathrm{Frac}(R)$ of characteristic zero.

PROBLEM 1: (Overconvergence) Recall that we defined

$$R < T_1, \ldots, T_n >^{\dagger} = \{\sum_{\alpha} c_\alpha T^\alpha \in R[[T_1, \ldots, T_n]] \, ; \, |c_\alpha| \rho^{|\alpha|} \stackrel{|\alpha| \to \infty}{\longrightarrow} 0 \text{ for some } \rho > 1\},$$

$$K < T_1, \ldots, T_n >^{\dagger} = \{\sum_{\alpha} c_\alpha T^\alpha \in K[[T_1, \ldots, T_n]] \, ; \, |c_\alpha| \rho^{|\alpha|} \stackrel{|\alpha| \to \infty}{\longrightarrow} 0 \text{ for some } \rho > 1\}.$$

(a) For $n = 1$, give full details of the proof of the Key-Lemma: $\frac{\partial}{\partial T} : K < T >^{\dagger} \to K < T >^{\dagger}$ is surjective.
(b) Let A be a wcfg-R-algebra (i.e. a quotient of some $R < T_1, \ldots, T_n >^{\dagger}$), let $\widehat{A}$ be its π-adic completion. Show: A is the subring of $\widehat{A}$ consisting of all expressions $z = \sum_{j=0}^{\infty} P_j(x_1, \ldots, x_m) \in \widehat{A}$ with $x_1, \ldots, x_m \in A$, with m-variate polynomials $P_j \in (\pi)^j R[X_1, \ldots, X_m]$ such that $\deg(P_j) \le c(j+1)$ for all j, for some constant $c = c(z)$.
(c) Can you find intermediate rings $K < T_1, \ldots, T_n >^{\dagger} \subsetneq B \subsetneq K < T_1, \ldots, T_n >$, characterized (similarly as $K < T_1, \ldots, T_n >^{\dagger}$) by some convergence condition ? If $n = 1$: do you find one respected by $\frac{\partial}{\partial T}$ and on which $\frac{\partial}{\partial T}$ is surjective ?

PROBLEM 2: (de Rham cohomology of rigid spaces) Recall that for any K-scheme $\mathcal{X}$ of finite type there is an associated K-rigid space $\mathcal{X}^{\mathrm{an}}$, its analytification. For any K-rigid space X, just as in algebraic geometry, one defines the de Rham complex $\Omega_X^\bullet = (\Omega_X^\bullet, d) = (\wedge^\bullet \Omega_X^1, d)$ on X (e.g. if X is an admissible open subset of $\mathrm{Spec}(K[T_1, \ldots, T_n])^{\mathrm{an}}$), then Ω_X^1 is a free coherent $\mathcal{O}_X$-module sheaf of rank n in the symbols $dT_1, \ldots, dT_n$). For X smooth one puts $H^*_{dR}(X) = \mathbf{H}^*(X, \Omega_X^\bullet)$, its de Rham cohomology. If X is quasi-Stein then this means

$$H^i_{dR}(X) = \frac{\ker(\Omega_X^i(X) \to \Omega_X^{i+1}(X))}{\mathrm{im}(\Omega_X^{i-1}(X) \to \Omega_X^i(X))}.$$

Examples of quasi-Stein spaces are $X = \mathcal{X}^{\mathrm{an}}$ for a smooth *affine* K-scheme $\mathcal{X}$ of finite type, or $X = X_{s,t}$ for some $0 \le s < t \le \infty$, the one dimensional open annulus of outer radius t, inner radius s, i.e. its $\hat{\overline{K}}$-valued points ($\hat{\overline{K}}$ the completion of an algebraic closure $\overline{K}$ of K) are

$$X_{s,t}(\hat{\overline{K}}) = \{y \in \hat{\overline{K}} \quad | \quad s < |y| < t\}.$$

(a) Let $\mathcal{W} = \mathrm{Spec}(A)$ be a smooth affine R-scheme, let $\overline{W} = \mathcal{W} \otimes_R k$. Let $\widehat{A}$ denote the π-adic completion of A so that $\widehat{A} \otimes_R K$ is a K-affinoid algebra. We may view $W = \mathrm{Sp}(\widehat{A} \otimes_R K)$ as an admissible open subset of $\mathcal{W}_K^{\mathrm{an}}$. Let $\mathcal{C}$ denote the set of all strict neighbourhoods of W in $\mathcal{W}_K^{\mathrm{an}}$, i.e. the set of admissible open subsets X of $\mathcal{W}_K^{\mathrm{an}}$

such that $\mathcal{W}_K^{\rm an} = (\mathcal{W}_K^{\rm an} - W) \cup X$ is an admissible open covering of $\mathcal{W}_K^{\rm an}$. Show that there is a natural isomorphism

$$H^*_{MW}(\overline{W}) \cong \lim_{\substack{\rightarrow \\ X \in \mathcal{C}}} H^*_{dR}(X).$$

(b) Compute $H^*_{dR}(X_{s,t})$.

(c) Let $\mathcal{Y}/R$ be a proper smooth connected R-scheme of relative dimension 1, let $\mathcal{Z} \subset \mathcal{Y}$ be an R-flat divisor such that $\mathcal{W} = \mathcal{Y} - \mathcal{Z}$ is affine. Let $\overline{W} = \mathcal{W} \otimes_R k$. Show that there is a canonical isomorphism

$$H^*_{MW}(\overline{W}) \cong H^*_{dR}(\mathcal{W}_K^{\rm an}).$$

[Hints: As $\mathcal{Y}/R$ is smooth we find an isomorphism $\mathcal{W}_K^{\rm an} - W \cong \coprod X_{0,1}$ (finite disjoint union of copies of $\coprod X_{0,1}$ and with W defined as in (b)), and this in such a way that the sets $W \cup \coprod X_{s,1}$ for $0 < s < 1$ form a cofinal system in the set of all strict neighbourhoods of W in $\mathcal{W}_K^{\rm an}$. (Here $W \cup \coprod X_{s,1}$ is set theoretically an admissible open subset of $\mathcal{W}_K^{\rm an}$, but the notation does *not* indicate an *admissible covering* of this open subset, of course.) To any such s one has a long exact cohomology sequence associated with the admissible covering $\mathcal{W}_K^{\rm an} = (\mathcal{W}_K^{\rm an} - W) \cup (W \cup \coprod X_{s,1})$. Now use (a) and (b).]

PROBLEM 3: Let $F_1, \ldots, F_r \in k[X_1, \ldots, X_n]$, put $d_i = \deg(F_i)$.

Theorem 1: (Chevalley-Warning) *If $n > \sum_{i=1}^r d_i$ then the number of solutions (in k^n) to the system*

$$F_1(x_1, \ldots, x_n) = \ldots = F_r(x_1, \ldots, x_n) = 0 \qquad (*)$$

is a multiple of $p = \mathrm{char}(k)$.

Theorem 2: (Ax-Katz) *If b denotes the smallest integer with*

$$b \geq \frac{n - \sum d_i}{\max\{d_i\}}$$

then the number of solutions to the system $(*)$ is divisible by q^b.

Find a simple and elementary proof (without p-adic methods) for Theorem 1. Theorem 2 can be derived from the last theorem of our course (but this is *not* meant to be part of this present problem).

Problem 4: (Dwork's trace formula) Let $B = B(X) = (a_{ij}(X))$ be an $r \times r$-matrix over $R < X_1^{\pm}, \ldots, X_n^{\pm} >$, i.e.

$$a_{ij}(X) \in R < X_1^{\pm}, \ldots, X_n^{\pm} >= \{ \sum_{u \in \mathbb{Z}^n} a_u X^u \mid a_u \in R, \lim_{|u| \to \infty} a_u = 0 \}.$$

(Here $|(u_1, \ldots, u_n)| = |u_1| + \ldots + |u_n|$.) Alternatively we may write $B = B(X) = \sum_u b_u X^u$ with $b_u \in \mathrm{Mat}(r \times r, R)$ such that $\lim_{|u| \to \infty} b_u = 0$. The L-function of B on the torus $\mathbb{G}_m^n/k$ is

$$L(B, \mathbb{G}_m^n/k, t) = \prod_{\overline{x} \in |\mathbb{G}_m^n|} \frac{1}{\det(I - t^{d(x)} B(x^{q^{d(x)-1}}) \cdots B(x^q) B(x))}$$

where $|\mathbb{G}_m^n|$ denotes the set of closed points of $|\mathbb{G}_m^n|$, where $x \in (\overline{K}^{\times})^n$ (in fact $x \in (W(\overline{k})^{\times})^n$) is the Teichmüller lifting of $\overline{x} \in |\mathbb{G}_m^n|$, where $x^q = (x_1^q, \ldots, x_n^q)$ and where $d(x)$ denotes the degree of $\overline{x}$ over $k = \mathbb{F}_q$. This infinite product is a well defined power series with coefficients in R because there are only finitely many closed points of a given degree.

Remark: If B is invertible in $\mathrm{Mat}(r \times r, K \otimes_R R < X_1^{\pm}, \ldots, X_n^{\pm} >)$ then it defines (the Frobenius matrix of) an F-crystal on $\mathbb{G}_m^n/k$. The above L-function is then Katz' L-function of this F-crystal on $\mathbb{G}_m^n/k$. If B is invertible even in $\mathrm{Mat}(r \times r, R < X_1^{\pm}, \ldots, X_n^{\pm} >)$ then the crystal on $\mathbb{G}_m^n/k$ is a unit-root F-crystal, hence corresponds by a well known equivalence of categories to a p-adic representation $\rho : \pi_1 \to \mathrm{GL}_r(R)$ of the arithmetic fundamental group π_1 of $\mathbb{G}_m^n/k$, or equivalently: to an étale p-adic sheaf $\mathcal{F}$ on $\mathbb{G}_m^n/k$. In this case, the above L-function is the Artin L-function of ρ, or the Grothendieck L-function of $\mathcal{F}$.

(a) (Dwork's trace formula, additive form) For $k \in \mathbb{N}$ consider the character sum

$$S_k(B) = \sum_{\overline{x} \in \mathbb{G}_m^n(\mathbb{F}_{q^k})} \mathrm{tr}(B(x^{q^{d(x)-1}}) \cdots B(x^q) B(x))$$

with tr denoting the trace of a matrix. Writing $B = \sum_{u \in \mathbb{Z}^n} b_u X^u$, consider the infinite matrix $F_B = (b_{qu-v})_{u,v \in \mathbb{Z}^n}$. Since $\lim_{|u| \to \infty} b_u = 0$, one checks immediately that the power F_B^k is well defined for each $k \in \mathbb{N}$. Show

$$S_k(B) = (q^k - 1)^n \mathrm{tr}(F_B^k).$$

(Suggestion: first try the case $k = 1$.)

(b) (Dwork's trace formula, multiplicative form) Show

$$L(B, \mathbb{G}_m^n/k, t) = \exp(\sum_{k=1}^{\infty} \frac{t^k}{k} S_k(B))$$

and

$$L(B, \mathbb{G}_m^n/k, t) = \prod_{i=0}^{n} \det(I - q^i t F_B)^{(-1)^i \binom{n}{i}}.$$

Problem 5: (Nuclear operators) Let M be a K-vector space and $L \in \mathrm{End}_K(M)$. An *eigenvalue* of L is a $\lambda \in \overline{K}$ (here $\overline{K}$ denotes the algebraic closure of K) such that $\ker(g_\lambda(L)) \neq 0$ for the minimal polynomial g_λ of λ. We say that L is *nuclear* if
(i) for every eigenvalue $\lambda \neq 0$ there exists an L-stable K-vector space decomposition $M = A_\lambda \oplus B_\lambda$ with $B_\lambda = \cup_{n\geq 1}\ker(g_\lambda(L)^n)$ and $\dim_K(B_\lambda) < \infty$, and $g_\lambda(L)$ bijective on A_λ, and
(ii) the non-zero eigenvalues of L form a finite set or a sequence converging to 0.

In this situtation we define the trace of L as $\mathrm{tr}(L) = \sum_{\lambda\neq 0}\mathrm{tr}(L|_{B_\lambda})$. Moreover, with L also L^s for $s \in \mathbb{N}$ is nuclear, and we put $\det(1-tL) = \prod_{\lambda\neq 0}\det(1-tL|_{B_\lambda}) = \exp(-\sum_{s\geq 1}\frac{\mathrm{tr}(L^s)t^s}{s})$.
If M, N are K-Banach spaces, we say that $L : M \to N$ is *completely continuous* if it is the uniform limit of linear maps finite rank. It is a classical result of Serre that any completely continuous endomorphism of a K-Banach space is nuclear. Recall that an orthogonal (countable) basis of a K-Banach space V is a subset $\{b_n\}_{n\in\mathbb{N}} \subset V$ such that each $x \in V$ can uniquely be written as $x = \sum_n \gamma_n b_n$ with $\gamma_n \in K$ such that $|\gamma_n| \overset{n\to\infty}{\to} 0$ and $|x| = \sup_n|\gamma_n b_n|$.

Let A be a wcfg-R-algebra and $F : A \to A$ a lift of the q-power Frobenius on $\overline{A} = A \otimes_R k$. Let M be a finitely generated A-module. An R-linear map $\theta : M \to M$ is called a *Dwork operator* if $\theta(F(a)m) = a\theta(m)$ for all $a \in A$, all $m \in M$.

(a) For a sequence of continuous maps of K-Banach spaces $U \overset{f}{\to} V \overset{u}{\to} W \overset{g}{\to} X$, if u is completely continuous, then so is $g \circ u \circ f$.

(b) Suppose $A = R < T_1, \ldots, T_n >^\dagger$. For $\rho > 1$ with $\rho \in |\overline{K}|$ let

$$A_K(\rho) = \{\sum_\alpha c_\alpha T^\alpha \in K[[T_1, \ldots, T_n]]\,;\, |c_\alpha|\rho^{|\alpha|} \overset{|\alpha|\to\infty}{\longrightarrow} 0\}.$$

This is a K-Banach space with respect to the norm $|\sum_\alpha c_\alpha T^\alpha| = \max|c_\alpha|\rho^{|\alpha|}$. Let M be a finite free A-module with basis $e_1, \ldots, e_r$ and let $M_K(\rho) = \sum_{j=1}^r A_K(\rho)e_j$. This is then also a K-Banach space. Show that for $1 < \rho_1 < \rho_2$ the inclusion $M_K(\rho_2) \subset M_K(\rho_1)$ is completely continuous.

(c) For a wcfg-R-algebra A and a finitely generated A-module M, any Dwork-operator $\theta : M \to M$ induces a nuclear map $\theta : M \otimes_R K \to M \otimes_R K$.

Hints: We may suppose $A = R < T_1, \ldots, T_n >^\dagger$. Then M admits a finite free resolution $0 \to M_s \to \ldots \to M_0 \to M \to 0$ and on each M_i one can construct a Dwork operator such that the obvious diagrams commute. This allows us to assume that M is A-free. Fix an A-basis $e_1, \ldots, e_r$ of M and let $M_K(\rho) = \sum_{j=1}^r A_K(\rho)e_j$. For sufficiently small ρ one has $\theta(M_K(\rho)) \subset M_K(\rho^q)$.

[This is the central point: *Dwork operators improve convergence.* To estimate $\theta(M_K(\rho))$ observe that any element in $M_K(\rho)$ may be written as

$$\sum_{\substack{\alpha<(q,\ldots,q)\\ j=1,\ldots,r}} F(a_{\alpha,j})t^\alpha e_j$$

with suitable $a_{\alpha,j} \in A_K(\rho^q)$ (decrease ρ if necessary). The defining rule for being a Dwork operator then shows $\theta(M_K(\rho)) \subset \sum_{\alpha<(q,\ldots,q),j} A_K(\rho^q)\theta(t^\alpha e_j)$ (decrease ρ if necessary).] Use (a) and (b) to deduce that the composition

$$\theta_\rho : M_K(\rho) \xrightarrow{\theta} M_K(\rho^q) \longrightarrow M_K(\rho)$$

is completely continuous, hence nuclear. Its trace may be calculated with respect to any orthogonal R-basis $\{b_t \mid t \geq 1\}$ of $M_K(\rho)$ as follows: if $\theta_\rho(b_t) = \sum_{m=1}^{\infty} \lambda_{t,m} b_m$, then $\operatorname{tr}(\theta_\rho) = \sum_{m=1}^{\infty} \lambda_{m,m}$. Now the set $\{T^\alpha e_j \,;\, j = 1, \ldots, r \text{ and } \alpha \in \mathbb{N}_0^n\}$ is an orthogonal basis for any of the $M_K(\rho)$, hence $\operatorname{tr}(\theta_\rho)$ does not depend on ρ, and similarly for the $\operatorname{tr}(\theta_\rho^m)$ (for $m \geq 1$) and hence for $\det(1 - t\theta_\rho)$. Deduce that $\theta = \lim_\rho \theta_\rho : M \otimes_R K = \cup_\rho M_K(\rho) \to M \otimes_R K$ is nuclear, too, and $\det(1 - t\theta_\rho) = \det(1 - t\theta)$ for ρ sufficiently small.