Minerva School 2009, Jerusalem
Rigid Cohomology (EgK)

Let $R$ be a complete discrete valuation ring, $\pi \in R$ a uniformizer, $R/(\pi) = k = \mathbb{F}_q$ the field with $q$ elements, $K = \text{Frac}(R)$ of characteristic zero.

**Problem 1:** (Overconvergence) Recall that we defined

$$R < T_1, \ldots, T_n >^\dagger = \{ \sum c_\alpha T^n \in R[[T_1, \ldots, T_n]] : |c_\alpha|^\rho |\alpha|^{-\infty} 0 \text{ for some } \rho > 1 \},$$

$$K < T_1, \ldots, T_n >^\dagger = \{ \sum c_\alpha T^n \in K[[T_1, \ldots, T_n]] : |c_\alpha|^\rho |\alpha|^{-\infty} 0 \text{ for some } \rho > 1 \}.$$

(a) For $n = 1$, give full details of the proof of the Key-Lemma: $\frac{\partial}{\partial T} : K < T >^\dagger \rightarrow K < T >^\dagger$ is surjective.

(b) Let $A$ be a wcfg-$R$-algebra (i.e. a quotient of some $R < T_1, \ldots, T_n >^\dagger$), let $\hat{A}$ be its $\pi$-adic completion. Show: $\hat{A}$ is the subring of $\hat{A}$ consisting of all expressions $z = \sum_{j=0}^\infty P_j(x_1, \ldots, x_m) \in \hat{A}$ with $x_1, \ldots, x_m \in A$, with $m$-variate polynomials $P_j \in (\pi)^jR[X_1, \ldots, X_m]$ such that $\deg(P_j) \leq c(j + 1)$ for all $j$, for some constant $c = c(z)$.

(c) Can you find intermediate rings $K < T_1, \ldots, T_n >^\dagger \subseteq B \subseteq K < T_1, \ldots, T_n >$, characterized (similarly as $K < T_1, \ldots, T_n >^\dagger$) by some convergence condition? If $n = 1$: do you find one respected by $\frac{\partial}{\partial T}$ and on which $\frac{\partial}{\partial T}$ is surjective?

**Problem 2:** (de Rham cohomology of rigid spaces) Recall that for any $K$-scheme $\mathcal{X}$ of finite type there is an associated $K$-rigid space $\mathcal{X}_{\text{an}}$, its analytification. For any $K$-rigid space $X$, just as in algebraic geometry, one defines the de Rham complex $\Omega^\bullet_X = (\Omega^\bullet_X, d) = (\wedge^\bullet \Omega^1_X, d)$ on $X$ (e.g. if $X$ is an admissible open subset of $\text{Spec}(K[T_1, \ldots, T_n])_{\text{an}}$, then $\Omega^1_X$ is a free coherent $\mathcal{O}_X$-module sheaf of rank $n$ in the symbols $dT_1, \ldots, dT_n$). For $X$ smooth one puts $H^\bullet_{\text{dR}}(X) = H^\bullet(X, \Omega^\bullet_X)$, its de Rham cohomology. If $X$ is quasi-Stein then this means

$$H^\bullet_{\text{dR}}(X) = \frac{\ker(\Omega^\bullet_X(X) \rightarrow \Omega^{\bullet+1}_X(X))}{\text{im}(\Omega^{\bullet-1}_X(X) \rightarrow \Omega^\bullet_X(X))}.$$

Examples of quasi-Stein spaces are $X = \mathcal{X}_{\text{an}}$ for a smooth affine $K$-scheme $\mathcal{X}$ of finite type, or $X = X_{s,t}$ for some $0 \leq s < t \leq \infty$, the one dimensional open annulus of outer radius $t$, inner radius $s$, i.e. its $\hat{K}$-valued points ($\hat{K}$ the completion of an algebraic closure $\bar{K}$ of $K$) are

$$X_{s,t}(\hat{K}) = \{ y \in \hat{K} \mid s < |y| < t \}.$$

(a) Let $W = \text{Spec}(A)$ be a smooth affine $R$-scheme, let $\hat{W} = \hat{W} \otimes_R k$. Let $\hat{A}$ denote the $\pi$-adic completion of $A$ so that $\hat{A} \otimes_R K$ is a $K$-affinoid algebra. We may view $W = \text{Sp}(\hat{A} \otimes_R K)$ as an admissible open subset of $W^\text{an}_K$. Let $C$ denote the set of all strict neighbourhoods of $W$ in $W^\text{an}_K$, i.e. the set of admissible open subsets $X$ of $W^\text{an}_K$.
such that $\mathcal{W}_K^{an} = (\mathcal{W}_K^{an} - W) \cup X$ is an admissible open covering of $\mathcal{W}_K^{an}$. Show that there is a natural isomorphism

$$H^*_MW(\mathcal{W}) \cong \lim_{\longleftarrow} H^*_dR(X).$$

(b) Compute $H^*_dR(X_{s,t})$.

(c) Let $Y/R$ be a proper smooth connected $R$-scheme of relative dimension 1, let $Z \subset Y$ be an $R$-flat divisor such that $W = Y - Z$ is affine. Let $\mathcal{W} = \mathcal{W} \otimes_R k$. Show that there is a canonical isomorphism

$$H^*_MW(\mathcal{W}) \cong H^*_dR(\mathcal{W}_K^{an}).$$

[Hints: As $Y/R$ is smooth we find an isomorphism $\mathcal{W}_K^{an} - W \cong \coprod X_{0,1}$ (finite disjoint union of copies of $\coprod X_{0,1}$ and with $W$ defined as in (b)), and this in such a way that the sets $W \cup \coprod X_{s,1}$ for $0 < s < 1$ form a cofinal system in the set of all strict neighbourhoods of $W$ in $\mathcal{W}_K^{an}$. (Here $W \cup \coprod X_{s,1}$ is set theoretically an admissible open subset of $\mathcal{W}_K^{an}$, but the notation does not indicate an admissible covering of this open subset, of course.) To any such $s$ one has a long exact cohomology sequence associated with the admissible covering $\mathcal{W}_K^{an} = (\mathcal{W}_K^{an} - W) \cup (W \cup \coprod X_{s,1})$. Now use (a) and (b).]

Problem 3: Let $F_1, \ldots, F_r \in k[X_1, \ldots, X_n]$, put $d_i = \deg(F_i)$.

**Theorem 1:** (Chevalley-Warning) If $n > \sum_{i=1}^r d_i$ then the number of solutions (in $k^n$) to the system

$$F_1(x_1, \ldots, x_n) = \ldots = F_r(x_1, \ldots, x_n) = 0 \quad (\ast)$$

is a multiple of $p = \text{char}(k)$.

**Theorem 2:** (Ax-Katz) If $b$ denotes the smallest integer with

$$b \geq \frac{n - \sum d_i}{\max\{d_i\}}$$

then the number of solutions to the system $(\ast)$ is divisible by $q^b$.

Find a simple and elementary proof (without $p$-adic methods) for Theorem 1. Theorem 2 can be derived from the last theorem of our course (but this is not meant to be part of this present problem).
Problem 4: (Dwork’s trace formula) Let $B = B(X) = (a_{ij}(X))$ be an $r \times r$-matrix over $R < X_1^\pm, \ldots, X_n^\pm >$, i.e.

$$a_{ij}(X) \in R < X_1^\pm, \ldots, X_n^\pm > = \{ \sum_{u \in \mathbb{Z}^n} a_u X^u \mid a_u \in R, \lim_{|u| \to \infty} a_u = 0 \}.$$  

(Here $|u_1, \ldots, u_n| = |u_1| + \ldots + |u_n|$.) Alternatively we may write $B = B(X) = \sum_b b_u X^u$ with $b_u \in \text{Mat}(r \times r, R)$ such that $\lim_{|u| \to \infty} b_u = 0$. The $L$-function of $B$ on the torus $\mathbb{G}_m^n/k$ is

$$L(B, \mathbb{G}_m^n/k, t) = \prod_{\pi \in |\mathbb{G}_m^n|} \frac{1}{\det(I - t^{d(x)}B(x)^{q(q-1)\ldots(q^{q-1})-1}) \ldots B(x^q)B(x)}$$

where $|\mathbb{G}_m^n|$ denotes the set of closed points of $|\mathbb{G}_m^n|$, where $x \in (\overline{K}^\times)^n$ (in fact $x \in (W^{(k)})^n$) is the Teichmüller lifting of $\pi \in |\mathbb{G}_m^n|$, where $x^q = (x_1^q, \ldots, x_n^q)$ and where $d(x)$ denotes the degree of $x$ over $k = \mathbb{F}_q$. This infinite product is a well defined power series with coefficients in $R$ because there are only finitely many closed points of a given degree.

Remark: If $B$ is invertible in $\text{Mat}(r \times r, K \otimes R) < X_1^\pm, \ldots, X_n^\pm >$ then it defines (the Frobenius matrix of) an $F$-crystal on $\mathbb{G}_m^n/k$. The above $L$-function is then Katz’ $L$-function of this $F$-crystal on $\mathbb{G}_m^n/k$. If $B$ is invertible even in $\text{Mat}(r \times r, R < X_1^\pm, \ldots, X_n^\pm >)$ then the crystal on $\mathbb{G}_m^n/k$ is a unit-root $F$-crystal, hence corresponds by a well known equivalence of categories to a $p$-adic representation $\rho : \pi_1 \to GL_r(R)$ of the arithmetic fundamental group $\pi_1$ of $\mathbb{G}_m^n/k$, or equivalently: to an étale $p$-adic sheaf $\mathcal{F}$ on $\mathbb{G}_m^n/k$. In this case, the above $L$-function is the Artin $L$-function of $\rho$, or the Grothendieck $L$-function of $\mathcal{F}$.

(a) (Dwork’s trace formula, additive form) For $k \in \mathbb{N}$ consider the character sum

$$S_k(B) = \sum_{\pi \in |\mathbb{G}_m^n|} \text{tr}(B(x)^{q(q-1)\ldots(q^{q-1})-1}) \ldots B(x^q)B(x)$$

with $\text{tr}$ denoting the trace of a matrix. Writing $B = \sum_{u \in \mathbb{Z}^n} b_u X^u$, consider the infinite matrix $F_B = (b_{q^k-1} u, v \in \mathbb{Z}^n)$. Since $\lim_{|u| \to \infty} b_u = 0$, one checks immediately that the power $F_B^k$ is well defined for each $k \in \mathbb{N}$. Show

$$S_k(B) = (q^k - 1)^n \text{tr}(F_B^k).$$

(Suggestion: first try the case $k = 1$.)

(b) (Dwork’s trace formula, multiplicative form) Show

$$L(B, \mathbb{G}_m^n/k, t) = \exp\left(\sum_{k=1}^{\infty} \frac{t^k}{k} S_k(B)\right)$$

and

$$L(B, \mathbb{G}_m^n/k, t) = \prod_{i=0}^{n} \det(I - q^i t F_B^{-1})^{-1}(n).$$
PROBLEM 5: (Nuclear operators) Let $M$ be a $K$-vector space and $L \in \text{End}_K(M)$. An eigenvalue of $L$ is a $\lambda \in \overline{K}$ (here $\overline{K}$ denotes the algebraic closure of $K$) such that $\ker(g_\lambda(L)) \neq 0$ for the minimal polynomial $g_\lambda$ of $\lambda$. We say that $L$ is nuclear if

(i) for every eigenvalue $\lambda \neq 0$ there exists an $L$-stable $K$-vector space decomposition $M = A_\lambda \oplus B_\lambda$ with $B_\lambda = \bigcup_{n \geq 1} \ker(g_\lambda(L)^n)$ and $\dim_K(B_\lambda) < \infty$, and $g_\lambda(L)$ bijective on $A_\lambda$, and

(ii) the non-zero eigenvalues of $L$ form a finite set or a sequence converging to 0.

In this situation we define the trace of $L$ as $\text{tr}(L) = \sum_{\lambda \neq 0} \text{tr}(L|_{B_\lambda})$. Moreover, with $L$ also $L^s$ for $s \in \mathbb{N}$ is nuclear, and we put $\det(1 - tL|_{B_\lambda}) = \exp(-\sum_{s \geq 1} \frac{\text{tr}(L^s)}{s})$.

If $M, N$ are $K$-Banach spaces, we say that $L : M \rightarrow N$ is completely continuous if it is the uniform limit of linear maps finite rank. It is a classical result of Serre that any completely continuous endomorphism of a $K$-Banach space is nuclear.

Recall that an orthogonal (countable) basis of a $K$-Banach space $V$ is a subset $\{b_n\}_{n \in \mathbb{N}} \subset V$ such that each $x \in V$ can uniquely be written as $x = \sum_n \gamma_n b_n$ with $\gamma_n \in K$ such that $|\gamma_n|^{\frac{n}{\rho} \rightarrow \infty}$ and $|x| = \sup_n |\gamma_n b_n|$. Let $A$ be a wcfg-$R$-algebra and $F : A \rightarrow A$ a lift of the $q$-power Frobenius on $\overline{A} = A \otimes_R k$. Let $M$ be a finitely generated $A$-module. An $R$-linear map $\theta : M \rightarrow M$ is called a Dwork operator if $\theta(F(a)m) = a\theta(m)$ for all $a \in A$, all $m \in M$.

(a) For a sequence of continuous maps of $K$-Banach spaces $U \xrightarrow{f} V \xrightarrow{u} W \xrightarrow{g} X$, if $u$ is completely continuous, then so is $g \circ u \circ f$.

(b) Suppose $A = R < T_1, \ldots, T_n >$. For $\rho > 1$ with $\rho \in |\overline{K}|$ let

$$A_K(\rho) = \left\{ \sum_{\alpha} c_{\alpha} T^\alpha \in K[[T_1, \ldots, T_n]] \mid |c_{\alpha}|_{|\rho|} \rightarrow 0 \right\}.$$ 

This is a $K$-Banach space with respect to the norm $|\sum_{\alpha} c_{\alpha} T^\alpha| = \max|c_{\alpha}|_{|\rho|}$. Let $M$ be a finite free $A$-module with basis $e_1, \ldots, e_r$ and let $M_K(\rho) = \sum_{j=1}^r A_K(\rho) e_j$. This is then also a $K$-Banach space. Show that for $1 < \rho_1 < \rho_2$ the inclusion $M_K(\rho_1) \subset M_K(\rho_2)$ is completely continuous.

(c) For a wcfg-$R$-algebra $A$ and a finitely generated $A$-module $M$, any Dwork-operator $\theta : M \rightarrow M$ induces a nuclear map $\overline{\theta} : M \otimes_R K \rightarrow M \otimes_R K$.

Hints: We may suppose $A = R < T_1, \ldots, T_n >$. Then $M$ admits a finite free resolution $0 \rightarrow M_s \rightarrow \cdots \rightarrow M_0 \rightarrow M \rightarrow 0$ and on each $M_i$ one can construct a Dwork operator such that the obvious diagrams commute. This allows us to assume that $M$ is $A$-free. Fix an $A$-basis $e_1, \ldots, e_r$ of $M$ and let $M_K(\rho) = \sum_{j=1}^r A_K(\rho) e_j$. For sufficiently small $\rho$ one has $\theta(M_K(\rho)) \subset M_K(\rho^\delta)$.

[This is the central point: Dwork operators improve convergence. To estimate $\theta(M_K(\rho))$ observe that any element in $M_K(\rho)$ may be written as

$$\sum_{a \in \{q, \ldots, q\}^\delta} F(a_{\alpha,j}) t^\alpha e_j$$

with suitable $a_{\alpha,j} \in A_K(\rho^\delta)$ (decrease $\rho$ if necessary). The defining rule for being a Dwork operator then shows $\theta(M_K(\rho)) \subset \sum_{a \in \{q, \ldots, q\}^\delta} A_K(\rho^\delta) \theta(t^\alpha e_j)$ (decrease $\rho$ if
necessary).] Use (a) and (b) to deduce that the composition

$$\theta_\rho : M_K(\rho) \xrightarrow{\theta} M_K(\rho^q) \longrightarrow M_K(\rho)$$

is completely continuous, hence nuclear. Its trace may be calculated with respect to any orthogonal $R$-basis $\{b_t \mid t \geq 1\}$ of $M_K(\rho)$ as follows: if $\theta_\rho(b_t) = \sum_{m=1}^{\infty} \lambda_{t,m} b_m$, then $\text{tr}(\theta_\rho) = \sum_{m=1}^{\infty} \lambda_{m,m}$. Now the set $\{T^{\alpha}e_j : j = 1, \ldots, r \text{ and } \alpha \in \mathbb{N}_0^n\}$ is an orthogonal basis for any of the $M_K(\rho)$, hence $\text{tr}(\theta_\rho)$ does not depend on $\rho$, and similarly for the $\text{tr}(\theta^m_\rho)$ (for $m \geq 1$) and hence for $\det(1 - t\theta_\rho)$. Deduce that $\theta = \lim_\rho \theta_\rho : M \otimes_R K = \cup_\rho M_K(\rho) \rightarrow M \otimes_R K$ is nuclear, too, and $\det(1 - t\theta_\rho) = \det(1 - t\theta)$ for $\rho$ sufficiently small.