

p -adic families of modular forms II

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Leftovers from last time

Lemma

$$G_k^* = (1 - p^{k-1} V_p) G_k$$

Proof.

We have

$$\sum_{p|d|np} d^{k-1} = p^{k-1} \sum_{d|n} d^{k-1}$$

So equality holds in all coefficients except possibly the constant. But then $G_k^* - (1 - p^{k-1} V_p) G_k$ is a constant but has weight k hence is 0. □

$$\Rightarrow a_0(G_k^*) = \frac{\zeta^*(1-k)}{2}, \quad \zeta^*(1-k) := (1 - p^{k-1}) \zeta(1-k)$$

is the zeta value with Euler factor at p removed.

The theta operator and the weight 2 Eisenstein series

$$\theta f(q) := qdf/dq, Q = \frac{1}{12} - 2 \sum \sigma_1(n)q^n$$

Notice:

$$\theta\left(\sum a_n q^n\right) = \sum n a_n q^n$$

Lemma

Q is a weight 2 p-adic modular form.

Proof.

Q is Const · G₂ (but G₂ is not a classical modular form)

$$G_2^* = (1 - pV)G_2 \Rightarrow G_2 = \sum p^n V^n G_2^*$$



Coleman, Gouvea and Jochnowitz: Q is not overconvergent.

Definition

If f is a generalized eigenform for U_p with eigenvalue λ we call $v(\lambda)$ the slope of f .

In *Classical and overconvergent modular forms*, Inventiones Mathematicae 124, Coleman proved the following result

Theorem

If $f \in M_{k+2}^\dagger(N)$ has slope $< k + 1$ then f is in the image of $M_{k+2}(Np)$.

The key ingredient in the proof is the comparison result of Baldassarri and Chiarellotto between algebraic de Rham and rigid cohomology [*Algebraic versus rigid cohomology with logarithmic coefficients*, in Barsotti Symposium in Algebraic Geometry] and the theory of the θ operator.

modular forms inside de Rham cohomology

$$X = X_0(N)$$

inside $\mathcal{E} \xrightarrow{\pi} X$ we have $\tilde{C} := \pi^{-1}C$, where C denotes the divisor of cusps.

We have

$$\mathcal{H} := \mathbb{R}^1 \pi_* \Omega_{\mathcal{E}/X}^\bullet(\log C)$$

an algebraic vector bundle. Gauss-Manin connection

$$\nabla : \mathcal{H} \rightarrow \mathcal{H} \otimes \Omega_X^1(\log C)$$

de Rham filtration

$$\mathcal{H} = F^0 \supset F^1 = \omega$$

and

$$\mathcal{H}/F^0 \cong \omega^{-1}$$

Follows from Poincaré duality.

Direct sum decomposition

$$\mathcal{H} = \omega \cdot \omega \oplus \omega^{-1} \cdot \eta, \quad \omega := \frac{dx}{y}, \quad \eta = \frac{xdx}{y}$$

Explicit formulas for ∇ :

$$\nabla(\omega) = (\eta - Q\omega) \frac{dq}{q}$$

$$\nabla(\eta) = (Q\eta - (Q^2 - \theta(Q))\omega) \frac{dq}{q}$$

Exercise

- 1 Find formulas for the Gauss-Manin connection on an elliptic curve $y^2 = 4x^3 + a(q)x + b(q)$ for general a, b .
- 2 See how these compare with the ones listed above.

The Kodaira-Spencer map

$$\text{KS} : \omega^2 \rightarrow \Omega_X^1(\log C)$$

Defined by

$$\text{KS}(\alpha \otimes \beta) = \langle \alpha, \nabla \beta \rangle$$

Exercise

- 1 Prove that the de Rham cup product of ω and η is 1.
- 2 Prove that for the Tate curve

$$\text{KS}(\omega^2) = \frac{dq}{q}$$

symmetric powers

The induced connection on $\mathcal{H}_k := \text{Symm}^k \mathcal{H}$

$$\nabla\left(\sum g_i \omega^i \eta^{k-i}\right) = \sum f_i \omega^i \eta^{k-i} \frac{dq}{q}$$

with

$$f_i = (i+1)g_{i+1} + \theta(g_i) + (k-2i)g_i Q - (k+1-i)g_{i-1}(Q^2 - \theta(Q))$$

Here, g_i has weight $2i - k$, f_i has weight $2i + 2 - k$.

Corollary

The operator $g \mapsto \theta(g) - kQg$ maps modular forms of weight k to modular forms of weight $k + 2$.

Proof.

Take $g_k = g$, $g_i = 0$ for $i < k$.

$$f_k = \theta(g) - kQg$$



Try to compute the first cohomology, i.e., the kernel of ∇ . Easy to see: Can start with any g_0 , solve for other g_i to get $f_i = 0$ for $i < k$. Coleman finds the following miracle:

$$(-1)^k k! f_k = \theta^{k+1} g_0$$

Proposition

- 1 The map θ^{k+1} sends M_{-k}^\dagger to M_{k+2}^\dagger .
- 2 $\text{coker}(\nabla : \mathcal{H}_k(W^\dagger) \rightarrow \Omega^1(\log C) \otimes \mathcal{H}_k(W^\dagger))$ is isomorphic to $M_{k+2}^\dagger / \theta^{k+1} M_{-k}^\dagger$.

If g is p -adic modular of weight k then $\theta(g)$ is p -adic modular of weight $k + 2$ (but in general not overconvergent).

Small slope forms are not in the image of θ^{k+1}

Proposition

If $f \in M_{k+2}^\dagger$ with $U_p f = \lambda f$ and $v(\lambda) < k + 1$, then $f \notin \theta^{k+1} M_{-k}^\dagger$

Proof.

The condition on $f = \sum a_n q^n$ is $a_{pn} = \lambda a_n$. If $f = \theta^{k+1} \sum b_n q^n$ we have $a_n = n^{k+1} b_n$ and

$$\lambda n^{k+1} b_n = \lambda a_n = a_{pn} = (pn)^{k+1} b_{pn}$$

so

$$b_{pn} = \frac{\lambda}{p^{k+1}} b_n$$

$v(\lambda) < k + 1$ implies that $v(b_{np^m}) \rightarrow -\infty$ contradicting boundedness. □

$\theta \circ U_p = pU_p \circ \theta \Rightarrow$ generalization to generalized eigenforms.

Rigid and algebraic cohomology

$\text{coker}(\nabla : \mathcal{H}_k(W^\dagger) \rightarrow \Omega^1(\log C) \otimes \mathcal{H}_k(W^\dagger))$ is the rigid cohomology of $X \otimes \mathbb{Z}/p - SS$ with coefficients in the isocrystal induced by \mathcal{H}_k . (but with log singularities at the cusps. Baldassarri and Chiarellotto \Rightarrow

$$= H^1(X, \Omega_X^\bullet(\log C \cup \tilde{SS}) \otimes \mathcal{H}_k)$$

where \tilde{SS} is a lift of SS .

Let $S_{k+2}^0 \subset M_{k+2}^\dagger$ subspace of f with

- 0 at the cusps
- trivial residue at the supersingular annuli

Coleman shows:

$$S_{k+2}^0 / \theta^{k+1} M_{-k}^\dagger \cong H_{k,\text{par}} := H_{\text{par}}^1(Y, \mathcal{H}^k)$$

Shimura isomorphism: $H_{k,\text{par}} \cong S_k(N) \oplus \overline{S_k(N)}$.

In particular $\dim H_{k,\text{par}} = 2 \dim S_k(N)$.

$S_{k+2,\text{cl}}^0 = p$ -old part of $S_k(Np)$

$\dim S_{k+2,\text{cl}}^0 = 2 \dim S_k(N)$

Coleman shows: $S_{k+2,\text{cl}}^0 \subset S_{k+2}^0$

Last Proposition $\Rightarrow (S_{k+2,\text{cl}}^0)_\alpha :=$ slope α part $\hookrightarrow (H_{k,\text{par}})_\alpha$ for $\alpha < k + 1$.

Duality \Rightarrow also for $\alpha = k + 1$.

Dimension count $\Rightarrow (S_{k+2,\text{cl}}^0)_\alpha \cong (H_{k,\text{par}})_\alpha$

This factors via the injection $(S_{k+2}^0)_\alpha \rightarrow (H_{k,\text{par}})_\alpha$ so

$(S_{k+2,\text{cl}}^0)_\alpha = (S_{k+2}^0)_\alpha$.

For non-cusp forms need some more dimension counting.

Determinants of completely continuous operators

p -adic version of Fredholm theory developed by Serre, [*Endomorphismes completement continus des especes des Banach p -adiques*, Publ. Math IHES 12, 1962] following work by Dwork. Extended to “Banach modules” by Coleman.
 K - Complete non-archimedean valuation field, \mathcal{O}_K -valuation ring

Definition

A Banach space over K is a vector space V over K s.t.:

- $\|\alpha v\| = |\alpha| \|v\|$
- $\|v + w\| \leq \max(\|v\|, \|w\|)$

Assumption: Image of $\|\bullet\| \subset$ closure of image of $|\bullet|$.

Example

For a set I $c(I) := \{f : I \rightarrow K, \lim_{i \rightarrow \infty} |f(i)| = 0\}$ with sup norm.

Proposition

If K is discretely valued, any Banach space is of this form.

More generally, $c(J, V)$ for V a Banach space.

If V, W are Banach spaces, then the space $\mathcal{L}(V, W)$ of continuous linear maps from V to W is Banach w.r.t.

$$\|T\| = \sup \frac{\|Tv\|}{\|v\|}$$

Definition

The space of completely continuous operators $\mathcal{C}(V, W) \subset \mathcal{L}(V, W)$ is the closure of the space of operators with finite dimensional image.

Exercise

Suppose $W = c(J)$. $T \in \mathcal{L}(V, W)$ is given by a sequence of functionals $(v'_i)_{i \in I}$. Show that this induces an isometry

$$\mathcal{C}(V, W) \cong c(I, V')$$

The Fredholm determinant

Suppose $V = c(I)$ and $S \in \mathcal{C}(V, V)$. We define the *Fredholm determinant* of T

$$\det(I - tS) = \sum_{m=0}^{\infty} c_m t^m, \quad c_m \rightarrow 0$$

First step:

L be any ring. I a set

$\sigma : L^I \rightarrow L^I$ L -linear \rightarrow matrix (I_{ij}) .

Assume: $\sigma(L^I) \subset$ a finitely generated submodule $\Rightarrow \sigma(L^I) \subset L^{I_0}$,
 $I_0 \subset I$ finite.

Set

$$\det(I - t\sigma) = \det(I - t\sigma)|_{L^{I_0}} = \sum c_m t^m \text{ Independent of } I_0$$

(and independent of the presentation of the module L^I).

In terms of the matrix l_{ij}

$$c_m = (-1)^m \sum \text{all } m \text{ minors}$$

Second step:

Assume $\|S\| \leq 1$. Then S preserves the \mathcal{O}_K -module $V_0 := \{\|v\| \leq 1\}$.

For $r < 1$ let $\mathfrak{a} = \mathfrak{a}_r := \{|k| \leq r\}$ and

$$\sigma_r = S \pmod{\mathfrak{a}} : V_0/\mathfrak{a} \rightarrow V_0/\mathfrak{a}$$

$S \in \mathcal{C} \Rightarrow \sigma_r$ of the type above.

$$\det(1 - tS) = \lim_{r \rightarrow 0} \det(1 - t\sigma_r)$$

Step 3: modify if $\|S\| > 1$.

S is represented by a matrix (s_{ij}) . If $r_j =$ norm of column j then $r_j \rightarrow 0$. We see

$$c_m = (-1)^m \sum \text{all } m \text{ minors}$$

This makes sense because if r_j are ordered decreasingly

$$|m \text{ minor}| \leq r_1 \cdots r_m$$

In fact

$$|c_m| \leq r_1 \cdots r_m$$

so $c_m r^m \rightarrow 0$ for any r .

Thus

Proposition

$\det(1 - tS)$ converges everywhere.

Exercise

- 1 Show that $S \mapsto \det(I - tS)$ is continuous w.r.t. pointwise convergence of coefficients on the right.
- 2 Show that for any $t \in K$,

$$\det((1 - tS_1)(1 - tS_2)) = \det(1 - tS_1) \det(1 - tS_2)$$

The Fredholm resolvent

Definition

The Fredholm resolvent of $S \in \mathcal{C}(V, V)$ is the formal power series with coefficients in $\mathcal{C}(V, V)$

$$P(t, S) := \frac{\det(I - tS)}{I - tS} = \sum v_m t^m$$

(a formal adjoint operator).

Theorem

$P(t, S)$ converges everywhere.

Theorem

Let $H(t) = \det(1 - tS)$. Then for $a \in K$ $I - aS$ is invertible iff $H(a) \neq 0$.

Exercise

Prove this.

Theorem

If $a \in K$ is a zero of order h of $\det(I - tS)$ then there is a direct sum decomposition $V = N(a) + F(a)$ where

- 1 $\dim N(a) = h$ and $I - aS$ is nilpotent on $N(a)$.*
- 2 $I - aS$ is invertible on $F(a)$.*

Coleman generalizes the above spectral theory to operators on Banach modules over a commutative Banach algebra.

Let $p \geq 5$

$$E = 1 + \frac{2}{L_p(0, \mathbf{1})} \sum_{n=1}^{\infty} \left(\sum_{p \nmid d|n} \tau^{-1}(d) \right) q^n$$

$\tau =$ Teichmüller character \circ reduction modulo p .

E is a weight 1 modular form on $X_1(p)$ and its q expansion is $\equiv 1 \pmod{p}$. For $X_1(Np)$ it plays a similar role to E_{p-1} .

$M_k = \omega(W^\dagger)$, W^\dagger a strict neighborhood of the tube of the component of $X_1(Np) \otimes \mathbb{Z}/p$ containing the cusp at ∞ .

$$f \rightarrow f \cdot E^k, M_0 \xrightarrow{\sim} M_k$$

$U_{(k)} = U_p$ on M_k .

$E^{-k} U_{(k)}(fE^k) = U_{(0)}(e^k f)$ where $e = E/V_p E$.

Interpolated operators on M_0

$$U_s(f) := U_{(0)}(e^s f)$$

For s in some ball B in \mathbb{C}_p .

U_s give an operator for the $A(B)$ module $A(B \times W^\dagger)$

Fredholm theory $\Rightarrow H(t, s) \in A(\mathbb{C}_p \times B)$.

For $s = k$, $H(t, k) = \det(1 - tU_{(k)})$.

Zeros of $H(t, k)$ correspond to generalized eigenvalues of $U_{(p)}$.

Suppose f is an eigenform of weight k and $U_p f = \lambda f$. Then $H(\lambda, k) = 0$. For $s \in B'$ a small ball around k we extend $\tilde{\lambda}(s)$ and use Riesz theory to construct $N(\tilde{\lambda})$. Use Hecke operators to cut down a family of eigenforms passing through f .